

LECTURE 2 (12-02-04)

Exercise 1. Suppose we are given two vector bundles: $\pi_i : E_i \rightarrow B$.

We set

$$E_1 \oplus E_2 = \{(u_1, u_2) \in E_1 \times E_2; \pi_1(u_1) = \pi_2(u_2)\}$$

and define the projection map $\pi : E_1 \oplus E_2 \rightarrow B$ as

$$\pi(u_1, u_2) = \pi_1(u_1) = \pi_2(u_2).$$

We see that the fibers of this projection are the direct sum of the fibers of E_1 and E_2 . It remains to verify the local triviality property. First we remark that if we have a vector bundle $p : E \rightarrow B$ and a subspace $A \subset B$, then $p : p^{-1}(A) \rightarrow A$ defines a vector bundle over A , called the restriction of E over A . Now, if we are given two vector bundles $\pi_i : E_i \rightarrow B_i$, then $\pi_1 \times \pi_2 : E_1 \times E_2 \rightarrow B_1 \times B_2$ is a vector bundle with fibers $\pi_1^{-1}(b_1) \times \pi_2^{-1}(b_2)$; this is because if we have local trivializations $g_\alpha : \pi_1^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^n$ and $g_\beta : \pi_2^{-1}(U_\beta) \rightarrow U_\beta \times \mathbb{C}^m$ for E_1 and E_2 , and then $g_\alpha \times g_\beta$ is a local trivialization for $E_1 \times E_2$. Therefore if E_1 and E_2 have the same base space B , the restriction of the product $E_1 \times E_2$ over the diagonal $B = \{(b, b) \in B \times B\}$ is $E_1 \oplus E_2$.

Note that such vector bundles E_1, E_2 are isomorphic if there is a continuous map $h : E_1 \rightarrow E_2$ taking each fiber π_1^{-1} to the corresponding fiber π_2^{-1} by a linear isomorphism (such a map is automatically a homeomorphism - c.f. M. Karoubi "K-theory" I.2.7).

Finally, we show that if $E \simeq E'$ and $F \simeq F'$, then $(E \oplus F) \simeq (E' \oplus F')$. To see this, consider the following diagrams

$$\begin{array}{ccc}
 E & \xrightarrow{\varphi_1} & E' \\
 \searrow \pi_1 & & \swarrow \pi'_1 \\
 & B &
 \end{array}
 \qquad
 \begin{array}{ccc}
 F & \xrightarrow{\varphi_2} & F' \\
 \searrow \pi_2 & & \swarrow \pi'_2 \\
 & B &
 \end{array}$$

and define the map $\varphi_1 \times \varphi_2 : E \oplus F \rightarrow E' \oplus F'$ as $(\varphi_1 \times \varphi_2)(v, w) := (\varphi_1(v), \varphi_2(w))$ for $(v, w) \in E \oplus F$; this map is well defined because $\pi_1' \varphi_1(v) = \pi_2' \varphi_2(w)$ and $\pi_1(v) = \pi_2(w)$ yields that $(\varphi_1(v), \varphi_2(w)) \in E' \oplus F'$. The continuity of $\varphi_1 \times \varphi_2$ follows from the continuity of φ_1 and φ_2 . To see that this is in fact an isomorphism between vector bundles, it suffices to show that the map $(\varphi_1 \times \varphi_2)|_{\pi^{-1}(b)} : \pi^{-1}(b) \rightarrow \pi'^{-1}(b)$ is a linear isomorphism for every $b \in B$. But this is true since by assumption and by the definition of direct sum of vector bundles, we have that $\pi^{-1}(b) = \pi_1^{-1}(b) \oplus \pi_2^{-1}(b)$ and $\pi'^{-1}(b) = \pi_1'^{-1}(b) \oplus \pi_2'^{-1}(b)$ are isomorphic as vector spaces.

Exercise 2. For Exercises 2 and 3 we prove the following proposition.

Proposition. Given a map $\varphi : X \rightarrow Y$ and a vector bundle $\pi : E \rightarrow Y$, there exists a vector bundle $\pi' : E' \rightarrow X$ with a map $\varphi' : E' \rightarrow E$ taking the fiber of E' over each point $x \in X$ isomorphically onto the fiber of E over $\varphi(x)$, and such a vector bundle E' is unique up to isomorphism.

Proof. We define

$$E' = \varphi^*(E) = \{(v, x) \in E \times X \mid \pi(v) = \varphi(x)\}.$$

This subspace of $E \times X$ fits into the following commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\varphi'} & E \\ \downarrow \pi' & & \downarrow \pi \\ X & \xrightarrow{\varphi} & Y \end{array}$$

where $\varphi'(v, x) = v$ and $\pi'(v, x) = x$. If we let

$$\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in X\},$$

the graph of φ , then π' factors as the composition $E' \rightarrow \Gamma_\varphi \rightarrow X$. The first of these two maps is the restriction of the vector bundle $\mathbb{I} \times \pi : X \times E \rightarrow X \times Y$ over the graph Γ_φ , where $\mathbb{I} : X \rightarrow X$ is the "zero" vector bundle (that is, $\mathbb{I}(x) = x$), so it is a vector bundle, and the second map is a homeomorphism, so their composition $\pi' : E' \rightarrow X$ defines a vector bundle over X . The map φ' is continuous and obviously takes the fiber E' over x isomorphically onto the fiber of E over $\varphi(x)$.

For the uniqueness statement, we can construct an isomorphism from an arbitrary E' satisfying the conditions in the proposition above to the particular one just constructed by mapping $v' \in E'$ to the pair $(\pi(v'), \varphi'(v'))$. This map takes each fiber of E' to the corresponding fiber of $\varphi^*(E)$ by a vector space isomorphism, so it is an isomorphism of vector bundles.

Exercise 3. We have the following commutative diagrams:

$$\begin{array}{ccc} \varphi^*(E) & \xrightarrow{\varphi_1} & E \\ \downarrow \pi' & & \downarrow \pi \\ X & \xrightarrow{\varphi} & Y \end{array}$$

$$\begin{array}{ccc} \varphi^*(F) & \xrightarrow{\varphi_2} & F \\ \downarrow p' & & \downarrow p \\ X & \xrightarrow{\varphi} & Y \end{array}$$

(a) Since $\pi' : \varphi^*(E) \rightarrow X$ and $p' : \varphi^*(F) \rightarrow X$ are vector bundles, by Exercise 1 we have that $q' : \varphi^*(E) \oplus \varphi^*(F) \rightarrow X$, $(z_1, z_2) \mapsto \pi'(z_1) = p'(z_2)$ is a vector bundle.

(b) The map $\psi := (\varphi_1 \times \varphi_2) : \varphi^*(E) \oplus \varphi^*(F) \rightarrow E \oplus F$ defined by

$$\psi(z_1, z_2) := (\varphi_1(z_1), \varphi_2(z_2))$$

takes the fiber of $\varphi^*(E) \oplus \varphi^*(F)$ over each point $x \in X$ isomorphically onto the fiber of $E \oplus F$ over $\psi(x)$, because φ_1 (respectively, φ_2) takes the fibers of $\varphi^*(E)$ (respectively, $\varphi^*(F)$) over each point $x \in X$ isomorphically onto the fiber of E (respectively, F) over $\varphi(x)$ (by the previous exercise).

(c) Since the above diagrams are commutative, we have

$$(\varphi \circ q')(z_1, z_2) = (\varphi(q'(z_1, z_2))) = \varphi(\pi'(z_1)) = (\pi \circ \varphi_1)(z_1).$$

On the other hand

$$[q \circ (\varphi_1 \times \varphi_2)](z_1, z_2) = q(\varphi_1(z_1), \varphi_2(z_2)) = \pi(\varphi_1(z_1)),$$

where $q : E \oplus F \rightarrow Y$ is the projection. So the following diagram is commutative:

$$\begin{array}{ccc} \varphi^*(E) \oplus \varphi^*(F) & \xrightarrow{\psi} & E \oplus F \\ \downarrow q' & & \downarrow q \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Therefore by the uniqueness property of the pullback up to isomorphism (proved in the previous exercise) we conclude that

$$\varphi^*(E \oplus F) \cong \varphi^*(E) \oplus \varphi^*(F).$$

Exercise 4. Let $\pi : E \rightarrow \{pt\}$ be a vector bundle. By definition, the vector bundle $E = \pi^{-1}(\{pt\})$ is a vector space of dimension n and hence $[E]$ is the set of all n -dimensional vector spaces. Defining $V(\{pt\}) := \{[E] : E \text{ is vector bundle over } \{pt\}\}$ and $D : V(\{pt\}) \rightarrow \mathbb{N}$ by $D([E]) := \dim(E)$, we have that D is well defined and it is an isomorphism of semigroups. Therefore $V(\{pt\}) \cong \mathbb{N}$ and $K^0(\{pt\}) \cong \mathbb{Z}$.

Exercise 5. Since $[0, 1]$ is contractible, it is homotopically equivalent to $\{pt\}$, hence $K^*([0, 1]) \simeq K^*(\{pt\})$, where $*$ = 0, 1. We write $(0, 1] = [0, 1] \setminus \{0\}$ and apply excision: we obtain an exact sequence

$$0 \rightarrow K^0((0, 1]) \rightarrow K^0([0, 1]) \rightarrow K^0(\{0\}) \rightarrow K^1((0, 1]) \rightarrow 0.$$

The inclusion $\{0\} \hookrightarrow [0, 1]$ and the constant map $[0, 1] \rightarrow \{0\}$ are homotopy equivalences, hence $\{0\} \hookrightarrow [0, 1]$ induces an isomorphism in K -theory, and we see that $K^i((0, 1]) = 0$, $i = 0, 1$.

Next, we write $(0, 1) = (0, 1] \setminus \{1\}$ and by excision it easily follows that $K^0((0, 1)) = 0$ and $K^1((0, 1)) \simeq \mathbb{Z}$. We have that \mathbb{R} is homeomorphic to $(0, 1)$, so $K^*(\mathbb{R}) = K^*((0, 1))$. By Bott periodicity, $K^{even}(\mathbb{R}) = 0$ and $K^{odd}(\mathbb{R}) = \mathbb{Z}$. Now,

$$\begin{aligned} K^0(\mathbb{R}^{2n}) &= K^0(\mathbb{R} \times \mathbb{R}^{2n-1}) = K^{2n-1}(\mathbb{R}) = \mathbb{Z} \text{ and} \\ K^0(\mathbb{R}^{2n+1}) &= K^0(\mathbb{R} \times \mathbb{R}^{2n}) = K^{2n}(\mathbb{R}) = 0, \text{ hence} \\ K^1(\mathbb{R}^{2n}) &= K^0(\mathbb{R}^{2n} \times \mathbb{R}) = 0, \text{ and} \\ K^1(\mathbb{R}^{2n+1}) &= K^0(\mathbb{R}^{2n+1} \times \mathbb{R}) = \mathbb{Z}. \end{aligned}$$

We compute now the K -groups of the sphere S^k . Since $S^k \times (0, \infty)$ is homeomorphic to $\mathbb{R}^{k+1} \setminus \{0\}$, we have that

$$\begin{aligned} K^0(\mathbb{R}^{k+1} \setminus \{0\}) &= K^0(S^k \times \mathbb{R}) = K^1(S^k) \text{ and} \\ K^1(\mathbb{R}^{k+1} \setminus \{0\}) &= K^1(S^k \times \mathbb{R}) = K^0(S^k \times \mathbb{R}^2) = K^2(S^k) = K^0(S^k). \end{aligned}$$

For $k = 2n$, excision gives $K^0(\mathbb{R}^{2n+1} \setminus \{0\}) = 0$ and $K^1(\mathbb{R}^{2n+1} \setminus \{0\})$ fits into an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow K^1(\mathbb{R}^{2n+1} \setminus \{0\}) \rightarrow \mathbb{Z} \rightarrow 0.$$

Since \mathbb{Z} is a projective \mathbb{Z} -module, this sequence splits and we obtain

$$K^1(\mathbb{R}^{2n+1} \setminus \{0\}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

For $k = 2n + 1$, excision gives a long exact sequence

$$0 \rightarrow K^0(\mathbb{R}^{2n+2} \setminus \{0\}) \rightarrow K^0(\mathbb{R}^{2n+2}) \rightarrow K^0(\{0\}) \rightarrow K^1(\mathbb{R}^{2n+2} \setminus \{0\}) \rightarrow 0.$$

By definition of compactly supported K -theory, $K^0(\mathbb{R}^{2n+2})$ is the kernel of the map $K^0(S^{2n+2}) \rightarrow K^0(\infty)$, where ∞ is the compactification point that turns \mathbb{R}^{2n+2} into S^{2n+2} . Hence the map $K^0(\mathbb{R}^{2n+2}) \rightarrow K^0(\infty)$ is the zero map. Finally, ∞ and "point", which is the point zero in \mathbb{R}^{2n+2} , are path connected so $K^0(\mathbb{R}^{2n+2}) \rightarrow K^0(\{0\})$ is the zero map as well. This finishes the computation.

LECTURE 3 (26-02-04)

Exercise 1. Let $\varphi : A \rightarrow B(H)$ be an injective representation of a C^* -algebra A and $\varphi_n : M_n(A) \rightarrow B(H^n)$ be the injective representation of $M_n(A)$, induced by φ in the obvious way. By definition

$$\|a\|_{M_n(A)} := \|\varphi_n(a)\|_{B(H^n)} = \sup_{\|v\|=1} \|\varphi_n(a)v\|_{H^n}.$$

For simplicity, since φ is isometric, we can assume $A \subseteq B(H)$ and φ, φ_n given by inclusion. Now let $a = (a_{ij}) \in M_n(A)$, $v = (v_i) \in H^n$ with $\|v\| = (\sum_{i=1}^n \|v_i\|^2)^{\frac{1}{2}} \leq 1$. We have:

$$\begin{aligned} \|av\| &= \left(\sum_{i=1}^n \left\| \sum_{j=1}^n a_{ij}v_j \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \|a_{ij}\| \|v_j\| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \|a_{ij}\|. \end{aligned}$$

For the other inequality, suppose that $v \in H$ with $\|v\| \leq 1$. We denote by $E_j(v)$ the element in H^n whose j 'th component is v and whose other components are zero. For $i, j = 1, \dots, n$, we have:

$$\|a_{ij}v\| \leq \left(\sum_{i=1}^n \|a_{ij}v\| \right)^{\frac{1}{2}} = \|aE_j(v)\| \leq \|a\|.$$

Therefore, $\|a_{ij}\| \leq \|a\|$ for $1 \leq i, j \leq n$, $i, j \in \mathbb{N}$.

Exercise 2. Let

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a one-dimensional projection (non-trivial) in $M_2(\mathbb{C})$. Since $P = P^*$, a, d are real numbers and $b = \bar{c}$. Also, the condition $P^2 = P$ implies that $0, 1$ are the roots of the polynomial $\lambda^2 - (a + d)\lambda + (ad - bc)$.

Therefore $ad - bc = 0$ and $a + d = 1$. By a simple computation, we obtain $|b|^2 + (a - \frac{1}{2})^2 = \frac{1}{4}$. If we put $b = \frac{1}{2}(y + iz)$ and $a = \frac{1}{2}(1 + x)$, with $x, y, z \in \mathbb{R}$ then

$$P = p(x, y, z) := \frac{1}{2} \begin{pmatrix} 1 + x & y + iz \\ y - iz & 1 - x \end{pmatrix}$$

with $x^2 + y^2 + z^2 = 1$. Conversely, it is easy to check that every $p(x, y, z)$ defined as above is a projection.

- Exercise 3.** (i) Since $p = pp^* = p^*p$, we have $p \overset{\text{MvN}}{\sim} p$.
(ii) Let $p \overset{\text{MvN}}{\sim} q$. By definition there exists $x \in A$ such that $p = xx^*$ and $q = x^*x$. We have $p = (x^*)^*x^*$ and $q = x^*(x^*)^*$ and hence $q \overset{\text{MvN}}{\sim} p$.
(iii) Suppose that $p \overset{\text{MvN}}{\sim} q$ and $q \overset{\text{MvN}}{\sim} r$. There exist $x, y \in A$ such that $p = xx^*$, $q = x^*x$ and $q = yy^*$, $r = y^*y$. If we put $z = xy$ then

$$zz^* = xyy^*x = xqx = x^*xx^*x = p^2 = p$$

$$z^*z = y^*x^*xy = y^*qy = y^*yy^*y = r^2 = r.$$

Therefore, $p \overset{\text{MvN}}{\sim} r$.

Exercise 4 (a). We write two different solutions for (a).

1. Since every C^* -algebra is isomorphic, and therefore isometric, to a closed $*$ -subalgebra of $B(H)$ for some Hilbert space H , we can assume that p, q are two projection in $B(H)$. Let $H_0 = (p(H) + q(H))^\perp$, $H_1 = p(H) \cap q(H)$, $H_2 = p(H) \cap (H_1)^\perp$ and $H_3 = q(H) \cap (H_1)^\perp$. It is easy to see $p(H) + q(H) = H_1 + H_2 + H_3$ and since the H_i 's are mutually orthogonal and closed, $p(H) + q(H)$ is a closed subspace of H . Therefore, we can write

$$H = H_0 \oplus (pH + qH).$$

For any $w \in H$, there exist $w_0 \in H_0$, $w_1 \in pH + qH$ such that $w = w_0 + w_1$. Noting that $pH + qH = (p - q)H$, we have:

$$\|(p - q)(w)\| = \|(p - q)(w_1)\| = \|w_1\| \leq \|w\|$$

and therefore $\|p - q\| \leq 1$.

2. We may assume that A has a unit, if not we pass to the unitization. Since $p - q$ is self adjoint, there is a state ω on A such that $|\omega(p - q)| = \|p - q\|$. Since p, q are projections in A , hence positive elements, we have $0 \leq \omega(p) \leq \|\omega\|\|p\| = 1$, $0 \leq \omega(q) \leq \|\omega\|\|q\| = 1$ hence $\|p - q\| = |\omega(p - q)| = |\omega(p) - \omega(q)| \leq 1$.

(b) We have $p \perp q$ iff $p + q$ is a projection, so if $p \neq 0$ or $q \neq 0$, $\|p + q\| = 1$. Now we can write $\|p - q\|^2 = \|(p - q)^2\| = \|p + q\| = 1$.

(c) We give two proofs of (c). Let A_1 be the unitization of A and p, q be projections in A .

1. Consider the path

$$a_t := tp + (1 - t)q \in A, \quad t \in [0, 1].$$

This is a continuous path of self-adjoint elements, with $a_0 = p$, $a_1 = q$. We use continuous functional calculus in A_1 to construct a path of projections (in A). First we show that for each a_t , $\sigma(a_t)$ is disconnected. For this we prove the following:

Lemma 1. Let A be unital, $a \in A$ be self-adjoint, $p \in A$ a projection, and set $\delta := \|a - p\|$. Then $\sigma(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$.

Proof. Let $s \in \mathbb{R}$ be such that $\text{dist}(s, \{0, 1\}) > \delta$. Since $\sigma(p) \subset \{0, 1\}$, $p - s1$ is invertible. Now,

$$\|(a - s1)(p - s1)^{-1} - 1\| \leq \|(p - s1)^{-1}\|\|a - p\| < 1,$$

since $\|(p - s1)^{-1}\| = \max\{|s|^{-1}, |1 - s|^{-1}\} < 1/\delta$. Therefore, $(a - s1)(p - s1)^{-1}$, and also $a - s1$, is invertible, which proves our claim.

Since $\delta := \min\{\|a_t - p\|, \|a_t - q\|\} = \min\{t, (1 - t)\}\|p - q\| < 1/2$, we have that for all $t \in [0, 1]$, $\sigma(a_t) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$ is disconnected. Let $f : [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$ be such that $f = 0$ on $[-\delta, \delta]$ and $f = 1$ on $[1 - \delta, 1 + \delta]$. Then f is continuous, $f(a_t) \in A$ (since $a_t \in A$), $f(a_t)$ is a projection (since $f = \bar{f} = f^2$) with $f(a_0) = f(p) = p$, $f(a_1) = f(q) = q$ and the path $[0, 1] \rightarrow A$, $t \mapsto f(a_t)$ is continuous (this is seen approximating f by polynomials). We conclude that $p \overset{h}{\sim} q$.

2. First we prove the following lemma.

Lemma 2. If there exists an invertible element $z \in A_1$ such that $q = zpz^{-1}$, then $p \overset{u}{\sim} q$.

Proof. Since z is invertible so is z^*z , and we can consider the self-adjoint element $a = (z^*z)^{-1}$. By Theorem 8.1.3 (Lecture Notes - Part I) there is an isomorphism $\Phi : C(\sigma(a)) \rightarrow C^*(a, 1)$. Since $\sigma(a) \subseteq \mathbb{R}^+$ and $f : \sigma(a) \rightarrow \mathbb{C}$ defined by $f(t) = t^{1/2}$ is continuous, there exists a unique element of $C^*(a, 1)$ denoted by $|z|^{-1} := \Phi(f)$ such that

$$(|z|^{-1})^2 = (z^*z)^{-1}.$$

Define an unitary $u := z|z|^{-1}$, and note that p commutes with $|z|^{-1}$. Indeed, we have $zp = qz$ and $pz^* = z^*q$, and hence $pz^*z = z^*qz = z^*zp$. On the other hand we know that $\Phi(f) = |z|^{-1}$ is the limit of a sequence of polynomials in $(z^*z)^{-1}$ and since p commutes with z^*z , hence also with $(z^*z)^{-1}$, it commutes with $|z|^{-1}$. We obtain now

$$upu^* = z|z|^{-1}p|z|^{-1}z^* = zp|z|^{-2}z^* = qz|z|^{-2}z^* = q$$

So we proved our claim.

Now, define two selfadjoint elements in A_1 by $v_p := 2p-1$, $v_q := 2q-1$ and let $z_q := v_q v_p + 1$. Since $\|p - q\| < 1$ we have

$$\begin{aligned} \|z_q - 2\| &= \|v_q v_p - 1\| = \|v_q(v_p - v_q)\| \\ &\leq \|v_q\| \|v_p - v_q\| = \|v_p - v_q\| = 2\|p - q\| < 2 \end{aligned}$$

So by Lemma 3.4 (Lecture Notes - Part I) z_q is invertible in A_1 . On the other hand we can write

$$qz_q = q(2q - 1)(2p - 1) + q = 2qp = (2q - 1)(2p - 1)p + p = z_q p.$$

So $q = z_q p z_q^{-1}$, hence by the previous lemma we have that $q = u_q p u_q^*$ with $u_q := z_q |z_q|^{-1}$. Next we show that z_q , hence u_q , is homotopic to the identity, from which it will follow that $p \stackrel{h}{\sim} q$.

At first we define the following path:

$$\begin{aligned} \alpha : [0, 1] &\longrightarrow GL(A_1) \\ \alpha(t) &:= tz_q + 2 - 2t, \end{aligned}$$

which is homotopy leading from 2 to z_q . As $\|z_{q_1} - z_{q_2}\| = \|v_{q_1} v_p - v_{q_2} v_p\| \leq \|v_{q_1} - v_{q_2}\| \|v_p\| = 2\|q_1 - q_2\|$, we see that the map $q \mapsto z_q$ is continuous and since the map $\varphi : GL(A_1) \longrightarrow \mathcal{U}(A_1)$ defined by $\varphi(z_q) = z_q |z_q|^{-1}$ is continuous so the map $q \mapsto u_q$ is continuous on the set $\{q \in A : q = q^* = q^2, \|p - q\| < 1\}$. Now we can define the following path

$$\begin{aligned} \beta : [0, 1] &\longrightarrow \mathcal{U}(A_1) \\ \beta(t) &:= [(\varphi \circ \alpha)(t)] p [(\varphi \circ \alpha)(t)]^*, \end{aligned}$$

which is a homotopy leading from p to q .

Note. We have proved in particular that

$$\|p - q\| < 1 \Rightarrow p \overset{u}{\sim} q \text{ in } \mathcal{U}_0(A_1) \Rightarrow p \overset{h}{\sim} q,$$

where $\mathcal{U}_0(A_1) \subset \mathcal{U}(A_1)$ is the set of unitaries homotopic to 1.

Exercise 5. Suppose $p \overset{u}{\sim} q$ and let $p = uqu^*$ with u unitary. Put $v = qu^*$ and $w = (1 - q)u^*$; then $p = v^*v$, $q = vv^*$ and $1 - p = w^*w$, $1 - q = ww^*$, that is, $p \overset{MvN}{\sim} q$, $1 - p \overset{MvN}{\sim} 1 - q$.

Conversely, let $p = v^*v$, $q = vv^*$, $1 - p = w^*w$, $1 - q = ww^*$. We have $v - vv^*v = (1 - q)v$ and $((1 - q)v)^*(1 - q)v = 0$, hence $v = vv^*v$, so

$$v = qv = vp \text{ and } v^* = v^*q = pv^*.$$

Similarly one obtains $w = (1 - q)w = w(1 - p)$ and $w^* = w^*(1 - q) = (1 - p)w^*$. We have $v^*w = v^*q(1 - q)w = 0$ and $vw^* = vp(1 - p)w^* = 0$, hence $u := v^* + w^*$ is unitary and it follows that $p = uqu^*$.

LECTURE 4 (04-03-04)

Exercise 1. Sufficiency: we can find a unitary element $u \in M_\infty(A)$ such that $q = upu^*$. If $u \in M_N(A)$, left multiplication by u induces an isomorphism from pA^N to qA^N .

Necessity: Let $\alpha : pA^n \rightarrow qA^m$ be an isomorphism. Since $A^n = pA^n \oplus (1-p)A^n$, we can extend α to a homomorphism $A^n \rightarrow A^m$ by taking $\alpha = 0$ on $(1-p)A^n$ and by viewing the image qA^m as embedded in A^m . Similarly, we extend α^{-1} to a homomorphism $\beta : A^m \rightarrow A^n$ which is 0 on $(1-q)A^m$. Then α is given by left multiplication by a $n \times m$ matrix u and β is given by left multiplication by a $m \times n$ matrix v . We have the relations $uv = p$, $vu = q$, $u = pu = uq$, $v = qv = vp$. Put $N = n + m$ and notice that

$$\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

thus $\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix}$ is invertible, and we have also that

$$\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \sim \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, Lemma 2 in Exercise 3.4.(c) applied to the C^* -algebra $M_N(A)$ finishes the proof.

Exercise 2. To see that $p \oplus q \sim q \oplus p$, set

$$u = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$

Then $p \oplus q = u^*u$ and $q \oplus p = uu^*$.

Exercise 3. By definition, $[p] + [q] = [p \oplus q]$. If we put

$$u = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix},$$

then

$$uu^* = \begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix}, \quad u^*u = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Therefore, $p + q \sim p \oplus q$, and $[p] + [q] = [p + q]$.

Exercise 4. Let p and q be two projections in $M_\infty(A)$ such that $[p] = [q]$ in $K_0(A)$. We may assume that there exists $n \in \mathbb{N}$ such that $p, q \in M_n(A)$. By definition of the equivalence relation in $K_0(A)$, there exists a projection r such that $p \oplus r \sim q \oplus r$. Therefore, $p \oplus r \oplus (1-r) \sim q \oplus r \oplus (1-r)$. Since $r \perp (1-r)$, $r \oplus (1-r) \sim 1_m$ and hence $p \oplus 1_m \sim q \oplus 1_m$.

Exercise 5. Let $V(A) = P_\infty(A)/\sim$ and denote by $[p]$ the equivalence class of an element $p \in P_\infty(A)$. Let $\mathbb{Z}[V(A)]$ be the free abelian group on the set $V(A)$ and let M be the subgroup generated by $[p] + [q] - [p \oplus q]$, where $p, q \in P_\infty(A)$. Definition (a) gives $K_0(A)$ as $\mathbb{Z}[V(A)]/M$ and that is the object we have to consider.

Let $i : V(A) \rightarrow \mathbb{Z}[V(A)]$ be the inclusion and $\pi : \mathbb{Z}[V(A)] \rightarrow \mathbb{Z}[V(A)]/M$ be the projection. Note that πi is additive. We show that $(\mathbb{Z}[V(A)]/M, \pi i)$ satisfies the universal property of the Grothendieck group construction on the abelian semigroup $V(A)$.

Let G be an abelian group and let $f : V(A) \rightarrow G$ be additive. By the universal property of the free abelian group $\mathbb{Z}[V(A)]$, there is a

unique group homomorphism $g : \mathbb{Z}[V(A)] \rightarrow G$ such that $gi = f$. Since $M = \text{Ker}(\pi) \subseteq \text{Ker}(g)$ trivially, there is a unique homomorphism $h : \mathbb{Z}[V(A)]/M \rightarrow G$ such that $h(\pi) = g$; then $h(\pi i) = f$ and one easily checks that h is unique with respect to this property.

LECTURE 5 (11-03-04)

Exercise 1. (a) We first make some general remarks. Since $\pi s = \pi$, $[p] - [s(p)]$ is indeed an element of $K_0(A)$. For $(a, z) \in \dot{A}$, we have $s(a, z) = z1_{\dot{A}}$. If $p, q \in P_\infty(A)$ are equivalent projections, we claim that $s(p)$ and $s(q)$ are equivalent too: indeed, let p have size m , q size n and let $u \in M_{n,m}(\dot{A})$ be such that $p = u^*u$, $q = uu^*$. Denote by u_s the matrix obtained by applying s to each entry of u ; then $s(p) = u_s^*u_s$ and $s(q) = u_s u_s^*$.

Suppose that $p \oplus 1_k \sim q \oplus 1_l$, for some k, l . Then $[p \oplus 1_k] = [q \oplus 1_l]$ and by the above considerations we have $[s(p \oplus 1_k)] = [s(q \oplus 1_l)]$. Since $[p \oplus 1_k] - [s(p \oplus 1_k)] = [p] - [s(p)]$ and, similarly, $[q \oplus 1_l] - [s(q \oplus 1_l)] = [q] - [s(q)]$, the conclusion follows.

Conversely, if $[p] - [s(p)] = [q] - [s(q)]$, then $[p \oplus s(q)] = [q \oplus s(p)]$ and by Exercise 4.4 there is $k \in \mathbb{N}$ such that $p \oplus s(q) \oplus 1_k \sim q \oplus s(p) \oplus 1_k$. Since p is a projection, $s(p)$ can be identified with a projection in $M_m(\mathbb{C})$, hence there is $l \in \mathbb{N}$ such that $s(p) \sim 1_l$. A similar result holds for q , and we are done.

(b) Suppose $[p] - [s(p)] = 0$. This means $[p] = [s(p)]$ in $K_0(\dot{A})$, so by Exercise 4.4, there is $k \in \mathbb{N}$ such that $p \oplus 1_k \sim s(p) \oplus 1_k$. Conversely, suppose that $p \oplus 1_k \sim s(p) \oplus 1_k$ for some k . Therefore, $[p] + [1_k] = [s(p)] + [1_k]$ which implies $[p] - [s(p)] = 0$.

(c) By the definition of $\dot{\varphi}_*$, we have $\dot{\varphi}_*([p] - [s(p)]) = [\dot{\varphi}_\infty(p)] - [\dot{\varphi}_\infty(s(p))]$. On the other hand, $(s \circ \dot{\varphi})(a, z) = s(\varphi(a), z) = (0, z)$ and $\dot{\varphi} \circ s(a, z) = \dot{\varphi}(0, z) = (0, z)$, hence $s \circ \dot{\varphi} = \dot{\varphi} \circ s$. So for some $n \in \mathbb{N}$, we have

$$\dot{\varphi}_\infty(s(p)) = \dot{\varphi}_n(s_n(p)) = (\dot{\varphi} \circ s)_n(p)$$

$$s(\dot{\varphi}_\infty(p)) = s_m(\dot{\varphi}_n)(p) = (s \circ \dot{\varphi})_n(p)$$

and the equality follows.

Exercise 2. Let H be a Hilbert space. The *trace class operators* is the class

$$L^1(H) := \{u \in B(H) : \|u\|_1 := \sum_{e \in \mathcal{B}} |\langle ue, e \rangle| < \infty\},$$

where \mathcal{B} is a basis of H . For any $u \in L^1(H)$ we define

$$\text{Tr}(u) := \sum_{e \in \mathcal{B}} \langle ue, e \rangle.$$

We have that this sum is independent of the choice of the basis, and that $L^1(H)$ is a normed algebra under the norm $\|\cdot\|_1$ with $L^1(H) \subseteq B_0(H)$. We show that every finite rank operator is in $L^1(H)$: let $u \in B(H)$ be a finite rank operator and let $\{e_1, e_2, \dots, e_n\}$ be a basis for $u(H)$. We can extend it to a basis for H , namely \mathcal{E} . We have then:

$$\begin{aligned} \text{Tr}(u) &= \sum_{e \in \mathcal{E}} \langle ue, e \rangle = \sum_{i=1}^n \langle ue_i, e_i \rangle \\ &= \sum_{i=1}^n \langle e_i, e_i \rangle = n = \dim(u(H)), \end{aligned}$$

that is, $u \in L^1(H)$ and $\text{Tr}(u) = \dim(u(H))$.

Now let p be a projection in $B_0(H)$; then $p(H)$ is closed, hence complete. Since

$$\text{id}_{p(H)} = p|_{p(H)} : p(H) \longrightarrow p(H)$$

is compact, because p is compact, we have that $\dim(p(H)) < \infty$, that is, p has finite rank, and necessity follows.

Since every finite rank operator is in $B_0(H)$, the converse is trivial.

Exercise 3. We show first that:

Lemma. If $p, q \in P_\infty(B_0(H))$, then $p \sim q$ in $P_\infty \dot{B}_0(H)$ if and only if $\dim(p(H)) = \dim(q(H))$.

Proof. Let $\dim(p(H)) = \dim(q(H)) < \infty$, by Exercise 2, and $\mathcal{B}_1 = \{e_1, e_2, \dots, e_n\}$ and $\mathcal{B}_2 = \{e'_1, e'_2, \dots, e'_n\}$ be two bases for $p(H)$ and $q(H)$, respectively. Suppose that $\{e_1, e_2, \dots, e_n\} \cup \{e_\alpha\}_{\alpha \in \Gamma}$ and $\{e'_1, e'_2, \dots, e'_n\} \cup \{e'_\alpha\}_{\alpha \in \Gamma}$ are two bases for H , which extend \mathcal{B}_1 and \mathcal{B}_2 , respectively. We define $u : H \rightarrow H$ by $u(e_i) = e'_i$ for $i = 1, 2, \dots, n$ and $u(e_\alpha) = e'_\alpha$ for $\alpha \in \Gamma$. Therefore $u \in B(H)$ is unitary and $q = upu^*$; writing $v := up$, we have then that $v \in B_0(H)$ and $q = vv^*$, $p = v^*v$, that is $p \sim q$. The converse is trivial.

It follows that one has an inclusion

$$\mathbb{N} \longrightarrow V(B_0(H)), \quad n \longmapsto [p],$$

where p is a projection in $B_0(H)$ with $\dim(p(H)) = n$.

Now let $p \in P_\infty(\dot{B}_0(H))$, and write $p = p_0 + z$, where $p_0 \in M_\infty(B_0(H))$, $z \in M_\infty(\mathbb{C})$. From $p^2 = p$, we get that $z \in P_\infty(\mathbb{C})$ and hence, if z is nonzero, $z \sim \oplus^n 1_H$, for some $n \in \mathbb{N}$. If $z = 0$ then $p = p_0 \in P_\infty(B_0(H))$; we can therefore assume that $p = p_0 + \oplus^n 1_H$, and we see that p_0 has to satisfy

$$p_0^2 + 2p_0 = p_0 \Leftrightarrow p_0^2 = -p_0,$$

that is, $-p_0 \in P_\infty(B_0(H))$. Since a projection p with $\dim(p(H)) < \infty$ and $\oplus^n 1_H$ are not even equivalent in $P_\infty(B(H))$, they are not equivalent in $P_\infty(\dot{B}_0(H))$ and the same stands for p and $\oplus^n 1_H - q$, with q a projection such that $\dim(q(H)) < \infty$. We conclude that projections

in $P_\infty(B_0(H))$ are always equivalent either to $p \in P_\infty(B_0(H))$ or to $\oplus^n 1_H - q$, with $q \in P_\infty(B_0(H))$.

Now for $n > 1$, $\oplus^n 1_H$ and 1_H are not equivalent and neither is $\oplus^n 1_H$ equivalent to $\oplus^m 1_H$, $n \neq m$. For this, if $\oplus^n 1_H$ is equivalent to 1_H , there exists a partial isometry v such that $1_h = v^*v$ and $\oplus^n 1_H = vv^*$. So v is an isometry and hence $n = 1$. In particular, $\oplus^n 1_H - q \sim \oplus^m 1_H - q$ iff $n = m$. Also, the same reasoning as in the lemma above gives that $\oplus^n 1_H - q \sim \oplus^n 1_H - p$ iff $\dim(p(H)) = \dim(q(H))$. In general, one has that

$$\oplus^n 1_H - q \sim \oplus^m 1_H - p \text{ iff } m - \dim(p(H)) = n - \dim(q(H))$$

Thus one obtains a second inclusion

$$\mathbb{Z} \longrightarrow V(B_0(H)),$$

so that $V(B_0(H)) \cong \mathbb{N} \oplus \mathbb{Z}$, and hence $K_0(B_0(H)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

By definition of K_0 for non-unital C*-algebras and using the fact that $\pi_*([p]) = 0$ and $\pi_*(\oplus^n 1_H - q) = n$, for $p, q \in P_\infty(B_0(H))$, we obtain finally that $K_0(B_0(H)) := \ker(\pi_*) \cong \mathbb{Z}$.

LECTURE 6 (18-03-04)

Exercise 1. For this exercise we need a notation and a lemma:

Notation. Suppose that x, y are elements of a Hilbert space H . We define the operator $x \otimes y$ on H by

$$(x \otimes y)(z) = \langle z, y \rangle x.$$

Clearly, $\|x \otimes y\| = \|x\| \|y\|$. The rank of $x \otimes y$ is one if x and y are nonzero. If $u \in B(H)$, then

$$u(x \otimes y) = u(x) \otimes y$$

$$(x \otimes y)u = x \otimes u^*(y).$$

Lemma. Suppose that H is a Hilbert space, $u \in B_{\text{oo}}(H)$ and $\mathcal{B} = \{e_\alpha : \alpha \in \Gamma\}$ is an orthogonal basis for H . Then $u(e_\alpha) = 0$ for all α but finitely many α .

Proof. The real and imaginary part ($\frac{u+u^*}{2}$ and $\frac{u-u^*}{2i}$, respectively) of u are in $B_{\text{oo}}(H)$, since $B_{\text{oo}}(H)$ is self adjoint, so we may assume that u is hermitian. Now $u = u^+ - u^-$, by polar decomposition $|u| \in B_{\text{oo}}(H)$, so u^+ and u^- belong to $B_{\text{oo}}(H)$. Hence, we may assume that $u \geq 0$. The range $u(H)$ is finite dimensional, and therefore it is Hilbert space with an orthogonal basis e'_1, e'_2, \dots, e'_n . Let $p = \sum_{i=1}^n e'_i \otimes e'_i$, so p is the projection of H onto $u(H)$. Then $u = pu = u^{\frac{1}{2}} p u^{\frac{1}{2}}$ and hence $u = \sum_{i=1}^n x_i \otimes x_i$, where $x_i = u^{\frac{1}{2}}(e'_i)$. We can assume that there exist $e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_m} \in \mathcal{B}$ such that for any $1 \leq j \leq n$,

$$x_j = \sum_{k=1}^n \beta_{jk} e_{\alpha_k},$$

where $\beta_{jk} \in \mathbb{C}$. We have $u = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} e_{\alpha_i} \otimes e_{\alpha_j}$ and if $\alpha \neq \alpha_j$, then $u(e_\alpha) = 0$.

Let $a = (a_{ij}) \in M_n(\mathbb{C})$. For $(x_i) \in \ell^2$, $\pi(a)(x_i) = \sum_{i=1}^n (\sum_{j=1}^n a_{ij}x_j)e_i$.

It easily follows from the definition of φ_n :

- i) $\pi_n(a)$ is a bounded linear map on ℓ^2 with finite dimensional rank;
- ii) the map $a \mapsto \pi_n(a)$ is *-homomorphism from $M_n(\mathbb{C})$ into $B_{\circ\circ}(\ell^2) \subseteq B_{\circ}(\ell^2)$;
- iii) if $\pi_n(a) = \pi_m(b)$ with $m \geq n$ then $b = \varphi_m\varphi_{m-1}\dots\varphi_n(a)$.

We show that for any $u \in B_{\circ\circ}(\ell^2)$, there exists $a \in M_n(\mathbb{C})$ such that $u = \pi_n(a)$. For this, consider the standard basis of ℓ^2 , $\{e_i : i \in \mathbb{N}\}$. By the above lemma there exists $n \in \mathbb{N}$ such that $u(e_i) = 0$ for $i > n$. We may choose n large enough such that $u(H)$ is generated by e_1, e_2, \dots, e_n . Therefore for $1 \leq i, j \leq n$, there exists a_{ij} 's with $u(e_i) = \sum_{j=1}^n a_{ij}e_j$. So $u = \pi_n(a)$ where $a = (a_{ij})$.

By the definition of π_n and φ_n the diagram

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{\varphi_n} & M_{n+1}(\mathbb{C}) \\ \pi_n \downarrow & \swarrow \pi_{n+1} & \\ B_{\circ}(\ell^2) & & \end{array}$$

is commutative. Suppose that A is a C^* -algebra with morphisms $\pi'_n : M_n(\mathbb{C}) \rightarrow A$ such that the diagram

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{\varphi_n} & M_{n+1}(\mathbb{C}) \\ \pi'_n \downarrow & \swarrow \pi'_{n+1} & \\ A & & \end{array}$$

is commutative. We define $\rho : B_{\circ\circ}(\ell^2) \rightarrow A$ as follows: for $u \in B_{\circ\circ}(\ell^2)$ there exists $a \in M_n(\mathbb{C})$ such that $u = \pi_n(a)$. We put $\rho(u) = \pi'_n(a)$. If there exists $b \in M_m(\mathbb{C})$ such that $u = \pi_m(b)$. We may assume that $m \geq n$, so by (iii), $b = \varphi_m\varphi_{m-1}\dots\varphi_n(a)$. Therefore, $\pi'_m(b) =$

$\pi'_m \varphi_m \varphi_{m-1} \dots \varphi_n(a) = \pi'_n(a)$. This means that ρ is well-defined. Also, $\|\rho(u)\| = \|\pi'_n(a)\| \leq \|\pi'_n\| \|\pi_n(a)\| = \|\pi'_n\| \|u\|$. So we can extend ρ to $\tilde{\rho} : B_0(\ell^2) \rightarrow A$ and we have $\tilde{\rho}\pi_n = \pi'_n$. Suppose that ρ' is a *-homomorphism with $\rho'\pi_n = \pi'_n = \tilde{\rho}\pi_n$ for all $n \in \mathbb{N}$. Since $B_{00}(\ell^2)$ is dense in $B_0(\ell^2)$, the continuity of $\tilde{\rho}$ and ρ' implies to $\tilde{\rho} = \rho'$.

Exercise 2. Let (A_n, φ_n) be an inductive sequence of abelian groups. For each $m, n \leq 1$, we define $\varphi_{m,n} : A_m \rightarrow A_n$ as $\varphi_{n,n} = 1_{A_n}$, $\varphi_{m,n} = 0$ if $m > n$ and $\varphi_{m,n} = \varphi_{n-1} \dots \varphi_m$ if $m < n$. On the disjoint union $\coprod_{n \geq 1} A_n$ we define an equivalence relation by $a_i \sim a_j$ ($a_i \in A_i, a_j \in A_j$) if there exists $k \geq i, j$ with $\varphi_{i,k}(a_i) = \varphi_{j,k}(a_j)$. The equivalence class of a_i is denoted by $[a_i]$. We put an abelian group structure on the quotient $X = \coprod_{n \geq 1} A_n / \sim$ as follows: $n[a_i] = [na_i]$ ($n \in \mathbb{Z}$) and $[a_i] + [b_j] = [a_k + b_k]$, where $k \geq i, j$, $a_k = \varphi_{i,k}(a_i)$ and $b_k = \varphi_{j,k}(b_j)$. We denote by i_n the composite $A_n \rightarrow \coprod_{n \geq 1} A_n \xrightarrow{\pi} X$, hence every element of X can be represented by $i_n(a_n)$ for some $n \geq 1$ and $a_n \in A_n$. We claim that (X, i_n) is the inductive limit of the given inductive sequence. Indeed, one can easily check that $i_{n+1}\varphi_n = i_n$. If

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow f_n & \swarrow f_{n+1} \\ & & A \end{array}$$

($n \geq 1$) is a cone (*i.e.*, a commutative triangle), then $f_n \varphi_{m,n} = f_m$ for all $m \leq n$. We define $f : X \rightarrow A$ as $f([a_m]) := f_m(a_m)$. The fact that f is well defined follows from the definition of \sim . The given group structure on X shows that f is a homomorphism and we also have $f i_n = f_n$ for $n \geq 1$. The uniqueness of f with this property is immediate.

Exercise 3. For $n \geq 1$ we define $A_n := \mathbb{Z}$ and $\varphi_n : A_n \rightarrow A_{n+1}$ by $\varphi_n(1) := n$. To give a cone

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow f_n & \swarrow f_{n+1} \\ & & A \end{array}$$

is to give a sequence $\{a_n\}_{n \geq 1}$ of elements of A such that $a_1 = n!a_{n+1}$. We define $i_n : A_n \rightarrow \mathbb{Q}$ by $i_n(1) = \frac{1}{(n-1)!}$ and we claim that (\mathbb{Q}, i_n) is the inductive limit of the given inductive sequence. Indeed, a morphism $f : \mathbb{Q} \rightarrow A$ such that $f i_n = f_n \forall n \geq 1$ is uniquely determined if we require that $f(\frac{1}{(n-1)!}) = a_n$ for all $n \geq 1$.

Exercise 4. We keep the notations of the previous exercise. The map φ_n is now $1 \mapsto 2$. We denote by $\mathbb{Z}[\frac{1}{2}]$ the additive subgroup of \mathbb{Q} whose elements are of the form $\frac{k}{2^n}$ ($k \in \mathbb{Z}, n \in \mathbb{N}$), and for each $n \geq 1$ we define a map $i_n : A_n \rightarrow \mathbb{Z}[\frac{1}{2}]$ by $1 \mapsto \frac{1}{2^n}$. We claim that $(\mathbb{Z}[\frac{1}{2}], i_n)$ is the inductive limit of the given inductive sequence: indeed, to give a cone

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow f_n & \swarrow f_{n+1} \\ & & A \end{array}$$

($n \geq 1$) is to give a sequence $\{a_n\}_{n \geq 1}$ of elements of A such that $a_1 = 2^{n-1}a_n$. We define $f : \mathbb{Z}[\frac{1}{2}] \rightarrow A$ by sending $\frac{1}{2^n}$ to a_n . The claim follows.

To prove of the continuity of K_0 we need some lemmas:

Lemma 1. Let a be a self-adjoint element of a C^* -algebra such that $\|a - a^2\| < \frac{1}{4}$. Then there is a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$.

Proof. Since a is selfadjoint, we may assume that A is abelian (replace A by C^* -algebra generated by a if necessary) and may therefore suppose that $A = C_0(X)$ for some locally compact Hausdorff space X . The hypothesis implies that $\frac{1}{2}$ is not in the range of $|a|$, and therefore the set $E = |a|^{-1}((\frac{1}{2}, \infty))$ is open and compact (it is compact because it is equal to $\{x \in X : |a(x)| \geq \frac{1}{2}\}$ which is compact by definition of $C_0(X)$). Hence, $p = \chi_E$ is projection in A . Suppose that $x \notin E$, then $|a(x) - p(x)| = |a(x)| < \frac{1}{2}$. Now suppose that $x \in E$. Since $|a(x)||a(x) - 1| < \frac{1}{4}$ and $|a(x)| > \frac{1}{2}$, $|a(x) - p(x)| < \frac{1}{2}$. Hence, $|a(x) - p(x)| < \frac{1}{2}$ for all $x \in X$. We claim that $\|a - p\| < \frac{1}{2}$. For this we extend $a - p$ to a continuous function f on the one point-compactification X^* by $f(*) = 0$, where $*$ is the point at infinity. Since $|a(x) - p(x)| < \frac{1}{2}$ for all $x \in X$, then $\|a - p\| = \|f\| < \frac{1}{2}$ because the continuous function f attains its upper bound on the compact space X^* .

Lemma 2. Let p, q be projections in a C^* -algebra A and suppose that there is an element $u \in A$ such that $\|p - uu^*\|$ and $\|p - u^*u\|$ are less than one and $u = qup$. Then $p \sim q$.

Proof. Since p is unit of the C^* -algebra pAp , by Lemma 3.4 of the Lecture Notes - Part I, the inequality $\|p - u^*u\| < 1$ implies that u^*u is invertible in the C^* -algebra pAp and, similarly, the inequality $\|p - uu^*\| < 1$ implies that uu^* is invertible in qAq . We introduced the concept of absolute value in the solution of exercise 3.4(b); with the same notations, let $z = |u|^{-1}$ in pAp , i.e. $z^2 = (u^*u)^{-1}$. So, similar to the solution of the exercise 3.4(b), defining the unitary $w := uz$, we obtain $w^*w = zu^*uz = z|u|^2z = p$. Also $uu^*ww^* = uu^*uz^2u^* =$

$u|u|^2z^2u^* = uu^*$, so $w^* = q$ by invertibility of uu^* in qAq . Thus, $p \overset{u}{\sim} q$.

Lemma 3. Let $(A_n, \varphi_n)_{n \geq 1}$ be an inductive system of C^* -algebras with inductive limit (A, μ_n) and let B be the direct limit of the corresponding system $M_k(A_1) \xrightarrow{\varphi_1} M_k(A_2) \xrightarrow{\varphi_2} M_k(A_3) \xrightarrow{\varphi_3} \dots$, where k is a fixed integer. Denote by $\psi_n : M_k(A_n) \rightarrow B$ the natural morphisms ($n \geq 1$). Then

$$M_k(A) = B$$

Proof. Since the diagram

$$\begin{array}{ccc} M_k(A_n) & \xrightarrow{\varphi_n} & M_k(A_{n+1}) \\ & \searrow \mu_n & \downarrow \mu_{n+1} \\ & & M_k(A) \end{array}$$

commutes for each n , it follows from the definition of the direct limit that there is a unique $*$ -homomorphism $\pi : B \rightarrow M_k(A)$ such that the diagram

$$\begin{array}{ccc} M_k(A_n) & \xrightarrow{\psi_n} & B \\ & \searrow \mu_n & \downarrow \mu_{n+1} \\ & & M_k(A) \end{array}$$

commutes for each n . By the construction of the direct limit in the category of C^* -algebras, $A = cl(\cup \mu_n(A_n))$, so $M_k(A) = cl(\cup \mu_n(M_k(A_n)))$. By the same construction, $B = cl(\cup \psi_n(M_k(A_n)))$. So we can write

$$\cup \mu_n(M_k(A_n)) = \cup \pi \psi_n(M_k(A_n)) = \pi(\cup \psi_n(M_k(A_n))) \subseteq \pi(B)$$

and since the range of any $*$ -morphism is closed we have

$$M_k(A) = cl(\cup \mu_n(M_k(A_n))) \subseteq cl(\pi(B)) = \pi(B)$$

i.e. π is surjective.

To show that π is injective, first we show that it is injective when restricted to the C^* -subalgebras $\psi_n(M_k(A_n))$. For this suppose $\pi(\psi_n(a)) = 0$, where $a \in M_k(A_n)$. Let $\varepsilon > 0$. If $b \in A_n$ and $\mu_n(b) = 0$, then there exists $m \geq n$ such that $\|\varphi_{nm}(b)\| < \varepsilon$ (It follows directly from the construction of the direct limit). Applying this to the entries a_{ij} of matrix a , since $\mu_n(a) = 0$, so $\mu_n(a_{ij}) = 0$, hence there exists $m_{ij} \geq n$ such that $\|\varphi_{nm_{ij}}(a_{ij})\| < \varepsilon$, $1 \leq i, j \leq k$. If consider $m = \max_{i,j}(m_{ij})$, then $\|\varphi_{nm}(a_{ij})\| < \varepsilon$. Hence, by Exercise 3.1,

$$\|\varphi_{nm}(a)\| \leq \sum_{ij} \|\varphi_{nm}(a_{ij})\| < k^2\varepsilon$$

Consequently, $\|\psi_n(a)\| = \|\psi_m\varphi_{nm}(a)\| \leq \|\varphi_{nm}(a)\| < k^2\varepsilon$. Letting $\varepsilon \rightarrow 0$ we get $\|\psi_n(a)\| = 0$, so π is injective on $\psi_n(M_k(A_n))$.

Now since π is injective on C^* -subalgebras $\psi_n(M_k(A_n))$ (hence isometric on these subalgebras) and continuous it follows that π is isometric on B , i.e. π is injective on B .

Lemma 4. Let $(A_n, \varphi_n)_{n \geq 1}$ be an inductive system of unital C^* -algebras with inductive limit (A, μ_n) .

(1) If p is projection in A , then there is an integer n and a projection $q \in A_n$ such that p is unitary equivalent to $\mu_n(q)$ in A .

(2) If n is given and p, q are projection in A_n such that $\mu_n(p) \sim \mu_n(q)$ in A , then there is an integer $m \geq n$ such that $\varphi_{nm}(p) \sim \varphi_{nm}(q)$ in A_m .

Proof. Let p be a projection in A . Since $A = \overline{\cup_{n=1}^{\infty} \mu_n(A_n)}$ there is a sequence $(\mu_{n_k}(a_k))_{k=1}^{\infty}$ in $\cup_{n=1}^{\infty} \mu_n(A_n)$ converging to p . As $p = p^*$ we may assume that each a_k is self-adjoint (replacing a_k with $\operatorname{Re}(a_k)$ if necessary). Since $p = p^2$, the sequence $(\mu_{n_k}(a_k^2))_{k=1}^{\infty}$ also converges to p and therefore $(\mu_{n_k}(a_k - a_k^2))_{k=1}^{\infty}$ converges to 0. Hence, there exist

an integer number m and a selfadjoint element $a \in A$ such that $\|p - \mu_m(a)\| < \frac{1}{2}$ and $\|\mu_m(a - a^2)\| < \frac{1}{4}$. It follows that there exists $n \geq m$ such that $\|\varphi_{mn}(a - a^2)\| < \frac{1}{4}$. Set $b = \varphi_{mn}$. Then b is a selfadjoint element of A_n such that $\|b - b^2\| < \frac{1}{4}$, and therefore by Lemma 1, there is a projection $q \in A_n$ such that $\|b_q\| < \frac{1}{2}$. Since $b = \varphi_{mn}(a)$, $\mu_m(a) = \mu_n(b)$ and we have

$$\begin{aligned} \|p - \mu_n(q)\| &\leq \|p - \mu_m(a)\| + \|\mu_m(a) - \mu_n(b)\| + \|\mu_n(b) - \mu_n(q)\| \\ &\leq \|p - \mu_m(a)\| + \|b - q\| \\ &\leq \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

so by Exercise 3.4(c), the projections p and $\mu_n(q)$ are unitary equivalent in A .

Now suppose that n is a given integer, and that p, q are projections in A_n such that $\mu_n(p) \sim \mu_n(q)$ in A . Then, there exists a partial isometry u in A such that $\mu_n(p) = u^*u$ and $\mu_n(q) = uu^*$. Now u is the limit of a sequence $(\mu_{n_k}(v_k))_{k=1}^{\infty}$ where $v_k \in A_{n_k}$ and $n_k \geq n$, and, since $u = uu^*u = \mu_n(q)u = u\mu_n(p)$, we may assume that $v_k = \varphi_{nn_k}(q)v_k\varphi_{nn_k}(p)$ (since $\mu_{n_k}\varphi_{nn_k}(p) = \mu_n(p)$ and $\mu_{n_k}\varphi_{nn_k}(q) = \mu_n(q)$, replace v_k by $\varphi_{nn_k}(q)v_k\varphi_{nn_k}(p)$ if necessary). Clearly, $\mu_n(p) = \lim_{k \rightarrow \infty} \mu_{n_k}(v_k^*v_k)$ and $\mu_n(q) = \lim_{k \rightarrow \infty} \mu_{n_k}(v_kv_k^*)$. Hence, there exists an integer $k \geq n$ and $v \in A_k$ such that $v = \varphi_{nn_k}(q)v\varphi_{nn_k}(p)$ and $\|\mu_k(\varphi_{nk}(p) - v^*v)\| < 1$ and $\|\mu_k(\varphi_{nk}(q) - vv^*)\| < 1$. It follows that there is an integer $m \geq k$ such that $\|\varphi_{km}(p)(\varphi_{nk}(p) - v^*v)\| < 1$ and $\|\varphi_{km}(q)(\varphi_{nk}(q) - vv^*)\| < 1$. Therefore, if $w = \varphi_{km}(v)$, then $w \in A_m$ and we have $\|\varphi_{nm}(p) - w^*w\| < 1$ and $\|\varphi_{nm}(q) - ww^*\| < 1$ and $\varphi_{nm}(p)w\varphi_{nm}(q) = w$. Hence, by Lemma 2, the projections φ_{nm} and $\varphi_{nm}(q)$ are equivalent in A_m . This proves condition (2).

Exercise 5. Let $(A_n, \varphi_n)_{n \geq 1}$ be an inductive system of unital C^* -algebras with inductive limit (A, μ_n) and let G be the direct limit of the corresponding system $K_0(A_1) \xrightarrow{(\varphi_1)^*} K_0(A_2) \xrightarrow{(\varphi_2)^*} K_0(A_3) \xrightarrow{(\varphi_3)^*} \dots$ in the category of abelian groups. Denote by $\mu_n : A_n \rightarrow A$ and $\tau_n : K_0(A_n) \rightarrow G$ the natural morphism ($n \geq 1$). For each integer n the diagram

$$\begin{array}{ccc} K_0(A_n) & \xrightarrow{(\varphi_n)^*} & K_0(A_{n+1}) \\ & \searrow & \downarrow (\mu_{n+1})^* \\ & (\mu_n)^* & K_0(A) \end{array}$$

commutes, so by the definition of direct limit there is a unique morphism $\tau : G \rightarrow K_0(A)$ such that for each n the diagram

$$\begin{array}{ccc} K_0(A_n) & \xrightarrow{\tau_n} & G \\ & \searrow & \downarrow \tau \\ & (\mu_n)^* & K_0(A) \end{array}$$

commutes.

For each integer k , let B_k be the direct limit of the system $M_k(A_1) \xrightarrow{\varphi_1} M_k(A_2) \xrightarrow{\varphi_2} M_k(A_3) \xrightarrow{\varphi_3} \dots$, and for each n , let $\mu_n^k : M_k(A_n) \rightarrow B_k$ be the natural morphism. By Lemma 3, there is a unique $*$ -isomorphism $\pi_k : B_k \rightarrow M_k(A)$ such that for each n the diagram

$$\begin{array}{ccc} M_k(A_n) & \xrightarrow{\mu_n^k} & B_k \\ & \searrow & \downarrow \pi_k \\ & \mu_n & M_k(A) \end{array}$$

commutes.

We show first that τ is surjective. Let $p, q \in P_\infty(A)$. Then $p, q \in M_k(A)$ for some integer k . Hence, $p = \pi_k(p')$ and $q = \pi_k(q')$ for some projection $p', q' \in B_k$. By Lemma 4, condition(1), there are projections

$r \in M_k(A_n)$ and $r' \in M_k(A_{n'})$ for some n and n' such that $p' \sim \mu_n^k(r)$ and $q' \sim \mu_{n'}^k(r')$. Now if $n > n'$ since $\mu_n^k \varphi_{n-1} \cdots \varphi_{n'+1} \varphi_{n'}(r') = \mu_{n'}^k(r')$, by choosing $r'' := \varphi_{n-1} \cdots \varphi_{n'+1} \varphi_{n'}(r')$, we have $p \sim \mu_n(r)$ and $q \sim \mu_n(r'')$ in $M_k(A)$, since $\pi_k \mu_n^k = \mu_n$. Consequently, $[p] = [\mu_n(r)]$ and $[q] = [\mu_n(r'')]$ so $[p] - [q] = [\mu_n(r)] - [\mu_n(r'')] = (\mu_n)_*([r] - [r'']) = \tau \tau_n([r] - [r''])$, since $(\mu_n)_* = \tau \tau_n$. Therefore τ is surjective.

Now we show that τ is injective. Suppose that $x \in \ker(\tau)$. Since $G = \cup \tau_n(K_0(A_n))$ (by the construction of the direct limit in the category of abelian groups.), we can write $x = \tau_n([p] - [q])$ for some projections p, q in $P_\infty(A_n)$. We may suppose that $p, q \in M_k(A_n)$. Since $\tau(x) = 0$, we have $[\mu_n(p)] - [\mu_n(q)] = (\mu_n)_*([p] - [q]) = 0$, as $\tau \tau_n = (\mu_n)_*$. Thus, by the exercise 4 of lecture 4, there is an integer l such that $\mu_n(p) \oplus 1_l \sim \mu_n(q) \oplus 1_l$; that is $\mu_n(p \oplus 1_l) \sim \mu_n(q \oplus 1_l)$ in the C^* -algebra $M_{l+k}(A)$, since μ_n^l s are unital. Applying the $*$ -isomorphism π_{l+k}^{-1} , we get $\mu_n^{l+k}(p \oplus 1_l) \sim \mu_n^{l+k}(q \oplus 1_l)$ in B_{l+k} . Hence by Lemma 4, condition(2), there is an integer $m \geq n$ such that $\varphi_{nm}(p \oplus 1_l) \sim \varphi_{nm}(q \oplus 1_l)$ in $M_{l+k}(A_m)$ where φ_{nm} means $\varphi_m \varphi_{m-1} \cdots \varphi_n$. Therefore, $\varphi_{nm}(p) \oplus 1_l \sim \varphi_{nm}(q) \oplus 1_l$. Thus, by exercise 4 of lecture 4, $(\varphi_{nm})_*([p] - [q]) = [\varphi_{nm}(p)] - [\varphi_{nm}(q)] = 0$, and if we apply τ_m to both sides and observe that $\tau_m(\varphi_{nm})_* = \tau_n$, we get $x = \tau_n([p] - [q]) = 0$. This shows that τ is injective and completes the proof.

LECTURE 7 (25-03-04)

Exercise 1. Let us first show that

$$(1) \quad 0 \rightarrow C_0((0, 1)) \xrightarrow{\iota} C([0, 1]) \xrightarrow{\pi} \mathbb{C} \oplus \mathbb{C} \rightarrow 0$$

is a short exact sequence, where we have defined $\iota : C_0((0, 1)) \rightarrow C([0, 1])$ by $\iota(f) = f$ on $(0, 1)$ and $\iota(f) = 0$ on $\{0, 1\}$. It is clear that ι is a well-defined homomorphism and that it is injective. Next, let $\pi : C([0, 1]) \rightarrow \mathbb{C} \oplus \mathbb{C}$ be defined by $f \mapsto (f(0), f(1))$. This is clearly a surjective $*$ -homomorphism. We claim that $Im(\iota) = Ker(\pi)$. Indeed, if $\tilde{f} \in Im(\iota)$, then $\tilde{f}(0) = \tilde{f}(1) = 0$ and $\tilde{f} \in Ker(\pi)$; conversely if $g \in Ker(\pi)$, then $g(0) = g(1) = 0$ and defining $f := g|_{(0,1)}$, then $f \in C_0((0, 1))$ such that $\iota(f) = g$. Therefore (1) is a SES.

Applying K_0 to the SES (1) and using Proposition 7.1, we have the following exact sequence

$$0 \xrightarrow{\iota_*} \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \oplus \mathbb{Z}.$$

Clearly, this is not a SES, since $\pi_* : n \mapsto (n, n)$ is not surjective. The claim is proved. Note that we have used that $K_0(C([0, 1])) \simeq K^0([0, 1]) \simeq \mathbb{Z}$ and $K_0(\mathbb{C} \oplus \mathbb{C}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. We sketch a proof for the last isomorphism. Let

$$0 \longrightarrow \mathbb{C} \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{\pi_1} \end{array} \mathbb{C} \oplus \mathbb{C} \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{i_2} \end{array} \mathbb{C} \longrightarrow 0$$

be the obvious split exact sequence. We apply the functor K_0 to it; we obtain a split exact sequence of abelian groups:

$$0 \longrightarrow K_0(\mathbb{C}) \begin{array}{c} \xrightarrow{i_{1*}} \\ \xleftarrow{\pi_{1*}} \end{array} K_0(\mathbb{C} \oplus \mathbb{C}) \begin{array}{c} \xrightarrow{\pi_{2*}} \\ \xleftarrow{i_{2*}} \end{array} K_0(\mathbb{C}) \longrightarrow 0 ,$$

so $K_0(\mathbb{C} \oplus \mathbb{C}) \simeq K_0(\mathbb{C}) \oplus K_0(\mathbb{C})$.

Exercise 2. (a) Let $a \in M_n(\dot{A})$, $a = \{(x_{ij}, z_{ij})\}_{i,j}$. By definition, $\dot{\pi}_n(a) = (\pi(x_{ij}), z_{ij})_{i,j}$ and hence $\dot{\pi}_n(a) = s_n(\dot{\pi}_n(a)) \Leftrightarrow \pi(x_{ij}) = 0, \forall i, j \Leftrightarrow \exists a_{ij} \in J$ such that $i(a_{ij}) = x_{ij}, \forall i, j \Leftrightarrow a \in \text{ran}(i_n)$.

(b) Let $[\dot{\pi}_n(p)] - [s(\dot{\pi}_n(p))] = 0$ in $K_0(\dot{B})$, where $p \in P_n(\dot{A})$. Then there exists $k \in \mathbb{N}$ such that $\dot{\pi}_n(p) \oplus 1_k \sim s(\dot{\pi}_n(p)) \oplus 1_k$. Writing $q := p \oplus 1_k$, we have that $q \in P_{n+k}(\dot{A})$ and $\dot{\pi}_{n+k}(q) \sim s_n(\dot{\pi}_{n+k}(q))$ in $M_{n+k}(\dot{B})$. By Lemma 2.6, there is a unitary element $u \in M_{2(n+k)}(\dot{B})$ such that

$$u \dot{\pi}_{2(n+k)}(q) u^* = s(\dot{\pi}_{2(n+k)}(q)).$$

We will use the following result:

Lemma. Let $\varphi : A \rightarrow B$ be a surjective $*$ -homomorphism between unital C^* -algebras. Then each unitary element $u \in M_\infty B$, which is homotopic to the identity can be lifted to a unitary $v \in M_\infty(A)$ which is homotopic to the identity.

Proof. The proof relies on the following characterization of unitaries that are homotopic to the identity:

$$u \in \mathcal{U}_0(B) \text{ iff } u = \exp(w_1) \dots \exp(w_p),$$

for some self-adjoint $w_i, i = 1, \dots, p$ (see Rordam, Prop. 2.1.6). Now, for such $u \in \mathcal{U}_0(B)$, let $x_i \in A$ be such that $\varphi(x_i) = w_i$ and define $v_i := \frac{x_i + x_i^*}{2}$. Then v_i is self-adjoint and $\varphi(v_i) = \varphi(x_i)$ (since w_i is self-adjoint) and, moreover, $u = \exp \circ \varphi(v_1) \dots \exp \circ \varphi(v_p)$. From the continuous functional calculus, $\exp \circ \varphi = \varphi \circ \exp$: since φ is multiplicative, it commutes with polynomials and hence it commutes with \exp as well.

Set $v := \exp(v_1)\dots\exp(v_p) \in \mathcal{U}_0(A)$ to prove our claim.

Since, by Lemma 10.1, $\text{diag}(u, u^*)$ is homotopic to $1_{4(n+k)}$, we have that there is a unitary $v \in M_{4(n+k)}(\dot{A})$ such that $\dot{\pi}_{4(n+k)}(v) = \text{diag}(u, u^*)$. Define $p' := v(q \oplus 0_{2(n+k)})v^*$ and remark that $p' \in P_{4(n+k)}(\dot{A})$, with $p' \sim q$ in $P_\infty(\dot{A})$. We claim that $\dot{\pi}_{4(n+k)}(p') = s(\dot{\pi}_{4(n+k)}(p'))$: indeed,

$$\dot{\pi}_{4(n+k)}(p') = \text{diag}(s(\dot{\pi}_{2(n+k)}(q)), 0_{2(n+k)})$$

and applying s on the right hand side does not change anything.

Exercise 3. (a) Let X be a locally compact Hausdorff space. As we know, there is an isomorphism

$$\Delta(C_0(X)) \cong X, \quad x \longmapsto \delta_x,$$

where $\delta_x(f) = f(x)$. Now suppose that Y is a locally compact space and $T : C_0(Y) \longrightarrow C_0(X)$ is a homomorphism. Let $T' : (C_0(X))' \longrightarrow (C_0(Y))'$ be the dual map; then, since T is a homomorphism, we see that $T'(\omega)$ is multiplicative whenever ω is, and therefore it defines a weak*-continuous map

$$T' : \Delta(C_0(X)) \cup \{\theta\} \longrightarrow \Delta(C_0(Y)) \cup \{\theta'\}$$

with θ, θ' the zero functionals of $C_0(X)$ and $C_0(Y)$, respectively. Under the isomorphism above, this map induces a continuous map

$$\tilde{\psi} : X \cup \{\infty_X\} \longrightarrow Y \cup \{\infty_Y\},$$

where ∞_X and ∞_Y denote the points at infinity (mapped to θ, θ'). Set $Z = \{x \in X : \tilde{\psi}(x) \neq \infty_Y\}$ and suppose that K is a compact subset of Y . Since $\tilde{\psi}$ is continuous $(\tilde{\psi})^{-1}(K)$ is a closed subset of

$X \cup \{\infty_X\}$ and hence it is compact. Since $(\tilde{\psi})^{-1}(K)$ is a subset of Z , $\psi := \tilde{\psi}|_Z : Z \rightarrow Y$ is proper. Therefore ψ induces a morphism

$$\psi^* : C_0(Y) \rightarrow C_0(Z).$$

For arbitrary $f \in C_0(Y)$, $x \in Z$:

$$(Tf)(x) = \delta_x(Tf) = T'(\delta_x)(f) = \delta_{\tilde{\psi}(x)}(f) = f(\psi(x)) = \psi^*(f)(x).$$

If $x \in X - Z$, then $T'(\delta_x) = \theta'$ and hence $(Tf)(x) = \theta'(f) = 0$. Therefore $T = i \circ \psi^*$.

(b) It easily follows from (a) that T is nondegenerate if and only if $Z = X$.

Exercise 4. (a) The canonical embedding $i_{n,A} : A \rightarrow M_n(A)$ induces a group homomorphism $(i_{n,A})_* : K_0(A) \rightarrow K_0(M_n(A))$. We show first that the non-unital case can be derived from the unital case. For this purpose let A be a (non-unital) C^* -algebra. Then

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \dot{A} & \xrightarrow{\pi} & \mathbb{C} \longrightarrow 0 \\ & & \downarrow i_{n,A} & & \downarrow i_{n,\dot{A}} & & \downarrow i_{n,\mathbb{C}} \\ 0 & \longrightarrow & M_n(A) & \longrightarrow & M_n(\dot{A}) & \xrightarrow{\pi_n} & M_n(\mathbb{C}) \longrightarrow 0 \end{array}$$

is commutative diagram with split exact rows. It follows that

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(\dot{A}) & \xrightarrow{\pi} & K_0(\mathbb{C}) \longrightarrow 0 \\ & & \downarrow (i_{n,A})_* & & \downarrow (i_{n,\dot{A}})_* & & \downarrow (i_{n,\mathbb{C}})_* \\ 0 & \longrightarrow & K_0(M_n(A)) & \longrightarrow & K_0(M_n(\dot{A})) & \xrightarrow{\pi_n} & K_0(M_n(\mathbb{C})) \longrightarrow 0 \end{array}$$

is commutative diagram with (split) exact rows. The five lemma now shows that $(i_{n,A})_*$ is an isomorphism if $(i_{n,\dot{A}})_*$ and $(i_{n,\mathbb{C}})_*$ are isomorphisms. We therefore need only prove the proposition when A is unital.

Now we prove $(i_{n.A})_*$ is bijective when A is unital. To do this we show that i_* is the identity map. Let p be a projection in $P_\infty(A)$, hence $p \in M_r(A)$ for some $r \in \mathbb{N}$. So if $p = (a)_{rs}$, ($1 \leq r, s \leq r$), then $i_*([(a)_{rs}]) = [i_r((a)_{rs})] = [(i(a_{rs}))_{kl}]$ where $1 \leq k, l \leq rn$. On the other hand, we know that the elementary matrix operations leave the homotopy classes invariant, so that $[(i(a_{rs}))_{kl}] = [(a)_{rs}]$ and this finishes the first of the proof. For naturality, suppose $\varphi : A \longrightarrow B$ is a morphism between C^* -algebras. It is clear that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & M_n(A) \\ \downarrow \varphi & & \downarrow \varphi \\ B & \xrightarrow{i} & M_n(B) \end{array}$$

is commutative. Now by functoriality of K_0 , the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{i_*} & K_0(M_n(A)) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ K_0(B) & \xrightarrow{i_*} & K_0(M_n(B)) \end{array}$$

is commutative.

(b) By the continuity of K_0 and the previous part, we can write:

$$K_0(B_0(A)) = K_0(\varinjlim M_n(A)) = \varinjlim (K_0(M_n(A))) = \varinjlim (K_0(A)) = K_0(A).$$

LECTURE 8 (01-04-04)

Exercise 1. Let $[p, q, v] \in K_0(A, A/J)$ with $p, q \in P_n(A)$. Then by definition $\pi_n(p)$ and $\pi_n(q)$ are MvN equivalent in A/J . Hence, in $K_0(A/J)$ we have $0 = [\pi_n(p)] - [\pi_n(q)] = \pi_*([p] - [q])$.

Conversely, let $x \in \text{Ker}(\pi_*)$ and write $x = [p] - [q]$ for some $p, q \in P_n(A)$. Since $[\pi_n(p)] = [\pi_n(q)]$, we can find $m \in \mathbb{N}$ such that $\pi_n(p) \oplus 1_m \stackrel{\text{MvN}}{\sim} \pi_n(q) \oplus 1_m$, i.e., there is $v \in M_{n+m}(A/J)$ such that $\pi_n(p) \oplus 1_m = v^*v$ and $\pi_n(q) \oplus 1_m = vv^*$. If we put $p' = p \oplus 1_m$, $q' = q \oplus 1_m$ and v' such that $v = \pi_{n+m}(v')$, then the triple (p', q', v') is a relative K-cycle and $[p'] - [q'] = x$.

Exercise 2. We first see that the map

$$\alpha : K_0(A, A/J) \rightarrow \text{ker}(\pi_*), \quad \alpha([p, q, v]) := [p] - [q]$$

is surjective, i.e., that $\text{Im}(\alpha) = \text{ker}(\pi_*)$. This is the exactness of $K_0(A, A/J) \xrightarrow{\alpha} K_0(A) \rightarrow K_0(A/J)$ in the middle, which we proved in the previous exercise.

To prove injectivity, let $\lambda : A/J \rightarrow A$ be a splitting of π . Consider the homomorphism $\beta : K_0(A) \rightarrow K_0(A, A/J)$ induced by

$$\beta([p]) := [p, \lambda\pi(p), p].$$

We will show that the composite

$$K_0(A, A/J) \xrightarrow{\alpha} K_0(A) \xrightarrow{\beta} K_0(A, A/J)$$

is the identity. We check this on elements of type $[p, q, p]$ and then prove in a lemma that every class in $K_0(A, A/J)$ can be written in this form.

We first remark that if (p, q, v) and (p, q, w) are relative K -cycles such that $\pi(v) = \pi(w)$, then $[p, q, v] = [p, q, w]$. Indeed, the two cycles can be connected by the linear homotopy of K -cycles

$$(p, q, tv + (1 - t)w), \quad t \in [0, 1].$$

In particular, we always have $[p, q, v] = [p, q, \lambda\pi(v)]$. Next, note that

$$(\beta\alpha)([p, q, p]) = [p \oplus \lambda\pi(q), \lambda\pi(p) \oplus q, p \oplus q]$$

and, on the other hand, since $[\lambda\pi(q), \lambda\pi(p), \lambda\pi(q)]$ is degenerate,

$$\begin{aligned} [p, q, p] &= [p, q, \lambda\pi(p)] = [p \oplus \lambda\pi(q), q \oplus \lambda\pi(p), \lambda\pi(p) \oplus \lambda\pi(q)] \\ &= [p \oplus \lambda\pi(q), q \oplus \lambda\pi(p), p \oplus q]. \end{aligned}$$

and therefore

$$[p, q, p] = [p \oplus \lambda\pi(q), \lambda\pi(p) \oplus q, p \oplus q].$$

We are now left with proving:

Lemma. For any $[p, q, v] \in K_0(A, A/J)$, one has $[p, q, v] = [p, q', p]$, where q' is homotopic to q .

Proof. Let

$$u := \begin{pmatrix} \pi(v) & 1 - \pi(v)\pi(v^*) \\ \pi(v^*)\pi(v) - 1 & \pi(v^*) \end{pmatrix}.$$

Then u is a unitary matrix over A/J such that $\pi(q \oplus 0) = u\pi(p \oplus 0)u^*$. Moreover, u is connected through a path of unitaries to the identity matrix: the path

$$h(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t)\pi(v) & 1 - (1 - \sin(\frac{\pi}{2}t))\pi(v)\pi(v)^* \\ (1 - \sin(\frac{\pi}{2}t))\pi(v)^*\pi(v) - 1 & \cos(\frac{\pi}{2}t)\pi(v)^* \end{pmatrix}$$

consists of unitaries and connects u to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is itself connected by a similar path (take $v = 1$) to the identity matrix. By the lemma in Exercise 7.2.(b), there is a unitary matrix u_1 over A which is connected through a path u_t of unitaries to the identity matrix and such that $\pi(u_1) = u$. We claim that

$$[p, q, v] = [p, q, u_1 p].$$

For this it suffices to show that $\pi(u_1 p) = \pi(v)$. This follows from the general fact that for any partial isometry v in a unital C^* -algebra we have $v = vv^*v$ (see the proof in Exercise 3.5). Now,

$$(p, u_t u_1^* q u_1 u_t^*, u_t p)$$

is a path of relative K -cycles that connects $(p, u_1^* q u_1, p)$ to $(p, q, u_1 p)$. We conclude that $[p, q, v] = [p, q', p]$, with $q' := u_1^* q u_1$ homotopic to q .

Exercise 3. Suppose that X is a compact Hausdorff space and let $J \subseteq A = C(X)$ be a closed ideal; then there is a closed subset Y of X such that

$$J = \{f \in C(X) : f = 0 \text{ on } Y\}.$$

It is easy to see that $A/J \cong C(Y)$ and $J \cong C_0(X \setminus Y)$. By these facts we have:

$$K^0(X \setminus Y) \cong K_0(C_0(X \setminus Y)) \cong K_0(J).$$

On the other hand, the relative K -group $K^0(X, Y)$ in topological K -theory is defined as the group of equivalence classes of triples $[E, F, \sigma]$, where E, F are vector bundles over X and $\sigma : E|_Y \rightarrow F|_Y$ is an isomorphism. By the Serre-Swan Theorem, given projections $p, q \in$

$P_\infty(A)$, with $A = C(X)$, there are vector bundles E, F , unique up to isomorphism, such that

$$pA^n \cong \Gamma(E), \text{ and } qA^n \cong \Gamma(F).$$

Noting that $\pi : A = C(X) \rightarrow A/J \cong C(Y)$ is given by restriction, and that, from Exercise 4.1, $\pi(p) \sim \pi(q)$ iff $\pi(p)C(Y)^n \cong \pi(q)C(Y)^m$, we have that

$$\begin{aligned} \pi(p) \sim \pi(q) &\Leftrightarrow \pi(pC(X)^n) \cong \pi(qC(X)^m) \\ &\Leftrightarrow \pi(\Gamma(E)) \cong \pi(\Gamma(F)) \Leftrightarrow E|_Y \cong F|_Y. \end{aligned}$$

The map

$$K_0(A, A/J) \rightarrow K^0(X, Y), [p, q, v] \mapsto [E, F, \sigma],$$

with E, F as above and σ is the isomorphism induced by the equivalence of $\pi(p), \pi(q)$ given by $\pi(v)$, is an isomorphism. Excision in C^* -algebraic K -theory then yields

$$K^0(X \setminus Y) \cong K_0(J) \cong K_0(A, A/J) \cong K^0(X, Y).$$

Exercise 4. As in the lecture notes, we regard \dot{J} as a subalgebra of A via the map $(a, z) \mapsto a + z1_A$. By the commutativity of

$$\begin{array}{ccc} \dot{J} & \longrightarrow & A(M_n(A)) \\ \downarrow & & \downarrow \\ \dot{J}/J & \longrightarrow & A/J \end{array}$$

we can identify $\mathbb{C} \simeq \dot{J}/J$ in the pair (\dot{J}, \mathbb{C}) with $\mathbb{C}1_{A/J}$. We will show that the map $K_0(\dot{J}, \mathbb{C}) \rightarrow K_0(A, A/J)$, given by $[p, q, v] \mapsto [p, q, v]$ is an isomorphism.

Surjectivity: let $[p, q, v] \in K_0(A, J)$ with $p, q \in P_n(A), v \in M_n(A)$. Then $[p, q, v] = [p \oplus (1-p), q \oplus (1-p), v \oplus (1-p)]$ and by Exercise 4.3

we have $p \oplus (1 - p) \sim 1$. Defining $p' := p \oplus (1 - p)$, $q' := q \oplus (1 - p)$, $v' := v \oplus (1 - p)$, then we can find a unitary element u in A such that $p'' = up'u^*$, where $p'' = 1$ has entries in \dot{J} . By Lemma 10.1, $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$

is homotopic to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ via a path of unitaries. Hence, replacing p' by $p' \oplus 0$, u by $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$, we may assume that u is homotopic to the identity matrix via some homotopy u_t . It follows that

$$[p, q, v] = [p', q', v'] = [p'', q', v'u],$$

since the last two cycles are connected by the path of K -cycles $(u_t p' u_t^*, q', v' u_t)$.

Now, from the lemma in Exercise 2, there is $q'' \in A$ such that

$$[p'', q', v'u] = [p'', q'', p'']$$

and q'' has entries in \dot{J} since $\pi(q'') = \pi(p'') = 1$. Surjectivity is proved.

Note. We have in fact proved that for each $[p, q, v]$, $p, q, v \in A$ there exist $p'', q'' \in \dot{J}$ such that (p'', q'', p'') is homotopic to (p', q', v') , where $p' := p \oplus (1 - p)$, $q' := q \oplus (1 - p)$, $v' := v \oplus (1 - p)$.

Injectivity: let $[p, q, v] \in K_0(\dot{J}, \mathbb{C})$ be such that $[p, q, v] = 0$ in $K_0(A, A/J)$. Recall that the zero class is the class of $[e, e, e]$, for any projection e in A ; in particular, we can assume that $e \in \dot{J}$. From the definition of equivalence in $K_0(A, A/J)$, we need to check that:

(i) if there are degenerate cycles $[r, s, w], [r', s', w'] \in K_0(A, A/J)$ such that $[p, q, v] \oplus [r, s, w] = [r', s', w']$, then $[p, q, v] = 0$ in $K_0(\dot{J}, \mathbb{C})$.

For this we note that since the map is surjective, we can assume that $r, s, w \in \dot{J}$, $r', s', w' \in \dot{J}$, and the cycles are degenerate in $K_0(\dot{J}, \mathbb{C})$.

We need to check also that

(ii) if there is a homotopy of cycles $[p_t, q_t, v_t]$ in $K_0(A, A/J)$ such that $[p_0, q_0, v_0] = [p, q, v]$ and $[p_1, q_1, v_1] = [e, e, e]$, with $p, q, v, e \in \dot{J}$, then $[p, q, v] = 0$ in $K_0(\dot{J}, \mathbb{C})$.

Let $p'_t := p_t \oplus (1 - p_t)$, $q'_t := q_t \oplus (1 - p_t)$, $v'_t := v_t \oplus (1 - p_t)$; clearly, the path $t \mapsto (p'_t, q'_t, v'_t)$ is continuous. As we have noted when proving surjectivity, we know that there are $p''_t, q''_t \in \dot{J}$ such that, for each t , (p''_t, q''_t, p''_t) is homotopic to (p'_t, q'_t, v'_t) . We conclude that the path $t \mapsto (p''_t, q''_t, p''_t)$ is also continuous (in A , hence in \dot{J}) and therefore $[p''_0, q''_0, p''_0] = [p''_1, q''_1, p''_1]$ in $K_0(\dot{J}, \mathbb{C})$. But

$$[p, q, v] = [p'_0, q'_0, v'_0] = [p''_0, q''_0, p''_0]$$

and

$$[p''_1, q''_1, p''_1] = [p'_1, q'_1, v'_1] = [e \oplus (1 - e), e \oplus (1 - e), e \oplus (1 - e)] = 0$$

in $K_0(\dot{J}, \mathbb{C})$. Injectivity is proved.

LECTURE 9 (08-04-04)

Exercise 1. We first make some general remarks, to be used later on. For a locally compact Hausdorff space X and a normed vector space V , we consider the normed vector space $C_0(X, V)$ with the obvious definition. For $f \in C_0(X)$ and $v \in V$, we define $fv \in C_0(X, V)$ by $(fv)(x) = f(x)v$. We have:

Lemma. The subspace V_0 generated by elements fv as above is dense in $C_0(X, V)$.

Proof. Let $f \in C_0(X, V)$ and let $\epsilon > 0$. We can find a compact set $K \subseteq X$ such that $\|f(x)\| < \epsilon$ for all $x \in X - K$.

We can also find an open cover $\{U_i\}_{i=1}^n$ of K such that $\|f(x) - f(y)\| < \epsilon$ whenever x and y belong to U_i for some i . Let $\{h_i\}$ be a partition of unity on X subordinate to $\{U_i\}_{i=1}^{n+1}$ where $u_{n+1} := X - K$, consisting of compactly supported functions. For each $1 \leq i \leq n$, let $x_i \in U_i$ and put $v_i = f(x_i)$. One can easily check that $\|f - \sum_{i=1}^{i=n} h_i v_i\| < \epsilon$. The lemma is proved.

Consider now a SES $0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} A/J \longrightarrow 0$ of C^* -algebras. The injectivity of i_* is clear. Let $f \in \ker(\pi_*)$; then $f(t) = i(g(t))$ for some $g : [0, 1] \longrightarrow A$, and the fact that i is an isometry implies that $g \in SJ$. Also, π_* is surjective: since π is surjective and $\pi_*(f(a)) = f(\pi(a))$, we have $A_0 \subseteq \text{Im}\pi_*$, where A_0 is the set considered in the previous lemma.

Exercise 2. K_n is a functor since it is the composite of $n + 1$ functors. K_n is homotopy invariant: it follows from the fact that S preserves homotopy equivalences.

K_n is continuous: for this it is enough to show that S is continuous.

Let $(A_n, \varphi_n)_{n \geq 1}$ be an inductive system of C^* -algebras with inductive limit (A, μ_n) . Let $(\varinjlim SA_n, \lambda_n)$ be the inductive limit of the inductive system $(SA_n, S\mu_n)_{n \geq 1}$. By the universal property of inductive limits, there is a unique

$$\gamma : \varinjlim SA_n \longrightarrow SA$$

such that $\gamma\lambda_n = S\mu_n$ for every $n \geq 1$. We show that γ is an isomorphism. To prove surjectivity, it suffices to show that $Im(\gamma)$ contains the set V_0 considered in the lemma in Exercise 1. So let $f \in C_0((0, 1))$ and $a \in A$. Using the standard description of the inductive limit of C^* -algebras (see Lecture Notes, page 16, and [9] Proposition 6.2.4), namely that $A = cl(\cup \mu_n(A_n))$, one can write $fa = \gamma(x)$ for some $x \in \varinjlim SA_n$. By the same standard description,

$$ker \lambda_n = \{f \in SA_n \mid \lim_{m \rightarrow \infty} \|S\varphi_{m,n}(f)\| = 0\}$$

where $\varphi_{m,n} = \varphi_{m-1} \dots \varphi_n$ and γ is injective iff $ker(S\mu_n) \subseteq ker(\lambda_n)$ for each n . Let $f \in ker S\mu_n$. Then $\mu_n(f(t)) = 0$ for every $t \in (0, 1)$, hence

$$\lim_{m \rightarrow \infty} \|\varphi_{m,n}(f(t))\| = 0,$$

by the description of $ker \mu_n$, and notice that $S\varphi_{m,n}(f)(t) = \varphi_{m,n}(f(t))$.

K_n is stable: from Exercise 3.1, it follows that a function $f : [0, 1] \longrightarrow M_n(A)$ is continuous iff each entry $f_{ij} : [0, 1] \longrightarrow A$ is continuous. Therefore the functors SM_n and M_nS are naturally isomorphic. The conclusion follows, since, as we have just proved, S is continuous.

Exercise 3. Since B is contractible, i_* is surjective. Moreover, SB is contractible: if $\varphi : B \longrightarrow B$ is a contraction of B then $f \mapsto \varphi_t \circ f$ is

contraction of SB . We decompose $i : J \rightarrow A$ as $i = p\alpha$, where

$$p : C_\pi(A, B) \longrightarrow A, (a, f) \mapsto a \text{ and } \alpha : J \longrightarrow C_\pi(A, B), x \mapsto (x, 0).$$

We have the following SESs:

$$0 \longrightarrow SB \longrightarrow C_\pi(A, B) \xrightarrow{p} A \longrightarrow 0$$

$$0 \longrightarrow J \xrightarrow{\alpha} C_\pi(A, B) \longrightarrow CB \longrightarrow 0,$$

hence p_* is injective. We show that α_* is injective as well. Let $E = \{f \in C([0, 1], A) : f(0) \in i(J)\}$. We have an exact sequence

$$0 \longrightarrow CJ \xrightarrow{u} E \xrightarrow{v} C_\pi(A, B) \longrightarrow 0,$$

where $u(f)(t) = i(f(1-t))$ and $v(g) = (g(1), \pi g)$, hence v_* is injective.

We decompose α as $v\mathbf{ct}$ where $\mathbf{ct} : i(J) \longrightarrow E$ is $\mathbf{ct}(x)(t) = i(x)$.

Observe that $\mathbf{ev} : E \longrightarrow i(J)$ defined as $\mathbf{ev}(f) = f(0)$ is a homotopy inverse to \mathbf{ct} , hence \mathbf{ct}_* is an isomorphism .

LECTURE 10 (22-04-04)

Exercise 1. (a) By definition, $S(A/J) = C_0((0, 1), A/J)$. Clearly,

$$S(A/J) \cong \{f \in C(I, A/J) : f(0) = f(1) = 0\}$$

(to see this, consider the map $\varphi : S(A/J) \rightarrow \{f \in C(I, A/J) : f(0) = f(1) = 0\}$ defined by $f \mapsto \tilde{f}$ where $\tilde{f}(x) = f(x)$ for every $x \in (0, 1)$ and $\tilde{f}(0) = \tilde{f}(1) = 0$). Now we show that $\dot{S}(A/J) = \{f \in C(I, A/J) : f(0) = f(1) \in \mathbb{C}1_{A/J}\}$. Consider the map

$$\psi : \dot{S}(A/J) \rightarrow \{f \in C(I, A/J) : f(0) = f(1) \in \mathbb{C}1_{A/J}\}$$

defined by $\psi(f, \lambda) = f + \lambda 1_{A/J}$. We have that ψ is an isomorphism. We show only that it is surjective. Suppose that $g \in \{f \in C(I, A/J) : f(0) = f(1) \in \mathbb{C}1_{A/J}\}$; then there is $\lambda \in \mathbb{C}$ such that $g(0) = g(1) = \lambda 1_{A/J}$. It is obvious that $\psi(g - g(0), \lambda) = g$.

(b) By Proposition 4.3,

$$K_1(A/J) = K_0(S(A/J)) = \{[p] - [s(p)] : p \in P_\infty(\dot{S}(A/J))\}.$$

By the previous part

$$P_\infty(\dot{S}(A/J)) = \{p \in C(I, P_\infty(A/J)) : p(0) = p(1) \in P_\infty(\mathbb{C}1_{A/J})\},$$

and under the isomorphism ψ from (a), we get $s(p) = p(0)$ (regarded as a constant function). On the other hand, the scalar parts of p and $s(p)$ are equal, i.e. $p(0) = s(p)(0)$. Since any $[p] - [q]$ can be written as $[p'] - [s(p')]$, with $p'(0) = p(0) \oplus (1 - q(0)) = p(1) \oplus (1 - q(1)) = p'(1)$, we can summarize everything as follows

$$K_1(A/J) = \{[p] - [q] : p, q \in C(I, P_\infty(A/J)), p(0) = p(1) = q(0) = q(1)\}.$$

(c) To show that

$$\dot{C}_\pi(A, A/J) = \{(a, f) \in A \oplus C(I, A/J) : f(1) = \pi(a), f(0) = \mathbb{C}1_{A/J}\}$$

it suffices to consider the following map

$$\Phi(((a, f), \lambda)) := (a + \lambda 1_A, \tilde{f} + \lambda) \in A \oplus C(I, A/J),$$

where $(a, f) \in C_\pi$, $\lambda \in \mathbb{C}$, and $\tilde{f} : I \rightarrow A/J$ is defined as $\tilde{f}(x) = f(x)$ for every $x \in (0, 1]$ and $\tilde{f}(0) = 0$.

(d) The map $K_1(A/J) \rightarrow K_0(C_\pi(A, A/J))$ is obtained from the following map

$$j_* : K_0(\dot{S}(A/J)) \rightarrow K_0(\dot{C}_\pi(A, A/J))$$

where $j : \dot{S}(A/J) \rightarrow \dot{C}_\pi(A, A/J)$ is obtained from the map $\iota : S(A/J) \rightarrow C_\pi(A, A/J)$, with $\iota(f) = (0, f)$. So at first we describe the map j . Suppose $f \in \dot{S}(A/J)$. If $f(0) = \lambda$, then by (a), $\psi(f - f(0), \lambda) = f$, so that $i((f - f(0), \lambda)) = ((0, f - f(0)), \lambda)$. Then we have

$$j(f) = \Phi \circ i \circ \psi^{-1}(f) = (0 + \lambda 1_A, f - f(0) + f(0))$$

i.e. $j(f) = (\hat{f}(0), f)$ where $\hat{f}(0)$ is defined by replacing $1_{A/J}$ in $f(0)$ by 1_A . Therefore the map $K_0(\dot{S}(A/J)) \rightarrow K_0(\dot{C}_\pi(A, A/J))$ is given by

$$[p] - [q] \mapsto [(\hat{p}(1), p)] - [(\hat{q}(1), q)],$$

where $\hat{p}(1)$ is defined by replacing $1_{A/J}$ in $p(1)$ by 1_A .

(e) The map $j \mapsto (j, 0)$ from J to $C_\pi(A, A/J)$ induces a map $\dot{J} \rightarrow \dot{C}_\pi(A, A/J)$ as follows:

$$e = (h, \lambda) \mapsto ((h, 0), \lambda) \mapsto (h + \lambda 1_A, 0 + \lambda 1_{A/J}) = (e, \pi(e))$$

So the map $K_0(\dot{J}) \rightarrow K_0(\dot{C}_\pi(A, A/J))$ is given by

$$[e] - [f] \mapsto [e, \pi(e)] - [f, \pi(f)]$$

where $e, f \in P_\infty(\dot{J})$. Now the claim follows from Exercise 5.1.(c).

(f) First we prove that the restriction of π from the space of projections in A to the space of projections in A/J has the path-lifting property. This means that if p_t is a path of projections in A/J , and P_0 is a projection in A with $\pi(P_0) = p_0$, then there is a path of projections P_t in A which lifts p_t (that is, $\pi(P_t) = p_t$ for all t) and begins with P_0 . We reduce to the unitary case: we show that a path u_t of unitaries in A/J can be lifted to a path of unitaries in A with $U_0 = 1$.

Since $\pi : A \rightarrow A/J$ is a surjective, unital *-homomorphism, we know from the lemma in Exercise 7.2.(b) that every unitary homotopic to the identity can be lifted, and a simple adaptation of the proof yields the result. A full proof can be given as follows (see [Higson-Roe], Prop. 4.3.14, page 95). For sufficiently small t , the number -1 does not belong to the spectrum of u_t , using the functional calculus we can find a path of selfadjoint elements $x_t \in A/J$ with $\exp(ix_t) = u_t$ and $x_0 = 0$. Now since the restriction of π to a map from the space of selfadjoint elements of A to the space of selfadjoint elements of A/J has the path-lifting property ([Higson-Roe, Lemma 4.3.13, page 95]), x_t can be lifted to a path X_t of selfadjoint elements of A with $X_0 = 0$. Then $U_t = \exp(iX_t)$ defines the desired path of unitaries. This proves the path-lifting property for unitaries.

The corresponding property for projections is a consequence, since it follows from Exercise 3.4.(c) that every path of projections is the

conjugate of a fixed projection by a path of unitaries beginning at the identity.

Now, we can consider the loop $p'(t) = p(1-t) \in C(I, P_\infty(A/J))$ so in the lifting we have the beginning point x which lifts $p(1)$, by the definition of the path-lifting property, we are allowed choose x arbitrary with the only property that $\pi(x) = p(1)$. So we choose $x = \hat{p}(1)$.

(g) We have $\pi \circ \hat{p}(0) = p(0) \in P_\infty(\mathbb{C}1_{A/J})$ so if for a moment we consider $\hat{p}(0)$ and $p(0)$ as the corresponding entries, then $\hat{p}(0) + J = \lambda 1_A + J$, i.e. $\hat{p}(0) - \lambda 1_A \in J$. On the other hand, since we can consider \dot{J} as $j + \lambda 1_A$ where $j \in J$ and $\lambda \in \mathbb{C}$, we have $(\hat{p}(0) - \lambda 1_A) + \lambda 1_A \in \dot{J}$, i.e. $\hat{p}(0) \in \dot{J}$. Therefore, $\hat{p}(0) \in P_\infty(\dot{J})$. To show that $[\hat{p}(0)] - [\hat{p}(1)] \in K_0(J)$, we must show that $[\hat{p}(0)] - [\hat{p}(1)]$ belongs to kernel of the map introduced in (e). But this is clear because $\hat{p}(0) \stackrel{h}{\sim} \hat{p}(1)$ and $p(0) = p(1)$.

Now let \hat{p}' be another lift for p . By the definition of $\hat{p}(1)$ and $\hat{p}'(1)$, we have $\hat{p}(1) = \hat{p}'(1)$. On the other hand, $\hat{p}'(0) \sim \hat{p}'(1)$ and $\hat{p}(0) \sim \hat{p}(1)$ so $\hat{p}'(0) \sim \hat{p}(0)$, and $Twist(p)$ is indeed independent of the chosen lift.

(h) By (e), we can write

$$\begin{aligned} Twist(p) - Twist(q) &= [\hat{p}(0)] - [\hat{p}(1)] - ([\hat{q}(0)] - [\hat{q}(1)]) \mapsto \\ &[\hat{p}(0), p(0)] - [\hat{p}(1), p(1)] - ([\hat{q}(0), q(0)] - [\hat{q}(1), q(1)]). \end{aligned}$$

On the other hand, since $p(1) = q(1)$, we have $\hat{p}(1) = \hat{q}(1)$ by the definition of $\hat{p}(1)$ and $\hat{q}(1)$. Therefore

$$\begin{aligned} &[\hat{p}(0), p(0)] - [\hat{p}(1), p(1)] - ([\hat{q}(0), q(0)] - [\hat{q}(1), q(1)]) = \\ &[\hat{p}(0), p(0)] - [\hat{p}(1), p(1)] - ([\hat{q}(0), q(0)] - [\hat{p}(1), p(1)]) = \\ &[\hat{p}(0), p(0)] - [\hat{q}(0), q(0)]. \end{aligned}$$

(i) The desired homotopy is $t \mapsto (\hat{p}(t), p_t)$, where $p_t(s) := p(ts)$.

(j) By part (d), we have

$$\partial_1([p] - [q]) = [\hat{p}(1), p] - [\hat{q}(1), q].$$

By (i), $(\hat{p}(1), p) \stackrel{h}{\sim} (\hat{p}(0), p(0))$ in $P_\infty(C_\pi)$. On the other hand, when we have $[p] - [q] \in K_1(A/J)$, we can assume in fact that $q = s(p)$; hence, by Exercise 5.1.(a), we can write

$$[\hat{p}(1), p] - [\hat{q}(1), q] = [\hat{p}(0), p(0)] - [\hat{q}(0), q(0)].$$

Now, ∂_1 is the composite of the map introduced in (d) and the inverse of the map introduced in (e). By part (h) and the fact that the map $K_0(J) \rightarrow K_0(C_\pi(A, A/J))$ introduced in part (e) is an isomorphism (Lemma 9.1), we have

$$\partial_1([p] - [q]) = \text{Twist}(p) - \text{Twist}(q).$$

Exercise 2. (a) Let b be any lift of u . We put $a = bf(b^*b)$ where f is defined on $[0, \infty]$ by $f(x) = \frac{1}{x}$ if $x \geq 1$ and $f(x) = 1$ if $0 \leq x \leq 1$. Since $\|xf\|_\infty = \|f\|_\infty = 1$, $\|b^*bf(b^*b)\| = \|f(b^*b)\| = 1$. So $\|a\|^2 = \|f(b^*b)b^*bf(b^*b)\| \leq \|f(b^*b)\| \|b^*bf(b^*b)\| \leq 1$. Also, $\pi(a) = \pi(b)f(\pi(b^*b)) = uf(1) = u$.

(b) For any polynomial P , we have $aP(a^*a) = P(aa^*)a$. Since $\sigma(aa^*) \cup \{0\} = \sigma(a^*a) \cup \{0\} \subseteq [0, 1]$ and the function $x \mapsto (1-x)^{\frac{1}{2}}$ is the limit of a sequence of polynomials on $[0, 1]$, we have $a(1-a^*a)^{\frac{1}{2}} = (1-aa^*)^{\frac{1}{2}}a$. Similarly, $a^*(1-aa^*)^{\frac{1}{2}} = (1-a^*a)^{\frac{1}{2}}a^*$. Now, to prove that w is unitary is just a simple calculation. The equality $\pi((1-aa^*)^{\frac{1}{2}}) = \pi((1-a^*a)^{\frac{1}{2}}) = 0$ gives $\pi(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. We define $\overline{w(t)} =$

$$\begin{pmatrix} ta & -(1-t^2aa^*)^{\frac{1}{2}} \\ (1-t^2a^*a)^{\frac{1}{2}} & ta^* \end{pmatrix}$$
 where $t \in [0, 1]$. So $\overline{w(t)}$ is a path of unitaries which connects w to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \stackrel{h}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, there exists a path $w(t)$ of unitaries such that $w(0) = w$ and $w(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So the path $u(t) = \pi(w(t))$ is a path in $U_2(\frac{A}{J})$ which connects $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Using (a) and (b), we obtain that p is a projection. Since $1 - aa^*$ and $1 - a^*a$ are in J we can regard p with a projection in $P_{2n}(\dot{J})$. To show that $[p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$ is in $K_0(J)$, it is enough to show that this element is in the kernel of $\lambda_* : K_0(\dot{J}) \rightarrow \mathbb{Z}$ where λ_* is induced by $\lambda : \dot{J} \rightarrow \mathbb{C}$ where $\lambda(j, \alpha) = \alpha$. We have

$$\lambda_*([p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = 0$$

The image of $[u]$ under the isomorphism $K'_1(A/J) \rightarrow K_1(A/J)$ is $[e] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$, where $e : t \mapsto u(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t)^*$ and the second term is the constant loop $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The loop e has the lift $t \mapsto w(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w(t)^*$ and the constant loop $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has

the lift $t \xrightarrow{q} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So, since twist is independent of the particular lift chosen,

$$\text{twist}(p) = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

and

$$\text{twist}(q) = 0.$$

By Exercise 1(j) and the diagram

$$\begin{array}{ccc} K_1(A/J) & \xrightarrow{\partial_1} & K_0(J) \\ \uparrow \cong & \nearrow \partial'_1 & \\ K'_1(A/J) & & \end{array}$$

we have finally:

$$\partial'_1 = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

LECTURE 11 (29-04-04)

We first make some general remarks. Recall that the isomorphism $\mathcal{F} : L^2(\mathbb{T}) \xrightarrow{\sim} l^2(\mathbb{Z})$ is given by $f \mapsto \{\hat{f}(n)\}_{n \in \mathbb{Z}}$, where

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt, \quad n \in \mathbb{Z}$$

are the Fourier coefficients of f . The set $\{u_n\}_{n \in \mathbb{Z}}$, with $u_n(t) := e^{int}$, is an orthogonal basis for $L^2(\mathbb{T})$.¹ A trigonometric polynomial is a function of the form $p(t) = \sum_{n=-N}^N c_n u_n(t)$. The set of trigonometric polynomials is dense in $C(\mathbb{T})$ with respect to the sup norm.

Let $i : H^2(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ be the inclusion and, for $f \in L^\infty(\mathbb{T})$, let $M_f : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ be the induced multiplication operator on $L^2(\mathbb{T})$. Then $T_f = pM_f i$, and we can see that $T_f^* = T_{\bar{f}}$:

$$\begin{aligned} \langle T_f^*(u), v \rangle &= \langle u, pM_f i(v) \rangle = \langle u, M_f(i(v)) \rangle = \langle M_{\bar{f}}(u), i(v) \rangle \\ &= \langle M_{\bar{f}}(u), p i(v) \rangle = \langle pM_{\bar{f}}(u), v \rangle = \langle T_{\bar{f}}(u), v \rangle. \end{aligned}$$

We have that $\|T_f\| \leq \|p\| \|M_f\| \|i\| = \|f\|_\infty$ (see, e.g., Exercise 2.4 - Part I), and $T : C(\mathbb{T}) \rightarrow B(H^2), f \mapsto T_f$ is a linear and bounded *-map.

We also remark that $s^*(e_0) = 0$ and $s^*(e_n) = e_{n-1}$ for $n \geq 1$.

Exercise 1. One can see that $T_{u_1} : H^2(\mathbb{T}) \rightarrow H^2(\mathbb{T})$ (respectively, $T_{u_{-1}}$) is the operator which corresponds to $s : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ (respectively, s^*) under the isomorphism \mathcal{F} .

¹Note that functions on \mathbb{T} are in 1-1 correspondence with functions on \mathbb{R} that are periodic with period 2π , writing $f(t) = f(e^{it}), t \in \mathbb{R}$. On \mathbb{T} we have then that $u_n(z) = z^n$.

To show that \mathcal{T}' and \mathcal{T} are isomorphic it suffices to find a $*$ -isomorphism between the sets $\{T_p; p \text{ trigonometric polynomials}\}$ and the set of all finite linear combinations on the set $\{T_1 \dots T_n | n \in \mathbb{N}^*, T_k = s \text{ or } s^*\}$. Since $T_{u_n} = (T_{u_1})^n$ for $n \geq 0$ and $T_{u_n} = (T_{u_{-1}})^{|n|}$ for $n < 0$, we may send T_{u_1} to s and $T_{u_{-1}}$ to s^* and extend this map by linearity.

Exercise 2. Since $T : C(\mathbb{T}) \rightarrow B(H^2)$, $f \mapsto T_f$ is linear and bounded, we reduce to the case $f = u_{m'}$, $g = u_n$ for some $m', n \in \mathbb{Z}$. Without loss of generality we can assume that $m' = -m$ for $m \geq 1$ and $n \geq 0$. Then, for $T_{u_{-m}}T_{u_n} - T_{u_n}T_{u_{-m}}$, we have

$$s^{*m}s^n - s^n s^{*m} = s^{*m}s^n(1 - s^m s^{*m})$$

since $s^{*m}s^n s^m = s^n$. This is a finite rank operator since $1 - s^m s^{*m}$ is the projection onto $\{e_0, \dots, e_{m-1}\}$. The isomorphism between \mathcal{T} and \mathcal{T}' yields the conclusion.

Exercise 3. Let $T \in B_0(l^2(\mathbb{N}))$. Since the finite rank operators are dense in $B_0(l^2(\mathbb{N}))$, to show that P_n is an approximate unit for $B_0(l^2(\mathbb{N}))$, it is enough to show that $\|F - FP_n\| \rightarrow 0$ as $n \rightarrow \infty$ for an operator F of finite rank (and similarly, one sees that $\|F - P_n F\| \rightarrow 0$ as well). Such an F is a linear combination of rank one projections $x \otimes y$, where $x \otimes y(z) := \langle z, y \rangle x$. So suppose that $F = \sum_{i=0}^m x_i \otimes y_i$ and let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $l^2(\mathbb{N})$. Then $P_n = \sum_{j=0}^n e_j \otimes e_j$ and

$$\begin{aligned} \|F - FP_n\| &= \left\| \sum_i x_i \otimes y_i - \sum_{i,j} \langle e_j, y_j \rangle (x_i \otimes e_j) \right\| \\ &= \left\| \sum_i x_i \otimes y_i - \sum_i x_i \otimes \left(\sum_j \langle e_j, y_i \rangle e_j \right) \right\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Let now $T \in B(l^2(\mathbb{N}))$. For $i, j \in \mathbb{N}$, consider the operator E_{ij} on $l^2(\mathbb{N})$ given by $E_{ij}(x) = \langle x, e_j \rangle e_i$. Then one can see that $P_n = \sum_{i=1}^n E_{ii}$ and $P_n T P_n \in \text{span}\{E_{ij}\}_{0 \leq i, j \leq n}$. We have $P_1 = E_{11} = 1 - ss^*$ and $E_{ij} = s^{i-1} P_1 s^{*(j-1)}$, hence $E_{ij} \in \mathcal{T}$, so $P_n T P_n \in \mathcal{T}$.

If $T \in B_0(l^2(\mathbb{N}))$, then $T - P_n T P_n = T - T P_n + (T - P_n T) P_n \rightarrow 0$ as $n \rightarrow \infty$, since $P_n T \rightarrow T$ by the above considerations and since $\|P_n\| = 1$.

Exercise 4. Consider the composite

$$\alpha : C(\mathbb{T}) \rightarrow B(H^2(\mathbb{T})) \rightarrow B(H^2(\mathbb{T}))/B_0(H^2(\mathbb{T})), \quad \alpha(f) := \pi(T_f).$$

The map α is a *-homomorphism, since $T_f^* = T_{\bar{f}}$ and since $T_{fg} - T_f T_g$ is compact. This can be shown using the same argument as in Exercise 2. Since $\{T_f\}$ generates \mathcal{T}' , the image of α is $\mathcal{T}'/B_0(H^2(\mathbb{T}))$. It remains to show that α is injective. For this it suffices to show: T_f compact implies $f = 0$. We observe that $T_{u_{-1}} T_f T_{u_1} = T_{u_{-1} f T_{u_1}} = T_f$, hence

$$T_{u_{-1}}^m T_f T_{u_1}^m = T_f \quad \text{for all } m \geq 1.$$

We conclude, in particular, that $\|T_f\| \leq \|T_{u_1}^{*m} T_f\|$, so it is enough to show that $\|T_{u_1}^{*m} T_f\| \rightarrow 0$ as $m \rightarrow \infty$. Here is where we use compactness of T_f . It is enough to check the relation for operators of finite rank. Since any operator of finite rank is a linear combination of rank one projections, we reduce the problem to rank-one projections. For each $z \in l^2(\mathbb{N})$, we have $\|s^{*m}(z)\|^2 = \left\| \sum_{n \geq m} \langle z, u_n \rangle u_{n-m} \right\|^2 = \sum_{n \geq m} |\langle z, u_n \rangle|^2 \rightarrow 0$ as $m \rightarrow \infty$. This implies the claim for rank one projections and we are done.

Exercise 5. Let $f \in C(\mathbb{T})$ be a nowhere vanishing function. We

show first that T_f is a Fredholm operator. First, recall that T_f is a Fredholm operator iff $T_f + B_0(H^2)$ is invertible in the Calkin algebra $B(H^2)/B_0(H^2)$. Hence, by Exercise 4, T_f is Fredholm if and only if $T_f + B_0(H^2)$ is invertible in $\mathcal{T}'/B_0(H^2) \simeq C(\mathbb{T})$, that is, T_f is Fredholm if and only if f is invertible in $C(\mathbb{T})$.

Next, we show the Gohberg-Krein theorem for $f(z) = z^n$, $z \in \mathbb{T}$. With the notations introduced above, we have $f(u_1(t)) = u_n(t) = e^{i\pi nt}$ and $T_f = T_{u_n} = (T_{u_1})^n$. Recall that, using the isomorphism \mathcal{F} of Exercise 1, T_{u_n} corresponds to the shift operator in $l_2(\mathbb{N})$ and so we can compute the $Index(T_{u_n})$ by remarking that $\dim(\text{Ker}(s^n)) = 0$ (for $x \in l^2(\mathbb{N})$ written as $x = \sum_{i \geq 0} x_i e_i$, if $s^n(x) = \sum_{i \geq 0} x_i e_{i+n} = 0$ then $x = 0$) and that $\dim(\text{Ker}(s^{*n})) = n$ (recalling the definition of s^* , the condition $s^{*n}(x) = x_i e_{i-n} + a_{i+1} e_{i+1-n} + \dots = 0$ for $i \geq n$ implies $x_i = 0$ for $i \geq n$). We conclude that $Index(T_{u_n}) = -n$. On the other hand $\omega(f) = n$. Indeed, that follows directly from the definition of ω , because if the homotopy class of $[z \mapsto z]$ corresponds to 1 in \mathbb{Z} , the class of $[z \mapsto z^n]$ corresponds to n . we conclude that, for $f(z) = z^n$, $Index(T_f) = -\omega(f)$.

Now we use the continuity of the index (for a proof we refer to [6,ch2,2.1.6]) to see that for an arbitrary nowhere vanishing function $f \in C(\mathbb{T})$, the index of the operator T_f only depends on the homotopy class of the function $f \in \pi(\mathbb{T})$ (since $f \mapsto T_f$ is continuous). We claim that any such (non-vanishing) function is homotopic to u_n , for some n . To do that, we first show the following lemma:

Lemma If f is an invertible function in $C(\mathbb{T})$, then there exists a unique $n \in \mathbb{Z}$ such that $f(u_1(t)) = u_n(t) e^{\psi(u_1(t))}$ for some $\psi \in C(\mathbb{T})$.

Proof. As remarked above, the set of trigonometric polynomials is dense in $C(\mathbb{T})$. So it is enough to show the statement for $p(t) = \sum_{|n| \leq N} c_n u_n(t)$ for some $n \in \mathbb{Z}$ and some $c_n \in \mathbb{C}$. Moreover, we can only consider $p(t)$ being of the form $p(t) = \sum_{n \geq 0} c_n u_n(t)$ (multiplying by $u_N(t)$, if needed). In this form, p is a polynomial in u_1 and so a product of a constant and factors of the form $u_1 - \lambda$ where $\lambda(t) \notin \mathbb{T}$ for all t . Therefore we can further reduce the problem to the case $p = (u_1 - \lambda)$ with $\lambda(t) \neq 1$. If $|\lambda| < 1$, then $\|p - u_1\| = |\lambda| < 1 = \|u_1^{-1}\|^{-1}$. So $p(t)$ is of the form $u_1(t)e^{\psi(u_1(t))}$, for some $\psi \in C(\mathbb{T})$ (recall that if $\|1 - p\| < 1$, then $p = e^\psi$ for some $\psi \in C(\mathbb{T})$ because we can take $\psi = \ln \circ p$ where $\ln : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ is the principal branch of the logarithm function). Otherwise, if $|\lambda| > 1$, then $\|(1 - \lambda u_1)^{-1}\| < 1$ so $1 - \lambda^{-1}u_1$ is of the form e^ψ for some $\psi \in C(\mathbb{T})$, hence $p = -\lambda e^{\psi(u_1)} = e^{\psi'(u_1)}$ for some $\psi' \in C(\mathbb{T})$. \square

Now, consider the homotopy $H : [0, 1] \times \mathbb{T} \rightarrow \mathbb{C}^*$, $(s, z) \mapsto z^n e^{s\psi(z)}$, which connects $f(u_1(t))$ and $u_1^n(t)$. Therefore every invertible function $f \in C(\mathbb{T})$ is homotopic to u_1^n for some (unique) n . The continuity of the index gives now that $\text{Index}(T_f) = \text{Index}(T_{u_n}) = n$, and, since ω is also an homotopy invariant, $\omega(f) = \omega(u_n) = -n$. This finishes the proof.

Alternatively, one could argue as follows. The continuity of the index map implies that $\text{Index}(T_{e^\psi}) = 0$. Indeed, consider $A_t = T_{e^{t\psi(u_1(t))}}$ which is a path of Fredholm operators connecting Id and T_{e^ψ} . The continuity of index implies that $\text{Index}(T_{e^{t\psi}})$ is constant on this path and hence $\text{Index}(T_{e^\psi}) = \text{Index}(Id) = 0$. Now, $T_f = T_{u_n e^\psi} = T_{u_n} T_{e^\psi} + \text{compact}$ and therefore, by the invariance of the index under compact

perturbations,

$$\text{Index}(T_f) = \text{Index}(T_{u_n} T_{e^\psi}) = \text{Index}(T_{u_n}) + \text{Index}(T_{e^\psi}) = n.$$

From this considerations we are done, because in order to compute the index of an arbitrary T_f we only need to calculate the Index of $T_{e^{\omega(f)}}$.