

## Answers to Take-Home Exam Part I (2003-2004)

**Exercise 1** 1. Check that the commutant of any set in an algebra  $A$  is a subalgebra of  $A$ .

2. Check that, if  $A$  is a  $*$ -algebra and  $B \subset A$  is involutive (that is, if  $b \in B$  then  $b^* \in B$ ), then the commutant  $B'$  is a  $*$ -subalgebra of  $A$ .

3. For any  $B \subset A$  and  $M := B'$ , show that  $M'' = M$ .

*Answer:* Trivial

**Exercise 2** 1. Show that  $(M \otimes 1_d)' = M' \otimes M_d(\mathbb{C})$ .

2. Show that  $(M \otimes M_d(\mathbb{C}))' = M' \otimes 1_d$ .

*Answer:* Easy

**From now on,  $H$  is a finite-dimensional Hilbert space.**

**Proposition 1** Let  $M$  be a unital  $*$ -subalgebra of  $B(H)$ . Then  $M'' = M$ .

**Lemma 1** Let  $M$  be a  $*$ -algebra in  $B(H)$ , take a nonzero vector  $v \in H$ , and let  $p$  be the (orthogonal) projection onto  $Mv$  (defined by  $pw = w$  if  $w \in Mv$  and  $pw = 0$  if  $w \in (Mv)^\perp$  with respect to the inner product on  $H$ ). Then  $p \in M'$ .

**Exercise 3** 1. Prove this lemma.

2. Prove Proposition 1 from this lemma.

3. Conclude from Proposition 1 and Exercise 1 that any involutive unital subset  $M \subset B(H)$  is a  $*$ -subalgebra of  $B(H)$  iff  $M'' = M$ .

*Answer to 1:* If  $a \in M$  then  $apH \subseteq pH$  by definition of  $p$ . Hence  $p^\perp ap = 0$  hence  $ap = pap$  with  $p^\perp = 1 - p$ . If  $a = a^*$ , then

$$(ap)^* = pa = (pap)^* = pap = ap,$$

so that  $[a, p] = 0$ . Since any element of a  $*$ -algebra is the sum of two selfadjoint ones, this is true for all  $a \in M$ . ■

*Answer to 2:* Let  $H \cong \mathbb{C}^n$ . The idea of the proof is to take  $n$  arbitrary vectors  $v_1, \dots, v_n$  in  $H$ , and, given  $a \in M''$ , construct a matrix  $b \in M$  such that  $av_i = bv_i$  for all  $i = 1, \dots, n$ . Hence  $a = b$ , so  $a \in M$ .

First, choose some  $v = v_1 \in H$ , and form the linear subspace  $Mv$  of  $H$ , with projection  $p$ . By Lemma 1,  $p \in M'$ . Hence  $a \in M''$  commutes with  $p$ . Since  $1 \in M$ , we therefore have  $v = v \in Mv$ , so  $v = pv$ , and  $av = apv = pav \in Mv$ . Hence  $av = bv$  for some  $b \in M$ .

Now use the same argument with the substitutions  $H \rightsquigarrow H^n$ ,  $M \rightsquigarrow M \otimes 1_n$ , and  $v \rightsquigarrow \oplus_i v_i$ . By Exercise 2, we know that

$$(M \otimes 1_n)'' = (M' \otimes M_n)' = M'' \otimes 1_n.$$

Hence given  $\oplus_{i=1}^n v_i$  and  $a \in M'' \otimes 1_n$ , there exists  $b \in M \otimes 1_n$  such that

$$(a \otimes 1_n) \oplus_i v_i = (b \otimes 1_n) \oplus_i v_i.$$

It follows that  $av_i = bv_i$  for all  $i = 1, \dots, n$ , so that  $a = b$ . ■

*Answer to 3:* Trivial

**Lemma 2** *Let  $M \subset B(H)$  be a unital  $*$ -algebra. Then  $M$  acts irreducibly on  $H$  if and only if  $M = B(H)$ .*

**Exercise 4** *Prove this lemma, using Proposition 1.*

*Answer:* By Proposition 1,  $M = L(H)$  iff  $M' = \mathbb{C}1$ . If  $M' \neq \mathbb{C}1$ , then  $M'$  must contain a self-adjoint element  $a \neq 1$ . By standard linear algebra, the spectral projections  $p_i$  of  $a$  lie in  $M'$  as well. Hence  $p_i H$  is stable under  $M$  for all  $i$ , so  $M$  cannot be irreducible.

Conversely, when  $H$  contains a proper subspace  $K$  stable under  $M$ , the the projection onto  $K$  lies in  $M'$ , so that  $M' \neq \mathbb{C}1$ . ■

**Theorem 1** *Let  $H$  be a finite-dimensional Hilbert space and suppose that  $M \subset B(H)$  is a unital  $*$ -algebra of matrices on  $H$ . Then there exist Hilbert spaces  $H_i$ , unique up to unitary equivalence, unique integers  $d_i$ , and a unitary transformation*

$$u : H \rightarrow \bigoplus_i H_i \otimes \mathbb{C}^{d_i}$$

such that

$$uMu^* = \bigoplus_i B(H_i) \otimes 1_{d_i}.$$

*In words,  $M$  is unitarily equivalent to a direct sum of matrix algebras with possible multiplicity.*

**Exercise 5** 1. *Prove this theorem, using Schur's lemma and Proposition 1.*

2. *Infer that  $M$  is isomorphic to a direct sum of matrix algebras.*

3. *Conclude that any finite-dimensional  $C^*$ -algebra is isomorphic to a direct sum of matrix algebras.*

*Answer to 1:* If a  $*$ -algebra  $M \subset L(H)$  acts irreducibly on  $H$ , we are ready by Schur's lemma. When the action is reducible,  $H$  has a proper subspace  $K$  stable under  $M$ . It follows that  $K^\perp$  is stable as well. Repeating this process, one eventually obtains a complete reduction  $H = \bigoplus_{i \in I} H_i$ , where each  $H_i$  is irreducible under  $M$ : the finite-dimensionality of  $H$  guarantees that this process indeed terminates. By Schur's lemma,  $M \upharpoonright H_i = L(H_i)$ .

For any pair  $i, j \in I$ , consider the intertwiner space

$$L_{ij} := \{c \in L(H_i, H_j) \mid ca_i = a_jc \forall a \in M\},$$

where  $a_i := a \upharpoonright H_i$ . Now  $L(H_i, H_j)$  is clearly a vector space, and there are only two possibilities: it is either zero-dimensional or one-dimensional. For if  $c \in L(H_i, H_j)$ , then  $c^* \in L(H_j, H_i)$  since  $M$  is a  $*$ -algebra. Hence for any  $c, d \in L(H_i, H_j)$  one has  $c^*d \in L(H_i, H_i)$  and  $cd^* \in L(H_j, H_j)$ . Hence  $c^*d = \lambda 1$  for some  $\lambda \in \mathbb{C}$  by Schur's lemma. Multiplying with  $c$  and noting  $cc^* = \mu 1$ ,  $c$  and  $d$  must be proportional.

If  $L(H_i, H_j)$  is one-dimensional, it contains a unitary  $u$ . For  $c^*c = \lambda 1$  for some  $\lambda \geq 0$ , and there must be a  $c$  for which  $\lambda > 0$ ; otherwise, one would be in the zero-dimensional case. Then  $u = c/\sqrt{\lambda}$  is unitary. It follows that  $M \upharpoonright H_i$  and  $M \upharpoonright H_j$  are spatially isomorphic, and after the above unitary transformation they are even the same, i.e.  $a \upharpoonright H_i = a \upharpoonright H_j$  for all  $a \in M$ . Let the isomorphism class of  $M_i$  contain  $d_i$  elements; one joins the corresponding  $H_i$  together to form  $H_i \otimes \mathbb{C}^{d_i}$ , on which the restriction of  $M$  is evidently unitarily equivalent to  $L(H_i) \otimes 1_{d_i}$ . One then has  $H \cong \oplus_{i \in [I]} H_i$ , where  $[I]$  contains one element of each isomorphism class in  $I$ , and  $\cong$  denotes unitary equivalence.

We now compute  $M'$ . Let  $p_i$  be the projection onto  $H_i$ . It is clear that  $p_i \in M'$ . Hence for any  $c \in M'$  one has  $p_i c p_j \in M'$ . If  $L(H_i, H_j)$  is zero-dimensional, then  $p_i c p_j = 0$ . Hence  $p_i c p_j$  can be nonzero only when  $i$  and  $j$  are equivalent, so that

$$M' \cong \oplus_{i \in [I]} (L(H_i) \otimes 1_{d_i})' = \oplus_{i \in [I]} 1 \otimes M_{d_i}$$

by Exercise 1 and Schur's lemma. Taking the commutant once again, and using  $M'' = M$  (cf. Proposition 1), one proves the claim.  $\blacksquare$

*Answer to 2:* The map

$$a \mapsto \oplus_{i \in [I]} a \upharpoonright H_i$$

is a  $*$ -isomorphism from  $M$  to  $\oplus_{i \in [I]} L(H_i)$ . (Hence the multiplicity spaces  $\mathbb{C}^{d_i}$  are important for spatial isomorphism, but can be ignored when one is merely interested in a  $*$ -isomorphism.)  $\blacksquare$

*Answer to 3:* Start from the universal representation  $\pi_u(A)$  on  $H_u$  used in the proof of the Gelfand–Naimark theorem. Take  $v_1 \in H_u$  and consider the representation  $\pi_1(A)$ , defined as the restriction of  $\pi_u(A)$  to the Hilbert subspace  $H_1 := \pi_u(A)v_1$  of  $H_u$ . Note that the GNS-construction implies that  $\pi_1$  cannot be identically zero. If  $\pi_1$  is injective, we stop here. If not, let  $A_1 := \ker(\pi_1)$ , and find  $v_2 \in H_u$  for which  $\pi_2(A)$ , defined as the restriction of  $\pi_u(A)$  to  $H_2 := \pi_u(A)v_2$ , has the property that  $\pi_2(A_1)$  is not identically zero. If  $\pi_2(A_1)$  is injective, then  $\pi_1 \oplus \pi_2$  on  $H_1 \oplus H_2$  is injective. If not, consider  $A_2 := \ker(\pi_2(A_1))$ , and iterate this procedure until one has found an injective representation  $\pi(A) := \oplus_{i=1}^n \pi_i(A)$  on  $H := \oplus_{i=1}^n H_i$ . This iteration has a finite number of steps because  $A$  is finite-dimensional, so at some stage the sequence  $A_0 = A \supset A_1 \supset A_2 \supset \dots$  must end at zero. Since  $A$  is finite-dimensional, each  $H_i$ , and hence  $H$ , is finite-dimensional.

This construction guarantees that  $H$  does not contain a nonzero subspace  $K$  that is annihilated by  $\pi(A)$ . Consequently,  $\pi$  is nondegenerate (in the sense that

$\pi(A)H = H$ ), and since  $\pi(A) \cong A$  is finite-dimensional this implies that  $\pi(A)$  must contain the unit operator on  $H$  (so that  $A$  must be unital itself). One way to see this is to use the general fact that a  $C^*$ -algebra has an approximate unit; in a nondegenerate representation this strongly converges to 1, but since in the finite-dimensional case  $\pi(A)$  is closed in all relevant topologies, the limit must lie in  $\pi(A)$ . Another method is to consider the center of  $A$ ; this is commutative, hence of the form  $C_0(X)$ , but since  $A$  is finite-dimensional  $X$  must be a finite set, so that  $C(X)$  contains 1.

Consequently, we have shown that  $A$  is isomorphic to a unital  $*$ -algebra acting on a finite-dimensional Hilbert space, so the claim follows from the previous exercise.