## Lecture 2 (2-10-03)

1.(a) $C(X)$ is a vector space, defining

$$(f+g)(x) := f(x) + g(x)$$

$$(\lambda f)(x) := \lambda f(x)$$

for $f, g \in C(X)$, $\lambda \in \mathbb{C}$, $x \in X$.

(b) Clearly $\|f\|_\infty \geq 0$ and $\|f\|_\infty = 0$ iff $f(x) = 0$ for all $x \in X$, that is, iff $f = 0$ in $C_b(X)$. It is also obvious that $\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty$, for any $\lambda \in \mathbb{C}$. We are left with showing that the triangle inequality holds. But, since $|f(x) + g(x)| \leq |f(x)| + |g(x)|$, for any $f, g \in C_b(X)$, $x \in X$, we also have that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C_b(X)$. Then for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for any $n, m > N$,

$$\|f_n - f_m\|_\infty < \epsilon \Leftrightarrow |f_n(x) - f_m(x)| < \epsilon, \text{ for all } x \in X. \tag{1}$$

In particular, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy, for any $x \in X$, and therefore converges in $\mathbb{C}$. Let $f(x) := \lim f_n(x)$. Letting $m \to \infty$ in (1), we have that for all $n > N$,

$$|f_n(x) - f(x)| < \epsilon, \text{ for all } x \in X \Rightarrow \|f_n - f\|_\infty < \epsilon.$$

We then have that $f$ is bounded, since, for some fixed $n > N$ and any $x \in X$,

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \epsilon + \|f_n\|_\infty < M,$$

and continuous, since $f_n$ is and

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)|.$$

We conclude that $f \in C_b(X)$ and $f_n \to f$ in $C_b(X)$, and hence $C_b(X)$ is complete.

(d) Let $f, g, h \in C_b(X)$. Then,

$$(fg)(x) = (gf)(x)$$

$$(fg)h(x) = f(x)g(x)h(x) = f(gh)(x)$$

$$(f+g)h(x) = f(x)h(x) + g(x)h(x) = (fh+gh)(x)$$

that is, multiplication is commutative, associate and distributive with respect to addition. Therefore $C_b(X)$ is a commutative algebra.

(e) We have that, for any $f, g \in C_b(X)$,

$$\|f\,g\|_\infty = \sup_{x\in X} |f(x)g(x)| = \sup_{x\in X} |f(x)|\,|g(x)| \le \sup_{x\in X} |f(x)| \sup_{x\in X} |g(x)| = \|f\|_\infty \|g\|_\infty.$$

(f) For $f, g \in C_b(X)$, $\lambda \in \mathbb{C}$:

$$(fg)^*(x) = \overline{(fg)(x)} = \overline{f(x)g(x)} = f^*(x)g^*(x) = g^*(x)f^*(x)$$

$$f^{**}(x) = \overline{\overline{f(x)}} = f(x)$$

$$(\lambda f)^*(x) = \overline{\lambda f(x)} = \overline{\lambda} f^*(x).$$

(g) Just note that $|f^*(x)\,f(x)| = |\overline{f(x)}f(x)| = |f(x)|^2$, and therefore

$$\|f^*f\|_\infty = \sup_{x\in X} |f(x)|^2 = \left(\sup_{x\in X} |f(x)|\right)^2 = \|f\|_\infty^2.$$

(h) Need to show that $C(X) \subset C_b(X)$, that is that every continuous function on a compact space $X$ is bounded. But the image of a compact set under a continuous function is compact, that is, for any $f \in C(X)$, $f(X) \subset \mathbb{C}$ is compact and hence bounded.

2.(a) $M_n(\mathbb{C})$ is a vector space with

$$a + b := (a_{ij} + b_{ij}), \quad \lambda a := (\lambda a_{ij})$$

for $a = (a_{ij}), b = (b_{ij}) \in M_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and it is clearly an algebra with respect to the usual matrix multiplication

$$(ab)_{ij} := \sum_{k=1}^n a_{ik}b_{kj}, \quad i, j = 1, ..., n.$$

(b) Write $(ab)^* = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^n \overline{a_{jk}b_{ki}} = \sum_{k=1}^n a^*_{kj}b^*_{ik} = (b^*a^*)_{ij}$$

and we have $(ab)^* = b^* a^*$, for all $a, b \in M_n(\mathbb{C})$. Also,

$$(\lambda a)^*_{ij} = \overline{\lambda a_{ji}} = \overline{\lambda} a^*_{ij}$$

$$a^{**}_{ij} = (\overline{a_{ji}})^* = \overline{\overline{a_{ij}}} = a_{ij}.$$

(c) (i) We clearly have $\|a\| \geq 0$, for $a \in M_n(\mathbb{C})$, and $\|a\| = 0$ iff $az = 0$ for all $z \in \mathbb{C}$, that is, iff $a = 0$. Also, since $\|(\lambda a)z\| = |\lambda| \, \|az\|$, for any $a \in M_n(\mathbb{C})$, $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, we have $\|\lambda a\| = |\lambda| \, \|a\|$. The triangle inequality follows from the fact that $\|(a+b)z\| \leq \|az\| + \|bz\|$, for any $a, b \in M_n(\mathbb{C})$, $z \in \mathbb{C}^n$.

To check the last assertion, write $\|z\|^2 = (z, z)$, where $(z, w) := \sum_{k=1}^n \overline{z_k} w_k$ is the usual inner product in $\mathbb{C}^n$. It is easy to check that for any $a \in M_n(\mathbb{C})$, $z, w \in \mathbb{C}^n$, $(az, w) = (z, a^* w)$, with $a^*$ defined as in (b). Then,

$$\|az\|^2 = (az, az) = (z, a^* az)$$

and if $\lambda$ is an eigenvalue of $a^* a$ and $v$ an eigenvector, with $\|v\| = 1$, we have

$$\|a\|^2 \geq \|av\|^2 = \lambda(v, v) = \lambda \Rightarrow \|a\|^2 \geq \max \lambda$$

(and in particular we get that $\lambda \in \mathbb{R}^+$). On the other hand, let $\{v_k\}$ be a basis of orthonormal eigenvectors of $a^* a$, with respect to eigenvalues $\lambda_k$, $k = 1, ..., n$ (such a basis exists since $a^* a$ is self-adjoint). Writing $z = \sum_{k=1}^n c_k v_k$, we have

$$\|az\|^2 = \sum_{j=1}^n \sum_{k=1}^n \overline{c_k} \, c_j (v_k, \lambda_j v_j) = \sum_{k=1}^n |c_k|^2 \lambda_k \leq \max \lambda_k \, \|z\|$$

and therefore

$$\|a\|^2 \leq \max \lambda_k.$$

(ii) Let $(a_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in $M_n(\mathbb{C})$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k, m > N$,

$$\|a_k - a_m\| < \epsilon \Leftrightarrow |a_k z - a_m z| < \epsilon, \text{ for all } z \in \mathbb{C}^n, \ \|z\| = 1. \tag{2}$$

In particular, the sequence $(a_k z)_{k \in \mathbb{N}}$ is Cauchy, for any $z \in \mathbb{C}^n$, $\|z\| = 1$, and therefore converges in $\mathbb{C}$. For $\|z\| = 1$, let $az := \lim a_k z$; this defines $a \in M_n(\mathbb{C})$. Letting $m \to \infty$ in (2), we have that for all $k > N$,

$$|a_k z - az| < \epsilon, \text{ for all } z \in \mathbb{C}^n, \ \|z\| = 1 \Rightarrow \|a_k - a\| < \epsilon.$$

We conclude that $a_k \to a$ in $M_n(\mathbb{C})$, and hence that $M_n(\mathbb{C})$ is complete.

(iii) Have, for any $z \in \mathbb{C}^n$, $\|abz\| \leq \|a\|\,\|bz\| \leq \|a\|\,\|b\|\,\|z\|$, and therefore

$$\|ab\| = \sup_{\|z\|=1} \|abz\| \leq \|a\|\,\|b\|.$$

(iv) For $a \in M_n(\mathbb{C})$, $z \in \mathbb{C}^n$,

$$\|az\|^2 = (az, az) = (z, a^*az) \leq \|z\|\|a^*az\| \leq \|z\|^2\|a^*a\|$$

and therefore

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\,\|a\| \tag{3}$$

and we have then that

$$\|a\| \leq \|a^*\|.$$

Since $a^{**} = a$, equality follows.

(v) From $\|a\| = \|a^*\|$ and (3)

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\,\|a\| = \|a\|^2$$

and therefore $\|a\|^2 = \|a^*a\|$, for all $a \in M_n(\mathbb{C})$.

3.(a) $B(H)$ is a vector space with

$$(a+b)z := az + bz,\ (\lambda a)z := \lambda az$$

for $a, b \in B(H)$, $\lambda \in \mathbb{C}$, $z \in H$, and an algebra endowed with composition of operators.

(b) Let $a, b \in B(H)$, $z, w \in H$. Have that

$$(z, b^*a^*w) = (bz, a^*w) = (abz, w) \Rightarrow (ab)^* = b^*a^*;$$

$$(z, a^{**}w) = (a^*z, w) = \overline{(w, a^*z)} = \overline{(aw, z)} = (z, aw) \Rightarrow a^{**} = a;$$

$$(z, (\lambda a)^*w) = (\lambda az, w) = \overline{\lambda}(az, w) = (z, \overline{\lambda}a^*w) \Rightarrow (\lambda a)^* = \overline{\lambda}a^*.$$

(c) (i) Showing that $\|\ \|$ is indeed a norm is done in exactly the same way as in Ex.2.

**Note** It is no longer true that $\|a\|^2$ coincides with the norm of the largest eigenvalue of $a^*a$; in fact, $a^*a$ may not have any eigenvalues (see Exercise

4, Lecture 5). We have however that $\|a\|^2 = r(a^*a)$, where $r(a^*a)$ is the spectral radius of $a^*a$ (see Exercise 1, Lecture 5).

(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(H)$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n, m > N$,

$$\|a_n - a_m\| < \epsilon \Leftrightarrow |a_n z - a_m z| < \epsilon, \text{ for all } z \in H, \|z\| = 1. \tag{4}$$

In particular, the sequence $(a_n z)_{n\in\mathbb{N}}$ is Cauchy in $H$, for any $z \in H$, $\|z\| = 1$, and it converges since $H$ is complete. For $\|z\| = 1$, let $az := \lim a_n z$; this defines a linear operator $a : H \to H$. Letting $m \to \infty$ in (4), we have that for all $n > N$,

$$|a_n z - az| < \epsilon, \text{ for all } z \in H, \|z\| = 1 \Rightarrow \|a_n - a\| < \epsilon.$$

In particular, since $a_n$ is bounded, we see that $a$ is also bounded, and we conclude that $a_n \to a$ in $B(H)$, and hence that $B(H)$ is complete.

(iii) (Exactly as in Ex.2.)

(iv) (Exactly as in Ex.2.)

(v) (Exactly as in Ex.2.)

4.(a) For any $\psi \in L^2(X;\mu)$,

$$\|\pi(f)\psi\|_2^2 = \int_X d\mu(x)|f(x)|^2\|\psi(x)|^2 \leq (\|f\|_\infty^\mu)^2 \|\psi\|_2^2$$

and hence $\pi(f)$ is bounded, with $\|\pi(f)\| \leq \|f\|_\infty^\mu$. To prove equality, let $L < \|f\|_\infty^\mu$ be arbitrary. Then, by definition of $\| \ \|_\infty^\mu$,

$$\mu\left(\{x \in X : |f(x)| > L\}\right) > 0$$

and since $\mu$ is $\sigma$-finite, there is a measurable $Y \subset \{x \in X : |f(x)| > L\}$ with $0 < \mu(Y) < \infty$. Let $\chi_Y \in L^2(X;\mu)$ be the characteristic function of $Y$. We have that

$$\|\pi(f)\chi_Y\|_2^2 = \int_X d\mu(x)|f(x)|^2\|\chi_Y(x)|^2 \geq L^2\|\chi_Y\|_2^2.$$

We conclude that $\|\pi(f)\| \geq L$, for all $L < \|f\|_\infty^\mu$ and therefore that $\|\pi(f)\| = \|f\|_\infty^\mu$.

(b) It is clear that $\pi$ is linear and that $\pi(fg) = \pi(f)\pi(g)$, for any $f, g \in C_b(X)$. Moreover,

$$(\pi(f^*)\psi, \phi) = \int_X d\mu(x)\pi(f^*)\psi(x)\overline{\phi(x)} = \int_X d\mu(x)\psi(x)\overline{f(x)\phi(x)} = (\psi, \pi(f)\phi),$$

for all $\psi, \phi \in L^2(X; \mu)$, and hence $\pi(f)^* = \pi(f^*)$.

(c) We show first that $\|f\|_\infty^\mu = \|f\|_\infty$ for all $f \in C_b(X)$ iff $\mu(U) > 0$, for any open $U \subset X$, $U \neq \emptyset$. Let then $U \subset X$ be open, $U \neq \emptyset$, and $V \subset U$ be closed. By Urysohn's lemma, there exists $f \in C_b(X)$ such that $0 \leq f \leq 1$, $f = 1$ on $V$ and $\operatorname{supp} f \subset U$. Then $\|f\|_\infty = 1$ and if $\|f\|_\infty^\mu = \|f\|_\infty$, we conclude that $\mu(U) > 0$ (if $\mu(U) = 0$ then $\|f\|_\infty^\mu = 0$, since $f = 0$ on $X \backslash U$). Conversely, assume that any non-empty open set in $X$ has non-zero measure and let $f \in C_b(X)$. (Note that one always has $\|f\|_\infty^\mu \leq \|f\|_\infty$.) For any $\epsilon > 0$, the set

$$U_\epsilon := \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$$

is open and $U_\epsilon \neq \emptyset$. Hence $\mu(U_\epsilon) > 0$ and therefore we have that $\|f\|_\infty^\mu > \|f\|_\infty - \epsilon$,[1] for any $\epsilon > 0$, that is, $\|f\|_\infty^\mu \geq \|f\|_\infty$.

We can now see that this condition is equivalent to $\operatorname{supp}(\mu) = X$. Recall that the *support* of a measure $\mu$ is the smallest closed subset $A \subset X$ such that $\mu(E) = \mu(E \cap A)$, for all measurable $E \subset X$. Note that in this case, $\mu(X \backslash A) = 0$ and $X \backslash A$ is open. Hence, if every non-empty open set in $X$ has positive measure, we conclude that $X \backslash A = \emptyset$, that is, $\operatorname{supp} \mu = X$. Now if $\operatorname{supp} \mu = X$, we have that for any open $U \subset X$, $U \neq \emptyset$, there is a measurable set $E \subset X$ such that $\mu(E \cap U) > 0$, since $\mu(E) = \mu(E \cap U) + \mu(E \cap (X \backslash U))$ and $X \backslash U$ is not the support of $\mu$. Therefore $\mu(U) \geq \mu(E \cap U) > 0$.[2]

(d) We have in this case $\|\pi(f)\| = \|f\|_\infty^\mu = \|f\|_\infty$ and therefore $\pi(f) = 0$ iff $\|\pi(f)\| = 0$ iff $\|f\|_\infty = 0$ iff $f = 0$, that is, $\pi$ is injective.

(e) Let $\mu$ be a measure such that the condition in (c) is satisfied and let $A := \pi(C_b(X))$. From (b), we have that $\pi$ is a $*$-homomorphism, that is, $A$ is a $*$-subalgebra of $B(H)$, and, since $\pi$ is bijective, $A$ and $C_b(X)$ are isomorphic as $*$-algebras. Now, since $\pi$ is isometric, $A$ is closed and $C_b(X) \cong A$

[1] Note that $\|f\|_\infty^\mu$ is the smallest $c$ such that $\mu(\{x \in X : |f(x)| > c\}) = 0$.

[2] It can be easily seen that the Lebesgue measure on $\mathbb{R}^n$ satisfies this condition and it can in fact be shown that such a measure always exists on arbitrary locally compact spaces.

as $C^*$-algebras.

## Lecture 3 (9-10-03)

1.(a) Let $f, g \in L^1(\mathbb{R})$. We have

$$\int_{\mathbb{R}} dx\, |f * g(x)| = \int_{\mathbb{R}} dx \left| \int_{\mathbb{R}} dy\, f(x-y)g(y) \right| \leq \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, |f(x-y)|\, |g(y)|.$$

Since $(x, y) \mapsto f(x-y)g(y)$ is measurable on $\mathbb{R}^2$, and

$$\begin{aligned} \int_{\mathbb{R}} dy \int_{\mathbb{R}} dx\, |f(x-y)||g(y)| &= \int_{\mathbb{R}} dy\, |g(y)| \int_{\mathbb{R}} dx\, |f(x-y)| \\ &= \int_{\mathbb{R}} dy\, |g(y)| \int_{\mathbb{R}} dz\, |f(z)| = \|g\|_1 \|f\|_1 < \infty, \end{aligned}$$

we have from Fubini's theorem that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|g\|_1 \|f\|_1. \tag{5}$$

It is easy to check that $L^1(\mathbb{R})$ is an algebra, and, noting that the Lebesgue measure on $\mathbb{R}$ is invariant under translations and transformations $x \mapsto -x$,

$$g * f\,(x) = \int_{\mathbb{R}} dy\, g(x-y)f(y) = \int_{\mathbb{R}} dz\, g(z)f(x-z) = f * g\,(x),$$

that is, $L^1(\mathbb{R})$ is commutative. Since it is complete and (5) holds, we have finally that $L^1(\mathbb{R})$ is a commutative Banach algebra.

(b) First note that any functional $w \in \Delta(L^1(\mathbb{R}))$ is bounded, with $\|w\| \leq 1$ [1], and therefore $\Delta(L^1(\mathbb{R})) \subset (L^1(\mathbb{R}))^*$. There is an isomorphism $(L^1(\mathbb{R}))^* \cong L^\infty(\mathbb{R})$ [2], and therefore for each $w \in \Delta(L^1(\mathbb{R}))$ there exists a uniquely defined $\widehat{w} \in L^\infty(\mathbb{R})$ such that

$$w(f) = \int_{\mathbb{R}} dx f(x)\widehat{w}(x). \tag{6}$$

We will show that $\widehat{w}(x) = e^{itx}$, for some $t \in \mathbb{R}$ and that the correspondence $t \mapsto e^{itx}$ defines an isomorphism between $\mathbb{R}$ and $\Delta(L^1(\mathbb{R}))$.

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[1] This is true for unital algebras, and the result on the non-unital case follows easily; see also Exercise 5, Lecture 6.

[2] See for instance Rudin, Real and complex analysis, page 127.

Let $f, g \in L^1(\mathbb{R})$ and $w \in \Delta(L^1(\mathbb{R}))$, $\widehat{w} \in L^\infty(\mathbb{R})$ be given as in (6). We have that

$$\begin{aligned} w(f * g) &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy f(x-y) g(y) \widehat{w}(x) \\ &= \int_{\mathbb{R}} dy\, g(y) \int_{\mathbb{R}} dx f(x-y) \widehat{w}(x) \\ &= \int_{\mathbb{R}} dy\, g(y) w(f_y), \end{aligned}$$

where $f_y(x) := f(x-y) \in L^1(\mathbb{R})$. On the other hand,

$$w(f)w(g) = \int_{\mathbb{R}} dy\, g(y) \widehat{w}(y) w(f),$$

and since $w$ is multiplicative, we have that

$$\widehat{w}(y) w(f) = w(f_y), \text{ a.e.}(dy). \tag{7}$$

Now $w : L^1(\mathbb{R}) \to \mathbb{C}$ is bounded and for each fixed $f \in L^1(\mathbb{R})$, the map $y \in \mathbb{R} \to f_y \in L^1(\mathbb{R})$ is continuous. Taking $f$ such that $w(f) \neq 0$, we conclude that $\widehat{w}$ is continuous on $\mathbb{R}$.[1]

We now show that $\widehat{w}(x+y) = \widehat{w}(x)\widehat{w}(y)$. Note first that for any $f \in L^1(\mathbb{R})$ such that $w(f) \neq 0$, $w(f_y)w(f)^{-1} = \widehat{w}(y)$ does not depend on $f$, and therefore $w(f_y)w(f)^{-1} = w(g_y)w(g)^{-1}$, for all $y \in \mathbb{R}$, $f, g \in L^1(\mathbb{R})$ such that $w(f) \neq 0$, $w(g) \neq 0$. In particular, since $w$ is multiplicative, we get $w(f_y * g) = w(f * g_y)$. Also, for any such $f$, $w(f_y) \neq 0$ for any $y \in \mathbb{R}$, since $w(f)^2 = w(f * f) = w(f_y * f_{-y}) = w(f_y)w(f_{-y}) \neq 0$. In this case, we have

$$\widehat{w}(x+y) = w(f_{x+y})w(f)^{-1} = w((f_x)_y))w(f_x)^{-1}w(f_x)w(f)^{-1}, \text{ a.e.},$$

and therefore that

$$\widehat{w}(x+y) = \widehat{w}(x)\widehat{w}(y) \text{ a.e}, \tag{8}$$

and since $w$ is continuous, equality holds for all $x, y \in \mathbb{R}$.

**Notes** 1. Property (8) could have been derived directly from the multiplicativity of $w$: note that for any $f, g \in L^1(\mathbb{R})$,

$$\begin{aligned} w(f * g) &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy f(x-y) g(y) \widehat{w}(x) \\ &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy f(x) g(y) \widehat{w}(x+y) \end{aligned}$$

[1] That is, $w$ has a continuous representative and we can therefore assume that $w \in C(\mathbb{R})$.

and on the other hand

$$w(f)w(g) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, f(x)g(y)\widehat{w}(x)\widehat{w}(y).$$

We then have that for any $g \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} dy\, g(y)\,(\widehat{w}(x+y) - \widehat{w}(x)\widehat{w}(y)) = 0 \text{ a.e}(dx),$$

and hence that $\widehat{w}(x+y) = \widehat{w}(x)\widehat{w}(y)$ a.e.

2. It is true that a measurable function satisfying (8) is necessarily continuous; the direct proof of this result is however far from trivial. This is the reason we proceed in this fashion.

Now we know that $\widehat{w}$ is bounded and we have that, for any $n \in \mathbb{Z}$,

$$\widehat{w}(nx) = \widehat{w}(x)^n$$

(note that $\widehat{w}(0) = 1$ and therefore $\widehat{w}(-x) = \widehat{w}(x)^{-1}$). We conclude that $|\widehat{w}(x)| = 1$ for all $x \in \mathbb{R}$. We have then shown that $\widehat{w}$ is a continuous map $\mathbb{R} \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, which satisfies $\widehat{w}(x+y) = \widehat{w}(x)\widehat{w}(y)$.[1] At this point, we could use the known result that any character of $\mathbb{R}$ is of the form $e^{itx}$ for some $t \in \mathbb{R}$; we give here a proof of this fact. Let $V := \{e^{i\theta} : |\theta| < \pi/2\} \subset \mathbb{T}$; since $\widehat{w} : \mathbb{R} \to \mathbb{T}$ is continuous, we can take an open $U \subset \mathbb{R}$ such that $U \subset \widehat{w}^{-1}(V)$ and can assume that $U = (-L, L)$, $L > 0$. Now $arg(e^{i\theta}) = \theta$ defines a homeomorphism $arg : V \to (-\pi/2, \pi/2)$ and for $x, y \in U$ such that $x+y \in U$, write $\widehat{w}(x) = e^{i\theta_x} \in V$, $\widehat{w}(y) = e^{i\theta_y} \in V$. Then $\widehat{w}(x+y) = \widehat{w}(x)\widehat{w}(y) = e^{i(\theta_x + \theta_y)} \in V$ and we have

$$arg(\widehat{w}(x+y)) = \theta_x + \theta_y = arg(\widehat{w}(x)) + arg(\widehat{w}(y)).$$

Therefore, $arg \circ \widehat{w}$ is a continuous function on $\mathbb{R}$ such that $(arg \circ \widehat{w})(nx) = n\ arg \circ \widehat{w}(x)$, for any $x \in U$, $n \in \mathbb{Z}$ with $nx \in U$ and by density, we have that $arg \circ \widehat{w}(x) = cx$ for all $x \in U$. Since $arg$ is a homeomorphism, we have then that $\widehat{w}(x) = e^{cx}$, for $x \in U$, and since $|\widehat{w}(x)| = 1$, we get $c = it$, for some $t \in \mathbb{R}$. But $U = (-L, L) \subset \mathbb{R}$ generates $\mathbb{R}$ as an additive group and hence, from (8), we have $\widehat{w}(x) = e^{itx}$, for all $x \in \mathbb{R}$.

We conclude finally that any $w \in \Delta(L^1(\mathbb{R}))$ is given by

$$w(f) = \int_{\mathbb{R}} dx\, f(x)\, e^{itx}, \tag{9}$$

[1] Such a map is called a *character* of $\mathbb{R}$.

for some $t \in \mathbb{R}$. It is easy to check that, for any $t \in \mathbb{R}$, $w_t : L^1(\mathbb{R}) \to \mathbb{C}$ defined by (9) yields an element of $\Delta(L^1(\mathbb{R}))$ and that $w_t = w_s$ iff $t = s$. We thus have a well-defined, bijective map

$$\Phi : \mathbb{R} \to \Delta(L^1(\mathbb{R})), \ \ t \mapsto w_t$$

$$w_t(f) = \int_{\mathbb{R}} dx f(x) e^{itx}, \ f \in L^1(\mathbb{R}).$$

We now check that this map is a homeomorphism. If $t_n \to t$, with $t_n, t \in \mathbb{R}$ then, for any $f \in L^1(\mathbb{R})$, $w_{t_n}(f) \to w_t(f)$, by the Lebesgue Dominated Convergence Theorem, and therefore $w_{t_n} \to w$ in the weak*-topology. We conclude that $\Phi$ is continuous. To check that it is bicontinuous, let $w_{t_i}$ be a net in $\Delta(L^1(\mathbb{R}))$ such that $w_{t_i} \to w_t$ in the weak*-topology. Then for any $f \in L^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} dx f(x) e^{it_i x} \to \int_{\mathbb{R}} dx f(x) e^{itx}.$$

Let $f(x) := \chi\, e^x$, where $\chi$ is the characteristic function of $(-\infty, 0)$. Then $f \in L^1(\mathbb{R})$ and, for any $s \in \mathbb{R}$,

$$\int_{\mathbb{R}} dx f(x) e^{isx} = \int_{-\infty}^{0} dx e^{(is+1)x} = \frac{1}{is+1}.$$

Hence,

$$\frac{1}{it_i + 1} \to \frac{1}{it+1} \ \Leftrightarrow \ t_i \to t$$

and $\Phi^{-1}$ is also continuous.

(c) The Gelfand transform is the map

$$L^1(\mathbb{R}) \to C(\Delta(L^1(\mathbb{R})), \ \ f \mapsto \widehat{f}$$

where $\widehat{f}(\omega) = \omega(f)$. Identifying $\Delta(L^1(\mathbb{R}))$ with $\mathbb{R}$ as in (b), we then have that $\widehat{f}(t) = \int_{\mathbb{R}} dx f(x) e^{itx}$, which is nothing else but the Fourier transform of $f$.

2.(a) The space $L^1(\mathbb{Z})$ is the space of sequences $(a_n)_{n\in\mathbb{Z}}$ of complex numbers such that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$; it is a Banach space endowed with the norm $\|a\|_1 = \sum_{n\in\mathbb{Z}} |a_n|$.

Define convolution in $L^1(\mathbb{Z})$ by $(a*b)_n := \sum_{m\in\mathbb{Z}} a_{n-m}b_m$, for $a,b \in L^1(\mathbb{Z})$. We have

$$\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}} a_{n-m}b_m\right| \leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |a_{n-m}||b_m| = \sum_{m\in\mathbb{Z}} |b_m| \sum_{n\in\mathbb{Z}} |a_{n-m}| = \|a\|_1\|b\|_1$$

and therefore $a*b \in L^1(\mathbb{Z})$, with $\|a*b\|_1 \leq \|a\|_1\|b\|_1$. It is clear that $L^1(\mathbb{Z})$ equipped with $*$ is a commutative algebra, and, since it is complete, it is a Banach algebra. It is easy to check that the element $e_0 \in L^1(\mathbb{Z})$ such that $(e_0)_n = 0$, for $n \neq 0$ and $(e_0)_0 = 1$ is an identity for $*$ and $L^1(\mathbb{Z})$ is unital.

(b) We will show that $\Delta(L^1(\mathbb{Z})) \cong \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Let $e_k \in L^1(\mathbb{Z})$, for $k \in \mathbb{Z}$, be such that $(e_k)_n = \delta_{kn}$. For any $a \in L^1(\mathbb{Z})$ we can write $a = \sum_{n\in\mathbb{Z}} a_n e_n$, and for any $w \in \Delta(L^1(\mathbb{Z}))$ we have that

$$w(a) = \sum_{n\in\mathbb{Z}} a_n w(e_n).\text{[1]} \tag{10}$$

It is easy to check that $e_k * e_m = e_{m+k}$, for all $k,m \in \mathbb{Z}$. In particular, $e_k = (e_1)^k$ and, since $w$ is multiplicative, we get

$$w(a) = \sum_{n\in\mathbb{Z}} a_n \left(w(e_1)\right)^n.$$

We conclude that each $w \in \Delta(L^1(\mathbb{Z}))$ is uniquely determined by $w(e_1)$ (in the sense that if $w, w' \in \Delta(L^1(\mathbb{Z}))$ and $w(e_1) = w'(e_1)$, then $w = w'$). We now show that $w(e_1) \in \mathbb{T}$. Since $\|e_k\| = 1$, for any $k \in \mathbb{Z}$ and $\|w\| = 1$ (because $w \in \Delta(L^1(\mathbb{Z}))$, we have

$$|w(e_k)| \leq \|w\|\|e_k\| = 1, \text{ for all } k \in \mathbb{Z}.$$

Noting that $w(e_{-1}*e_1) = w(1) = 1$, we have also that $|w(e_1)| = |w(e_{-1})|^{-1} \geq 1$ and therefore $|w(e_1)| = 1$ and $w(e_1) \in \mathbb{T}$. We thus have a well-defined, injective map

$$\Phi : \Delta(L^1(\mathbb{Z})) \to \mathbb{T}, \quad w \mapsto w(e_1).$$

On the other hand, for each $z \in \mathbb{T}$ one can define

$$w_z(a) = \sum_{n\in\mathbb{Z}} a_n z^n, \; a \in L^1(\mathbb{Z})$$

[1] Note that we showed that any $w \in (L^1(\mathbb{Z}))^*$ is given by $w(a) = \sum a_n \widehat{w}_n$, where $\widehat{w} \in L^\infty(\mathbb{Z})$; compare with Exercise 1.

and it is easy to check that $w_z \in \Delta(L^1(\mathbb{Z}))$ with $w_z(e_1) = z$. Hence $\Phi$ is also surjective. Now, directly from the definition of the weak-$*$ topology, we have that $\Phi$ is continuous [1]. Since $\Delta(L^1(\mathbb{Z}))$ is compact (and $\mathbb{T}$ is Hausdorff), we conclude that $\Phi$ is a homeomorphism.

(c) The Gelfand transform is the map

$$L^1(\mathbb{Z}) \to C(\Delta(L^1(\mathbb{Z})), \;\; a \mapsto \widehat{a},$$

where $\widehat{a}(w) = w(a)$, $w \in \Delta(L^1(\mathbb{Z})$. Under the isomorphism given in (b) we have that $z \in \mathbb{T} \mapsto w_z \in \Delta(L^1(\mathbb{Z}))$ and we can write $\widehat{a} \in C(\mathbb{T})$ such that

$$\widehat{a}(z) = w_z(a) = \sum_{n \in \mathbb{Z}} a_n z^n.$$

Now if $\widehat{a}(n)$ denotes the $n$-th Fourier coefficient of $\widehat{a}$, we have

$$\widehat{a}(n) = \frac{1}{2\pi} \int_0^{2\pi} \widehat{a}(e^{it}) e^{-int} dt = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} a_m \int_0^{2\pi} e^{i(m-n)t} dt = a_n.$$

The Gelfand transform is then the map that to each $a \in L^1(\mathbb{Z})$ associates the continuous function $\widehat{a}$ on $\mathbb{T}$ whose Fourier coefficients are given by $a$.

**Note** Functions on $\mathbb{T}$ can be identified with periodic functions on $\mathbb{R}$: if $f \in L^1(\mathbb{T})$, then $\widetilde{f}(t) := f(e^{it})$ defines $\widetilde{f} \in L^1(\mathbb{R})$ which is periodic and the converse is also true. In this light, given $g \in L^1(\mathbb{R})$, periodic, such that $a_g := (\widehat{g}(n))_{n \in \mathbb{N}} \in L^1(\mathbb{Z})$ (where $\widehat{g}(n)$ denotes the $n$th Fourier coefficient of $g$), we have that the Gelfand transform $\widehat{a_g} \in C(\mathbb{R})$ is given by $\widehat{a_g}(t) = \sum_{n \in \mathbb{Z}} \widehat{g}(n) e^{int}$, the Fourier series of $g$.

3.(a) First note that a basis of neighbourhoods of $w_0 \in \Delta(A)$ on the weak$^*$-topology is given by

$$U_{a,\epsilon} := \{w \in \Delta(A) \,|\, |w(a) - w_0(a)| < \epsilon\},$$

$a \in A$, $\epsilon > 0$. Now let $w_1, w_2 \in \Delta(A)$ be such that $w_1 \neq w_2$; then there is $a \in A$ such that $w_1(a) \neq w_2(a)$. Let $\epsilon < |w_1(a) - w_2(a)|/2$ and $U_i := \{w \in \Delta(A) : |w(a) - w_i(a)| < \epsilon\}$, for $i = 1, 2$. It is clear that $U_i$ is a neighbourhood of $w_i$ and $U_1 \cap U_2 = \emptyset$ and therefore $\Delta(A)$ is Hausdorff.

[1] Strictly speaking, we know only that $\Delta(L^1(\mathbb{Z})) \to \mathbb{C}, \;\; w \mapsto w(e_1)$ is continuous, but since $\Delta(L^1(\mathbb{Z}))$ is in fact mapped to $\mathbb{T}$, this yields the continuity of $\Phi$.

One could also use the fact that a space is Hausdorff iff no filter on it has more than one limit point. Now if $(\omega_\lambda)_{\Lambda\in I}$ is a filter in $\Delta(A)$ such that $\omega_\lambda \to \omega$ and $\omega_\lambda \to \omega'$ then, by the definition of the weak*-topology, for any $a \in A$,

$$|\omega_\lambda(a) - \omega(a)| \to 0$$

$$|\omega_\lambda(a) - \omega'(a)| \to 0$$

hence $\omega(a) = \omega'(a)$ for any $a \in A$, that is $\omega = \omega'$ and $\Delta(A)$ is Hausdorff.

Alternatively, one could also use the fact that the map $\phi_a : \Delta(A) \to \mathbb{C}$ is continuous with the weak*-topology. Considering neighbourhoods $U_i \subset \mathbb{C}$ of $w_i(a)$, $i = 1, 2$, such that $U_1 \cap U_2 = \emptyset$, one gets neighbourhoods $\phi_a^{-1}(U_i) \subset \Delta(A)$ of $w_i$, $i = 1, 2$, with $\phi_a^{-1}(U_1) \cap \phi_a^{-1}(U_2) = \emptyset$.

(b) Have $w(a) = w(a.1) = w(a)w(1)$, for all $a \in A$. Since $w$ is non-zero, there is $a \in A$ with $w(a) \neq 0$, and hence $w(1) = 1$.

(c) Let $w_n \in \Delta(A)$, $n \in \mathbb{N}$, be such that $w_n \to w$ in the weak*-topology, that is, we have $w_n(a) \to w(a)$, for all $a \in A$. Then $w \in A^*$ and we need only show that $w$ is multiplicative and non-zero. Let $a, b \in A$. Then $w_n(ab) \to w(ab)$ and also, $w_n(a)w_n(b) \to w(a)w(b)$, we have that $w(ab) = w(a)w(b)$. Since $w_n(1) = 1$ for all $n \in \mathbb{N}$, have that $w(1) = 1$ and therefore $w \in \Delta(A)$.

(d) If $A$ is non-unital then $\Delta(A)$ is not necessarily closed (one may now have $w_n \in \Delta(A)$ and $w_n \to 0 \notin \Delta(A)$). We give two simple examples:

– $A = \{f \in C([-1, 1]) : f(0) = 0\}$. It is easy to check that $A$ is a $C^*$-subalgebra of $C([-1, 1])$, without unit, and that for any $t \in [-1, 1]$, with $t \neq 0$, $w_t(f) := f(t)$ defines an element $w_t \in \Delta(A)$. Now let $w_n := w_{1/n}$; for any $f \in A$ we have

$$w_n(f) = f(1/n) \to f(0) = 0$$

and hence $w_n \to 0$ in the weak*-topology. But $0 \notin \Delta(A)$ and $\Delta(A))$ is not closed.

– $A = C_0(X)$, where $X$ is a locally compact space that is not compact. If $\Delta(A)$ were closed, then by the Banach-Alaoglu theorem it would be compact. But we know that $\Delta(A) \cong X$ (see Theorem 6.11 and Corollary 6.8) hence $\Delta(A)$ is not closed.

## Lecture 5 (23-10-03)

1. For $a \in A$, we have that, since $A$ is commutative and from the $C^*$-axiom $\|b^* b\| = \|b\|^2$, for any $b \in A$,

$$\|a\|^4 = \|a^* a\|^2 = \|(a^* a)^* a^* a\| = \|(a^2)^* a^2\| = \|a^2\|^2$$

that is, $\|a^2\| = \|a\|^2$. By induction on $p \in \mathbb{N}$, we have $\|a^n\| = \|a\|^n$, for $n = 2^p$. Therefore,

$$r(a) = \lim \|a^n\|^{1/n} = \lim \|a^{2^p}\|^{1/2^p} = \|a\|.$$

2. We have that $f \in C(X)$ is invertible iff $f(x) \neq 0$, for all $x \in X$ (in this case $f^{-1} = 1/f \in C(X)$). Hence, $\lambda \in \sigma(f)$ iff $f - \lambda 1$ is not invertible iff there exists $x \in X$ such that $(f - \lambda 1)(x) = 0$ iff there exists $x \in X$ such that $f(x) = \lambda$. Conclude that $\sigma(f) = \{f(x) : x \in X\}$.

3. We have that $A \in M_n(\mathbb{C})$ is invertible iff $\det(A) \neq 0$. Therefore, $\lambda \in \sigma(A)$ iff $\det(A - \lambda 1) = 0$, that is, iff $\lambda$ is an eigenvalue of $A$.

4. Examples of a bounded operator on a Hilbert space with no eigenvalues:

(1) Let $H$ be a (separable) Hilbert space with an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ and define a linear operator $s_r : H \to H$ by

$$s_r(e_n) = e_{n+1},\ n \in \mathbb{N}.$$

This is an injective bounded operator, with $\|s_r(x)\| = \|x\|$, for all $x \in H$. In particular, 0 is not an eigenvalue. Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $x = \sum x_n\, e_n$ be such that $s_r x = \lambda x$; have that

$$x_n = \lambda x_{n+1},\ \lambda x_1 = 0$$

and therefore $x_n = 0$, for all $n \in \mathbb{N}$, and $\lambda$ is not an eigenvalue. Conclude that $s_r$ does not have any eigenvalues.

A particular instance of this example is the so-called *right-shift operator* $s_r : l^2 \to l^2$ such that

$$s_r(x_1, x_2, ..., x_n, ...) = (0, x_1, x_2, ..., x_n, ...),$$

which as we have seen has no eigenvalues. [1]

[1] But one can show that $\sigma(s_r) = [0, 1]$.

(2) Let $H := L^2(X;\mu)$, where $X$ is a compact topological space, $f \in C(X)$ and define

$$M_f : L^2(X;\mu) \to L^2(X;\mu),\ g \mapsto fg.$$

Then

$$M_f(g) = \lambda g \Leftrightarrow f(x)g(x) = \lambda g(x)\ a.e. \Leftrightarrow (f(x) - \lambda)g(x) = 0\ a.e.$$

and one can see that $\lambda$ is an eigenvalue iff $\mu(f^{-1}(\lambda)) \neq 0$. In particular, if $f$ is injective and $\mu(\{x\}) = 0$, for $x \in X$, then $M_f$ has no eigenvalues. For instance, if $X \subset \mathbb{R}^n$, $\mu$ is the Lebesgue measure, and $f(x) := x$, then

$$M_f(g) = \lambda g \Leftrightarrow (x - \lambda)g(x) = 0,\ a.e\ \Rightarrow g(x) = 0,\ a.e.$$

and therefore $g = 0$ in $L^2(X)$. Hence $M_f$ has no eigenvalues (note that $M_f$ is self-adjoint).

Examples of a bounded operator on a Hilbert space with eigenvalues and other parts of the spectrum:

(3) Let $H$ be a (separable) Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ be an arbitrary bounded sequence of complex numbers. Setting

$$Ae_n := \lambda_n e_n$$

one defines a bounded operator $A \in B(H)$. By definition, $\lambda_n$ is an eigenvalue, for all $n \in \mathbb{N}$, and one can easily see that there are no other eigenvalues. On the other hand, since $\sigma(A)$ is compact, in particular it is closed and $\overline{(\lambda_n)_{n\in\mathbb{N}}} \subset \sigma(A)$. If $\lambda \notin \overline{(\lambda_n)_{n\in\mathbb{N}}}$ then $((\lambda_n - \lambda)^{-1})_{n\in\mathbb{N}}$ is a bounded sequence and defines an inverse for $A - \lambda I$. Hence

$$\overline{(\lambda_n)_{n\in\mathbb{N}}} = \sigma(A).$$

For instance, take $H = l^2$, $A : l^2 \to l^2$ such that

$$A(x) = (x_n/n)_{n\in\mathbb{N}}.$$

The eigenvalues of $A$ are given by $\lambda = 1/n$, $n \in \mathbb{N}$ and

$$\sigma(A) = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}.$$

(one can easily see that $A$ is not invertible in $B(l^2)$).

Also, taking $(\lambda_n)$ as an enumeration of the rationals on some compact $X \subset \mathbb{R}^n$, we have $\sigma(A) = X$ and we show in particular that any compact set is the spectrum of a bounded operator.
(4) Define the *left-shift operator* $s_l : l^2 \to l^2$ such that

$$s_l(x_1, x_2, ..., x_n, ...) = (x_2, x_3, ..., x_{n+1}, ...).$$

From $s_l(x) = \lambda x$ we get that $x_n = \lambda^{n-1} x_1$, $n \in \mathbb{N}$. Therefore $\lambda$ is an eigenvalue of $s_l$ iff $|\lambda| < 1$. On the other hand, we know that $\sigma(s_l)$ is compact, in particular it is closed and hence $\{\lambda \mid |\lambda| = 1\} \subset \sigma(s_l)$.

The spectrum of an operator $A \in B(H)$ can be decomposed into three disjoint subsets:
– the *discrete* (or *point*) spectrum:

$$\sigma_d(A) := \{\lambda \in \mathbb{C} \mid \ker(A - \lambda I) \neq 0\};$$

– the *continuous* spectrum:

$$\sigma_c(A) := \{\lambda \in \mathbb{C} \mid \ker(A - \lambda I) = 0,\ \overline{R(A - \lambda I)} = X, (A - \lambda I)^{-1} \text{ unbounded}\};$$

– the *residual* (or *essential*) spectrum:

$$\sigma_r(A) := \sigma(A) \backslash (\sigma_d(A) \cup \sigma_c(A)) = \{\lambda \in \mathbb{C} \mid \ker(A - \lambda I) = 0,\ \overline{R(A - \lambda I)} \neq X\}.$$

Not so much can be said in general about $\sigma_d(A)$, $\sigma_c(A)$, $\sigma_r(A)$. If $H$ is finite dimensional, then $\sigma(A) = \sigma_p(A)$, for all $A \in B(H)$ (see Exercise 3). In the general setting, if $A \in B(H)$ is compact, then $\sigma(A) \subset \sigma_p(A) \cup \{0\}$ (but $\sigma_c(A)$ and $\sigma_r(A)$ might be non-empty). If $A$ is normal (that is, $AA^* = A^*A$), then $\ker A = \ker A^*$, and, since $\ker A^* = \overline{R(A)}^{\perp}$, we see that $A$ is injective iff it has dense image; this yields that $\sigma_r(A) = \emptyset$. We give a few simple examples:
(i) For the right-shift operator $s_r : l^2 \to l^2$ defined in (1) we have:

$$\sigma_d(s_r) = \emptyset,\ \sigma_c(s_r) = \{\lambda \mid |\lambda| = 1\},\ \sigma_r(s_r) = \{\lambda \mid |\lambda| < 1\}.$$

On the other hand, for the left-shift operator $s_l : l^2 \to l^2$ defined in (4), have that

$$\sigma_d(s_l) = \{\lambda \mid |\lambda| < 1, \}\ \sigma_c(s_l) = \{\lambda \mid |\lambda| = 1\},\ \sigma_r(s_l) = \emptyset.$$

(Note that $s_l = s_r^*$.)
(ii) $A : l^2 \to l^2$ such that

$$A(x) = (x_n/n)_{n \in \mathbb{N}}.$$

We have seen in (3) that $\sigma_d(A) = \{1/n \mid n \in \mathbb{N}\}$. Now $A$ has dense image, but if $B$ is an inverse for $A$ then $Bx = (nx_n)_{n\in\mathbb{N}}$ and $B$ is not bounded. Conclude that $0 \in \sigma_c(A)$ and therefore, also from (3), that

$$\sigma_d = \{1/n \mid n \in \mathbb{N}\},\ \sigma_c(A) = \{0\},\ \sigma_r(A) = \emptyset.$$

(Note that $A$ is compact and self-adjoint.)

(iii) If $A \in B(H)$ is induced by a sequence $(\lambda_n)$ as in (3), we always have that $A$ is normal ($A^*Ae_n = AA^*e_n = |\lambda|^2$), hence $\sigma_r(A) = \emptyset$ and we have then

$$\sigma_d = \{\lambda_n \mid n \in \mathbb{N}\},\ \sigma_c(A) = \overline{\{\lambda_n \mid n \in \mathbb{N}\}}\backslash\{\lambda_n \mid n \in \mathbb{N}\}.$$

(iv) $A : l^2 \to l^2$ such that

$$A(x) = (0, x_1, x_2/2, ..., x_n/n, ...).$$

Then $\sigma_d(A) = \sigma_c(A) = \emptyset$, $\sigma_r(A) = \{0\}$.

(v) $A : l^2 \to l^2$ such that

$$A(x) = (x_2, x_3/2, ..., x_{n+1}/n, ...).$$

Then $\sigma_d(A) =\{0\}, \sigma_c(A) = \sigma_r(A) = \emptyset$.

## Lecture 6 (6-11-03)

1. It is clear that $C(X,Y)$ is an (algebraic) ideal of $C(X)$, for any $Y \subset X$, and $C(X,Y)$ is closed if $Y$ is closed.

Conversely, let $I$ be a closed ideal of $C(X)$ and $Y$ be its zero-set, that is,

$$Y := \{x \in X : f(x) = 0, \text{ for all } f \in I\}.$$

We have clearly that $I \subset C(X,Y)$. To prove the converse, let $g \in C(X,Y)$ and $\epsilon > 0$ be arbitrary. For any $x \notin Y$, there is $f_x \in I$ and a neighbourhood $U_x$ of $x$ such that such that $f_x(y) \neq 0$, for $y \in U_x$. Let

$$h_x(y) := f_x(y)\frac{g(x)}{f_x(x)},\ y \in X.$$

We have then that $h_x \in I$, and, for $y \in U_x, \notin Y$,

$$|g(y) - h_x(y)| = |f_x(y)| \left| \frac{g(y)}{f_x(y)} - \frac{g(x)}{f_x(x)} \right| \leq \|f_x\| \left| \frac{g(y)}{f(y)} - \frac{g(x)}{f(x)} \right| < \epsilon$$

on a sufficiently small neighbourhood of $x$, which we denote again by $U_x$, since $g/f$ is continuous. For $x \in Y$, take a small neighbourhood $U_x \subset Y$; then $g(y) - f(y) = 0$, for all $f \in I$, $y \in U_x$. Since $X$ is compact, can take $U_{x_i}$, $i = 1, ..., p$ such that $X \subset \cup_{i=1,...,p} U_{x_i}$. Let $u_i \in C(X)$ be such that $0 \leq u_i \leq 1$, $\text{supp}(u_i) \subset U_i$ and $\sum_{i=1,..,p} u_i = 1$, and define

$$f := \sum_{i=1}^{p} u_i h_{x_i} \in I,$$

where, for $x_i \in Y$, we put $h_{x_i} = h$, for some fixed $h \in I$. We have that $f \in I$, since $I$ is an ideal, and moreover

$$|g(x) - f(x)| = \left| \sum_{i=1}^{p} u_i(x)(g(x) - h_i(x)) \right| \leq \sum_{i=1}^{p} u_i(x) \left| g(x) - h_i(x) \right|$$

$$\|g - f\| = \sup_{x \in X} |g(x) - f(x)| \leq \sum_{i=1}^{p} \sup_{x \in U_{x_i}} |g(x) - h_i(x)| \leq p\,\epsilon.$$

We conclude then that $g \in \overline{I} = I$, since $I$ is closed, and therefore that $C(X,Y) \subset I$.

We sketch two alternative proofs of this last result.

1 – Define $I_{X-Y} := \{f_{|X-Y} \mid f \in I\}$, and show that $I_{X-Y} \subset C_0(X-Y)$ satisfies the conditions of the locally compact version of the theorem of Stone-Weierstrass to get that $I_{X-Y}$ is dense in $C_0(X-Y)$. Now every $g \in C(X,Y)$ defines by restriction a function $g_{|X-Y} \in C_0(X-Y)$, and hence for any $\epsilon > 0$ there is $f \in I$ such that $\sup_{x \in X-Y} |f(x) - g(x)| < \epsilon$. Since $g(x) = f(x) = 0$, for $x \in Y$, we have in fact $\|f - g\| < \epsilon$ and $g \in \overline{I} = I$, that is, $C(X,Y) \subset I$.

2 – This proof also uses the fact that $I$ is closed, but now in that in this case we have $(I^\perp)^\perp = I$. Define

$$I^\perp = \{F \in (C(X))^* \mid F(f) = 0, \text{ for all } f \in I\}$$

$$(I^\perp)^\perp = \{f \in C(X) \mid F(f) = 0, \text{ for all } F \in I^\perp\}.$$

Identifying $(C(X))^*$ with the space of all finite regular Borel signed measures on $X$, denoted here by $M(X)$, one has then that

$$I^\perp = \{\mu \in M(X) \mid \int_X d\mu\, f = 0, \text{ for all } f \in I\}.$$

One then checks that if $\mu \in I^\perp$ then $\mathrm{supp}(\mu) \subset X - Y$ (one sees that for all $f \in I$, $|f|^2 = 0$ on $\mathrm{supp}(\mu)$). Now for $g \in C(X,Y)$, $g = 0$ on $Y$ and therefore $\int_X d\mu\, g = 0$ for all $\mu \in I^\perp$, that is, $g \in (I^\perp)^\perp = I$, and therefore $C(X,Y) \subset I$.

**Note** The class of *maximal* ideals will then be given by

$$\{I_x := C(X, \{x\}) : x \in X\}$$

and is in one-to-one correspondence with $X$ (in fact, $I_x = \ker(\phi_x)$, where $\phi_x \in \Delta(C(X))$ is the evaluation functional, $\phi_x(f) = f(x)$).

2. First note that $C_0(X) \cong \{f \in C(\dot{X}) : f(\infty) = 0\}$. Indeed, the map $C_0(X) \to \{f \in C(\dot{X}) : f(\infty) = 0\}, f \mapsto \widetilde{f}$ such that

$$\widetilde{f}(x) = f(x),\ x \in X \text{ and } \widetilde{f}(\infty) = 0$$

is easily seen to be well-defined and bijective, and moreover $\|f\| = \|\widetilde{f}\|$.

Now let $I$ be an ideal of $C_0(X)$; then, under the isomorphism above, $I$ is an ideal of $C(\dot{X})$: just note that for any $\widetilde{f} \in C(\dot{X})$, $\widetilde{f} - \widetilde{f}(\infty) \in C_0(X)$ and therefore, for $g \in I$,

$$g\widetilde{f} = g(\widetilde{f} - \widetilde{f}(\infty)) + \widetilde{f}(\infty)g \in I.$$

On the other hand, if $J$ is an ideal in $C(\dot{X})$, we have also that $J \cap \{f \in C(\dot{X}) : f(\infty) = 0\}$ is an ideal in $C_0(X)$. We conclude that the ideals of $C_0(X)$ are in one-to-one correspondence with $J \cap \{f \in C(\dot{X}) : f(\infty) = 0\}$, where $J$ is an ideal in $C(\dot{X})$. [1] From Ex.1, we then have that the ideals of $C_0(X)$ are given by

$$\{f \in C_0(X) \,|\, \widetilde{f} \in C(\dot{X}, Y)\}$$

with $Y \in \dot{X}$ closed such that $\infty \in Y$. Since $Y_0 \subset X$ is closed iff $Y_0 \cup \{\infty\}$ is closed in $\dot{X}$, we have finally that every ideal of $C_0(X)$ is of the form

$$C_0(X, Y_0) := \{f \in C_0(X) \,|\, f(x) = 0, x \in Y_0\}$$

with $Y_0 \subset X$ closed.

3.(a) Have that $J_\omega = \omega^{-1}(0)$ is closed, since $\omega$ is continuous, and for any $a \in A$, $j \in J_\omega$,

$$\omega(aj) = \omega(a)\omega(j) = 0$$

which shows that $J_\omega$ is an ideal of $A$. To check that it is maximal, let $a \in A$ be arbitrary, and write

$$a = x + \omega(a)1, \text{ with } x := a - \omega(a)1 \in \ker\omega = J_\omega.$$

There is then a vector space isomorphism $A \cong J_\omega \oplus \mathbb{C}$, that is, $J_\omega$ has codimension 1 and is therefore maximal.

(b) Assume $J_{w_1} = J_{w_2}$. Then for any $a \in A$, $a - w_2(a)1 \in J_{w_1}$, since $w_2(a - w_2(a)1) = 0$. But

$$w_1(a - w_2(a)1) = w_1(a) - w_2(a)w_2(1) = w_1(a) - w_2(a) = 0$$

and hence $w_1(a) = w_2(a)$ for all $a \in A$.

4. Recall that $\dot{A}$ is defined as the $C^*$-algebra $A \oplus \mathbb{C}$ with

$$(a + \lambda)(b + \nu) := (ab + \nu a + \lambda b + \lambda\nu)$$

[1] This is just a particular instance of a general fact: if $A$ is a non-unital $C^*$-algebra and $\dot{A}$ is its unitization, then the ideals of $A$ are given by $J \cap A$, with $J$ an ideal of $\dot{A}$; see Ex.4.

$$\|a+\lambda\| := \|\rho(a)+\lambda I\|_{L(A)}$$

where $\rho : A \to Ł(A)$ is defined as $\rho(a)b = ab$ and $I$ is the identity in $L(A)$. In Ex.2, we have considered the isometric homomorphism $C_0(X) \to C(\dot{X})$, $f \mapsto \widetilde{f}$ such that $\widetilde{f}(x) = f(x)$, for $x \in X$ and $\widetilde{f}(\infty) = 0$. Define now $\Phi : \dot{A} \to C(\dot{X})$ by

$$\Phi(f+\lambda) := \widetilde{f} + \lambda;$$

it is immediate to check that $\Phi$ is an algebra homomorphism. If $\Phi(f+\lambda) = \Phi(g+\nu)$ then $\Phi(f+\lambda)(\infty) = \lambda = \Phi(g+\nu)(\infty) = \nu$ and also $\widetilde{f} = \widetilde{g}$, which gives $f = g$; we conclude that $\Phi$ is one-to-one. On the other hand, let $h \in C(\dot{X})$. Then $f := h - h(\infty) \in C_0(X)$ and, with $\lambda := h(\infty)$, $\Phi(f+\lambda) = \widetilde{f} + \lambda = h$. Hence $\Phi$ is a bijection. Since any (algebraic) isomorphism between $C^*$-algebras is isometric, we have finally that $\Phi$ is a homeomorphism. (In this case, one can check directly that

$$\begin{aligned}
\|\Phi(f+\lambda)\| &= \|\widetilde{f}+\lambda\|_\infty \\
&= \max\{\sup_{x\in X}|f(x)+\lambda|,\, \lambda|\} \\
&= \sup_{\|g\|=1}\sup_{x\in X}|(f(x)+\lambda)g(x)| \\
&= \|f+\lambda 1\|_{\dot{A}}.)
\end{aligned}$$

5.(a) Clearly, $\dot{\omega}$ is linear, since $\omega$ is, and since $\dot{\omega}(1) = 1$, $\dot{\omega}$ is not identically zero. We need to check that it is multiplicative:

$$\begin{aligned}
\dot{\omega}\left((a+\lambda 1)(b+\nu 1)\right) &= \dot{\omega}\left(ab+\nu a+\lambda b+\lambda\nu 1\right) \\
&= \omega(a)\omega(b)+\nu\omega(a)+\lambda\omega(b)+\lambda\nu \\
&= \dot{\omega}(a+\lambda 1)\dot{\omega}(b+\nu 1).
\end{aligned}$$

(b) We have that $\omega_\infty$ is clearly linear and, since $\omega_\infty(1) = 1$, $\omega_\infty$ is not identically zero. Now, computations as above give that

$$\omega_\infty\left((a+\lambda 1)(b+\nu 1)\right) = \lambda\nu = \omega_\infty(a+\lambda 1)\,\omega_\infty(b+\nu 1).$$

(c) Let $\eta \in \Delta(\dot{A})$. Since $\eta$ is linear and $\eta(1) = 1$, we have

$$\eta(a+\lambda 1) = \eta(a+0)+\eta(\lambda 1) = \eta_A(a)+\lambda.$$

It is immediate that the restriction $\eta_A$ is multiplicative in $A$ and therefore either $\eta_A \in \Delta(A)$, and $\eta$ is given by (a), or $\eta_A = 0$, and $\eta = \omega_\infty$ defined in (b).

(d) First note that the map $\Phi : \Delta(A) \to \Delta(\dot{A}),\ \omega \mapsto \dot{\omega}$, with $\dot{\omega}$ as in (a), is a continuous injection. Indeed, if $\{\omega_i\}_{i\in I}$ is a convergent net in $\Delta(A)$, $\omega_i \to w$, then by definition $\omega_i(a) \to \omega(a)$, for all $a \in A$ and therefore

$$\dot{\omega_i}(a + \lambda 1) = \omega_i(a) + \lambda \to \omega(a) + \lambda = \dot{\omega}(a + \lambda 1).$$

Now we check that $\Phi$ can be extended to a homeomorphism

$$\dot{\Phi} : \dot{\Delta(A)} \to \Delta(\dot{A}),\ \ \dot{\Phi}(\infty) := \omega_\infty,$$

where $\omega_\infty$ is as defined in (b). It is clear that $\dot{\Phi}$ is a one-to-one map and surjectivity follows from (c). If we prove that $\dot{\Phi}$ is continuous in $\dot{\Delta(A)}$, then since $\dot{\Delta(A)}$ is compact and $\Delta(\dot{A})$ is Hausdorff, we have that $\Phi^{-1}$ is continuous as well, and the proof is concluded. Let then $U_\infty \subset \Delta(\dot{A})$ be a neighbourhood of $\omega_\infty$, which we write as

$$U_\infty = \{\eta \in \Delta(\dot{A})\,|\,|\eta(a + \lambda 1) - \omega_\infty(a + \lambda 1)| < \epsilon\}.$$

We have then

$$U_\infty = \{\dot{\omega} \in \Delta(\dot{A})\,|\,|\,\omega(a)| < \epsilon\} \cup \{\omega_\infty\}$$

and therefore $\Phi^{-1}(U_\infty) = \{\infty\} \cup \{\omega \in \Delta(A)\,|\,|\omega(a)| < \epsilon\}$. Now $\Delta(A) - \Phi^{-1}(U_\infty) = \{\omega \in \Delta(A)\,|\,|\omega(a)| \geq \epsilon\}$ is closed and hence, by the Banach-Alaoglu theorem, it is compact. We conclude that $\Phi^{-1}(U_\infty)$ is indeed a neighbourhood of $\infty$ in $\dot{\Delta(A)}$ and $\Phi$ is continuous.

## Lecture 7 (13-11-03)

1. Let $C$ be a category according to Definition 6.1 and let $(f,g),(g,h) \in C^2$. Then $s(f) = t(g)$ and $s(g) = t(h)$, and, from 2 in Definition 6.1, we have

$$s(fg) = s(g) = t(h),$$
$$t(gh) = t(g) = s(f). \tag{11}$$

Therefore, $(fg,h) \in C^2$, $(f,gh) \in C^2$ and, from 3 in Definition 6.1, $(fg)h = f(gh)$. Conversely, assume that whenever $(f,g) \in C^2$ and $(g,h) \in C^2$ we have that $(fg)h$, $f(gh)$ are defined and equal. Let $(f,g) \in C^2$; Only need to show that $s(fg) = s(g)$ and $t(fg) = t(f)$. Take $h$ such that $(g,h) \in C^2$ (for instance, $h = i(s(g)))$; then we have $s(g) = t(h)$ and by assumption $(fg,h) \in C^2$. Hence

$$s(fg) = t(h) = s(g).$$

A similar argument shows that $t(fg) = t(f)$, and the two definitions are indeed equivalent.

2. From 1 in Definition 6.1, we have that $s(i(x)) = t(i(x)) = x$, for all objects $x$; hence $s \circ i$ and $t \circ i$ are bijections. Therefore, $s, t$ are surjective and $i$ is injective.

3. Let $C$ be a category as in Definition 6.1; we will show that the 'arrows only' definition holds. Let $C_1$ be the class of arrows, and $C_2 \subset C_1 \times C_1$ be the subclass of composable arrows, $m : C_2 \to C_1$ given by composition, $m(f,g) = fg$.

We first prove (a). If $fg$ and $gh$ are defined, then $s(f) = t(g)$ and $s(g) = t(h)$. Since $s(fg) = s(g) = t(h)$, have that $(fg)h$ is defined. On the other hand, if $(fg)h$ is defined, then in particular $fg$ is defined and $s(f) = t(g) = t(gh)$ and hence $f(gh)$ is defined. We can see in this way that $f(gh)$ is defined iff $(fg)h$ is defined and in this case clearly $fg$ and $gh$ are defined. We then have that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. Using 3. in Definition 6.1, we now have that $(i) \Rightarrow (iv)$ and therefore (a) holds ($(iv) \Rightarrow (iii)$ is trivial).

As for (b), let $\widetilde{s}(f) := i(s(f)) \in C_1$, for $f \in C_1$; then $(f, \widetilde{s}(f)) \in C_2$. If $(\widetilde{s}(f), h) \in C_2$, then $s((\widetilde{s}(f)) = s(i(s(f))) = s(f) = t(h)$ and therefore $\widetilde{s}(f)h = i(s(f))h = i(t(h))h = h$; similarly, if $(g, \widetilde{s}(f)) \in C_2$, then $s(g) = t(i(s(f)) = s(f)$ and $g\widetilde{s}(f) = gi(s(f)) = gi(s(g)) = g$. Conclude that $\widetilde{s}(f)$ is the required identity, and we can define in the same way an identity $\widetilde{t}(f)$ with $(\widetilde{t}(f), f) \in C_2$.

Conversely, assume that $C_1$ is a category according to the 'arrows only' definition. Define the class of objects

$$C_0 := \{e \in C_1 \mid e \,\text{is an identity}\}$$

and let $i : C_0 \to C_1$ be the inclusion map. By assumption, we know that to each $f \in C_1$ one can associate an identity $t(f)$ (respectively, $s(f)$) such that $(t(f), f)$ (respectively, $(f, s(f))$ is defined. We show that $t(f)$ (and similarly $s(f)$) is the unique identity with this property and that will yield the definition of the target (and source) map. Let $f \in C_1$ and $e, e'$ be identities such that $(e, f), (e', f) \in C_2$. Then $ef = f$ and therefore $e'(ef)$ is defined; by (a), we have that $e'e$ is defined and

$$e \,\text{is an identity} \;\Rightarrow e'e = e'$$

$$e' \,\text{is an identity} \;\Rightarrow e'e = e.$$

We have therefore well-defined source and target maps

$$s : C_1 \to C_0,\; f \mapsto s(f)$$

$$t : C_1 \to C_0,\; f \mapsto t(f).$$

We now show that $C_2 = \{(f, g) \in C_1 \times C_1 \mid s(f) = t(g)\}$ (where $C_2$ is given by the 'arrows only' definition). Let $f, g \in C_1$ such that $s(f) = t(g)$; then by definition of $t(g)$, $(s(f), g) \in C_2$, and since $(f, s(f)) \in C_2$, have from (a) that $(fs(f), g) = (f, g) \in C_2$. Conversely, let $(f, g) \in C_2$; it suffices to show that $s(f)g$ is defined, since in this case, because $s(f)$ is an identity, we have $s(f) = t(g)$. But $fs(f) = f$, and hence $(fs(f))g$ is defined; by (a), we have $s(f)g$ defined as well.

Finally we check that the axioms in Definition 6.1 are satisfied.

1. Want to show that for $x \in C_0$, $s(i(x)) = t(i(x)) = x$, that is, that for any identity $e$, $s(e) = t(e) = e$. It suffices to show that $ee$ is defined: in this case, since $e$ is an identity, have one one hand that $e = s(e)$ and similarly, $e = t(e)$. Now $(t(e), e) \in C_2$ and, since $t(e)e = e$, we have $(t(e)e, e) \in C_2$; from (a), conclude that $(e, e) \in C_2$ as well, that is, $ee$ is defined. [1]

2. Let $(f, g) \in C_2$; to check that $t(fg) = t(f)$ it suffices to show that $t(fg)f$ is defined, again since $t(fg)$ is an identity. But $t(fg)(fg)$ is defined,

[1] Moreover, it is clear that if $f \in C_1$ is such that $f = s(f) = t(f)$ then $f$ is an identity; we have then that $C_0 = \{f \in C_1 \mid s(f) = t(f) = f\}$. Also, note that we have shown in particular that $s$ and $t$ are surjective, that is, $C_0 = Im(s) = Im(t)$. One could then have started with any of these definitions for $C_0$ and proceed from there.

and hence, by (a), $t(fg)f$ is defined. Similarly, we have $gs(fg)$ defined and thus $s(fg) = s(g)$.

3. If $(f, g) \in C_2$ and $(g, h) \in C_2$, then by (a) we have $(fg)h = f(gh)$.

4. For any $f \in C_1$, $i(t(f)) \circ f = t(f)f = f$, since $i$ is the inclusion and $t(f)$ is an identity for $f$. Similarly, $f \circ i(s(f)) = f$.

4. Let $C$ be a category as in Definition 6.1. We show that the 'objects' definition holds. Let $C_0$ be the class of objects and for each $(x, y) \in C_0 \times C_0$ let

$$(x, y) := \{f \in C_1 \mid s(f) = y, t(f) = x\}.$$

Then, if $f \in (x, y)$, $g \in (y, z)$, have $s(f) = t(g)$, that is $(f, g) \in C_2$ and we define

$$m : (x, y) \times (y, z) \to (x, z),\ (f, g) \mapsto fg.$$

Now we verify (a),(b)(c).

(a) If $f \in (x, y) \cap (x', y')$ then $s(f) = y = y'$, $t(f) = x = x'$.

(b) For each $x \in C_0$, have $t(i(x)) = s(i(x)) = x$; hence $i(x) \in (x, x)$ and for $f \in (x, y), g \in (y, x)$, $i(x)f = i(t(f))f = f$ and $gi(x) = gi(s(g)) = g$.

(c) This is just a restatement of 3. in Definition 6.1.

Conversely, let $C$ be a category according to the 'objects' definition. Take $C_0$ for the class of objects and define

$$C_1 := \bigcup_{x,y \in C_0} (x, y)$$

$$s, t : C_1 \to C_0$$

such that if $f \in (x, y)$ then $s(f) := y$, $t(f) := x$ (note that $s, t$ are well-defined because of (a)). Then, from (b), we have a map

$$i : C_0 \to C_1,\ x \mapsto i(x)$$

and, defining $C_2 := \{(f, g) \in C_1 \times C_1 \mid s(f) = t(g)\}$ we have that $(f, g) \in C_2$ iff $f \in (x, y)$, $g \in (y, z)$, for $x = t(f)$, $y = s(f) = t(g)$, $z = s(g)$, that is, iff $fg$ is defined. We now only need to check that the axioms in Definition 6.1 are satisfied.

1. For $x \in C_0$, $i(x) \in (x, x)$; hence $s(i(x)) = t(i(x)) = x$.

2. Let $(f, g) \in C_2$. Then, as above, $f \in (x, y)$, $g \in (y, z)$ and therefore $fg \in (x, z)$; this gives $s(fg) = z = s(g)$, $t(fg) = x = t(f)$.

3. If $(f, g), (g, h) \in C_2$, then $f \in (x, y)$, $g \in (y, z), h \in (z, w)$, and by (c), $(fg)h = f(gh)$.

4. For $f \in C_1$, have that $f \in (t(f), s(f))$ and from (b), $i(t(f))f = f$, $fi(s(f)) = f$.

5. From (b), and since $s_G$ is surjective, we have that $\Phi_0$ is determined by $\Phi_1$, in that, for all $x \in G_0$, we have

$$\Phi_0(x) = \Phi_0(s_G(i_G(x)) = s_h \circ \Phi_1(i_G(x)).$$

Now assume that $\Phi_1$ is a functor according to the 'arrows' definition. Define

$$\Phi_0 : G_0 \to H_0,\ \Phi_0 := s_H \circ \Phi_1 \circ i_G.$$

We check that (a),(b),(c) hold.

(a) By definition, $i_H \circ \Phi_0 = i_H \circ s_H \circ \Phi_1 \circ i_G$. For any $x \in G_0$,

$$\Phi_1 \circ i_G(x)\ i_H(s_H(\Phi_1 \circ i_G(x))) = \Phi_1 \circ i_G(x);$$

on the other hand, by (a), $\Phi_1 \circ i_G(x)$ is an identity [1], since $i_G(x)$ is, hence, noting that $\Phi_1 \circ i_G(x)\, i_H \circ \Phi_0(x)$ is defined, have

$$\Phi_1 \circ i_G(x)\ i_H \circ \Phi_0(x) = i_H \circ \Phi_0(x).$$

Conclude that $i_H \circ \Phi_0(x) = \Phi_1 \circ i_G(x)$.

(b) For $f \in G_1$, have that $f = f\, i_G(s_G(f))$ and, from (b),

$$\Phi_1(f) = \Phi_1\left(f\, i_G(s_G(f))\right) = \Phi_1(f)\Phi_1\left(i_G(s_G(f))\right)$$

hence $s_H \circ \Phi_1(f) = s_H \circ \Phi_1 \circ i_G \circ s_G(f) = \Phi_0 \circ s_G(f)$.

Similarly, we have that $t_H \circ \Phi_1(f) = t_H \circ \Phi_1 \circ i_G \circ t_G(f)$; from (a), we know that $\Phi_1 \circ i_G \circ s_G(f)$ is an identity (since $i_G \circ s_G(f)$ is) and therefore its source and target coincide, that is, we have $t_H \circ \Phi_1(f) = s_H \circ \Phi_1 \circ i_G \circ t_G(f) = \Phi_0 \circ t_G(f)$.

(c) This is a restatement of (b) in the arrows definition.

We finally show that the 'usual' definition yields the 'arrows' definition. Noting that an identity $e$ is of the form $i_G(x)$ for some $x \in G_0$, have from (a) that $\Phi_1(e)$ is also an identity in $H$. On the other hand, if $(f, g) \in G_2$, then $(\Phi_1(f), \Phi_1(g)) \in H_2$ by (b), and $\Phi_1(fg) = \Phi_1(f)\Phi_1(g)$, by (c). Therefore the 'arrows' definition holds.

[1] Recall that an identity in a category $G$ is an element of the form $i_G(x)$, for some $x \in G_0$. This is coherent with the definition of identity given in Exercise 3: it is clear that any $i_G(x)$ satisfies those conditions and on the other hand if $e$ is an identity and $ef$ is defined, by unicity, we have $e = i_G(t_G(f))$, and any identity is of the form $i_G(s_G(f))$, $i_G(t_G(f))$.

## Lecture 8 (20-11-03)

1. Let $I : Vec \to Vec$ be the identity functor and $** : Vec \to Vec$ be defined on objects by $V \mapsto V^{**}$, where $V^{**}$ denotes the second dual of the vector space $V$, and on arrows by $\phi \mapsto \phi^{**}$ where $\phi^{**}(w) = w \circ \phi^*$, with $\phi^*(f) = f \circ \phi$, for $w \in V^{**}$, $f \in W^*$, $\phi : V \to W$.

It is clear that if $i_V : V \to V$ denotes the identity, then $(i_V)^{**} = i_{V^{**}}$, and that $**$ is compatible with the source and target maps. It is also easy to check that if $\phi$, $\psi$ are composable arrows then $(\phi \circ \psi)^{**} = \psi^{**} \circ \psi^{**}$, and we have then that $**$ is a (covariant) functor. We now show that $** \cong I$.

For each object $V$, define $\tau_V : V \to V^{**}$ by $\tau_V(v) := \widehat{v}$, where $\widehat{v}(f) := f(v)$, for $v \in V$, $f \in V^*$. We have that $\tau_V$ is injective, since $V^*$ separates the points of $V$, and therefore also surjective, since $\dim(V) = \dim(V^{**}) < \infty$. Hence $\tau_V$ is an isomorphism for any $V$. We check that the collection $\{\tau_V\}_{V \in Vec^0}$ defines a natural transformation between $I$ and $**$, that is, that for any arrow $\phi : V \to W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\tau_V} & V^{**} \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi^{**}} \\ W & \xrightarrow[\tau_W]{} & W^{**}. \end{array}$$

Indeed, for any $v \in V$, $f \in W^*$, we have

$$(\tau_W \circ \phi)(v)(f) = \widehat{\phi(v)}(f) = f \circ \phi(v),$$

$$(\phi^{**} \circ \tau_V)(v)(f) = (\tau_V(v) \circ \phi^*)(f) = \tau_V(v)(f \circ \phi) = f \circ \phi(v).$$

We conclude that $** \cong I$.

2. Assume that $G$ and $H$ are equivalent categories, that is, that there exist functors $\alpha : G \to H$ and $\beta : H \to G$ such that $\alpha\beta \cong id_H$, $\beta\alpha \cong id_G$. We want to show that $\alpha$ is essentially surjective and fully faithful.

For each $v \in H^0$, we have $\alpha\beta(v) \cong v$ and hence, taking $u := \beta(v)$, we see that $\alpha$ is essentially surjective. To show that it is fully faithful, note that to each $u \in G^0$, we have isomorphisms $\tau_u : u \to \beta^0\alpha^0(u)$ such that for any given arrow $\phi$, the following diagram commutes

$$\begin{array}{ccc} u & \xrightarrow{\tau_u} & \beta^0\alpha^0(u) \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \beta^1\alpha^(\phi)} \\ u' & \xrightarrow[\tau_{u'}]{} & \beta^0\alpha^0(u'). \end{array}$$

The arrow $\phi$ is then completely determined by $\alpha^1(\phi)$, in that we have $\phi = \tau_{u'}^{-1}(\beta^1\alpha^1(\phi))\tau_u$, and hence $\alpha$ is injective. (The same argument for $\alpha\beta$ shows that $\beta$ is injective as well.) Now let $\psi : \alpha^0(u) \to \alpha^0(u')$ be an arrow and define $\phi : x \to y$ be $\phi := \tau_{u'}^{-1}\beta^1(\psi)\tau_u$. Again using the commutativity of the above diagram, we have that

$$\phi = \tau_{u'}^{-1}(\beta^1\alpha^1(\phi))\tau_u = \tau_{u'}^{-1}\beta^1(\psi)\tau_u,$$

that is, we have

$$\beta^1\alpha^1(\phi) = \beta^1(\psi) \Leftrightarrow \alpha^1(\phi) = \psi,$$

since $\beta$ is injective, and we conclude then that $\alpha$ is also surjective.

Conversely, let $\alpha : G \to H$ be an essentially surjective, fully faithful functor. We want to show that $G$, $H$ are equivalent, that is, we want to define a functor $\beta : H \to G$ such that $\alpha\beta \cong id_H$, $\beta\alpha \cong id_G$.

**Note** In order to use the axiom of choice, we have to assume extra conditions on the categories $G$ and $H$ (since the axiom of choice is not defined on the universe of classes).

For each $v \in H^0$ let $G_v := \{u \in G^0 \,|\, \alpha^0(u) \cong v\}$. Since $\alpha$ is essentially surjective, $G_v \neq \emptyset$. Using the axiom of choice, we have that there is a map $\beta^0 : H^0 \to G^0$ such that $\beta^0(v) \in G_v$, that is, such that

$$\alpha^0\beta^0(v) \cong v,\ v \in H^0;$$

this defines the functor $\beta$ on objects.

To define $\beta$ on arrows, pick for each $v \in H_0$ an isomorphism $\tau_v : v \cong \alpha^0\beta^0(v)$. Each arrow $\phi \in (v, v')_H$ defines an arrow $\tau_v^{-1}\phi\tau_{v'} \in (\alpha^0(u), \alpha^0(u'))$, where $u = \beta^0(v)$, $u' = \beta^0(v')$. Since $\alpha^1 : (u, u')_G \to (\alpha^0(u), \alpha^0(u'))_H$ is bijective[1], there is a uniquely defined arrow $\psi \in (u, u')$ such that

$$\alpha^1(\psi) = \tau_v^{-1}\phi\tau_{v'}.$$

We define $\beta^1(\phi) := \psi$. Note that with these definitions we have immediately that the diagram

$$\begin{tikzcd}
v \arrow[r, "\tau_v"] \arrow[d, "\phi"'] & \alpha^0\beta^0(v) \arrow[d, "\alpha^1\beta^{(}\phi)=\alpha(\psi)"] \\
v' \arrow[r, "\tau_{v'}"'] & \alpha^0\beta^0(v')
\end{tikzcd}$$

[1] Note that $\alpha^1 : G^1 \to H^1$ is not necessarily bijective, and in fact, in this case, we would have that $\alpha^0$ is surjective – not just essentially surjective.

commutes, for any objects $v$, $v'$ and any arrow $\phi$, that is, once we prove that $\beta$ is indeed a functor, we have that $\alpha\beta \cong id_H$. Now, it is immediate that $\beta$ is compatible with the source and target maps; to check compatibility with the inclusion maps, note that, from the definition of a category, $\tau_v \circ i_H(v) = \tau_v = i_H(\alpha^0\beta^0(v)) \circ \tau_v$ and the following diagram is commutative

$$\begin{array}{ccc} v & \xrightarrow{\tau_v} & \alpha^0\beta^0(v) \\ {\scriptstyle i_H(v)}\big\downarrow & & \big\downarrow{\scriptstyle i_H(\alpha^0\beta^0(v))=\alpha^1(i_G(\beta^0(v))} \\ v & \xrightarrow[\tau_v]{} & \alpha^0\beta^0(v) \end{array}$$

where we used the fact that $\alpha$ is a functor; from the definition of $\beta^1$, we have that $\beta^1(i_H(v)) = i_G(\beta^0(v))$. A similar argument shows that $\beta^1(\psi\phi) = \beta^1(\psi)\beta^1(\phi)$, for all compatible arrows $\psi$, $\phi$, and $\beta$ is therefore a functor. To finish the proof, we need only show that $\beta\alpha \cong id_G$. Now for each arrow $\psi \in (u, u')_G$, we know that the diagram

$$\begin{array}{ccc} \alpha^0(u) & \xrightarrow{\tau_{\alpha^0(u)}} & \alpha^0\beta^0(\alpha^0(u)) \\ {\scriptstyle \alpha^1(\psi)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^1\beta^1(\alpha^1(\psi))} \\ \alpha^0(u') & \xrightarrow[\tau_{\alpha^0(u')}]{} & \alpha^1\beta^1(\alpha^1(u')) \end{array}$$

commutes. Since $\alpha^1 : (u, \beta^0\alpha^0(u))_G \to (\alpha^0(u), \alpha^0(\beta^0\alpha^0(u)))_H$ is injective, there is a unique arrow $\epsilon_u \in (u, \beta^0\alpha^0(u))_G$ such that $\alpha^1(\epsilon_u) = \tau_{\alpha^0(u)}$, and $\epsilon_u$ is an isomorphism (because $\tau_{\alpha_u}$ is). Since $\alpha$ is a functor, the commutativity of the above diagram yields

$$\alpha^1(\beta^1\alpha^1(\psi) \circ \epsilon_u) = \alpha^1(\epsilon_{u'} \circ \psi)$$

and by the injectivity of $\alpha^1 : (u, \beta^0\alpha^0(u'))_G \to (\alpha^0(u), \alpha^0(\beta^0\alpha^0(u')))_H$, we have that the diagram

$$\begin{array}{ccc} u & \xrightarrow{\epsilon_u} & \beta^0\alpha^0(u) \\ {\scriptstyle \psi}\big\downarrow & & \big\downarrow{\scriptstyle \beta^1\alpha^1(\psi)} \\ u' & \xrightarrow[\epsilon_{u'}]{} & \beta^0\alpha^0(u') \end{array}$$

also commutes, which proves finally that $\beta\alpha \cong id_G$.

3. For the first diagram, let $a \in A$, $\omega \in \Delta(B)$. If $\tau_A(a) := \widehat{a} \in C(\Delta(A))$, where $\widehat{a}$ is the Gelfand transform of $a$, then,

$$(\tau_B \circ \phi)(a)(\omega) = \tau_B(\phi(a))(\omega) = \widehat{\phi(a)}(\omega) = \omega \circ \phi(a)$$

$$(\phi^{**} \circ \tau_A(a))(\omega) = \tau_A(a)(\omega \circ \phi) = \widehat{a}(\omega \circ \phi) = \omega \circ \phi(a).$$

For the second diagram, let $x \in X$, $f \in C(Y)$. If $\epsilon_X(x) \in \Delta(C(X))$ is the evaluation functional, we have,

$$(\epsilon_Y \circ \psi)(x)(f) = \epsilon_Y(\psi(x))(f) = f \circ \psi(x)$$

$$(\psi^{**} \circ \epsilon_X)(x)(f) = (\epsilon_X(x) \circ \psi^*)(f) = \epsilon_X(x)(f \circ \psi) = f \circ \psi(x).$$

4. Let $A$, $B$ be non-unital $C^*$-algebras and $\phi : A \to B$ be a non-degenerate homomorphism. If we denote by $\dot{\phi}$ the extension of $\phi$ to the unitization $\dot{A}$, we know from the equivalence of the categories CH and $\text{CC}_1\text{A}$ that the diagram

$$\begin{tikzcd} \dot{A} \arrow[r, "\tau_{\dot{A}}"] \arrow[d, "\dot{\phi}"'] & C(\Delta(\dot{A})) \arrow[d, "\dot{\phi}^{**}"] \\ \dot{B} \arrow[r, "\tau_{\dot{B}}"'] & C(\Delta(\dot{A})) \end{tikzcd}$$

is commutative, where $\tau_{\dot{A}}$, $\tau_{\dot{B}}$ denote the Gelfand transform and $\dot{\phi}^{**}$ is the usual map induced by $\dot{\phi}$. Now, since $\phi$ is non-degenerate, $\phi^* : \Delta(B) \to \Delta(A)$ is proper and therefore induces a map $\phi^{**} : C_0(\Delta(A)) \to C_0(\Delta(B))$. It is easy to check that under the isomorphism $C(\Delta(\dot{A})) \cong C_0(\dot{\Delta}(A))$ [1] one obtains a unital map $C_0(\dot{\Delta}(A)) \to C_0(\dot{\Delta}(B))$, which coincides with the map $(\dot{\phi^{**}})$, the extension of $\phi^{**}$ to the unitization. We then obtain the following commutative diagram

$$\begin{tikzcd} \dot{A} \arrow[r, "\tau_{\dot{A}}"] \arrow[d, "\dot{\phi}"'] & C(\Delta(\dot{A})) \arrow[r, "\cong"] \arrow[d, "\dot{\phi}^{**}"] & C_0(\dot{\Delta}(A)) \arrow[d, "(\dot{\phi^{**}})"] \\ \dot{B} \arrow[r, "\tau_{\dot{B}}"'] & C(\Delta(\dot{B})) \arrow[r, "\cong"] & C_0(\dot{\Delta}(B)) \end{tikzcd}$$

which, since the maps on the top and bottom lines are (unital) isomorphisms, restricts to

$$\begin{tikzcd} A \arrow[r, "\tau_A"] \arrow[d, "\phi"'] & C_0(\Delta(A)) \arrow[d, "\phi^{**}"] \\ B \arrow[r, "\tau_B"'] & C_0(\Delta(A)) \end{tikzcd}$$

[1] given in Lecture 6, Ex.3

with $\tau_A$, $\tau_B$ the Gelfand transform on $A$, $B$ (again isomorphisms).

Now let $X, Y$ be locally compact spaces and $\psi : X \to Y$ a proper map. Again from the equivalence of the categories CH and $\mathrm{CC_1A}$, we have a commutative diagram

$$\begin{tikzcd} \dot{X} \arrow[r, "\epsilon_{\dot{X}}"] \arrow[d, "\dot{\psi}"'] & \Delta(C(\dot{X})) \arrow[d, "\dot{\psi}^{**}"] \\ \dot{Y} \arrow[r, "\epsilon_{\dot{Y}}"'] & \Delta(C(\dot{Y})) \end{tikzcd}$$

where $\dot{\psi}$ is the extension of $\psi$ to the one-point compactification (which is well-defined since $\psi$ is proper), $\dot{\psi}^{**}$ is the usual map induced by $\dot{\psi}$ and $\epsilon_{\dot{X}}$, which is known to be an isomorphism. Since $\psi$ is proper, it induces a non-degenerate morphism $\psi^* : C_0(X) \to C_0(Y)$ and this induces a proper map $\psi^{**} : \Delta(C_0(X)) \to \Delta(C_0(Y))$. Now, under the isomorphism $\Delta(C(\dot{X})) \cong \dot{\Delta(C_0(X))}$, we get a map $\dot{\Delta(C_0(X))} \to \dot{\Delta(C_0(Y))}$ which is easily seen to restrict to $\psi^{**}$, that is, we have the following commutative diagram

$$\begin{tikzcd} \dot{X} \arrow[r, "\epsilon_{\dot{X}}"] \arrow[d, "\dot{\psi}"'] & \Delta(C(\dot{X})) \arrow[r, "\cong"] \arrow[d, "\dot{\psi}^{**}"] & \dot{\Delta(C_0(X))} \arrow[d, "(\dot{\psi^{**}})"] \\ \dot{Y} \arrow[r, "\epsilon_{\dot{Y}}"'] & \Delta(C(\dot{Y})) \arrow[r, "\cong"] & \dot{\Delta(C_0(Y))} \end{tikzcd}$$

which, noting that $\epsilon_{\dot{X}}(\infty) = \omega_\infty$, restricts to

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] \arrow[d, "\psi"'] & \Delta(C_0(X)) \arrow[d, "\psi^{**}"] \\ Y \arrow[r, "\epsilon_Y"'] & \Delta(C_0(Y)) \end{tikzcd}$$

with $\epsilon_X$, $\epsilon_Y$ the evaluation maps (of course also isomorphisms).

## Lecture 9 (27-11-03)

1. Recall that a subspace $H_0 \subset H$ is called invariant for a representation $\pi$ if $\pi(A)H_0 \subset H_0$. In this case, $\pi(a)_{|H_0}$, $a \in A$, defines a representation of $A$ on $H_0$, a subrepresentation of $\pi$, denoted by $\pi_{H_0}(A)$.

For each $\omega \in H$, $H_\omega := \overline{\pi(A)\Omega} \subset H$ is an invariant subspace and the representation $\Omega$ is cyclic for the subrepresentation $\pi_{H_\Omega}(A)$, since $\pi_{H_\Omega}(A) = \pi(A)_{|H_\Omega}$ yields $\overline{\pi_{H_\Omega}(A)\Omega} = \overline{\pi(A)\Omega}$. Picking an arbitrary $\Omega \in H$ we can then write

$$H = H_\Omega \oplus H_\Omega^\perp, \; \pi(A) = \pi_{H_\Omega} \oplus \pi_{H_\Omega^\perp}.$$

We want repeat the argument for $H_\Omega^\perp$ so that eventually we are able to write $\pi(A)$ as a sum of cyclic representations, that is, we want to show that there is a maximal family $\{\Omega_i\}$ such that $\pi(A)\Omega_i \perp \pi(A)\Omega_j$. For this, we use Zorn's lemma. Let

$$\mathcal{F} := \{F \subset H \mid \pi(A)\Omega_i \perp \pi(A)\Omega_j, \; \Omega_i, \Omega_j \in F, i \neq j\}.$$

This is a set partially ordered by inclusion, and any chain $\{F_i\} \subset \mathcal{F}$ (that is, any totally ordered subset) has an upper bound, namely, $\cup F_i$. Therefore $\mathcal{F}$ has a maximal element $F = \{\Omega_i\}$. Let $H_i := \overline{\pi(A)\Omega_i}$. If $\omega \in H \setminus \oplus H_i$, then $(\Omega, \pi(a)\Omega_i) = 0$, for all $a \in A$, $i \in I$, and hence $(\pi(b)\Omega, \pi(a)\Omega_i) = 0$, for all $a, b \in A$, $i \in I$, which would contradict the maximality of $F$. Conclude that $H = \oplus H_i$ and, denoting by $\pi_i(A)$ the (cyclic) subrepresentation induced by $H_i$, we have

$$\pi(A) = \oplus \pi_i(A),$$

and the proof is concluded.

2. Need to show that if $v \in V$ is such that $(v, v) = 0$ then $(v, w) = 0$, for all $w \in V$. But, from the Cauchy-Schwarz inequality

$$|(v, w)|^2 \leq (v, v)(w, w) = 0.$$

Now let $\omega$ be a state in a $C^*$-algebra $A$ and $N_\omega = \{a \in A \,|\, \omega(a^*a) = 0\}$. Want to show that $N_\omega$ is a closed left-ideal in $A$. Since $\omega$ is linear, $N_\omega$ is a linear space. Define a semi-definite positive form on $A$ by

$$(a, b) := \omega(a^*b).$$

We check that it is hermitian; for this it suffices to show that $\omega(a^*) = \overline{\omega(a)}$, since $(b, a) = \omega((a^*b)^*)$. Now if $a$ is self-adjoint, we have from the continuous functional calculus that $\|a\|I \pm a \geq 0$[1] and hence $\omega(\|a\|1 \pm a) \geq 0$. Writing

$$\omega(a) = \frac{1}{2}(\omega(\|a\|1 + a) - \omega(\|a\| - a)),$$

we have then that $\omega(a) \in \mathbb{R}$, that is, $\omega(a) = \overline{\omega(a)}$, for $a$ selfadjoint. For arbitrary $a \in A$, the result follows if we write $a = a_1 + ia_2$ with $a_1, a_2$ selfadjoint.

Now, from above, we have then that $N_\omega = \{a \in A \,|\, \omega(b^* a) = 0, \text{for all } b \in A\}$. Now let $b \in A$, $a \in N_\omega$, then

$$\omega((ba)^*(ba)) = \omega(a^* b^* ba) = \omega((b^* ba)^* a) = 0.$$

Therefore $ba \in N_\omega$ and $N_\omega$ is a left-ideal. Since both $\omega$[2] and the map $a \mapsto a^*$ are continuous, $N_\omega$ is closed.

3. Since $\Omega$ is cyclic for $\pi$ and $\Omega_\omega$ is cyclic for $\pi_{\Omega_\omega}$, that is, since $\overline{\pi_{\Omega_\omega}(A)\Omega_\omega} = H_\omega$, $\overline{\pi_(A)\Omega} = H$, it suffices to show that we have an unitary bounded operator

$$U_0 : \pi_{\Omega_\omega}(A)\Omega_\omega \to \pi_(A)\Omega$$

such that $\pi(a)U_0 = U_0 \pi_{\Omega_\omega}(a)$, for $a \in A$, since in this case $U_0$ can be extended to an unitary map $U : H_\omega \to H$. Now, for any $a \in A$,

$$\begin{aligned} \|\pi(a)\Omega\|^2 &= (\pi(a)\Omega, \pi(a)\Omega) \\ &= (\Omega, \pi(a^* a)\Omega) \\ &= \omega_\Omega(a^* a) \\ &= (\Omega_\omega, \pi_{\Omega_\omega}(a^* a)\Omega_\omega) \\ &= \|\pi_{\Omega_\omega}(a)\Omega_\omega\|^2. \end{aligned}$$

Hence there is a well-defined map $U_0$ such that

$$U_0(\pi_{\Omega_\omega}(a)\Omega_\omega) := \pi(a)\Omega$$

and $U_0$ is an isometry and therefore it is unitary (it preserves the inner product). Now for any $a, b \in A$,

$$\begin{aligned} \pi(a)U_0(\pi_{\omega_\Omega}(b)\Omega_\omega) &= \pi(a)\pi(b)\Omega \\ &= U_0(\pi_{\omega_\Omega}(ab)\Omega_\omega) \\ &= U_0(\pi_{\omega_\Omega}(a)\pi_{\omega_\Omega}(b)\Omega_\omega) \end{aligned}$$

[1] See also Lecture 11.
[2] See Exercise 1, Lecture 11.

and therefore $\pi(a)U_0 = U_0\pi_{\Omega_\omega}(a)$.

4. From Exercise 3, if $\pi_1(A)$ and $\pi_2(A)$ are cyclic representations, then both $\pi_1(A)$ and $\pi_2(A)$ are unitarily equivalent to the GNS-representation and hence unitarily equivalent: if there are unitaries $U_1$, $U_2$ such that

$$\pi_1(a)U_1 = U_1\pi_{\Omega_\omega}(a),\ \pi_2(a)U_2 = U_2\pi_{\Omega_\omega}(a)$$

then $\pi_1(a)U_1U_2^{-1} = U_1U_2^{-1}\pi_2(a)$, and $U_1U_2^{-1}$ is unitary.

## Lecture 10 (4-12-03)

1. Assume that $\sum_{i\in I} f(i)$ is finite and let $J := \{i \in I \mid f(i) \neq 0\}$. We want to show that $J$ is countable. First note that for any $\epsilon > 0$, the set $F_\epsilon := \{i \in I \mid f(i) > \epsilon\}$ is finite; otherwise, for any $n \in \mathbb{N}$, one can find $S_n \subset F_\epsilon$ such that $(S) = n$ and in that case $\sum_{i\in S_n} f(i) > n\epsilon$ which contradicts the fact that $\sum_{i\in I} f(i) < \infty$. Now we clearly have

$$J = \bigcup_{n\in\mathbb{N}} F_{1/n}$$

and therefore, being a countable union of finite sets, $J$ is countable.

2. Let $(x_n)$ be a Cauchy sequence in $H_u$, that is, for each $n \in \mathbb{N}$, $x_n = \{x_n^\omega\}_{\omega\in S(A)}$ with $x_n^\omega \in H_\omega$ and we are assuming that for any $\epsilon > 0$ there is $N$ such that for $n, m > N$, $\|x_n - x_m\| < \epsilon$, that is, for any finite set $F \subset S(A)$,

$$\sum_{\omega\in F} \|x_n^\omega - x_m^\omega\|^2 < \epsilon^2. \tag{12}$$

In particular, for each fixed $\omega \in S(A)$, $(x_n^\omega)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_\omega$ and therefore it converges. Let $x^\omega := \lim x_n^\omega$ and $x := \{x^\omega\}_{\omega\in S(A)}$. Letting $m \to \infty$ in (12), we have that there is $N$ such that for $n > N$ and for any finite set $F$,

$$\sum_{\omega\in F} \|x_n^\omega - x^\omega\|^2 < \epsilon^2.$$

We conclude that $x_n - x \in H_u$, and therefore that $x \in H_u$, and moreover

$$\|x_n - x\|^2 = \sup_{F\subset S(A),\,\text{finite}} \sum_{\omega\in F} \|x_n^\omega - x^\omega\|^2 < \epsilon^2$$

which shows that $H_u$ is complete.

3. We first show that $a = a^*$; this is straightforward from

$$(v, av) - (v, a^*v) = (v, av) - (av, v) = (v, av) - \overline{(v, av)} = 2Im(v, av) = 0$$

since $(v, av) \in \mathbb{R}$. Now let $\lambda \in \sigma(a)$. Since $a$ is selfadjoint, we know that $\lambda \in \mathbb{R}$, so that $a - \lambda I$ is selfadjoint as well. We first show that a selfadjoint operator $b$ is invertible iff

$$\inf\{\|bx\| \mid \|x\| = 1\} > 0. \tag{13}$$

It is clear from (13) that $b$ is injective. Since $\ker(b)^\perp = \ker(b^*)^\perp = \overline{Im(b)}$, we have also that $Im(b)$ is dense. If $bx_n \to y$, then $(bx_n)$ is a Cauchy sequence in $H$ and (13) yields that $(x_n)$ is also a Cauchy sequence in $H$, therefore convergent, $x_n \to x$; since $b$ is bounded, $bx_n \to bx$ and hence $Im(b)$ is closed. We conclude then that $b$ is a bijective operator and (13) gives that $b^{-1}$ is bounded, that is, $b$ is invertible.

Now, since $a - \lambda I$ is selfadjoint, we therefore have that $\lambda \in \sigma(a)$ iff

$$\inf\{\|(a-\lambda I)x\| \mid \|x\| = 1\} = 0$$

that is, if there exists a sequence $(x_n)$ such that $\|x_n\| = 1$ and

$$\|(a-\lambda I)x_n\| \to 0.$$

In this case, $(x_n, (a-\lambda I)x_n) \to 0$ as well, and since $(x_n, \lambda x_n) = \lambda$ and $(x_n, ax_n) \geq 0$, we conclude that $\lambda \geq 0$, which finishes our proof.

**Notes** 1- One can check directly that, for $\lambda < 0$,

$$\|(a-\lambda I)x\|^2 = ((a-\lambda I)x, (a-\lambda I)x) = \|ax\|^2 + \lambda^2\|x\|^2 - 2\lambda(x, ax) \geq \lambda^2\|x\|^2$$

(we have used the fact that $(x, ax) \geq 0$), and therefore $a - \lambda I$ is invertible.

2 - We can give a proof of this result without using the fact $\sigma(a) \subset \mathbb{R}$ noting that the invertibility criterion given above for selfadjoint operators also holds for normal operators (such that $b^*b = bb^*$), since in this case one has always $\ker(b) = \ker(b^*)$. Note that if $a$ is self-adjoint, then $a - \lambda I$ is normal, for any $\lambda \in \mathbb{C}$.

4. To characterize $S(M_2(\mathbb{C}))$, we first identify $(M_2(\mathbb{C}))'$. To do this, we define a Hilbert space structure on $M_2(\mathbb{C})$ and use Riesz Representation theorem. For $a, b \in M_2(\mathbb{C})$ define

$$(a, b) := \mathrm{Tr}(a^*b).$$

(Recall that for $a = (a_{ij})$, $\mathrm{Tr}(a) = \sum a_{ii}$.) Since Tr is linear and $\mathrm{Tr}(a^*) = \overline{\mathrm{Tr}(a)}$, one easily checks that $(\ ,\ )$ is an hermitian bilinear form. Also, if $a$ is self-adjoint, one has $\mathrm{Tr}(a) = \sum \lambda_i$, where $\lambda_i$ are the eigenvalues of $a$[1] Therefore,

$$(a, a) = \mathrm{Tr}(a^*a) = \sum \lambda_i \geq 0,$$

[1]Since, in this case, one has $a = u^{-1}du$ where $u$ is unitary and $d$ is the matrix representation of $a$ on a basis of eigenvectors; hence $\mathrm{Tr}(a) = \mathrm{Tr}(u^{-1}du) = \mathrm{Tr}(u^{-1}ud) = \mathrm{Tr}(d)$.

since $a^*a$ is selfadjoint and positive, and if $(a,a) = 0$ then $\lambda_i = 0$, $i = 1,2$ and hence $a^*a = 0$ which gives $a = 0$. We conclude that $(\ ,\ )$ is a an inner product and that $M_2(\mathbb{C})$ is indeed an Hilbert space.[1]

Now by Riesz Representation theorem, every $\omega \in (M_2(\mathbb{C}))'$ is given by

$$\omega(a) = \mathrm{Tr}(a_\omega a)$$

for some (unique) matrix $a_\omega \in M_2(\mathbb{C})$. To characterize the state space $S(M_2(\mathbb{C})) \subset (M_2(\mathbb{C}))'$, we show the following:

(i) $\omega(I) = 1$ iff $\mathrm{Tr}(a_\omega) = 1$;
(ii) $\omega(a^*a) \geq 0$, for all $a \in A$ iff $a_\omega$ is positive in $M_2(\mathbb{C})$.

Only (ii) needs proof. Let $a_\omega \in M_2(\mathbb{C})$ be positive; then, for any $a \in A$, we have that $a_\omega a^*a$ is also positive and therefore $\mathrm{Tr}(a_\omega a^*a) \geq 0$, that is, $w(a^*a) \geq 0$. Conversely, we show that $a_\omega = a_\omega^*$ and that $\sigma(a_\omega) \subset \mathbb{R}^+$. To see that $a_\omega$ is selfadjoint, note that for selfadjoint $a$, we have $\mathrm{Tr}(a_\omega^* a) = \mathrm{Tr}(a_\omega a)$ and writing an arbitrary $a \in A$ as $a = b + ic$ with $b, c$ selfadjoint, we see that $\mathrm{Tr}(a_\omega^* a) = \mathrm{Tr}(a_\omega a)$ for all $a \in A$; by unicity, we have $a_\omega^* = a_\omega$. Replacing, if necessary, $a_\omega$ by $d_\omega$, the diagonal matrix representing $a_\omega$ on a basis of eigenvectors, one can see that there are positive matrices $b_1$, $b_2$ such that

$$\mathrm{Tr}(a_\omega b_i) = \mathrm{Tr}(d_\omega b_i) = \lambda_i.$$

where $\lambda_1$, $\Lambda_2$ are the eigenvalues of $a_\omega$. Hence, since $\omega$ is positive, $\omega(b_i) = \lambda_i \geq 0$ and we have that $a_\omega$ is positive.

We conclude that $S(M_2(\mathbb{C})) \cong \{a \in M_2(\mathbb{C}) \,|\, a \geq 0,\ \mathrm{Tr}(a) = 1\}$. (Note that this isomorphism is induced by $(M_2(\mathbb{C}))' \cong M_2(\mathbb{C})$ and is then obviously a homeomorphism.) One can easily check that $a \in M_2(\mathbb{C})$ is such that $a \geq 0$ and $\mathrm{Tr}(a) = 1$ iff

$$a = \begin{bmatrix} 1+x & y+iz \\ y-iz & 1-x \end{bmatrix}$$

for some $x, y, z \in \mathbb{R}$. Now this matrix is positive (that is, semi-definite positive) iff $1 + x \geq 0$ and $\det(a) \geq 0$, that is, iff $x^2 + y^2 + z^2 \leq 1$ and we conclude that $S(M_2(\mathbb{C})) \cong B^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

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[1]Since $M_2(\mathbb{C})$ is finite dimensional, all norms are equivalent and therefore it is complete with respect to the norm induced by the defined inner product.

## Lecture 11 (11-12-03)

1. (i) Let $\omega : A \to \mathbb{C}$ be a positive functional ($A$ not necessarily unital). We first show that it suffices to prove boundedness on $A^+$. Given an arbitrary $a \in A$, one can write $a = b + ic$, where $b, c$ are self-adjoint; every self-adjoint element $b$ can be written as $b = b^+ - b^-$, where $b^+, b^-$ are positive. Hence we can write $a = b^+ - b^- + i(c^+ - c^-)$, with $b^+, b^-, c^+, c^-$ positive elements, and it is clear that $\|u\| \leq \|a\|$, for $u = b^\pm$, $u = c^\pm$. We have then

$$|\omega(a)| \leq |\omega(b^+)| + |\omega(b^-)| + |\omega(c^+)| + |\omega(c^-)|.$$

If $\omega$ is bounded on $A^+$, that is, if there exists $C > 0$ such that for all positive $u \in A$, $|\omega(u)| \leq C\|u\|$, we conclude

$$|\omega(a)| \leq 4C\|a\|$$

and $\omega$ is bounded on $A$.

We now prove that $\omega$ is bounded by contradiction. Assume then that $\omega$ is not bounded. Then, for any $n \in \mathbb{N}$ we can find $a_n$, with $\|a_n\| = 1$, such that

$$|\omega(a_n)| > n^3$$

and by the argument above, we can assume that $a_n$ is positive. The series $\sum_{n\in\mathbb{N}} \frac{a_n}{n^2}$ obviously converges to some $a \in A$. The sequence $s_n := \sum_{k=1}^{n} \frac{a_n}{n^2}$ is such that $s_n \geq 0$ and $s_{n+1} \geq s_n$. By the little lemma we prove below, one has that $a = \lim s_n$ is positive and $a \geq s_n$ for all $n \in \mathbb{N}$. Since $\omega$ is positive,

$$\omega(a) \geq \omega(s_n) \geq \omega\left(\frac{a_n}{n^2}\right) \geq n, \text{ for all } n \in \mathbb{N},$$

which is absurd, because then $\omega(a) = \infty$. Therefore $\omega$ is bounded.

We still need to prove the following:

**Lemma**: Let $(a_n)$ be a sequence on a $C^*$-algebra $A$ such that $a_n \to a$, $a_n \geq 0$, and $a_{n+1} \geq a_n$, for all $n \in \mathbb{N}$. Then $a \geq 0$ and, for any $N \in \mathbb{N}$, $a \geq a_N$.

**Proof**: Can assume that $A$ is unital[1]. It is clear that $a$ is self-adjoint, since the involution is continuous. From

$$\sigma(a) = \{\widehat{a}(\omega) \,|\, \omega \in \Delta(C^*(a, 1))\},$$

[1]Since an element in a non-unital algebra is positive iff it is positive in the unitization; recall that on a non-unital $C^*$-algebra $A$, $\sigma_A(a)$ is defined as $\sigma_{\dot{A}}(a + 0)$.

one easily checks that $a \geq 0$ iff $\widehat{a}(\omega) \geq 0$ for all $\omega$. Now we know that the Gelfand transform is continuous, therefore $\widehat{a_n} \to \widehat{a}$; but since $a_n \geq 0$, $\widehat{a_n}(\omega) \geq 0$ and hence $\widehat{a}(\omega) \geq 0$, that is, $a \geq 0$.

To show that, for any fixed $N \in \mathbb{N}$, $a - a_N \geq 0$, just apply the result just proved to the sequence $b_n := a_{n+N} - a_N$. $\square$

(ii) We show that a functional $\omega : A \to \mathbb{C}$, $A$ unital, is positive iff it is continuous and $\|\omega\| = \omega(1)$. Assume then that $\omega$ is positive; we know from (i) that it is bounded. Since $A$ is unital, one can give a different proof that will yield $\|\omega\|$. Let first $a$ be self-adjoint. Then from the continuous functional calculus on $C^*(a, 1)$, one has that $-\|a\|1 \leq a \leq \|a\|1$, that is, $\pm a + \|a\|1 \geq 0$ (just take $f(t) := \pm t + \|a\|$, and use the fact that $f \geq 0$ to conclude that $f(a) \geq 0$). Therefore, since $\omega$ is positive,

$$|\omega(a)| \leq \omega(1)\|a\|.$$

For arbitrary $a \in A$, from $|w(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$, we have now

$$|\omega(a)|^2 \leq \omega(1)\omega(a^*a) \leq \omega(1)^2\|a^*a\| = \omega(1)^2\|a^2\|$$

and therefore $\|\omega\| \leq \omega(1)$. It is trivial that $\|\omega\| \geq \omega(1)$ and hence $\|\omega\| = \omega(1)$.

Conversely, let $\omega$ be a bounded functional such that $\|\omega\| = \omega(1)$ and $a \in A$ be positive. Again from the continuous functional calculus, and since $\sigma(a) \subset \mathbb{R}^+$, we have that there is an $s \in \mathbb{R}^+$ such that

$$\sigma(1 - sa) = \{1 - st \mid t \in \sigma(a)\} \subset [0, 1].$$

Since $\|1 - sa\| = r(1 - sa)$, the spectral radius of $1 - sa$, we then have $\|1 - sa\| \leq 1$. But then

$$|\omega(1) - s\omega(a)| = |\omega(1 - sa)| \leq \omega(1)\|1 - sa\| \leq \omega(1).$$

Hence $\omega(1) - s\omega(a) \leq \omega(1)$ and therefore $\omega(a) \geq 0$. We conclude that $\omega$ is positive.

(iii) If $A$ is unital, then $\omega$ is a state if it is positive and $\omega(1) = 1$. From (ii), $\omega$ is positive iff $\|\omega\| = \omega(1)$, and hence $\omega$ is a state iff $\|\omega\| = 1$.

On a non-unital $A$, $\omega$ is a state if $\dot{\omega}$ is positive in $\dot{A}$, the unitization of $A$, where $\dot{\omega}(a + \lambda) := \omega(a) + \lambda$. It is clear that we have in this case

$$\|\omega\| = \sup_{a \in A, \|a\|=1} |\omega(a)| \leq \sup_{a+\lambda \in \dot{A}, \|a+\lambda\|=1} ||\dot{\omega}(a + \lambda)| = \|\dot{\omega}\|.$$

From (ii), since $\dot{\omega}$ is positive, $\|\dot{\omega}\| = \dot{\omega}(0+1) = 1$ and therefore $\|\omega\| \leq 1$. To show that $\|\omega\| = 1$, we consider an approximate identity in $A$, that is, a net $\{I_\lambda\}_{\lambda\in I}$ of self-adjoint elements, with $\|I_\lambda\| = 1$, such that, for all $a \in A$, $\|I_\lambda a - a\| \to 0$ and $\|aI_\lambda - a\| \to 0$. Then, since $\omega$ is bounded, $\omega(I_\lambda a - a) = (\omega(I_\lambda) - 1)\,\omega(a) \to 0$, for any $a \in A$, that is,

$$\omega(I_\lambda) \to 1.$$

Therefore, for any $\epsilon > 0$, there is $\lambda \in I$ such that $\omega(I_\lambda) > 1-\epsilon$. We conclude that $\|\omega\| \geq 1$.

(iv) On a non-unital $C^*$-algebra $A$, we have by definition that $\omega$ is a state iff $\dot{\omega}$ is positive in $\dot{A}$, the unitization of $A$, where $\dot{\omega}(a+\lambda) := \omega(a) + \lambda$. We have then $\dot{\omega}(0+1) = 1$ and hence $\dot{\omega}$ is a state. (It is clear that $\dot{\omega}$ is the unique extension of $\omega$ to $\dot{A}$.)

(v) By definition, $\omega$ is a state on $A$ if its extension $\dot{\omega}$ to $\dot{A}$ is positive. Therefore the restriction of a state (hence a positive functional) on $\dot{A}$ to $A$ is a state.

2. Let $a \in A$ be self-adjoint and $X := \Delta(C^*(a,1))$. We have then the diagram

$$\begin{array}{ccc} C(\sigma(a)) & \xrightarrow{\Psi} & C^*(a,1) \\ {\scriptstyle \widehat{a}^*}\downarrow & & \downarrow{\scriptstyle \Phi} \\ C(X) & = & C(X), \end{array}$$

where the maps involved are as follows:

– $\widehat{a}^*$ is induced by the Gelfand transform of $a$ regarded as a map $\widehat{a} : X \to \sigma(a)$. It was proved in class that $\widehat{a}$ is a homeomorphism, and therefore $\widehat{a}^*$ is an isomorphism.

– $\Phi$ is the Gelfand transform on $C^*(a,1)$, which is an isomorphism by the Gelfand-Neumark theorem (since $C^*(a,1)$ is commutative).

– $\Psi$ is given by the continuous functional calculus, $\Psi(f) := f(a)$, $f \in C(\sigma(a))$. We know by the definition of $f(a)$ that $\|f(a\| = \|f\|_\infty$; to show that $\Psi$ is an isomorphism one therefore only need to show that it is surjective. Recall that $C^*(a,1) \subset A$ is defined as the closure in $A$ of the set of all polynomials in $a$. It is clear that $p(a) = \Psi(p)$ where $p(t)$ is any polynomial on $\sigma(a)$. If $b = \lim p_n(a)$, for some sequence of polynomials $p_n$, then $(p_n(a))$ is a Cauchy sequence in $A$ and therefore also $(p_n)$ is a Cauchy sequence in $C(\sigma(a))$, since $\Psi$ is isometric. Hence there is $f \in C(\sigma(a))$ such that $p_n \to f$

and then we have also $p_n(a) \to f(a)$, that is, $b = \Psi(f)$. Conclude that $\Psi$ is surjective, and therefore an isomorphism.

We now finally check that the diagram indeed commutes. Need to show that for any $f \in C(\sigma(a))$,

$$\widehat{f(a)} = \widehat{a}^*(f),$$

where, as usual, $\widehat{f(a)}$, $\widehat{a}$ denote the Gelfand transforms of $f(a)$, $a$. For any $\omega \in X$,

$$\widehat{f(a)}(\omega) = \omega(f(a))$$
$$\widehat{a}^*(f)(\omega) = (f \circ \widehat{a})(\omega) = f(\omega(a)).$$

If $f(a)$ is given by a polynomial in $a$, we have $\omega(f(a)) = f(\omega(a))$, since $\omega$ is multiplicative (and linear). The boundedness of $\omega$ then yields this equality for general $f$.