Part II

K-theory of C^*-algebras

1 Introduction

As already mentioned in the Introduction to Part I, the K-theory of C^*-algebras may be thought of in the following way. One starts with topological K-theory, which encodes the theory of vector bundles on compact Hausdorff spaces.[1] This theory primarily associates an abelian group $K^0(X)$ to a space X. Explicitly, $K^0(X)$ is the abelian group with one generator for each isomorphism class $[E]$ of vector bundles over X, and relations $[E] + [F] = [E \oplus F]$.

By the pullback construction for vector bundles,[2] the association $X \mapsto K^0(X)$ is contravariant, in that a map $\varphi : X \to Y$ induces a homomorphism $\varphi^* : K^0(Y) \to K^0(X)$ of abelian groups.[3] This enables one to extend the definition of K^0 to locally compact spaces, as follows: if $\dot{X}$ is the one-point compactification of X, the inclusion $\infty \overset{\iota}{\hookrightarrow} \dot{X}$ of the compactification point induces a homomorphism $\iota^* : K^0(\dot{X}) \to K^0(\mathrm{pt})$. For noncompact X, the group $K^0(X)$ is defined as the kernel of this homomorphism, so that it is a subgroup of $K^0(\dot{X})$. In fact, it is easy to see that $K^0(\mathrm{pt}) \cong \mathbb{Z}$.

It turns out to be useful to define 'higher' K-groups $K^n(X) := K^0(X \times \mathbb{R}^n)$, $n \in \mathbb{N}$, which together define topological K-theory. This theory has the following important properties.

1. Contravariant functoriality: a proper map $\varphi : X \to Y$ induces group homomorphisms $\varphi_n^* : K^n(Y) \to K^n(X)$.
2. Homotopy invariance: if φ is a proper homotopy equivalence,[4] then each φ_n^* is an isomorphism.
3. Bott periodicity:[5] one has isomorphisms $K^{n+2}(X) \cong K^n(X)$ for all n, natural in X.
4. Excision:[6] any closed subset $Y \subset X$ gives rise to a cyclic exact sequence[7]

$$\begin{tikzcd}
K^0(X\backslash Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\
K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X\backslash Y). \arrow[l]
\end{tikzcd}$$

Hence one only has two independent K-groups, $K^0(X)$ and $K^1(X)$. To see what sort of abelian groups are involved, some examples are listed in Table 1.

Subsequently, topological K-theory is reformulated in terms of the unital commutative C^*-algebra $C(X)$, and this reformulation may then be extended to the noncommutative case. This

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[1] Recall that a vector bundle E over X is an open surjection $\pi : E \to X$ with the property that each fiber $E_x := \pi^{-1}(x)$ is a vector space isomorphic to $\mathbb{C}^n$, and each $x \in X$ has a neighbourhood U_x such that $\pi^{-1}(U_x) \cong U_x \times \mathbb{C}^n$, the isomorphism $\pi^{-1}(U_x) \to U_x \times \mathbb{C}^n$ being linear on each fiber.

[2] Recall that given a vector bundle $\pi : E \to Y$ over Y and a map $\varphi : X \to Y$ one defines the pullback bundle $\varphi^* E$ over X by

$$\varphi^* E = E \times_Y X := \{(v, x) \in E \times X \mid \pi(v) = \varphi(x)\},$$

with projection $(v, x) \mapsto x$.

[3] Here a generator $[E]$ of $K^0(Y)$ is mapped into a generator $[\varphi^* E]$ of $K^0(X)$, and since the pullback construction preserves direct sums, this induces the desired homomorphism $\varphi^* : K^0(Y) \to K^0(X)$ (with some abuse of notation).

[4] Two maps $\varphi : Z \to T$ and $\psi : Z \to T$ are said to be homotopic, written $\varphi \overset{\mathrm{h}}{\sim} \psi$, if there is a map $\Phi : [0,1] \times Z \to T$ such that $\Phi_0 = \varphi$ and $\Phi_1 = \psi$, where $\Phi_t := \Phi(t, \cdot)$. Two locally compact spaces X, Y are then called properly homotopy equivalent when there are proper maps $X \xrightarrow{\varphi} Y \xrightarrow{\psi} X$ such that $\varphi \circ \psi \overset{\mathrm{h}}{\sim} \mathrm{id}_Y$ and $\psi \circ \varphi \overset{\mathrm{h}}{\sim} \mathrm{id}_X$. Both φ and ψ are then called a homotopy equivalence. Note that under this definition $\mathbb{R}$ or $(0, 1)$ are not homotopic to a point because of the properness condition, which is violated by the map $\mathbb{R} \to \mathrm{pt}$. If X and Y are compact this condition is automatically satisfied.

[5] Bott periodicity is sometimes claimed to be the only nontrivial theorem in topological K-theory.

[6] This is really a consequence of the excision property as defined in algebraic topology combined with Bott periodicity and the cohomological nature of topological K-theory.

[7] An exact sequence of groups is a chain of homomorphisms $\cdots \xrightarrow{\varphi_i} G_i \xrightarrow{\varphi_{i+1}} G_{i+1} \xrightarrow{\varphi_{i+2}} \cdots$ with the property $\ker(\varphi_{i+1}) = \mathrm{im}(\varphi_i)$ for all i.

X	$K^0(X)$	$K^1(X)$
pt	$\mathbb{Z}$	0
$[0,1]$	$\mathbb{Z}$	0
$(0,1]$	0	0
$(0,1)$	0	$\mathbb{Z}$
S^{2n}	$\mathbb{Z}\oplus\mathbb{Z}$	0
S^{2n+1}	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{R}^{2n}$	$\mathbb{Z}$	0
$\mathbb{R}^{2n+1}$	0	$\mathbb{Z}$
$\mathbb{T}^n$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$

Table 1: Table of topological K-groups

leads to groups $K_0(A)$, where A is a unital C^*-algebra. Explicitly, $K_0(A)$ is the abelian group with one generator for each equivalence class $[p]$ of projections[8] in $M_\infty(A)$, and relations $[p]+[q]=[p\oplus q]$. Here we say that $p \sim q$ when there is a $v \in M_\infty(A)$ such that $vv^* = p$ and $v^*v = q$. In the commutative case one then indeed has $K_0(C(X)) \cong K^0(X)$ in a natural way.

C^*-algebraic K-theory has the opposite functorial behaviour of its topological counterpart, because of the contravariant nature of the association $X \mapsto C(X)$. Indeed, a morhpism $\varphi : A \to B$ preserves projections, so that a generator $[e]$ of $K_0(A)$ is simply mapped to a generator $[\varphi(p)]$ of $K_0(B)$. Hence φ induces a group homomorphism $\varphi_* : K_0(A) \to K_0(B)$. Analogous to the topological case, this may be used to define $K^0(A)$ for nonunital A as well. This is done through the unitization $\dot{A}$ and the associated morphism $\pi : \dot{A} \to \dot{A}/A \cong \mathbb{C}$.[9] Since this morphism only involves unital C^*-algebras, one has an induced map $\pi_* : K_0(\dot{A}) \to K_0(\mathbb{C})$. One then defines $K_0(A)$ for nonunital A as the kernel of π_*, so that it is a subgroup of $K_0(\dot{A})$. Note that $K_0(\mathbb{C}) \cong K^0(\mathrm{pt}) \cong \mathbb{Z}$.

The higher K-groups are then defined via the so-called suspension $SA := C_0(\mathbb{R}, A)$.[10] This is a C^*-algebra, and the operation S may be iterated so as to define a whole sequence of C^*-algebras $S^nA = C_0(\mathbb{R}^n, A)$. We then define $K_n(A) := K_0(S^nA)$. One still has $K_n(C(X)) \cong K^n(X)$ for any n.

In complete analogy to the commutative case, the ensuing C^*-algebraic K-theory has the following properties:

1. Covariant functoriality: a morphism $\varphi : A \to B$ induces group homomorphisms $\varphi^n_* : K_n(A) \to K_n(B)$.[11]

2. Homotopy invariance: if φ is a homotopy equivalence,[12] then φ^*_n is an isomorphism for all n,

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[8] That is, elements p such that $p^* = p^2 = p$.

[9] Recall that $\dot{A}$ contains A as an ideal, so that π is simply the canonical projection.

[10] Similar to the case $A = \mathbb{C}$, for any locally compact space X the set $C_0(X, A)$ consists of all continuous functions from X to A such that for each $\epsilon > 0$ there is a compact subset $K \subset X$ such that $\|f(x)\| < \epsilon$ for all x outside K. It is not difficult to show that $C_0(X, A)$ is a C^*-algebra in the obvious pointwise operations and the norm $\|f\|_\infty := \sup\{\|f(x)\| \mid x \in X\}$.

[11] Note that φ is not required to be nondegenerate. The restriction of C^*-algebraic K-theory to commutative C^*-algebras therefore appears to be an extension of topological K-theory, because the C^*-algebraic theory appears to be functorial under a larger class of morphisms. Namely, recall from Part I that the pullback $\psi^* : C_0(Y) \to C_0(X)$ of a proper map $\psi : X \to Y$ is automatically nondegenerate. However, it turns out that the restriction of C^*-algebraic K-theory to commutative C^*-algebras is strictly equivalent to topological K-theory. Cf. the end of section 7.

[12] This time, two morphisms $\varphi : C \to D$ and $\psi : C \to D$ are said to be homotopic, still written $\varphi \overset{\mathrm{h}}{\sim} \psi$, if there is a continuous map $\Phi : [0,1] \times C \to D$ with the properties that for each $t \in [0,1]$ the map $\Phi_t : a \mapsto \Phi(t,a)$ is a morphism from C to D, while for each $c \in C$ the function $t \mapsto \Phi(t,c)$ is continuous from $[0,1]$ to D, such that $\Phi_0 = \varphi$ and $\Phi_1 = \psi$. Equivalently, Φ may be seen as a morphism $\tilde{\Phi} : C \to C([0,1], D)$, with $\tilde{\Phi}_t : C \to D$ defined by $\tilde{\Phi}_t(c) = (\tilde{\Phi}(c))(t)$. This second picture immediately arises from the ordinary notion of homotopy in topology through the correspondence $X \mapsto C(X)$.

In any case, two C^*-algebras A, B are then called homotopy equivalent when there are morphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} A$ such that $\varphi \circ \psi \overset{\mathrm{h}}{\sim} \mathrm{id}_B$ and $\psi \circ \varphi \overset{\mathrm{h}}{\sim} \mathrm{id}_A$.

A	$K_0(A)$	$K_1(A)$
$\mathbb{C}$	$\mathbb{Z}$	0
$M_n(\mathbb{C})$	$\mathbb{Z}$	0
$B_0(H)$	$\mathbb{Z}$	0
$B(H)$	0	0
$C(H)$	0	$\mathbb{Z}$

Table 2: Table of C^*-algebraic K-groups

natural in A.

3. Bott periodicity: one has isomorphisms $K_{n+2}(A) \cong K_n(A)$ for all n, natural in A.
4. Excision: any short exact sequence[13] $0 \to J \to A \to B \to 0$ gives rise to a cyclic exact sequence

$$\begin{tikzcd}
K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d] \\
K_1(B) \arrow[u] & K_1(A) \arrow[l] & K_1(J). \arrow[l]
\end{tikzcd}$$

However, C^*-algebraic K-theory has the following additional property, which cannot even be formulated in the commutative case.

5. Stability: for any n, one has $K_0(M_n(A)) \cong K_0(A)$, natural in A.[14]

Table 2 lists some K-groups of C^*-algebras. Here H is supposed to be infinite-dimensional, and $C(H) := B(H)/B_0(H)$.

What is the point of C^*-algebraic K-theory? Its restriction to the commutative case, topological K-theory, is an important construction in algebraic topology, which gives a great deal of information about a space X. In any case, it is pleasing that this construction may be extended to the noncommutative case. Although C^*-algebraic K-theory in fact predated noncommutative geometry, to this day it remains the most perfect example of its philosophy. However, where topological K-theory is hardly of any use in the classification of topological spaces, its C^*-algebraic counterpart is actually a very useful tool in the classification of C^*-algebras.[15] It was discovered by Elliott and others in the mid 1970s that so-called AF-algebras[16] are completely classified by their K-groups, and meanwhile a much larger class of simple C^*-algebras has been found to be completely described by K-theory and related invariants.

Another use of C^*-algebraic K-theory, apparently unrelated to the previous one, has been in index theory. From what follows, this may appear to be a small and technical corner of operator theory, but in fact it is a central area in modern mathematics, combining functional analysis, geometry, and topology. The central result, the Atiyah–Singer index theorem, counts as one of the deepest results in 20th century mathematics. The generalizations of this theorem due to Connes

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[13] An exact sequence of C^*-algebras is a chain of morphisms $\cdots \xrightarrow{\varphi_i} A_i \xrightarrow{\varphi_{i+1}} A_{i+1} \xrightarrow{\varphi_{i+2}} \cdots$ with the property $\ker(\varphi_{i+1}) = \mathrm{im}(\varphi_i)$ for all i. A short exact sequence starts and ends with zeros, so that the second arrow must be injective and the penultimate arrow has te be surjective. It follows that (the image of) J is an ideal in A, and that $B \cong A/J$.

[14] This is even true for $n = \infty$, in the sense that $K_0(B_0(H) \otimes A) \cong K_0(A)$, where $B_0(H)$ is the C^*-algebra of compact operators on any Hilbert space H.

[15] Recall from the Introduction of Part I that the functoriality properties of K-theory imply that $A \cong B$ implies $K^n(A) \cong K^n(B)$ for all n, so that, conversely, if $K^n(A)$ and $K^n(B)$ fail to be isomorphic for some n, A and B cannot be isomorphic.

[16] AF stands for Approximately Finite; an AF-algebra can be approximated by finite-dimensional C^*-algebras, hence by matrix algebras.

and others belong to the core of noncommutative geometry, both as results and as motivation for its techniques.

Let H be a Hilbert space. A bounded operator $F : H \to H$ is called **Fredholm** when $\ker(F)$ and $\ker(F^*)$ are finite-dimensional. The first result in the theory of such operators is that F is Fredholm iff its image in $C(H)$ is invertible. In other words, F is Fredholm iff it is invertible modulo compact operators.[17] Thus its **index**

$$\mathrm{index}(F) := \dim(\ker(F)) - \dim(\ker(F^*))$$

is a finite integer. If H is finite-dimensional, then any operator has index zero, so the index is a purely infinite-dimensional phenomenon.[18] The simplest example is the **shift operator** $s : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined on the canonical basis by $se_n := e_{n+1}$. It is easy to show that $\mathrm{index}(s) = -1$.[19] The Atiyah–Singer index theorem is concerned with more complicated Fredholm operators, namely elliptic pseudodifferential operators on compact manifolds.

The relationship between index theory and the K-theory of C^*-algebras is based on the following observations. Firstly, it can be shown that $K_1(A)$, originally defined as $K_0(SA)$, is equal to group $\pi_0(GL_\infty(A))$ of connected components of the topological group of invertible elements in $M_\infty(A)$. In particular, a Fredholm operator F defines an element $[F]$ of $K_1(C(H))$. Secondly, consider the short exact sequence

$$0 \to B_0(H) \to B(H) \to C(H) \to 0.$$

This gives rise to a cyclic exact sequence

$$\begin{tikzcd}
K_0(B_0(H)) \arrow[r] & K_0(B(H)) \arrow[r] & K_0(C(H)) \arrow[d] \\
K_1(C(H)) \arrow[u] & K_1(B(H)) \arrow[l] & K_1(B_0(H)). \arrow[l]
\end{tikzcd}$$

The map $K_1(B) \to K_0(J)$ in the general cyclic exact sequence is usually called ∂. Identifying $K_0(B_0(H))$ with $\mathbb{Z}$, it then turns out that

$$\partial([F]) = \mathrm{index}(F)$$

for any Fredholm operator F. In fact, substituting the K-groups from Table 2, one concludes that $\partial : K_1(C(H)) \to K_0(B_0(H))$ must be an isomorphism,[20] so that the Fredholm operators exhaust $K_1(C(H))$. However, the main point is that the index, originally defined to be a number, comes out as an object that is really K-theoretic in nature. This is useful both is proving the original Atiyah–Singer index theorem, as well as in generalizing it. The most powerful generalizations of the Atiyah–Singer theorem are indeed almost entirely based on the K-theory of C^*-algebras. One idea here is that operators that fail to be Fredholm, so that $\mathrm{index}(F)$ is undefined, should have well-defined 'higher' indices. These take values in the K-groups of other C^*-algebras than $B_0(H)$, and are therefore generally no longer numerical in nature.

2 Projections in matrix algebras

Throughout this section A is a unital C^*-algebra.

K-theory is mainly concerned with projections in certain *-algebras, that is, operators p such that $p^2 = p^* = p$. In the commutative case, $A = C(X)$ has nontrivial projections[21] only when

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[17] Hence there exist $P, Q \in B(H)$ such that $FP - 1_H$ and $QF - 1_H$ are compact operators.

[18] One may define an operator $F : H_1 \to H_2$ between two different Hilbert space to be Fredholm when it is invertible modulo compact operators, This again guarantees a finite index. This time a nonzero index may arise even in the finite-dimensional case, as for any linear map $F : \mathbb{C}^n \to \mathbb{C}^m$ one has $\mathrm{index}(F) = n - m$.

[19] And $\mathrm{index}(s^k) = -k$.

[20] This is, in fact, the way to compute the K-theory of $C(H)$ from that of $B_0(H)$ and $B(H)$.

[21] Trivial projections are $p = 1$ and $p = 0$.

X is not connected; for projections in function algebras are characteristic functions χ_Y for some $Y \subset X$, and such functions fail to be continuous when X is connected (except when $Y = X$ or $Y = \emptyset$). On the other hand, even the C^*-algebra $M_2(\mathbb{C})$ of 2×2 matrices has plenty of nontrivial projections, see below. Such projections are still contained in $M_2(A)$, since one has the embedding $M_2(\mathbb{C}) \hookrightarrow M_2(A)$ induced by $t \mapsto t1_A$, $t \in \mathbb{C}$. However, even if A fails to have nontrivial projections itself, $M_2(A)$ may possess many projections that are not of the above form.

A very pretty example occurs for $A = C(S^2)$, where S^2 is the two-sphere. For any space X, one may identify $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$ by identifying a matrix $(f)_{ij}$ of functions $f_{ij} \in C(X)$ with the matrix-valued function f on X given by $x \mapsto f_{ij}(x)$. It follows that f is a projection in $M_n(C(X))$ iff the matrix $f(x)$ is a projection in $M_n(\mathbb{C})$ for each $x \in X$. Passing to $X = S^2$ and $n = 2$, it can be shown that the one-dimensional projections in $M_2(\mathbb{C})$ are of the form

$$p(x,y,z) := \tfrac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}, \tag{1}$$

where $(x,y,z) \in \mathbb{R}^3$ satisfies $x^2 + y^2 + z^2 = 1$. Thus the one-dimensional projections in $M_2(\mathbb{C})$ are precisely parametrized by S^2, so that a particularly canonical projection in $M_2(C(S^2)) \equiv C(S^2, M_2(\mathbb{C}))$ is the function $\sigma \mapsto p(\sigma)$, where $\sigma \in S^2$. This projection is often written by identifying S^2 with the Riemann sphere Σ (i.e., the one-point compactification of the complex plane), in which case one has, for $z \in \Sigma$,

$$p(z) = \frac{1}{1+|z|^2} \begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix} \ (z \neq \infty); \ p(\infty) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

For fixed z, this is the element of $M_2(\mathbb{C})$ that projects onto the line through the vector $(1, z)$ for $z \neq \infty$, and onto $(0, 1)$ for $z = \infty$. This projection will play a decisive role in K-theory.

Such examples motivate one to look at projections in $M_n(A)$ for arbitrary n, although it should be kept in mind that in practice it is rarely needed to go beyond $n = 2$. First, let us equip $M_n(A)$ with the structure of a C^*-algebra. Its structure as a *-algebra should be clear: all operations are as if elements of $M_n(A)$ are ordinary matrices, but instead of complex numbers one now adds and multiplies elements of A. Also, the complex conjugation in the definition of the adjoint $(a^*)_{ij} := \overline{a_{ji}}$ of a complex matrix is now replaced by $(a^*)_{ij} := (a_{ji})^*$, where the star on the right-hand side is the adjoint in A. The norm on $M_n(A)$ is defined by first taking an arbitrary injective representation of A on some Hilbert space H, and then representing $M_n(A)$ on $H^n = H \oplus \cdots \oplus H$ (i.e., the direct sum of n copies of H) in the obvious way. This embeds $M_n(A)$ in $B(H^n)$, which equips $M_n(A)$ with the operator norm of $B(H^n)$. This is obviously a C^*-norm, and it is an interesting exercise to show that $M_n(A)$ is complete in it. Since the norm on a C^*-algebra is unique, it follows that this norm is independent of the chosen representation on H.

To obtain a well-behaved theory, it is necessary to take the limit $n \to \infty$ in one way or the other, and the group $K_0(A)$ will be defined in terms of equivalence classes of projections in $M_\infty(A)$. However, it is convenient to keep n finite for the moment, so that in the following results one may keep the choice $B = M_n(A)$ in mind.

Definition 2.1 *Let B be a unital C^*-algebra. Two projections p, q in B are called:*

- ***homotopy equivalent****, written $p \overset{\mathrm{h}}{\sim} q$, when there is a path*[22] *of projections $e(t)$ in B such that $e(0) = p$ and $e(1) = q$.*

- ***unitarily equivalent****, written $p \overset{\mathrm{u}}{\sim} q$, when there is a unitary $u \in B$ (i.e., $uu^* = u^*u = 1$) such that $q = upu^*$.*

- ***Murray-von Neumann equivalent****, written $p \overset{\mathrm{MvN}}{\sim} q$, when there is some element $v \in B$ such that $v^*v = p$ and $vv^* = q$.*

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[22] A path in a topological space X is a continuous map $[0, 1] \to X$.

The last relation is not obviously an equivalence relation, though in fact it is; this is an easy exercise.

Let us examine what these definitions mean. Firstly, it may be surprising that they do not coincide; we will show that

$$p \overset{\mathrm{h}}{\sim} q \Rightarrow p \overset{\mathrm{u}}{\sim} q \Rightarrow p \overset{\mathrm{MvN}}{\sim} q, \tag{3}$$

so that $\overset{\mathrm{h}}{\sim}$ is the strongest, and $\overset{\mathrm{MvN}}{\sim}$ is the weakest of the three.

The significance of $\overset{\mathrm{h}}{\sim}$ is nicely illustrated by an example.

Lemma 2.2 *For any projection $p \in A$ one has*

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \text{ in } M_2(A). \tag{4}$$

The path of projections connecting the two sides is given by

$$e(t) = p \begin{pmatrix} \cos^2(\pi t/2) & \cos(\pi t/2)\sin(\pi t/2) \\ \cos(\pi t/2)\sin(\pi t/2) & \sin^2(\pi t/2) \end{pmatrix}. \qquad \blacksquare$$

Note that the two sides are not homotopy equivalent in $A \oplus A$; the off-diagonal terms in $e(t)$ are crucial. Technically, the main point about $\overset{\mathrm{h}}{\sim}$ is the following lemma (the proof of which is left to the reader as an exercise).

Lemma 2.3

1. *Any pair of projections in a C^*-algebra satisfies $\|p-q\| \leq 1$.*
2. *If $p \perp q$ (that is, $pq = qp = 0$), then $\|p-q\| = 1$.*
3. *If $\|p-q\| < 1$, then $p \overset{\mathrm{h}}{\sim} q$.*

Note that the last two items are not mutually exclusive: it may well happen that $p \perp q$ and $p \overset{\mathrm{h}}{\sim} q$. Indeed, Lemma 2.2 with $A = \mathbb{C}$ and $p = 1$ provides an example.

The significance of $\overset{\mathrm{u}}{\sim}$ should be obvious. If $B = B(H)$, then $q = upu^*$ is simply the image of p under the isomorphism $u : H \to H$. The subtlety is that for $B \subset B(H)$, the connecting unitary has to lie in B. Hence $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are unitarily equivalent in $B = M_2(\mathbb{C})$, but not in $B = \mathbb{C} \oplus \mathbb{C}$.

The notion of Murray-von Neumann equivalence comes from the theory of von Neumann algebras, which can be roughly classified according to this notion. This classification predated C^*-algebraic K-theory by about 40 years. To see its significance, let us note that for $B = B(H)$ one has $p \overset{\mathrm{MvN}}{\sim} q$ iff $\dim(pH) = \dim(qH)$, where both sides may be infinite.[23] For in that case pH and qH are isomorphic as Hilbert spaces, so that there exists a unitary $u : pH \to qH$. Defining $v : H \to H$ by $v = 0$ on $(pH)^{\perp}$, by $v = u$ on pH, and by linearity elsewhere, one indeed has $v^*v = p$ and $vv^* = q$. Such an operator v, which is unitary as an operator from the orthogonal complement of its kernel to its image, is called a **partial isometry**.[24] Equivalently, a partial isometry v may be defined by saying that v^*v and vv^* are projections; the former is the projection onto the orthogonal complement of its kernel, and the second is the projection onto its image. Any unitary operator is obviously a partial isometry, with both associated projections equal to 1, and any projection p is itself a partial isometry, whose associated projections are both equal to p.[25] Once again, the stipulation that $v \in B$ is essential: the two projections in the previous paragraph are Murray-von Neumann equivalent in $M_2(\mathbb{C})$, but not in $\mathbb{C} \oplus \mathbb{C}$.

The relationship between $\overset{\mathrm{u}}{\sim}$ and $\overset{\mathrm{MvN}}{\sim}$ is very simple.

Lemma 2.4 *Let p, q be projections in a unital C^*-algebra B. Then $p \overset{\mathrm{u}}{\sim} q$ iff $p \overset{\mathrm{MvN}}{\sim} q$ and $1-p \overset{\mathrm{MvN}}{\sim} 1-q$.*

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[23] As long as they are of the same cardinality.

[24] An **isometry** v by definition satisfies $v^*v = 1$.

[25] This shows the reflexivity of Murray-von Neumann equivalence.

The proof is a good exercise. Consequently, $\overset{\mathrm{u}}{\sim}$ and $\overset{\mathrm{MvN}}{\sim}$ coincide for $B = M_n(\mathbb{C})$, since if $\dim(p\mathbb{C}^n) = \dim(q\mathbb{C}^n)$, then

$$\dim((1-p)\mathbb{C}^n) = n - \dim(p\mathbb{C}^n) = n - \dim(q\mathbb{C}^n) = \dim((1-q)\mathbb{C}^n).$$

But for $B(H)$ with infinite-dimensional H one may well have $p \overset{\mathrm{MvN}}{\sim}$ without having $p \overset{\mathrm{u}}{\sim} q$. Indeed, recall the shift operator $s : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, defined on the canonical basis by $se_n := e_{n+1}$. This operator satisfies $ss^* = 1$ and $s^*s = [e_2, \cdots]$, the projection onto the orthogonal complement of e_1. Hence s is a partial isometry, establishing the fact that $[e_2, \cdots] \overset{\mathrm{MvN}}{\sim} 1$ in $B(\ell^2)$. However, these projections are by no means unitarily equivalent.

Since Lemma 2.4 establishes the implication $p \overset{\mathrm{u}}{\sim} q \Rightarrow p \overset{\mathrm{MvN}}{\sim} q$, in order to prove (3) we are left with

Lemma 2.5 *Let p, q be projections in a unital C^*-algebra B. Then $p \overset{\mathrm{h}}{\sim} q \Rightarrow p \overset{\mathrm{u}}{\sim} q$.*

By definition of $\overset{\mathrm{h}}{\sim}$, it suffices to prove the lemma when $\|p - q\|$ is as small as we like. For if $e(t)$ is a path connecting p and q, the lemma follows if $e(t_i) \overset{\mathrm{u}}{\sim} e(t_j)$ for $t_i - t_j$ arbitrarily small, which by continuity in t implies $\|e(t_i) - e(t_j)\|$ arbitrarily small. Put $c := pq + (1-p)(1-q)$. Since p and q are close to each other, and $p^2 = p$ as well as $(1-p)^2 = (1-p)$, c should be close to $p + (1-p) = 1$, and indeed, one easily estimates $\|c - 1\| \leq 2\|p - q\|$. Hence for $\|p - q\| < \frac{1}{2}$ the element c is invertible by Lemma 3.4 of Part I. Since $pc = cq$, one has $q = c^{-1}pc$.

The proof now hinges on a very important fact about C^*-algebras, which we state without proof: if $b = cac^{-1}$ for some a, b, c in a unital C^*-algebra A, then there is a unitary $u \in A$ such that $b = uau^*$. ■

An example showing that $\overset{\mathrm{h}}{\sim}$ and $\overset{\mathrm{u}}{\sim}$ are not the same is given in [9], p. 26. The main point of the entire discussion, however, is the following result.

Lemma 2.6 *1. If $p \overset{\mathrm{MvN}}{\sim} q$ in A, then*

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathrm{u}}{\sim} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ in } M_2(A).$$

2. If $p \overset{\mathrm{u}}{\sim} q$, then

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ in } M_2(A).$$

It should be clear that, whatever $M_\infty(A)$ and the ensuing equivalence relations mean in detail (and the two definitions below will do), this lemma with (3) implies

Theorem 2.7 *The equivalence relations $\overset{\mathrm{h}}{\sim}$, $\overset{\mathrm{u}}{\sim}$, and $\overset{\mathrm{MvN}}{\sim}$ on the set $P_\infty(A)$ of projections in $M_\infty(A)$ coincide.*

To prove the first claim of Lemma 2.6, assume $p = v^*v$ and $q = vv^*$. Then

$$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* \text{ for } u = \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}.$$

To prove the second, assume $q = upu^*$, and define the unitary operators

$$u(t) := \begin{pmatrix} u\cos(\pi t/2) & 1_A \cdot \sin(\pi t/2) \\ -1_A \cdot \sin(\pi t/2) & u^*\cos(\pi t/2) \end{pmatrix}.$$

Since each $u(t)$ is unitary, each $u(t) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u(t)^*$ is a projection. One has

$$\begin{aligned} u(0) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u(0)^* &= \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}; \\ u(1) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u(1)^* &= \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}. \end{aligned}$$

Hence $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ are homotopy equivalent, but we know from Lemma 2.2 that the latter projection is homotopy equivalent with $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. ■

The limit $n \to \infty$ and the ensuing construction of $M_\infty(A)$ may be performed in a number of ways, all eventually yielding equivalent definitions of K-theory. Firstly, the simplest procedure is to define H^∞ as the Hilbert space of all infinite sequences $(v_1, v_2, \ldots)$, where $v_i \in H$, such that $\sum_i \|v_i\|^2 < \infty$. For any n, $M_n(A)$ then acts on H^∞ through the natural embedding $H^n \hookrightarrow H^\infty$, given by $(v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_n, 0, 0, \ldots)$. One then defines $M_\infty(A)$ as the union of all such $M_n(A)$; elements of $M_\infty(A)$ therefore have finitely many entries in the upper left corner. The operator norm on $B(H^\infty)$ restricts to a C^*-norm on $M_\infty(A)$, and it easily follows that each $M_n(A) \subset M_\infty(A)$ then inherits its original norm defined above (i.e., through its embedding in $B(H^n)$), so that also the norm on $M_\infty(A)$ thus constructed is independent of the choice of H. However, $M_\infty(A)$ is no longer complete in this norm. Its completion in $B(H^\infty)$ is, of course, a C^*-algebra, but it is not obvious that this is independent of H.[26] Fortunately enough, this completion is not needed in what follows. What has been achieved here is that all $M_n(A)$ act on a single Hilbert space, with automatic embeddings $M_n(A) \subset M_m(A)$ for $n < m$.[27]

A second way to proceed is to initially keep all $M_n(A)$ separate, starting with the disjoint union $\coprod_{n=1}^\infty M_n(A)$. We then define the direct sum of any $a \in M_k(A)$ and $b \in M_l(A)$ as the element

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{5}$$

of $M_{k+l}(A)$, and define an equivalence relation $\sim$ on $\coprod_{n=1}^\infty M_n(A)$ by saying that $p \sim q$, for $p \in M_n(A)$ and $q \in M_m(A)$, when $q = p \oplus 0_{m-n}$ (in case that $m > n$) or $p = q \oplus 0_{n-m}$ (when $n > m$), where 0_k is the $k \times k$ zero matrix. As a set, $M_\infty(A)$ is then defined as the quotient $(\coprod_{n=1}^\infty M_n(A))/\sim$. It becomes a *-algebra in the following way: any two equivalence classes $[a], [b]$ arise from elements $a, b \in M_n(A)$ for some n, so that one may define $[a]+[b] := [a+b]$, $[a]\cdot[b] := [ab]$, etcetera. Similary, one puts $\|[a]\| := \|a\|$. Each of these operations is independent of the choice of representatives, and in this way one obtains a *-algebra $M_\infty(A)$, which is isomorphic to the one denoted by the same symbol in the previous paragraph.[28] For this reason, we will often omit the brackets $[\cdot]$ in what follows.

From Definition 2.1, one may pass to similar equivalence relations on projections in $M_\infty(A)$. In our first realization of $M_\infty(A)$ as a subalgebra of $B(H^\infty)$, the relations $\overset{\mathrm{h}}{\sim}$ and $\overset{\mathrm{MvN}}{\sim}$ are immediately defined on the set $P_\infty(A)$ of projections in $M_\infty(A)$. The relation $\overset{\mathrm{u}}{\sim}$ is not defined as such, because $M_\infty(A)$ evidently has no unit, and therefore no unitary elements. However, since any two elements of $M_\infty(A)$ lie in $M_n(A)$ for some n, one may define $p \overset{\mathrm{u}}{\sim} q$ in $M_\infty(A)$ when $p \overset{\mathrm{u}}{\sim} q$ in $M_m(A)$ for some $m \geq n$. This is well defined.[29] Of course, the same comment applies to $\overset{\mathrm{h}}{\sim}$ and $\overset{\mathrm{MvN}}{\sim}$.

In our second realization of $M_\infty(A)$ as the quotient $(\coprod_{n=1}^\infty M_n(A))/\sim$, we define the three equivalence relations on $P_\infty(A)$ by simply composing $\sim$ with the ones in Definition 2.1 applied to $B = M_n(A)$ for arbitrary n. For example, let $\overset{\mathrm{h}}{\sim}_n$ denote the relation $\overset{\mathrm{h}}{\sim}$ in Definition 2.1 applied to $B = M_n(A)$. We then define $\overset{\mathrm{h}}{\sim}$ on $P_\infty(A)$ by saying that $[p] \overset{\mathrm{h}}{\sim} [q]$ when p is equivalent to q in $\coprod_{n=1}^\infty M_n(A)$ under the equivalence relation generated by $\sim$ and all $\overset{\mathrm{h}}{\sim}_n$, $n \in \mathbb{N}$. This is well defined, and leads to the same equivalence classes as in the previous realization.[30] A similar definition applies to $\overset{\mathrm{u}}{\sim}$ and $\overset{\mathrm{MvN}}{\sim}$.

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[26] In fact, it is. The completion of $M_\infty(A)$ is the so-called C^*-algebraic tensor product $A \hat{\otimes} B_0(\ell^2)$. The theory of such tensor products is very involved, since the algebraic tensor product $A \otimes_{\mathbb{C}} B$ can often be completed into a C^*-algebra in many different ways. However, $A \otimes_{\mathbb{C}} B_0(H)$ turns out to have a unique completion for any A and H.

[27] This avoids the use of so-called inductive limits (either algebraically or C^*-algebraically).

[28] This construction gives $M_\infty(A)$ as the algebraic direct limit of the $M_n(A)$. It can be refined to give the C^*-algebraic direct limit, which is isomorphic to $A \hat{\otimes} B_0(\ell^2)$.

[29] But should be treated with care. For example, as we shall see, one may have $p, q \in A$ and $p \overset{\mathrm{u}}{\sim} q$ in $M_2(A)$ but not in A. More generally, if $p, q \in M_n(A)$, then $p \overset{\mathrm{u}}{\sim} q$ in $M_\infty(A)$ might well be achieved through some $u \in M_m(A)$ for $m > n$.

[30] A similar comment applies here. It may well happen dat $p, q \in M_n(A)$ fail to be equivalent in any of the three

3 Definition of K_0

We recall that $P_\infty(A)$ is the set of projections in $M_\infty(A)$; similarly, we denote the set of projections in $M_n(A)$ by $P_n(A)$. The set $P_\infty(A)$ carries an equivalence relation $\sim$, which by Theorem 2.7 may equivalently be defined by the relations $\overset{\mathrm{h}}{\sim}$, $\overset{\mathrm{u}}{\sim}$, and $\overset{\mathrm{MvN}}{\sim}$. This final relation may be further motivated by the following interesting property. Let $A^n := A \oplus \cdots \oplus A$ (n times), which is a right A module under the action $(a_1, \ldots, a_n) \cdot a := (a_1 a, \ldots a_n a)$. Clearly, if $p \in P_n(A)$ and $(a_1, \ldots, a_n) \in pA^n$, that is, $p(a_1, \ldots, a_n) = (a_1, \ldots, a_n)$, then $(a_1, \ldots, a_n) \cdot a \in pA^n$, i.e., $p(a_1 a, \ldots, a_n a) = (a_1 a, \ldots, a_n a)$, since the left action of $p \in P_n(A)$ commutes with the right action of A. Thus pA^n is a right A module for any $p \in P_n(A)$.

Proposition 3.1 *Let A be a unital C^*-algebra, and let $p \in P_n(A)$ and $q \in P_m(A)$. Then $p \sim q$ in $P_\infty(A)$ iff pA^n and qA^m are isomorphic as right A modules.*

The proof is an easy exercise.[31] Now, the simplest definition of $K_0(A)$ for a unital C^*-algebra A is the one given in the Introduction:

Definition 3.2 *Let A be a unital C^*-algebra. $K_0(A)$ is the abelian group with one generator for each equivalence class $[p]$ of projections $p \in P_\infty(A)$ under the equivalence relation $\sim$, and relations $[p] + [q] = [p \oplus q]$ between these generators.*

Here we may assume that $p \in P_n(A)$ and $q \in P_m(A)$, so that $p \oplus q$ is defined as in (5). This definition means the following. One first takes the vector space W_A over $\mathbb{Z}$ spanned by the basis $\{[p]\}$. The group $K_0(A)$ is the quotient W_A/I_A of this vector space by the $\mathbb{Z}$-linear subspace I_A spanned by all vectors of the form $[p] + [q] - [p \oplus q]$. The group operation is just addition in this quotient vector space.

Another definition of $K_0(A)$ is often found in the literature. Here one starts with the semigroup $V(A)$, defined as $V(A) := P_\infty(A)/\sim$, and operation $+$ defined by $[p] + [q] := [p \oplus q]$. This is easily checked to be an abelian semigroup, for $p \oplus q \sim q \oplus p$. Note that in Definition 3.2 the equation $[p] + [q] = [p \oplus q]$ defines relations between generators in terms if an already defined operation $+$, whereas here the same equation is used to define the operation $+$ on the set $P_\infty(A)/\sim$. One then proceeds in an abstract way:

Definition 3.3 *Let $(S, +)$ be an abelian semigroup.*[32] *The* ***Grothendieck group*** *$G(S)$ is the abelian group with underlying set $(S \times S)/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ when there exists $z \in S$ for which $x_1 + y_2 + z = x_2 + y_1 + z$, group operation $[x_1, y_1] + [x_2, y_2] := [x_1 + x_2, y_1 + y_2]$, and inverse $[x, y]^{-1} := [y, x]$. One often writes $[x] - [y] := [x, y]$. The canonical map $S \xrightarrow{\iota} G(S)$ is defined by $\iota_S(x) := [x + y, y]$ for arbitrary $y \in S$. This map is a homomorphism of semigroups.*

The notation with "formal differences" $[x] - [y]$ is motivated by the simplest example of the Grothendieck group construction, namely $G(\mathbb{N}, +) \cong \mathbb{Z}$. This example unfortunately suggests that ι_S is automatically injective, which, however, in general is the case iff S has the cancellation property (i.e., if $x + z = y + z$ for some z, then $x = y$). We state the two main properties of $G(S)$ without proof.

Proposition 3.4 *The Grothendieck group construction has the following properties:*

- ***Functoriality:*** *if $\varphi : S \to T$ is a homomorphism of semigroups, then there is a unique homomorphism of groups $G(\varphi) : G(S) \to G(T)$ making the following diagram commutative.*

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possible ways of Definition 2.1 with $B = M_n(A)$, whereas there exists some k such that $p \oplus 0_k$ and $q \oplus 0_k$ are equivalent in $B = M_{n+k}(A)$ for one or more of these relations. In that case, p and q are equivalent in $\coprod_{n=1}^{\infty} M_n(A)$.

[31] The upshot of this result is that for unital C^*-algebras the C^*-algebraic K_0 group is isomorphic to the purely algebraic K_0 group. This is no longer the case for K_1.

[32] Here S does not necessarily have a unit.

Explicitly, one has $G(\varphi)([x,y]_S) = [\varphi(x), \varphi(y)]_T$.

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, "\iota_S"'] & T \arrow[d, "\iota_T"] \\
G(S) \arrow[r, "G(\varphi)"'] & G(T)
\end{tikzcd}$$

- ***Universality:*** *if* S *is an abelian semigroup and* H *is an abelian group, with* $\varphi : S \to H$ *a homomorphism of semigroups, then there is a unique homomorphism* $\psi : G(S) \to H$ *such that the following diagram commutes:*

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[dr, "\iota_S"'] & H \\
& G(S) \arrow[u, "\exists!\psi"']
\end{tikzcd}$$

The second property is may be used to prove the following result (exercise!).

Proposition 3.5 *The group* $K_0(A)$ *of Definition 3.2 is naturally isomorphic to* $G(V(A))$.

It should be pointed out that one may have $[p] \neq [q]$ in $V(A)$, but $[p] = [q]$ in $K_0(A)$. In fact, it may be shown that the latter equality holds for $p, q \in P_\infty(A)$ iff $p \oplus 1_m \sim q \oplus 1_m$ for some $m \in \mathbb{N}$.[33]

Let us compute two instructive examples from Table 2. First take $A = M_n(\mathbb{C})$. It should be clear that $M_k(M_n(\mathbb{C})) \cong M_{kn}(\mathbb{C})$, so that $M_\infty(M_n(\mathbb{C})) \cong M_\infty(\mathbb{C})$. Hence any $p, q \in P_\infty(M_n(\mathbb{C}))$ lie in some $P_m(\mathbb{C})$, so that $p \sim q$ when $p \overset{\text{MvN}}{\sim} q$, which, as we have seen, holds iff $\dim(p\mathbb{C}^m) = \dim(q\mathbb{C}^m)$. This number is, in turn, independent of m. It follows that (with $0 \in \mathbb{N}$)

$$V(M_n(\mathbb{C})) \cong \mathbb{N}. \tag{6}$$

Hence with $K_0(M_n(\mathbb{C})) := G(V(M_n(\mathbb{C})))$ one obtains

$$K_0(M_n(\mathbb{C})) \cong \mathbb{Z}. \tag{7}$$

It is instructive to prove this by a more general method. A **trace** on a C^*-algebra A is a positive linear map $\tau : A \to \mathbb{C}$ for which $\tau(ab) = \tau(ba)$ for all $a, b \in A$.[34] For $A = M_n(\mathbb{C})$ the usual trace $\mathrm{Tr} : a \mapsto \sum_i a_{ii}$ provides an example. A trace on A extends to a trace τ_n on $M_n(A)$ by $\tau_n(a) = \sum_i \tau(a_{ii})$, and thence to a trace τ_∞ on $M_\infty(A)$. Using the fact that the equivalence relation $\sim$ on $P_\infty(A)$ may be defined through $\overset{\text{MvN}}{\sim}$, it is clear that $p \sim q$ in $P_\infty(A)$ implies $\tau_\infty(p) = \tau_\infty(q)$. Also, one clearly has $\tau_\infty(p \oplus q) = \tau_\infty(p) + \tau_\infty(q)$. It follows that τ_∞ descends to a semigroup homomorphism $[\tau_\infty] : V(A) \to \mathbb{R}$ defined by $[\tau_\infty] : [p] \mapsto \tau_\infty(p)$. The universality of the Grothendieck group construction then leads to a group homomorphism $\tau_* : K_0(A) \to \mathbb{R}$, given by

$$\tau_*([p] - [q]) = \tau_\infty(p) - \tau_\infty(q). \tag{8}$$

Note that this is well defined not only on the equivalence classes $[\cdot]$ of $\sim$, but also on $K_0(A)$, since $[p] - [q] = [p'] - [q']$ in $K_0(A)$ iff there is r such that $p \oplus q' \oplus r \sim p' \oplus q \oplus r$, which implies $\tau(p) + \tau(q') + \tau(r) = \tau(p') + \tau(q) + \tau(r)$, and hence $\tau_*([p] - [q]) = \tau_*([p'] - [q'])$.

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[33] In topological K-theory this corresponds to stable equivalence of vector bundles.

[34] It follows that τ is continuous.

For $A = M_n(\mathbb{C})$ one takes $\tau = \mathrm{Tr}$, and it is well known[35] that $\mathrm{Tr}\,(p) = \dim(p\mathbb{C}^n)$. Hence, as before, we conclude that $\mathrm{Tr}\,(p) = \mathrm{Tr}\,(q)$ iff $p \overset{\mathrm{MvN}}{\sim} q$, so that $[\mathrm{Tr}_{\infty}] : V(P_\infty(M_n(\mathbb{C}))) \cong V(P_\infty(\mathbb{C})) \to \mathbb{C}$ is injective. Since $V(P_\infty(\mathbb{C})) \cong \mathbb{N}$ has the cancellation property, the map $\iota : V(P_\infty(\mathbb{C})) \to G(V(P_\infty(\mathbb{C}))) \cong K_0(M_n(\mathbb{C}))$ is injective, so that $\mathrm{Tr}_* : K_0(M_n(\mathbb{C})) \to \mathbb{C}$ remains injective. But Tr_* cleary takes values in $\mathbb{Z}$, and it is also evidently surjective, since for any $k \in \mathbb{Z}$ one may find a projection $p \in P_\infty(\mathbb{C})$ for which $\mathrm{Tr}\,(p) = k$, whence $\mathrm{Tr}_*(\pm[p]) = k$ (where $\pm = +$ for $k \geq 0$ and $\pm = -$ for $k < 0$). It follows that $\mathrm{Tr}_* : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is injective and surjective, and therefore an isomorphism.

We next turn to $A = B(H)$ for an infinite-dimensional separable Hilbert space H. As explained before, we realize $M_\infty(B(H))$ as a subalgebra of $B(H^\infty)$, but since H^∞ is itself a separable Hilbert space it follows that the map $[p] \mapsto \dim(pH^\infty)$ is still a bijection between equivalence classes of projections in $M_\infty(B(H))$ and the set $\mathbb{N} \cup \{\infty\}$. Hence

$$V(B(H)) \cong \mathbb{N} \cup \{\infty\}, \tag{9}$$

but the additional element ∞ has the dramatic consequence that $G(\mathbb{N} \cup \{\infty\}) = 0$! Indeed, one has $p \oplus q' \oplus r \sim p' \oplus q \oplus r$ for any quadruple (p, q, p', q') if one takes r to be any infinite-dimensional projection. Hence $[p] - [q] = [p'] - [q']$ for any (p, q, p', q'), so that $G(\mathbb{N} \cup \{\infty\})$ just has a single element. It follows that[36]

$$K_0(B(H)) = 0. \tag{10}$$

4 Functoriality of K_0

The main property of K_0 is its functoriality. We state it as a lemma here, since the full statement will refer to the nonunital case as well.

Lemma 4.1 *Each morphism $\varphi : A \to B$ between unital C^*-algebras induces a map $\varphi_* : K_0(A) \to K_0(B)$ such that $\varphi_*(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ and, if $\psi : B \to C$ is another morphism, $\psi_* \circ \varphi_* = (\psi \circ \varphi)_*$. In other words, the association $A \mapsto K_0(A)$ is covariantly functorial.*

The proof may be done in two ways. Starting from Definition 3.2, for $p \in P_n(A) \subset P_\infty(A)$ one expands $\varphi : A \to B$ to a morphism $\varphi_n : M_n(A) \to M_n(B)$ in the obvious (entry-wise) way, and notes that $\varphi_n(p)$ is a projection in $M_n(B)$. This leads to a map $\varphi_\infty : P_\infty(A) \to P_\infty(B)$, and subsequently to $\tilde{\varphi}_* : W_A \to W_B$ defined by $\mathbb{Z}$-linear extension of $\tilde{\varphi}_*([p]) := [\varphi_\infty(p)]$. Clearly, $\varphi_\infty(p) \oplus \varphi_\infty(q) = \varphi_\infty(p \oplus q)$, so that

$$\tilde{\varphi}_*([p]) + \tilde{\varphi}_*([q]) = [\varphi_\infty(p)] + [\varphi_\infty(q)] = [\varphi_\infty(p) \oplus \varphi_\infty(q)] = [\varphi_\infty(p \oplus q)] = \tilde{\varphi}_*([p \oplus q]) = \tilde{\varphi}_*([p] + [q]).$$

Hence $\tilde{\varphi}_*$ preserves the defining relations between the generators of K_0, which implies that $\tilde{\varphi}_*(I_A) \subseteq I_B$. Hence the quotient map $\varphi_* : W_A/I_A \to W_B/I_B$ is well defined. But by definition this is precisely $\varphi_* : K_0(A) \to K_0(B)$. The properties stated in the lemma then easily follow from the corresponding properties of φ_∞.

Starting from the definition $K_0(A) := G(V(A))$, the same calculation shows that $\tilde{\varphi}_* : V(A) \to V(B)$ is a homomorphism of semigroups, which by the functoriality of Grothendieck group construction induces a map $\varphi_* : K_0(A) \to K_0(B)$. ■

Lemma 4.1 allows a satisfactory extension of the functor K_0 to nonunital C^*-algebras. One could try to define $K_0(A)$ for nonunital A in precisely the same way as in the unital case, but on that definition K-theory would lose the important property of half-exactness. Namely, as we will show later, on the correct definition K-theory has the property that a short exact sequence (SES) $0 \to J \to A \to B \to 0$ of C^*-algebras induces an exact sequence $K_0(J) \to K_0(A) \to K_0(B)$. This suggests the following definition.

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[35] As first recognized by John von Neumann.

[36] This is still true for nonseparable H.

Definition 4.2 *Let* $0 \to A \to \dot{A} \xrightarrow{\pi} \mathbb{C} \to 0$ *be the SES given by the unitization of a nonunital* C^**-algebra,*[37] *where* $\pi : \dot{A} \to \dot{A}/A \cong \mathbb{C}$ *is the canonical projection. Then* $K_0(A) := \ker(\pi_*)$.

Hence $K_0(A)$ is defined as a certain subgroup of $K_0(\dot{A})$. Clearly, the sequence $K_0(A) \to K_0(\dot{A}) \xrightarrow{\pi_*} \mathbb{Z}$ is now exact by construction.[38]

One justification of Definition 4.2 is the complete functoriality of K-theory:

Theorem 4.3 *Each morphism* $\varphi : A \to B$ *between* C^**-algebras induces a map* $\varphi_* : K_0(A) \to K_0(B)$ *such that* $\varphi_*(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ *and, if* $\psi : B \to C$ *is another morphism,* $\psi_* \circ \varphi_* = (\psi \circ \varphi)_*$. *In other words, the association* $A \mapsto K_0(A)$ *is covariantly functorial.*

Assume that A and B are both nonunital (otherwise the proof is similar, since Definition 4.2 remains correct even for the unital case). Then $\varphi : A \to B$ extends to $\dot{\varphi} : \dot{A} \to \dot{B}$ by $\dot{\varphi}(a, z) := (\varphi(a), z)$, and the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & \dot{A} \arrow[r, "\pi^A"] \arrow[d, "\dot{\varphi}"] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & B \arrow[r] & \dot{B} \arrow[r, "\pi^B"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

By commutativity of the diagram one has $\pi^B \circ \dot{\varphi} = \pi^A$, whence by Lemma 4.1 $\pi^B_* \circ \dot{\varphi}_* = \pi^A_*$. Hence $\dot{\varphi}_*(\ker(\pi^A_*)) \subseteq \ker(\pi^B_*)$. Thus $\dot{\varphi}_*$ maps $K_0(A) = \ker(\pi^A_*) \subset K_0(\dot{A})$ into $\ker(\pi^B_*) = K_0(B)$, so that we define $\varphi_* : K_0(A) \to K_0(B)$ as this restriction. The functoriality properties of φ_* then follow from those of $\dot{\varphi}_*$. ■

To understand the situation a bit better, first note that the embedding $A \hookrightarrow \dot{A}$ induces $P_\infty(A) \hookrightarrow P_\infty(\dot{A})$ and hence $P_\infty(A) \to K_0(\dot{A})$. Elements of the form $[p]$ with $p \in P_\infty(A)$ clearly lie in the kernel of π_*, since $p \in \ker(\tilde{\pi}_\infty)$. Hence one has a natural map $P_\infty(A) \to K_0(A)$. However, $\ker(\pi_*)$ may have elements that are not of this form. Consider the morphism $\lambda : \mathbb{C} \to \dot{A}$, given by $\lambda(z) := (0, z)$.[39] Now for any $p \in P_\infty(\dot{A})$ one has $[p] - [s(p)] \in \ker(\pi_*)$, where $s : \dot{A} \to \dot{A}$ is given by

$$s := \lambda \circ \pi, \tag{11}$$

since $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$. The previous elements are a special case, since $s(p) = 0$ for $p \in P_\infty(A)$. Such difference elements in fact exhaust $K_0(A)$:

Proposition 4.4 *Let* A *be a nonunital* C^**-algebra with associated SES* $0 \to A \to \dot{A} \xrightarrow{\pi} \mathbb{C} \to 0$, *and define* $s : \dot{A} \to \dot{A}$ *by* $s(a, z) = (0, z) = z1_{\dot{A}}$. *Then*

$$K_0(A) = \{[p] - [s(p)], p \in P_\infty(\dot{A})\}.$$

To prove this, we use the definition $K_0(\dot{A}) = G(V(\dot{A}))$. Any $x \in K_0(\dot{A})$ is of the form $x = [p] - [q]$, where $p, q \in P_n(\dot{A})$ for some n. By definition of the Grothendieck group and of $V(\dot{A})$, we may write

$$[p] - [q] = \left[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right] = \left[\begin{pmatrix} p & 0 \\ 0 & 1_n - q \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}\right].$$

If $x \in \ker(\pi_*)$ then $x \in \ker(s_*)$, since $s := \lambda \circ \pi$, so $x \in K_0(A)$ implies $s_*([p] - [q]) = 0$. Hence

$$s_*\left(\left[\begin{pmatrix} p & 0 \\ 0 & 1_n - q \end{pmatrix}\right]\right) = s_*\left(\left[\begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}\right]\right) = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}\right].$$

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[37] See Part I, section 5.

[38] For an example showing that half-exactness fails on the naive definition, see [9], Example 3.3.9.

[39] This morphism splits the SES $0 \to A \to \dot{A} \xrightarrow{\pi} \mathbb{C} \to 0$, in that $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$; in general, an SES $0 \to J \to A \xrightarrow{\pi} B \to 0$ is called **split** if there is a morphism $\lambda : B \to A$ such that $\pi \circ \lambda = \mathrm{id}_B$.

So

$$[p]-[q]=\left[\begin{pmatrix} p & 0\\ 0 & 1_n-q\end{pmatrix}\right]-\left[s\begin{pmatrix} p & 0\\ 0 & 1_n-q\end{pmatrix}\right]. \tag{12}$$

This finishes the proof. ■

One has a similar situation in topological K-theory: if $\dot{X}$ is the one-point compactification of a locally compact Hausdorff space X, the inclusion $\infty \overset{\iota}{\hookrightarrow} \dot{X}$ of the compactification point induces a homomorphism $\iota^* : K^0(\dot{X}) \to K^0(\mathrm{pt} \cong \mathbb{Z})$. For noncompact X, the group $K^0(X)$ is defined as the kernel of this homomorphism. All elements of $K_0(X)$ are then of the form $[E]-[T]$, where E is some vector bundle over $\dot{X}$ and T is a trivial bundle over $\dot{X}$.

As an example, let us compute $K_0(C_0(\mathbb{R}^2))$. As in Lemma 5.4 of Part I, we identify $\dot{C_0(\mathbb{R}^2)}$ with $C(S^2)$, since $S^2 \cong \Sigma = \dot{\mathbb{R}}^2$, where Σ is the Riemann shpere. The embedding $C_0(\mathbb{R}^2) \hookrightarrow C(\Sigma)$ is given by extending $f \in C_0(\mathbb{R}^2)$ to $\dot{f} \in C(\Sigma)$ by putting $\dot{f}(\infty) = 0$, where ∞ is the north pole of Σ. Consequently, the projection $\pi : C(\Sigma)/C_0(\mathbb{R}^2) \to \mathbb{C}$ is given by $\pi(f) = f(\infty)$.

It can be shown from the classification of vector bundles that $K_0(C(\Sigma))$ has two generators, namely $1_\Sigma \in P_1(C(\Sigma))$ (corresponding to the trivial bundle) and $\sigma \mapsto p(\sigma) \in P_2(C(\Sigma))$ (corresponding to the so-called tautological bundle, realizing Σ as $\mathbb{CP}^1$), see (1) etc. We simply call the second generator p. We identify $K^0(pt)$ with $\mathbb{Z}$ under the map $\mathrm{Tr} : P_\infty(\mathbb{C}) \to \mathbb{Z}$; cf. the previous section with $n = 1$ in the computation of $K_0(M_n(\mathbb{C}))$. Clearly,

$$\begin{aligned}\pi_*([1_\Sigma]) &= \mathrm{Tr}\,(1_\Sigma(\infty)) = \mathrm{Tr}\,(1) = 1;\\ \pi_*([p]) &= \mathrm{Tr}\,(p(\infty)) = \mathrm{Tr}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix} = 1.\end{aligned}$$

Thus $K^0(\mathbb{R}^2) = \ker(\pi_*)$ has the single generator $[p]-[1_\Sigma]$; this is often written as

$$b := [p] - \left[\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right], \tag{13}$$

with p given by (2), and called the **Bott element**. It is clearly of the form described in Proposition 4.4.

For a maximally noncommutative nonunital example, let us compute $K_0(B_0(H))$, where $B_0(H)$ is the C^*-algebra of compact operators on an infinite-dimensional Hilbert space H. We give the main idea, leaving a precise formulation as an exercise. We start with the computation of $K_0(\dot{B_0(H)})$. Referring to the discussion preceding Proposition 4.4, we first compute the contribution of $P_\infty(B_0(H))$ to $K_0(\dot{B_0(H)})$. It is almost trivial that $M_n(B_0(H)) \cong B_0(H)$. Moreover, a projection p on H is compact iff $\dim(pH) < \infty$ (exercise). This essentially reduces the contribution of $P_\infty(B_0(H))$ to $K_0(\dot{B_0(H)})$ to our earlier discussion of $P_\infty(M_n(\mathbb{C}))$. Thus one has $p \sim q$ in $P_\infty(B_0(H))$ iff $\dim(pH) = \dim(qH) < \infty$, and the class $[p]$ in $P_\infty(\dot{B_0(H)})/\sim$ is charcterized by $m_+ := \mathrm{Tr}\,(p) < \infty$.

We now turn to contributions to $P_\infty(\dot{B_0(H)})$ involving 1_H. By analogous reasoning, one has $1_H - p \sim 1_H - q$ iff $\dim(pH) = \dim(qH) < \infty$, and p is never equivalent to $1_H - q$ whenever $\mathrm{Tr}\,(p) < \infty$ and $\mathrm{Tr}\,(q) < \infty$. Thus one has further classes in $P_\infty(\dot{B_0(H)})/\sim$ of the form $[1_H - p]$, characterized by $m_- := \mathrm{Tr}\,(p) < \infty$.

Finally, one has classes $[\oplus^n 1_H]$, where $\oplus^n := 1_H \oplus \cdots 1_H$ (n copies), which are characterized by an integer $0 < n < \infty$. These are disjoint from the previous ones, except for the case $m_- = 0$, which coincides with $n = 1$. To avoid double counting, we therefore assume $m_- > 0$. Studying the addition law $\oplus$, one finds that as a semigroup one has

$$V(\dot{B_0(H)}) \cong \mathbb{N} \oplus \mathbb{Z}; \tag{14}$$

cf. (6) and (9). Namely, the triple (n, m_+, m_-) as above defines the element $(n, m_+ - m_-) \in \mathbb{N} \oplus \mathbb{Z}$, and this gives precisely the right addition law. Since the Grothendieck group of an abelian semigroup that is already a group is the semigroup itself, with $K_0(\dot{B_0(H)}) := V(\dot{B_0(H)})$ one obtains

$$K_0(\dot{B_0(H)}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{15}$$

One clearly has $\pi_*(n,m) = m$, so that one finally obtains

$$K_0(B_0(H)) \cong \mathbb{Z}, \tag{16}$$

just as a naive calculation would suggest. The element $m_+ - m_- \in \mathbb{Z}$ may be realized as $[p]-[q]$, where $p, q \in P_1(B_0(H))$ such that $\mathrm{Tr}\,(p) = m_+$ and $\mathrm{Tr}\,(q) = m_-$. To bring this in a form suggested by Proposition 4.4), we rewrite this as

$$[p]-[q] = \left[\begin{pmatrix} p & 0 \\ 0 & 1_H - q \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_H \end{pmatrix}\right] = \left[\begin{pmatrix} p & 0 \\ 0 & 1_H - q \end{pmatrix}\right] - \left[s\begin{pmatrix} p & 0 \\ 0 & 1_H - q \end{pmatrix}\right]; \tag{17}$$

cf. (13) and (12), respectively.

5 Continuity and stability of K_0

Given the result $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, there is another way to compute $K_0(B_0(H))$, which uses the continuity of K-theory under so-called direct limits. In a sense to be made precise in this section, $B_0(H)$ for separable H is the limit of $M_n(\mathbb{C})$ as $n \to \infty$, and it so happens that K_0 is continuous under this type of limit. Hence $B_0(H) = \lim_n M_n(\mathbb{C})$ and

$$K_0(B_0(H)) = K_0(\lim_n M_n(\mathbb{C})) = \lim_n K_0(M_n(\mathbb{C})) = \lim_n \mathbb{Z} = \mathbb{Z}. \tag{18}$$

It is helpful to discuss the notion of a direct limit[40] in an abstract setting.[41]

Definition 5.1 *Let C be a category, and suppose one has a collection of objects $\{c_n \in C\}_{n\in\mathbb{N}}$ and arrows $\{\varphi_n : c_n \to c_{n+1}\}$. The **direct limit** of $c_1 \xrightarrow{\varphi_1} c_2 \xrightarrow{\varphi_2} c_3 \cdots$ is an object $c \in C$ with arrows $c_n \xrightarrow{\mu_n} c$ such that each diagram*

$$\begin{tikzcd}
c_n \arrow[r, "\varphi_n"] \arrow[dr, "\mu_n"'] & c_{n+1} \arrow[d, "\mu_{n+1}"] \\
& c
\end{tikzcd}$$

commutes, and that is universal in the sense that whenever an object c' exists with maps $c_n \xrightarrow{\mu'_n} c'$ satisfying the analogous commutative diagram, there is a unique map $c \to c'$ such that

$$\begin{tikzcd}
c_n \arrow[d, "\mu_n"'] \arrow[dr, "\mu'_n"] & \\
c \arrow[r, "\exists!\mu'"'] & c'
\end{tikzcd}$$

commutes. We then write $c = \lim_n c_n$.

It follows that the limit c is unique up to isomorphism when it exists. Inductive limits may or may not exist in a certain category. They do exist in the categories of C^*-algebras and of abelian group,s which is all that matters for us. For example, it is an exercise to prove that $B_0(\ell^2) \cong \lim_n M_n(\mathbb{C})$, and hence $B_0(H) \cong \lim_n M_n(\mathbb{C})$ for any separable H. Although $B(H)$ satisfies the first diagram, it fails to be universal, for there is a unique arrow $B_0(H) \to B(H)$ for given H, but not the other way round.

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[40] Sometimes called an **inductive limit** or a **colimit**.

[41] Though not the most abstract one, which involves arbitrary index sets.

More generally, if $\{A_n\}$ is an increasing set of subalgebras of a given C^*-algebra A (i.e., $A_n \subset A_{n+1} \subset A$ for all n), then $\lim_n A_n = A := \overline{\cup_n A_n}$, the norm closure of the union of all A_n. The maps $\mu_n : A_n \to A$ are simply given by the obvious embeddings $A_n \hookrightarrow A$, from which it is clear that the first part of Definition 5.1 is satisfied. As to the second condition, suppose there is a C^*-algebra A' with maps $\mu'_n : A_n \to A'$ such that $\mu'_{n+1} \circ \varphi_n = \mu'_n$. define $\mu' : \cup_n A_n \to A'$ by $\mu' | A_n := \mu'_n$. Then $\|\mu'\| = 1$ (provided at least one μ'_n is nonzero), so that μ' extends to the closure A by continuity. The uniqueness of this extension gives the uniqueness of μ' in the second diagram in Definition 5.1.

This construction suggests the construction of general direct limits in the category of C^*-algebras by reducing to this situation, as follows.

Proposition 5.2 *The category of C^*-algebras and morphisms has direct limits.*

We merely sketch the proof; see [9] for details. Given $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \cdots$, where all A_n are C^*-algebras and each μ_n is a morphism, one first forms $\ell_b(A) \subset \prod_{n\in\mathbb{N}} A_n$ (that is, the collection of maps $a : \mathbb{N} \to \coprod_n A_n$ for which $a(n) \in A_n$) as

$$\ell_b(A) := \{a : \mathbb{N} \to \coprod_n A_n \mid a(n) \in A_n, \sup_n \|a(n)\| < \infty\},$$

and its subspace

$$\ell_0(A) := \{a : \mathbb{N} \to \coprod_n A_n \mid a(n) \in A_n, \lim_n \|a(n)\| = 0\}.$$

It is easy to show that $\ell_b(A)$ is a C^*-algebra under pointwise operations and the norm $\|a\| := \sup_n \|a(n)\|$, and that $\ell_0(A)$ is an ideal in $\ell_b(A)$. Thus we may form the C^*-algebra $\ell_\infty(A) := \ell_b(A)/\ell_0(A)$, which, as its name suggests, purely involves the behaviour of a function a at infinity. As usual, we denote the canonical projection $\ell_b(A) \to \ell_\infty(A)$ by π. One has natural maps $\nu_n : A_n \to \ell_b(A)$ defined, for $a_n \in A_n$, by

$$\begin{aligned} \nu_n(a_n) \quad : \quad & k \mapsto 0 \text{ if } k < n; \\ & n \mapsto a(n); \\ & k \mapsto \varphi_{k-1} \circ \cdots \varphi_n(a_n) \text{ if } k > n. \end{aligned}$$

This yields further maps $\mu_n : A_n \to \ell_\infty(A)$ defined by $\mu_n := \pi \circ \nu_n$. With these constructions, one has achieved that all A_n may be seen as subalgebras of $\ell_\infty(A)$ through μ_n, with $\mu_n(A_n) \subset \mu_{n+1}(A_{n+1})$. One then puts

$$A := \overline{\cup_n \mu_n(A_n)},$$

and shows that A is the direct limit of the given system $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \cdots$. ■

The construction of direct limits in the category of abelian groups is based on the same idea (exercise!). Also, we leave the proof of the following result as an exercise to the reader.

Theorem 5.3 *The functor K_0 is continuous under direct limits; in other words, one has*

$$K_0(\lim_n A_n) = \lim_n K_0(A_n),$$

where, given a system $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \cdots$ in the category of C^*-algebras, the corresponding system in the category of abelian groups is given by $K_0(A_1) \xrightarrow{(\varphi_1)_*} K_0(A_2) \xrightarrow{(\varphi_2)_*} K_0(A_3) \cdots$.

As an application of this theorem, the reader is invited to prove the **stability** of K_0:

Theorem 5.4 1. *$K_0(M_n(A)) \cong K_0(A)$, natural in A, for any $n \in \mathbb{N}$;*

2. *$K_0(B_0(A)) \cong K_0(A)$, where $B_0(A)$ is defined as the direct limit of $M_n(A)$ as $n \to \infty$.*

6 Homotopy invariance of K_0

In this section we establish the homotopy invariance of K_0. We recall the notion of homotopy from topology: two maps $F : Z \to T$ and $G : Z \to T$ between spaces Z, T are said to be **homotopic**, written $F \overset{\mathrm{h}}{\sim} G$, when there is a map $\Phi : I \times Z \to T$ such that $\Phi_0 = F$ and $\Phi_1 = G$, where $\Phi_t := \Phi(t, \cdot)$, and $I := [0, 1]$. Two spaces X, Y are then called homotopy equivalent when there are maps $X \xrightarrow{A} Y \xrightarrow{B} X$ such that $A \circ B \overset{\mathrm{h}}{\sim} \mathrm{id}_Y$ and $B \circ A \overset{\mathrm{h}}{\sim} \mathrm{id}_X$. Both A and B are then called a homotopy equivalence.

Assuming that Z and T are compact Hausdorff spaces, we apply the Gelfand-Naimark functor $X \mapsto C(X)$ of Part I to this definition. This suggests a definition of homotopy of maps $\varphi : C(T) \to C(Z)$ and $\psi : C(T) \to C(Z)$: we declare such maps to be homotopic when there is a unital morphism $\tilde{\Phi} : C(T) \to C(I \times Z)$ for which $\tilde{\Phi}_0 = \varphi$ and $\tilde{\Phi}_0 = \psi$, where, for $t \in I$, $\tilde{\Phi}_t : C(T) \to C(Z)$ is defined by $\tilde{\Phi}_t(f) := (\tilde{\Phi}(f))(t, -)$. By Theorem 6.6 of Part I, it evidently follows that $F : Z \to T$ and $G : Z \to T$ are homotopic iff their pullbacks $\varphi = F^* : C(T) \to C(Z)$ and $\psi = G^* : C(T) \to C(Z)$ are homotopic in the sense just defined.

Now note that $C(I, Z) \cong C(I, C(Z))$ by the isomorphism $\tilde{f} \mapsto f$ with $(f(t))(z) := \tilde{f}(t, z)$. Thus we may define two maps $\varphi, \psi : A \to B$, where A and B are unital commutative C^*-algebras, to be homotopic when there is a unital morphism $\tilde{\Phi} : A \to C(I, B)$ for which $\tilde{\Phi}_0 = \varphi$ and $\tilde{\Phi}_0 = \psi$. This, in turn, amounts to saying that there is a path $\Phi_t : A \to B$ of unital morphisms, $t \in I$, for which $\Phi_t(1_A) = 1_B$ for all t and the function $t \mapsto \Phi_t(a) := (\Phi(a))(t)$ is in $C(I, B)$ for each $a \in A$, with again $\Phi_0 = \varphi$ and $\Phi_0 = \psi$. It turns out that C^*-algebraic K-theory is invariant under such maps even if the unitality condition is dropped. In the spirit of noncommutative geometry, we therefore generalize the situation to arbitrary C^*-algebras as follows.

Definition 6.1 *1. Let A and B be C^*-algebras. Two morphisms $\varphi, \psi : A \to B$ are called* ***homotopic****, written $\varphi \overset{\mathrm{h}}{\sim} \psi$, when there is a path $\Phi_t : A \to B$ of morphisms, $t \in I$, for which the function $t \mapsto \Phi_t(a)$ is in $C(I, B)$ for each $a \in A$, and $\Phi_0 = \varphi$, $\Phi_0 = \psi$.*

2. Two C^*-algebras A, B are said to be ***homotopy equivalent****, again written $A \overset{\mathrm{h}}{\sim} B$, when there are morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} A$ such that $\alpha \circ \beta \overset{\mathrm{h}}{\sim} \mathrm{id}_B$ and $\beta \circ \alpha \overset{\mathrm{h}}{\sim} \mathrm{id}_A$.*

3. A C^*-algebra A is called ***contractible*** *when $A \overset{\mathrm{h}}{\sim} 0$.*

It should be pointed out that in the commutative nonunital case this definition is not equivalent to the usual notion of homotopy. Firstly, $\varphi, \psi : C_0(T) \to C_0(Z)$ are not required to be nondegenerate, which, by Theorem 6.11 in Part I means that φ and ψ are not necessarily the pullbacks of any map from Z to T whatsoever. Secondly, when φ and ψ happen to be nondegenerate, then again by Theorem 6.11 in Part I, they are pullbacks $\varphi = F^*$ and $\psi = G^*$ of maps $F, G : Z \to T$ that are necessarily proper. The corresponding notion of **proper homotopy** between locally compact spaces is defined as in the compact case, where now all maps are required to be proper. For example, $\mathbb{R}$ or $(0, 1)$ are not properly homotopy equivalent to a point because of the properness condition, which is violated by the map $\mathbb{R} \to \mathrm{pt}$. Thus one has $C_0(X) \overset{\mathrm{h}}{\sim} C_0(Y)$ when X and Y are properly homotopy equivalent (which in case that X ad Y are both compact simply means that X and Y are homotopy equivalent in the usual sense). For example, $C_0(\mathbb{R})$ and $C_0((0, 1))$ are not contractible C^*-algebras, although the underlying spaces are contractible in the usual sense.

The following example turns out to be crucial for K-theory. Let A be a C^*-algebra. The **cone** of A is the C^*-algebra

$$CA := C_0((0, 1], A). \tag{19}$$

This is, of course, the subspace of $C(I, A)$ of functions vanishing at $t = 0$.[42]

Lemma 6.2 *The cone of any C^*-algebra is contractible.*

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[42] One also finds the isomorphic C^*-algebras $CA := C_0([0, 1), A)$ and $CA := C_0([0, \infty), A)$ in the literature.

We need to find morphisms $CA \xrightarrow{\alpha} 0 \xrightarrow{\beta} CA$ such that $\alpha \circ \beta \overset{\mathrm{h}}{\sim} \mathrm{id}_B$ and $\beta \circ \alpha \overset{\mathrm{h}}{\sim} \mathrm{id}_A$. The only possibility is $\alpha(f) = 0$ for all f and $\beta(0) = 0$, so that $\alpha \circ \beta = 0$ should be homotopic to id_{CA}. The other condition, $\beta \circ \alpha \overset{\mathrm{h}}{\sim} \mathrm{id}_0$ is automatic. Now take $\Phi_t : CA \to CA$ to be $(\Phi_t(f))(s) := f(st)$. ■

We now come to the homotopy invariance of K_0.

Proposition 6.3 *If $\varphi : A \to B$ and $\psi : A \to B$ are homotopic morphisms, then $\varphi_* = \psi_* : K_0(A) \to K_0(B)$.*

Denote the path connecting φ and ψ by Φ_t, as before. As explained before, this expands to $(\Phi_t)_\infty : M_\infty(A) \to M_\infty(B)$ in the obvious way, and, since each Φ_t is a morphism, subsequently restricts to $(\Phi_t)_\infty : P_\infty(A) \to P_\infty(B)$. Let $p \in P_\infty(A)$. The notion of continuity in the definition of homotopy implies that $(\Phi_0)_\infty(p) \overset{\mathrm{h}}{\sim} (\Phi_1)_\infty(p)$, or $\varphi_\infty(p) \overset{\mathrm{h}}{\sim} \psi_\infty(p)$. Hence $\tilde{\varphi}_* = \tilde{\psi}_* : V(A) \to V(B)$.

If A and B are unital this immediately implies the claim. In the general case, one first applies this to $K_0(\dot{A})$ and $K_0(\dot{B})$ and subsequently restricts to their subspaces $K_0(A)$ and $K_0(B)$. ■

The following observations are trivial consequences, which we list for future use.

Corollary 6.4 *1. If $A \overset{\mathrm{h}}{\sim} B$ then $K_0(A) \cong K_0(B)$. Specifically, if $\alpha : A \to B$ is a homotopy equivalence, then $\alpha_* : K_0(A) \to K_0(B)$ is an isomorphism, with inverse β_* for any $\beta : B \to A$ such that al $\circ\, \beta \overset{\mathrm{h}}{\sim} \mathrm{id}_B$ and $\beta \circ \alpha \overset{\mathrm{h}}{\sim} \mathrm{id}_A$.*

2. If $A \overset{\mathrm{h}}{\sim} 0$ then $K_0(A) = 0$.

7 Half-exactness of K_0

Ideally, one might hope that K-theory is **exact**, in the sense that an SES $0 \to J \to A \to B \to 0$ of C^*-algebras induces an SES $0 \to K_0(J) \to K_0(A) \to K_0(B) \to 0$ of abelian groups. With the given definition of K_0, this evidently cannot be the case: for example, K_0 maps the SES $0 \to B_0(H) \to B(H) \to C(H) \to 0$, where $\dim(H) = \infty$, into the sequence $0 \to \mathbb{Z} \to 0 \to 0 \to 0$, which clearly fails to be exact. Another counterexample to exactness, which we leave to the reader as an exercise, is the SES $0 \to C_0((0,1)) \to C([0,1]) \to \mathbb{C} \oplus \mathbb{C} \to 0$. However, K_0 enjoys the following weaker property.

Proposition 7.1 *The functor K_0 is **half-exact**, in that an SES $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ of C^*-algebras induces an exact sequence $K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(B)$ of abelian groups.*

The exactness of the latter sequence simply means that $\ker(\pi_*) = \mathrm{ran}(\iota_*)$. Here the inclusion $\mathrm{ran}(\iota_*) \subseteq \ker(\pi_*)$ or, in other words, $\iota_* \circ \pi_* = 0$, is immediate from the functoriality of K_0 and the exactness of the SES of C^*-algebras, which yields $\iota \circ \pi = 0$ and hence $\iota_* \circ \pi_* = (\iota \circ \pi)_* = 0$.

Thus we only need to prove the opposite inclusion $\ker(\pi_*) \subseteq \mathrm{ran}(\iota_*)$. Here a complication immediately arises. Since J will be nonunital (being isomorphic to an ideal in A), we need to define the K_0-groups by means of Definition 4.2. Hence we must deal with the unitization $\dot{J}$ of J; and for simplicity we will also replace A and B by $\dot{A}$ and $\dot{B}$, respectively, even if they are already unital; Definition 4.2 will still produce the correct K_0-groups. Now the problem is that the corresponding sequence $0 \to \dot{J} \xrightarrow{\dot{\iota}} \dot{A} \xrightarrow{\dot{\pi}} \dot{B} \to 0$ is not even exact in the middle: although $\iota \circ \pi = 0$, one has $\dot{\pi} \circ \dot{\iota}(0, z)_{\dot{J}} = (0, z)_{\dot{B}} \neq 0$. However, the following observation is easy to prove (exercise). Recall that a morphism $\varphi : A \to B$ defines a morphism $\varphi_n : M_n(A) \to M_n(B)$ by componentwise application of φ.

Lemma 7.2 *Let $0 \to M_n(\dot{J}) \xrightarrow{\dot{\iota}_n} M_n(\dot{A}) \xrightarrow{\dot{\pi}_n} M_n(\dot{B}) \to 0$ be the sequence of matrix-valued unitizations corresponding to a given SES $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ of C^*-algebras. Then $a \in \mathrm{ran}(\dot{\iota}_n)$ iff $\dot{\pi}_n(a) = s_n(\dot{\pi}(a))$, where $s : \dot{B} \to \dot{B}$ is defined by (11) applied to B.*

Now let $g \in K_0(A)$. According to Proposition 4.4, which remains valid in the unital case, we may write $g = [p] - [s(p)]$ for some $p \in P_n(\dot{A})$. If $g \in \ker \pi_*$ then clearly

$$\pi_*([p] - [s(p)]) = [\dot{\pi}_n(p)] - [s \circ \dot{\pi}_n(p)] = 0.$$

This does not imply that $\dot{\pi}_n(p) = s \circ \dot{\pi}_n(p)$, but the key point in the proof is that one may pick a projection $p' \sim p \oplus 1_k$ for some k (so that, in particular, $g = [p'] - [s(p')]$), for which this is true, i.e., $\dot{\pi}_n(p') = s \circ \dot{\pi}_n(p')$ (exercise). Lemma 7.2 then implies that $p' \in \text{ran}(\dot{\iota}_n)$, i.e., $p' = \dot{\iota}_n(q)$ for some $q \in P_n(\dot{J})$. Hence

$$g = [p] - [s(p)] = [p'] - [s(p')] = [\dot{\iota}_n(q)] - [s(\dot{\iota}_n(q))] = \iota_*([q] - [s(q)]).$$

Again by Proposition 4.4, $[q] - [s(q)]$, which a priori lies in $K_0(\dot{J})$, actually lies in $K_0(J)$. Hence $g \in \text{ran}(\iota_*)$, which proves Proposition 7.1. ■

Let us apply this to the commutative case. As shown in part I, any ideal J of $A = C_0(X)$ is of the form $J = C(X, Y) = C_0(X \backslash Y)$, where $Y \subset X$ is closed. We write $Z := X \backslash Y$ and note that $A/J \cong C_0(Y) = C_0(X \backslash Z)$, so that any SES of commutative C^*-algebras is of the form

$$0 \to C_0(Z) \xrightarrow{i} C_0(X) \xrightarrow{\pi} C_0(X \backslash Z) \to 0.$$

Here $i(f) := f$ on Z and $i(f) := 0$ on $X \backslash Z$, and $\pi(g)$ is simply the restriction of g to $X \backslash Z$. By Proposition 7.1, this induces an exact sequence $K^0(Z) \to K^0(X) \to K^0(X \backslash Z)$.

The unusual feature of this sequence is that one would expect the inclusion map $Z \overset{\iota}{\hookrightarrow} X$ to induce a map $K^0(X) \xrightarrow{\iota^*} K^0(Z)$ by contravariant functoriality of topological K-theory. It indeed does, but in addition one apparently has the map $K^0(Z) \cong K_0(C_0(Z)) \xrightarrow{i^*} K_0(C_0(X)) \cong K^0(X)$ (in the opposite direction!) that is induced by the covariant functoriality of C^*-algebraic K-theory.[43] One might therefore think that C^*-algebraic K-theory, when restricted to the commutative case, is more functorial than topological K-theory, but this is not the case. If one combines the functoriality of topological K-theory with its excision property, one precisely recovers the combination of these two properties of C^*-algebraic K-theory restricted to the commutative case.[44]

8 Excision in K-theory

We now take a deeper look at the half-exactness and related properties of K-theory, motivated by the notion of excision in algebraic topology. So far, we have studied K-theory as a functor K_0 whose argument is a C^*-algebra; the argument of the functor K^0 of topological K-theory was a locally compact Hausdorff space. However, in algebraic topology (co)homology theories are initially defined over pairs (X, Y), where Y is a closed subspace of X. A space X is then identified with a pair $(X, \emptyset)$. A morphism between two such pairs (X, Y) and (X', Y') is taken to be a map $\varphi : X \to X'$ for which $\varphi(Y) \subset Y'$. For example, in the ensuing category one has an SES

$$\emptyset \to Y \hookrightarrow X \to (X, Y) \to \emptyset, \tag{20}$$

and all short exact sequences are isomorphic to one of this type.

For example, topological K-theory should really start with an abelian group $K^0(X, Y)$. To define this group for compact X (and hence compact Y), one takes triples (E, F, σ), where $E \to X$ and $F \to X$ are vector bundles over X, and $\sigma : E|Y \to F|Y$ is an isomorphism. Let us call such a triple a relative K-cycle. A relative K-cycle (E, F, σ) is called degenerate when σ can be extended to an isomorphism $E \to F$. We now define an equivalence relation on the set of all K-cycles (for given (X, Y)). One firstly puts $(E, F, \sigma) \sim (E, F, \sigma')$ if σ is homotopic to σ', and secondly declares that $(E, F, \sigma) \sim (E \oplus E', F \oplus F', \sigma \oplus \sigma')$ whenever (E', F', σ') is a degenerate K-cycle.

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[43] This is the simplest example of a so-called shriek map in topological K-theory.

[44] This is because any morphism $\varphi : C_0(Y) \to C_0(X)$ factors as $\varphi = i \circ \psi^*$, where $Z \subset X$ is open and $\psi : Z \to Y$ is a continuous map.

The set of of equivalence classes $[E, F, \sigma]$ under these relations forms an abelian group under the obvious addition rule $[E, F, \sigma] + [E', F', \sigma'] := [E \oplus E', F \oplus F', \sigma \oplus \sigma']$. Note that $0 = [E, E, \mathrm{id}]$ for any E, and that $[E, F, \sigma]^{-1} = [F, E, \sigma^{-1}]$ (so that there is no need to perform a Grothendieck group construction, although that would do no harm). This construction is related to our previous definition of K-theory by the following result.

Proposition 8.1 *One has* $K^0(X, Y) \cong K^0(X\backslash Y)$.

We just sketch the proof. Let X/Y be the quotient of X by the equivalence relation $x \sim x$ for all $x \in X\backslash Y$ and $y \sim y'$ for all $y, y' \in Y$, and note that the one-point compactification of $X\backslash Y$ may be identified with X/Y; the point $[Y] \in X/Y$ is the compactification point ∞. The main technical point in the proof, which we omit, is to show that given a class $[E', F', \sigma']$ one may pick a representative triple (E, F, σ) for which $E|Y = F|Y = Y \times \mathbb{C}^n$, and $\sigma' = \mathrm{id}$. In that case, there is a bundle $\tilde{E} \rightarrow X/Y$, unique up to isomorphism, whose pullback $\pi^*\tilde{E} \rightarrow X$ under the projection $\pi : X \rightarrow X/Y$ is isomorphic to E. To see this, note that the triviality of $E|Y$ and the local triviality of E imply that Y has an open neighbourhood N_Y over which E is still isomorphic to a trivial bundle. One then has two bundles $E|(X\backslash Y)$ and $(N_Y/Y) \times \mathbb{C}^n$, over $X\backslash Y$ and N_Y/Y, respectively, whose restrictions to the intersection $(X\backslash Y) \cap (N_Y/Y) = (N_Y/Y)\backslash[Y]$ are isomorphic. By a standard result in the theory of vector bundles (cf. Prop. 11.23 in [10]), there then exists a bundle $\tilde{E}$ over $X/Y = (X\backslash Y) \cup (N_Y/Y)$ whose restriction to $X\backslash Y$ is isomorphic to $E|(X\backslash Y)$, and whose restriction to N_Y/Y is isomorphic to $(N_Y/Y) \times \mathbb{C}^n$. This is the desired bundle. A similar construction, of course, applies to F.

One therefore has a map $[E, F, \sigma] \mapsto [\tilde{E}] - [\tilde{F}]$ from $K^0(X, Y)$ to $K^0(X/Y)$, and in fact $[\tilde{E}] - [\tilde{F}] \in K^0(X\backslash Y)$. This ensuing map $K^0(X, Y) \rightarrow K^0(X\backslash Y)$ then turns out to be an isomorphism. ∎

Let us note that this construction covers topological K-theory for noncompact spaces, for one may take the pair $(\dot{Z}, \infty)$, so that $\dot{Z}\backslash\infty = Z$.[45] Applying Proposition 8.1 then yields $K^0(\dot{Z}, \infty) \cong K^0(Z)$; putting $Z = X\backslash Y$ and hence $\dot{Z} = X/Y$ in this equation yields

$$K^0(X/Y, [Y]) \cong K^0(X\backslash Y). \tag{21}$$

Combining this with Proposition 8.1 then gives **excision**:

$$K^0(X, Y) \cong K^0(X/Y, [Y]). \tag{22}$$

Generalizing this to C^*-algebraic K-theory, we follow our usual strategy of replacing X by $A = C(X)$, $X\backslash Y$ by $J = C_0(X\backslash Y) = C(X, Y)$, and Y by $C(Y) = A/J$. Thus we need to define so-called relative K-theory groups $K_0(A, A/J)$.

Definition 8.2 *Let A be a unital C^*-algebra with ideal $J \subset A$ and canonical projection $\pi : A \rightarrow A/J$. Let (p, q, v) be a triple where $p, q \in P_n(A)$ and $v \in M_n(A)$ satisfy $\pi_n(v)^*\pi_n(v) = \pi_n(p)$ and $\pi_n(v)\pi_n(v)^* = q$ (in other words, $\pi_n(p)$ and $\pi_n(q)$ are Murray-von Neumann equivalent in A/J via $\pi_n(v)$). Such a triple is called a relative K-cycle (for given A and J).*

We say that $(p, q, v) \sim (p', q', v')$ when there is a path $(p(t), q(t), v(t))$ of triples as above for which $p(0) = p$, $p(1) = p'$, etc., and in addition we put

$$(p, q, v) \sim (p, q, v) \oplus (p', q', v') := (p \oplus p', q \oplus q', v \oplus v')$$

when the triple (p', q', v') is degenerate in the sense that when $(v')^*v' = p'$ and $v'(v')^* = q'$ (in other words, when p' and q' are already Murray-von Neumann equivalent in A via v').

Then $K_0(A, A/J)$ is defined as the abelian group of equivalence classes $[p, q, v]$ with addition rule $[p, q, v] + [p', q', v'] := [p \oplus p', q \oplus q', v \oplus v']$.

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[45] Proposition 8.1 then leads to an alternative definition of $K^0(X)$ as the group defined by classes $[E, F, \sigma]$, where now E and F are bundles over X that are isomorphic via σ outside some compact subset of X.

This definition mimics the one of $K^0(X,Y)$ in the sense that the isomorphism $E|Y \cong F|Y$ has been generalized to $\pi_n(p) \overset{\mathrm{MvN}}{\sim} \pi_n(q)$; projecting $\pi : A \to A/J$ is like taking the restriction to Y of a vector bundle. The neutral element in $K_0(A, A/J)$ is $[p,p,p]$ for any projection p, and the inverse of $[p,q,v]$ is $[q,p,v^*]$, so again there is no need for the Grothendieck group.[46] Now the noncommutative generalization of Proposition 8.1 is

Proposition 8.3 *One has* $K_0(A, A/J) \cong K_0(J)$.

One proves this by first showing that

$$K_0(\dot{J}, \mathbb{C}) \cong K_0(J), \tag{23}$$

which is the analogue of (21), and subsequently proving that

$$K_0(A, A/J) \cong K_0(\dot{J}, \mathbb{C}), \tag{24}$$

which is the analogue of (22).

First note that there is a canonical homomorphism $K_0(A, A/J) \to K_0(A)$, given by

$$[p,q,v] \mapsto [p] - [q] \tag{25}$$

In the commutative case this would be the map $K^0(X,Y) \to K^0(X)$ defined by $[E,F,\sigma] \mapsto [E] - [F]$.

Lemma 8.4 *The ensuing sequence* $K_0(A, A/J) \to K_0(A) \xrightarrow{\pi_*} K_0(A/J)$ *is exact (i.e., in the middle).*

We leave the proof as an exercise. Note that combined with Proposition 8.3, this lemma provides a new proof of the fact that C^*-algebraic K-theory is half-exact (cf. Proposition 7.1). To prove (23), one first shows that if the SES $0 \to J \to A \xrightarrow{\pi} A/J \to 0$ splits, then $K_0(A, A/J) \cong \ker(\pi_*)$. Applying this result with $A = \dot{J}$ and $J = J$, hence $A/J = \mathbb{C}$, this immediately implies (23); the isomorphism $K_0(\dot{J}, \mathbb{C}) \to K_0(J)$ is just the map (25) from $K_0(\dot{J}, \mathbb{C})$ to $K_0(\dot{J})$ restricted to its image.

In addition, we have a natural map $K_0(\dot{J}, \mathbb{C}) \to K_0(A, J)$, given by $[p,q,v] \mapsto [p,q,v]$. This expression means that we extend the embedding $J \subset A$ to $\dot{J} \subset A$ by identifying the unit of $\dot{J}$ with the one in A, and in addition we identify $\mathbb{C} \cong \dot{J}/J$ in the pair $(\dot{J}, \mathbb{C})$ with $\mathbb{C}1_{A/J}$. The hard part in proving Proposition 8.3 is to show that this map is an isomorphism. We just sketch the main idea of the proof, leaving the details to the reader (cf. [6]). One has to show that every relative K-cycle (p,q,v) with $p,q \in P_n(A)$ and $v \in M_n(A)$ is equivalent to one of the form where the entries lie in $P_n(\dot{J})$ and and $M_n(\dot{J})$, respectively. To illustrate the procedure, we move p to $1 \in \dot{J}$. We add the degenerate triple $(1-p, 1-p, 1-p)$ to (p,q,v); the first term is

$$\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim 1. \qquad \blacksquare$$

A nice example is given by $A = B(H)$ and $J = B_0(H)$, so that $A/J = C(H)$. The ensuing isomorphism $K_0(B(H), C(H)) \cong K_0(B_0(H)) \cong \mathbb{Z}$ yields a map $K_0(B(H), C(H)) \to \mathbb{Z}$ with the following property. Let $u \in B(H)$ be an essentially unitary operator, in the sense that $u^*u - 1 \in B_0(H)$ and $uu^* - 1 \in B_0(H)$. It follows that $(1,1,u)$ is a relative K-cycle for the pair $(B(H), C(H))$. If p is the projection onto $\ker(u)$ and q is the projection onto $\ker(u^*)$, then

$$(1,1,u) = (p,q,0) \oplus (1-p, 1-q, u(1-p)),$$

since $up = 0$. Now $(1-p, 1-q, u(1-p))$ is homotopic to a degenerate relative cycle since $u : \mathrm{ran}(1-p) \to \mathrm{ran}(1-q)$ is invertible, so that $(1,1,u) \sim (p,q,0)$. Since u is Fredholm, p and q

---

[46] This follows, for example, by first defining $K_0(A, A/J)$ in terms of the Grothendieck group of the semigroup in question, and then using Proposition 8.3 to show that the inverse of $[p,q,v]$ is indeed $[q,p,v^*]$, so that the Grothendieck group construction was superfluous from the outset. A direct proof of this formula for the inverse would be desirable, however.

are finite-dimensional, so $(p, q, 0)$ is a relative K-cycle for $(B_0(H), \mathbb{C})$. Its image in $K_0(B_0(H))$ is $[p] - [q]$, which under the usual identification $K_0(B_0(H)) \cong \mathbb{Z}$ is mapped to $\mathrm{Tr}\,(p) - \mathrm{Tr}\,(q)$. But this is equal to

$$\mathrm{Tr}\,(p) - \mathrm{Tr}\,(q) = \dim(pH) - \dim(qH) = \dim(\ker(u)) - \dim(\ker(u^*)) = \mathrm{index}(u).$$

Hence the map $K_0(B(H), C(H)) \to \mathbb{Z}$ is given by $[1, 1, u] \mapsto \mathrm{index}(u)$; knowing it is an isomorphism, it follows that every relative K-cycle for $(B(H), C(H))$ may be brought into the form $[1, 1, u]$ for some essentially unitary operator u.

9 The long exact sequence in K-theory

Let us return to the category of pairs (X, Y) discussed at the beginning of the preceding section. A cohomology theory[47] consists of a sequence of contravariant[48] functors H^n, $n \in \mathbb{Z}$, from the category P of such pairs to the category of abelian groups, satisfying a number of axioms.[49] One such axiom is homotopy invariance. For our present purpose, the most important other such axiom states the existence of a long exact sequence, in the following sense. Consider the functor $R : P \to P$, defined by $R(X, Y) := Y \equiv (Y, \emptyset)$. For each n there should be a natural transformation $\partial^n : H^n \circ R \to H^{n+1}$, called a **connecting map**, such that for each pair (X, Y) the sequence

$$\cdots \to H^n(X, Y) \to H^n(X) \to H^n(Y) \xrightarrow{\partial^n} H^{n+1}(X, Y) \to H^{n+1}(X) \to H^{n+1}(Y) \to \cdots$$

is exact. If the excision property holds, this sequence clearly reads

$$\cdots \to H^n(X \backslash Y) \to H^n(X) \to H^n(Y) \xrightarrow{\partial^n} H^{n+1}(X \backslash Y) \to H^{n+1}(X) \to H^{n+1}(Y) \to \cdots .$$

This clearly implies that each H^n is half-exact, in the sense that the sequence $H^n(X, Y) \to H^n(X) \to H^n(Y)$ of abelian groups defined by functoriality from the SES (20) should be exact in the middle.

In a homology theory the connecting maps are $\partial_n : H_n \to H_{n-1} \circ R$. Formally, due to its covariance K-theory of C^*-algebras should be a homology theory, which as far as the connecting maps is concerned is incompatible with the aim of recovering topological K-theory, which is a cohomology theory, as its commutative case. As a compromise, we make the identification of the topological theory with the commutative C^*-algebraic theory by means of

$$K^n(X) \cong K_{-n}(C_0(X)), \tag{26}$$

and define the connecting maps in C^*-algebraic K-theory as $\partial_n : K_n \circ R \to K_{n-1}$. Here C^*-algebraic K-theory is seen as a two-variable theory as in the previous section, and $R(A, A/J) := A/J \equiv (A/J, 0)$. The axiom of the long exact sequence then states that the standard SES $0 \to J \to A \to A/J \to 0$ induces an exact sequence

$$\cdots \to K_n(A, A/J) \to K_n(A) \to K_n(A/J) \xrightarrow{\partial_n} K_{n-1}(A, A/J) \to K_{n-1}(A) \to K_{n-1}(A/J) \to \cdots . \tag{27}$$

Using excision as in Proposition 8.3, this of course reads

$$\cdots \to K_n(J) \to K_n(A) \to K_n(A/J) \xrightarrow{\partial_n} K_{n-1}(J) \to K_{n-1}(A) \to K_{n-1}(A/J) \to \cdots . \tag{28}$$

We will show that K-theory has an associated long exact sequence if we define the higher K-groups K_n for $n \geq 0$ in terms of the **suspension** SA of A, defined by

$$SA := C_0((0, 1), A). \tag{29}$$

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[47] Strictly speaking, what we discuss here is called an unreduced generalized cohomology theory, also called an exotic cohomology theory.

[48] Covariant functors define a homology theory. Because topological K-theory is a cohomology theory, we restrict ourselves to the contravariant case here.

[49] These axioms were first formulated by Eilenberg and Steenrod in 1952.

Let us note that S is a covariant functor from the category of C^*-algebras to itself: a morphism $\varphi : A \to B$ induces $S\varphi : SA \to SB$, given by $((S\varphi)(f))(t) := \varphi(f(t))$, where $f \in SA$. Furthermore, it is an exercise to show that the functor S is **exact**, in the sense that an SES $0 \to J \to A \to A/J \to 0$ is mapped into an SES $0 \to SJ \to SA \to S(A/J) \to 0$.

Now consider the SES

$$0 \to SB \xrightarrow{\iota} CB \xrightarrow{\pi} B \to 0, \tag{30}$$

where ι is the inclusion, and $\pi(f) := f(1)$. Anticipating that

$$K_n(CB) = 0 \ \forall n, \ \forall B, \tag{31}$$

as is to be expected from Corollary 6.4.2 with Lemma 6.2, the desired exact sequence (28) with $J = BA$, $A = CB$ and $A/J = B$ gives, using Lemma 6.2, the exact sequence

$$\cdots \to K_n(SB) \to 0 \to K_n(B) \xrightarrow{\partial_n} K_{n-1}(SB) \to 0 \to K_{n-1}(B) \to \cdots .$$

At least up to isomorphism, this forces the definition

$$K_n(A) := K_0(S^n A), \tag{32}$$

and similarly for the relative K-groups. It is then easy to show that each K_n is functorial, half-exact, continuous, homotopy invariant, and stable, and that (31) indeed holds.

The key technique in constructing the connecting maps $\partial_n : K_n(A/J) \to K_{n-1}(A, A/J)$ or $\partial_n : K_n(A/J) \to K_{n-1}(J)$ is the **mapping cone** $C_\varphi(A, B)$ of a morphism $\varphi : A \to B$ between arbitrary C^*-algebras. This is a subalgebra of the C^*-algebra $A \oplus CB$,[50]

$$C_\varphi(A, B) := \{(a, f) \in A \oplus CB \mid f(1) = \varphi(a)\}; \tag{33}$$

here the cone CB is defined in (19). One often simply writes C_φ for $C_\varphi(A, B)$. The mapping cone gives rise to an SES

$$0 \to SB \xrightarrow{i} C_\varphi(A, B) \xrightarrow{p} A \to 0, \tag{34}$$

where $i(f) := (0, f)$ and $p(a, f) := a$. The following lemma, in which $\pi : A \to A/J$ is the canonical projection, is fundamental.

Lemma 9.1 *The map $[p, q, v] \mapsto [p] - [q]$ from $K_0(A, A/J)$ to $K_0(C_\pi(A, A/J))$ is an isomorphism. Similarly (cf. Proposition 8.3), the map $j \mapsto (j, 0)$ from J to $C_\pi(A, A/J)$ induces an isomorphism $K_0(J) \xrightarrow{\cong} K_0(C_\pi(A, A/J))$.*

In addition, defining $K_n(A, A/J) := K_0(S^n A, S^n(A/J))$, one has $K_n(A, A/J) \cong K_n(C_\pi(A, A/J))$ and $K_n(J) \cong K_n(C_\pi(A, A/J))$ for any n.

Note that the second part follows from the first part, the exactness of the suspension functor S, and the isomorphism $SC_\varphi(A, B) \cong C_{S\varphi}(SA, SB)$: just apply the first part to $S^n\pi : S^n A \to S^n B$.

Before proving this lemma, we show how to construct the connecting maps from it. Consider the SES

$$0 \to S(A/J) \xrightarrow{i} C_\pi(A, A/J) \xrightarrow{p} A \to 0, \tag{35}$$

which is a special case of (34). Half-exactness of K_0 applied to (35) and Lemma 9.1 give a diagram

$$\begin{tikzcd}
K_1(A/J) \arrow[r] \arrow[dr, "\partial_1"'] & K_0(C_\pi(A, A/J)) \arrow[r] & K_0(A) \\
& K_0(A, A/J) \arrow[u, "\cong"] \arrow[ur] &
\end{tikzcd} .$$

---

[50] The direct sum $A \oplus B$ of two C^*-algebras A and B is defined through pointwise operations and the norm $\|a \oplus b\| := \sup\{\|a\|, \|b\|\}$.

Here ∂_1 is defined so as to make the left-hand triangle commute; the right-hand triangle commutes because the map $K_0(C_\pi(A,A/J)) \to K_0(A)$ is induced by $(a,f) \mapsto a$ from $C_\pi(A,A/J)$ to A, the map $K_0(A,A/J) \to K_0(C_\pi(A,A/J))$ is $[p,q,v] \mapsto [p]-[q]$ (see Lemma 9.1), and finally the map $K_0(A,A/J) \to K_0(A)$ is given by the same formula $[p,q,v] \mapsto [p]-[q]$, see (25). Hence the large outer triangle commutes as well. Lemma 8.4 applied to K_0 as well as to K_1 and the above diagram then give a sequence

$$K_1(A,A/J) \to K_1(A) \to K_1(A/J) \overset{\partial_1}{\to} K_0(A,A/J) \to K_0(A) \to K_0(A/J), \tag{36}$$

which happens to be exact: exactness at $K_0(A,A/J)$ is obvious from the exactness of the top row in the diagram, which is in turn a consequence of the SES (35) and the half-exactness of K_0, whereas exactness at $K_1(A/J)$ follows from a similar argument: we apply the SES (34) with $\varphi : A \to B$ given by $p : C_\pi(A,A/J) \to A$, i.e.,

$$0 \to SA \to C_p(C_\pi, A) \to C_\pi(A,A/J) \to 0.$$

Lemma 9.1 applied to the SES (35) yields $K_0(C_p(C_\pi, A)) \cong K_0(S(A/J)) = K_1(A/J)$, and using Lemma 9.1 as stated gives a commutative diagram

$$\begin{tikzcd}
K_1(A) \arrow[r] \arrow[dr, "(S\pi)_*"'] & K_0(C_p(C_\pi, A)) \arrow[r] & K_0(C_\pi(A,A/J)) \\
& K_1(A/J) \arrow[u, "\cong"] \arrow[r, "\partial_1"] & K_0(A,A/J) \arrow[u, "\cong"]
\end{tikzcd}$$

The exactness of (36) at $K_1(A/J)$ then follows from the exactness of the top row of this second diagram.

This argument may be iterated to prove exactness of the LES (27), as follows. Consider the following short exact sequences:

$$\begin{aligned}
&0 \to J \to A \overset{p_0}{\to} A/J \to 0; && (37)\\
&0 \to S(A/J) \to C_{p_0} \overset{p_1}{\to} A \to 0; && (38)\\
&0 \to SA \to C_{p_1} \overset{p_2}{\to} C_{p_0} \to 0; && (39)\\
&0 \to SC_{p_0} \to C_{p_2} \overset{p_3}{\to} C_{p_1} \to 0; && (40)\\
&0 \to SC_{p_1} \to C_{p_3} \overset{p_4}{\to} C_{p_2} \to 0; && (41)\\
&0 \to SC_{p_2} \to C_{p_4} \overset{p_5}{\to} C_{p_3} \to 0; && (42)\\
&\cdots . && (43)
\end{aligned}$$

Here $p_0 := \pi$ and $p_1 := p$. Each SES arises from the preceding one by applying the SES (34) to the last nontrivial arrow in it. Lemma 9.1 then gives the isomorphisms

$$\begin{aligned}
K_0(C_{p_0}) &\cong K_0(J); && (44)\\
K_0(C_{p_1}) &\cong K_0(S(A/J)) = K_1(A/J); && (45)\\
K_0(C_{p_2}) &\cong K_0(SA) = K_1(A); && (46)\\
K_0(C_{p_3}) &\cong K_0(SC_{p_0}) = K_1(C_{p_0}) \cong K_1(J); && (47)\\
K_0(C_{p_4}) &\cong K_0(SC_{p_1}) = K_1(C_{p_1}) \cong K_2(A/J); && (48)\\
K_0(C_{p_5}) &\cong K_0(SC_{p_2}) = K_1(C_{p_2}) \cong K_2(A); && (49)\\
\cdots &\cong \cdots . && (50)
\end{aligned}$$

Applying the half-exactness of K_0 to the SES (39) and using (44) and (45) then recovers the second diagram above (with $K_0(J) \cong K_0(A,A/J)$). Applying the half-exactness of K_0 to the SES

(40) and using (44) for K_1, (45), and (46) gives a commutative diagram

$$\begin{tikzcd}
K_1(C_{p_0}) \arrow[r] & K_0(C_{p_2}) \arrow[r] & K_0(C_{p_1}) \\
K_1(J) \arrow[r] \arrow[u, "\cong"] & K_1(A) \arrow[r] \arrow[u, "\cong"'] & K_1(A/J) \arrow[u, "\cong"']
\end{tikzcd}$$

This reconfirms the exactness of (36) at $K_1(A)$. Subsequently, we apply the half-exactness of K_0 to the SES (41) and use (45), (47), and (46) to find a commutative diagram

$$\begin{tikzcd}
K_1(C_{p_1}) \arrow[r] & K_0(C_{p_3}) \arrow[r] & K_0(C_{p_2}) \\
K_2(A/J) \arrow[r, "\partial_2"] \arrow[u, "\cong"] & K_1(J) \arrow[r] \arrow[u, "\cong"'] & K_1(A) \arrow[u, "\cong"']
\end{tikzcd}$$

which proves exactness of (36) at $K_1(J) \cong K_1(A, A/J)$, etcetera. This procedure eventually leads to the LES (28) and hence to (27). In general, the connecting map $\partial_n : K_n(A/J) \to K_{n-1}(J)$ is defined by taking the $(n-2)$'th suspension of the SES

$$0 \to SC_{p_n} \to C_{p_{n+2}} \to C_{p_{n+1}} \to 0$$

and applying suitable suspensions of the isomorphisms (48), (47), and (46), respectively; this yields the commutative diagram

$$\begin{tikzcd}
K_{n-1}(C_{p_n}) \arrow[r] & K_{n-2}(C_{p_{n+2}}) \arrow[r] & K_{n-2}(C_{p_{n+1}}) \\
K_n(A/J) \arrow[r, "\partial_n"] \arrow[u, "\cong"] & K_{n-1}(J) \arrow[r] \arrow[u, "\cong"'] & K_{n-1}(A) \arrow[u, "\cong"']
\end{tikzcd}$$

and proves exactness of the LES at $K_{n-1}(J)$.

Lemma 9.1 follows from the following proposition, whose proof we leave as an exercise.

Proposition 9.2 *Let* $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ *be an SES in which* B *is contractible (cf. Definition 6.1)). Then* $\iota_* : K_0(J) \to K_0(A)$ *is an isomorphism.*

To derive Lemma 9.1, take the SES $0 \to J \xrightarrow{e} C_\pi(A, A/J) \xrightarrow{q} C(A/J) \to 0$, where $e(j) := (j, 0)$ and $q(a, f) := f$. ■

It is helpful to give an explicit expression for the connecting map $\partial_1 : K_1(A/J) \to K_0(J)$ (where A is unital). The details of the various steps below are left to the reader as an exercise. As we have seen, $K_1(A/J) := K_0(S(A/J))$, and ∂_1 is defined as the composition $K_0(S(A/J)) \to K_0(C_\pi(A, A/J)) \to K_0(J)$, where the first map is induced by the map $S(A/J) \xrightarrow{i} C_\pi(A, A/J)$ with $i(f) := (0, f)$, and the second is the inverse of the isomorphism $K_0(J) \to K_0(C_\pi(A, A/J))$ induced by the map $j \mapsto (j, 0)$ from J to $C_\pi(A, A/J)$.

It is clear from its definition that, for any B, the suspension SB consists of loops in B through 0, that is, of maps $f : [0, 1] \to B$ satisfying $f(0) = f(1) = 0$. It follows that

$$\dot{S}B = \{f \in C(I, B) \mid f(0) = f(1) \in \mathbb{C}1_B\}. \tag{51}$$

Consequently,

$$K_1(B) = \{[p] - [q] \mid p, q \in C(I, P_\infty(B)) : p(0) = p(1) = q(0) = q(1) \in P_\infty(\mathbb{C}1_B)\}. \tag{52}$$

Similarly, for the unitization of the mapping cone we have

$$\dot{C}_\varphi(A,B) = \{(a,f) \in A \oplus C(I,B) \mid f(1) = \varphi(a)\}. \tag{53}$$

We now take $B = A/J$ and $\varphi = \pi$. The map $K_1(A/J) \to K_0(C_\pi(A, A/J))$ is given by

$$[p] - [q] \mapsto [(\hat{p}(1), p)] - [(\hat{q}(1), q)],$$

where p and q are as specified in (52), and $\hat{p}(1)$ is defined by replacing $1_{A/J}$ in $p(1) \in P_\infty(\mathbb{C}1_{A/J})$ by 1_A, etc. It follows that the map $K_0(J) \to K_0(C_\pi(A, A/J))$ is the restriction of the map $K_0(\dot{J}) \to K_0(\dot{C}_\pi(A, A/J))$ given by

$$[e] - [f] \mapsto [(e, \pi(e))] - [(f, \pi(f))], \tag{54}$$

where $e, f \in P_\infty(\dot{J})$. Now, it may be shown that a loop $p \in C(I, P_\infty(A/J))$ with $p(0) = p(1) \in P_\infty(\mathbb{C}1_{A/J})$ has a lift (with respect to $\pi : A \to A/J$) to a curve $\hat{p} \in C(I, P_\infty(A))$ with $\hat{p}(1)$ as defined above. In general, though, $\hat{p}(0)$ may differ from $\hat{p}(1)$. The difference is measured by the **twist**, defined as

$$\mathrm{twist}(p) := [\hat{p}(0)] - [\hat{p}(1)]. \tag{55}$$

This quantity is independent of the chosen lift $p \mapsto \hat{p}$, and a priori lies in $K_0(A)$. Now, any two lifts of a given element of $M_\infty(A/J)$ to $M_\infty(A)$ must differ by an element of $M_\infty(J)$. Consequently, since $\hat{p}(1)$ and $\hat{p}(0)$ are both lifts of $p(1) = p(0)$, and $\hat{p}(1) \in P_\infty(\mathbb{C}1_A) \subset P_\infty(\dot{J})$ (since 1_A is the unit of $\dot{J} \subset A$), it must be that $\hat{p}(0) \in P_\infty(\dot{J})$. Hence $\mathrm{twist}(p) \in K_0(\dot{J})$. Similarly, $s_\infty(\hat{p}(0)) = s_\infty(\hat{p}(1)) = \hat{p}(1)$, which implies that

$$\mathrm{twist}(p) \in K_0(J). \tag{56}$$

Under the map in (54) one has

$$\mathrm{twist}(p) - \mathrm{twist}(q) \mapsto [\hat{p}(0), p(0)] - [\hat{q}(0), q(0)].$$

Furthermore, one has $(\hat{p}(0), p(0)) \overset{h}{\sim} (\hat{p}(1), p)$ in $P_\infty(C_\pi)$. (Here $p(0)$ on the left-hand side is constant in $t \in I$, whereas p on the right-hand side is the given function on I.) It follows that the map $\partial_1 : K_1(A/J) \to K_0(J)$ is given by

$$\partial_1([p] - [q]) = \mathrm{twist}(p) - \mathrm{twist}(q). \tag{57}$$

Similarly, $\partial_1 : K_1(A/J) \to K_0(A, A/J)$ is

$$\partial_1([p] - [q]) = [\hat{p}(0) \oplus \hat{q}(1), \hat{p}(1) \oplus \hat{q}(0), \hat{p}(1) \oplus \hat{q}(1)]. \tag{58}$$

10 Another look at K_1

There exists an alternative but equivalent definition of K_1, which is quite useful in practical computations. We start with A unital,[51] and denote the set of unitary elements in $M_n(A)$ by $U_n(A)$. This is clearly a topological group. To let $n \to \infty$, we slightly modify the two equivalent procedures to define $P_\infty(A)$ described in Section 2. As to the first procedure, we now let $U_n(A)$ act on the Hilbert space H^∞ by

$$u : (v_1, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots) \mapsto (\sum_{j=1}^n u_{1j} v_j, \ldots, \sum_{j=1}^n u_{nj} v_j, v_{n+1}, v_{n+2}, \ldots);$$

in other words, we identify u with $u \oplus 1_{n+1,\infty}$, where $1_{n+1,\infty}$ is the projection on the orthogonal complement of $H^n \subset H^\infty$. One then defines $U_\infty(A) := \cup_{n=1}^\infty U_n(A)$.[52]

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[51] For nonunital A one puts $K_1'(A) := \mathfrak{K}_1'(\dot{A})$.

[52] This yields $U_\infty(A)$ as the algebraic direct limit of the groups $U_n(A)$ with respect to the maps $U_n(A) \to U_{n+1}(A)$ given by $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$.

In the second procedure, we start with the disjoint union $\coprod_{n=1}^{\infty} U_n(A)$, on which we define an equivalence relation $\sim$ by saying that $u \sim v$, for $u \in U_n(A)$ and $v \in U_m(A)$, when $v = u \oplus 1_{m-n,m}$ (in case that $m > n$) or $u = v \oplus 1_{n-m,n}$ (when $n > m$), in obvious notation. As a set, $U_\infty(A)$ is then defined as the quotient $(\coprod_{n=1}^{\infty} U_n(A))/\sim$. As a group, one defines $[u] \cdot [v] : [u' \cdot v']$, where u' and v' are representatives of u and v that both lie in a given $U_m(A)$. As a topological space, $U_\infty(A)$ is simply equipped with the quotient topology inherited from $\coprod_{n=1}^{\infty} U_n(A)$ and the norm topology on each $U_n(A)$. We then define

$$K_1'(A) := \pi_0(U_\infty(A)), \tag{59}$$

where, for any topological group G, the zeroth homotopy group is defined as $\pi_0(G) := G/G_0$, in which G_0 consists of all elements of G that are path-connected to the identity. It is almost trivial that G_0 is a normal subgroup of G, so that $\pi_0(G)$ is again a group. In the case at hand, $\pi_0(U_\infty(A))$ is commutative for the following reason.

Lemma 10.1 *One has* $\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$.

To prove this, one reduces to the case $u, v \in U_1(A)$ by replacing A by the C^*-algebra $M_n(A)$ if necessary. It is easy to see that if $u \in A$ is unitary with $\sigma(u) \neq \mathbb{T}$, then $u \in U_1(A)_0$. This yields $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix},$$

and the lemma follows. ■

Consequently, one may simply define the group operation on $K_1'(A)$ by

$$[u] \cdot [v] := [u \oplus v], \tag{60}$$

where the brackets refer to the class of $u \in U_n(A) \subset U_\infty(A)$ in the quotient $U_\infty(A)/U_\infty(A)_0$.

For example, since $U_n(\mathbb{C})$ is path-connected one has $K_1'(\mathbb{C}) = 0$. Namely, any unitary matrix is of the form $u = \exp(a)$ for some a with $a^* = -a$; the path $t \mapsto \exp(ta)$ connects u to the identity. Similarly, one finds

$$K_1'(M_n(\mathbb{C})) = K_1'(B_0(H)) = K_1'(B(H)) = 0. \tag{61}$$

In the proof of the last entry, a may be unbounded, but that doesn't change anything.

Proposition 10.2 *One has* $K_1'(A) \cong K_1(A) := K_0(SA)$*, natural in* A*.*

We start by constructing a map from $K_0(SA)$ to $K_1'(A)$. Using the description (52) of $K_0(SA)$, we have $[p] - [q] = [p] - [p(0)] - ([q] - [q(0)])$, where $p(0)$ is the constant loop with value $p(0)$, etc., and indeed $[p] - [p(0)] \in K_0(SA)$ and $[q] - [q(0)] \in K_0(SA)$ separately, so it suffices to define the image of $[p] - [p(0)]$ in $K_1'(A)$. Suppose that $p(t) \in P_n(A)$ for each $t \in I$; trivially, $p(t) \overset{\mathrm{h}}{\sim} p(1)$, so by Lemma 2.5 applied to fixed t there is a unitary $u(t) \in U_n(A)$ such that $p(t) = u(t)p(1)u(t)^*$. Since $p(1) = p(0)$, we have $p(0) = u(0)p(0)u(0)^*$, or $u(0)p(0) = p(0)u(0)$. Now, since $p(0) \in P_n(\mathbb{C}1_A \cong P_n(\mathbb{C})$, and each projection in $M_n(\mathbb{C})$ is equivalent to one of the form $\mathrm{diag}(1_k, 0_l)$ with $k + l = n$, it follows that we may assume that $p(0) = \mathrm{diag}(1_k 1_A, 0_l)$, so that $u(0) = \mathrm{diag}(v_p, w_p)$ for some $v_p \in U_k(A)$ and $w_p \in U_l(A)$. Our map $K_0(SA) \to K_1'(A)$ is now given by $[p] - [p(0)] \mapsto [v_p]$, and hence

$$[p] - [q] \mapsto [v_p] \oplus [v_q^*]. \tag{62}$$

Using (60), it is easy to show that this is a homomorphism. To show it is an isomorphism we construct an inverse $K_1'(A) \to K_0(SA)$. For any $v \in U_n(A)$, we know from Lemma 10.1 that

$\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix} \overset{\mathrm{h}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence there exists a path $u(t)_{t\in I}$ of unitaries with $u(0) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$ and $u(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The map $K_1'(A) \to K_0(SA)$ is then given by

$$[v] \mapsto \left[p : t \mapsto u(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(t)^*\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]. \tag{63}$$

We now give an explicit form of the boundary map $\partial_1' : K_1'(A/J) \to K_0(J)$. Let $u \in U_n(A/J)$. Lift u to $a \in M_n(A)$ (that is, $\pi_n(a) = u$), with $\|a\| \leq 1$. Then form

$$p = \begin{pmatrix} aa^* & a(1-a^*a)^{1/2} \\ a^*(1-aa^*)^{1/2} & 1-a^*a \end{pmatrix} \in P_{2n}(\dot{J}).$$

Finally, with $[u]$ the image of u in $U_{2n}(A/J)/U_{2n}(A/J)_0 \subset K_1'(A/J)$,

$$\partial_1'([u]) = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(J). \tag{64}$$

We leave the proof that this is the correct result as an exercise to the reader, and consider an important special case. Suppose we can lift u to a partial isometry a (i.e., a^*a and aa^* are projections). This is the case, for example, if $J = B_0(H)$, $A = B(H)$, $A/J = C(H)$. In that case, it follows (check this!) that

$$p = \begin{pmatrix} aa^* & 0 \\ 0 & 1-a^*a \end{pmatrix}.$$

If $A \subset B(H)$ and $u \in A/J$, this reads

$$p = \begin{pmatrix} \langle \mathrm{ran}(a) \rangle & 0 \\ 0 & \langle \ker(a) \rangle \end{pmatrix},$$

where $\langle K \rangle$ denotes the projection on a closed subspace $K \subset H$. Then

$$[p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 0 \\ 0 & \langle \ker(a) \rangle \end{pmatrix}\right] - \left[\begin{pmatrix} 1-\langle \mathrm{ran}(a) \rangle & 0 \\ 0 & 0 \end{pmatrix}\right] = [\langle \ker(a) \rangle] - [\langle \ker(a^*) \rangle],$$

since $1 - \langle \mathrm{ran}(a) \rangle = \langle \mathrm{ran}(a) \rangle^{\perp} = \langle \ker(a^*) \rangle$. So if $J = B_0(H)$, $A = B(H)$, $A/J = C(H)$, and $u = \pi(F)$ for some given Fredholm operator $F \in B(H)$, one obtains, using the fact that $K_0(B_0(H)) \cong \mathbb{Z}$ via $[p] - [q] \mapsto \dim(pH) - \dim(qH)$,

$$\partial_1'([\pi(F)]) = \mathrm{index}(F). \tag{65}$$

In fact, this is true for any Fredholm operator F, not merely those for which $\pi(F)$ is unitary.[53]

11 Bott periodicity

We now come to the highlight of the course, the **Bott periodicity theorem**.

Theorem 11.1 *One has isomorphisms $K_{n+2}(A) \cong K_n(A)$ for all n, natural in A.*

The naturality property means that for each morphism $\varphi : A \to C$ the diagram

$$\begin{tikzcd} K_{n+2}(A) \arrow[r, "\cong"] \arrow[d, "(S^{n+2}\varphi)_*"'] & K_n(A) \arrow[d, "(S^n\varphi)_*"] \\ K_{n+2}(C) \arrow[r, "\cong"] & K_n(C) \end{tikzcd}$$

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[53] This claim relies on the possibility of realizing $K_1(A)$ as $\pi_0(GL_\infty(A))$ instead of $\pi_0(U_\infty(A))$.

commutes.

It clearly suffices to prove this for $n = 0$, and we may also assume that A is unital. We will construct maps

$$\beta_A \quad : \quad K_0(A) \to K_2(A); \tag{66}$$

$$\alpha_A \quad : \quad K_2(A) \to K_0(A), \tag{67}$$

and show that they are inverse to each other. The construction of β_A is based on the existence of a pairing

$$\times : K_0(B) \otimes_{\mathbb{Z}} K_0(A) \to K_0(B\hat{\otimes}A), \tag{68}$$

in which we will write $x \times y$ for $\times(x, y)$. The meaning of the symbol $\hat{\otimes}$ on the right-hand side will be explained in a minute. As to the left-hand side, readers unfamiliar with the notion of a tensor product of vector spaces over $\mathbb{Z}$ (i.e., abelian groups) may simply think of $\times$ as a map from the Cartesian product $K_0(B) \times K_0(A)$ to $K_0(B\hat{\otimes}A)$ that satisfies $x \times (y + z) = (x \times y) + (x \times z)$, $(x + v) \times y = (x \times y) + (v \times y)$, and $nx \times y = x \times ny = n(x \times y)$ for all $n \in \mathbb{Z}$, etc.

The so-called **minimal tensor product** $B\hat{\otimes}A$ is the C^*-algebra defined as the completion of the algebraic tensor product $B \otimes_{\mathbb{C}} A$ in the norm $\| \sum_i a_i \otimes b_i \| := \|(\pi_1 \otimes \pi_2)(\sum_i a_i \otimes b_i)\|$, where $\pi_1(B)$ and $\pi_2(A)$ are injective representations of B and A on Hilbert spaces H_1, H_2, respectively, and the representation $(\pi_1 \otimes \pi_2)(B \otimes_{\mathbb{C}} A)$ on $H_1 \otimes H_2$ (the usual tensor product of Hilbert spaces) is defined by linear extension of $(\pi_1 \otimes \pi_2)(a \otimes b)(\psi \otimes \varphi) := \pi_1(a)\psi \otimes \pi_2(b)\varphi$. This norm turns out to be independent of the choice of π_1 and π_2 (as long as they are faithful), and since $(\pi_1 \otimes \pi_2)(B \otimes_{\mathbb{C}} A)$ is a *-algebra acting on $H_1 \otimes H_2$, it easily follows that $B\hat{\otimes}A$ is indeed a C^*-algebra.[54]

Now, for A and B unital the map (68) is defined by picking $p \in P_m(B)$, $q \in P_n(A)$, and putting $[p] \times [q] := [p \otimes q]$. Here $p \otimes q$ is a priori an element of $M_m(B) \otimes_{\mathbb{C}} M_n(A)$, which we identify with $M_{mn}(B \otimes_{\mathbb{C}} A) \subset M_{mn}(B\hat{\otimes}A)$; thus we reagard $p \otimes q$ as an element of $P_{mn}(B\hat{\otimes}A)$, so that $[p \otimes q] \in K_0(B\hat{\otimes}A)$. This extends to the desired pairing in the obvious way, that is,

$$([p] - [p']) \times ([q] - [q']) := [p \otimes q] - [p \otimes q'] - [p' \otimes q] + [p' \otimes q'].$$

If A and/or B do not have a unit, the pairing is defined in a similar way, by first unitizing A and/or B. For example, the case of relevance to us wil be the one where A is unital and B isn't. One then restricts the pairing on $K_0(\dot{B}) \otimes_{\mathbb{Z}} K_0(A)$ to its subgroup $K_0(B) \otimes_{\mathbb{Z}} K_0(A)$. One in principle has $\times : K_0(\dot{B}) \otimes_{\mathbb{Z}} K_0(A) \to K_0(\dot{B}\hat{\otimes}A)$, but $K_0(B\hat{\otimes}A)$ is canonically a subgroup of $K_0(\dot{B}\hat{\otimes}A)$,[55] and a small exercise shows that if $x \in K_0(B)$ and $y \in K_0(A)$, then $x \times y$ lies in this subgroup.

Taking $B = S^2\mathbb{C}$, so that $B\hat{\otimes}A \cong S^2A$, we therefore obtain a map $x : K_2(\mathbb{C}) \otimes_{\mathbb{Z}} K_0(A) \to K_2(A)$, since $(S^2\mathbb{C})\hat{\otimes}A \cong S^2A$. For any fixed $b \in K_2(\mathbb{C})$, this yields a map $b\times - : K_0(A) \to K_2(A)$. Using the identification $K_2(\mathbb{C}) \cong K_0(C_0(\mathbb{R}^2))$, we may take b to be the Bott element defined in (13).[56] We now put $\beta_A := b \times -$, or, explicitly,

$$\beta_A(y) = b \times y. \tag{69}$$

The reason we have used the minimal tensor product in the definition of the pairing is that it is functorial in the following sense: for any morphism $\varphi : A \to C$, one has an induced map

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[54] There is an extensive theory of tensor products of C^*-algebras; see, e.g., [11]. The point of this theory is that $B\hat{\otimes}A$ is by no means the only completion of $B \otimes_{\mathbb{C}} A$ as a C^*-algebra. There exists a family of norms on $B \otimes_{\mathbb{C}} A$ with the property that completion defines a C^*-algebra, of which $\hat{\otimes}$ is the smallest one. There is a largest one as well. For certain C^*-algebras A, called **nuclear**, there is a unique norm on $B \otimes_{\mathbb{C}} A$ or $A \otimes_{\mathbb{C}} B$ with this property, for any C^*-algebra B. Examples of nuclear C^*-algebras are commutative C^*-algebras and C^*-algebras of compact operators. Moreover, if in an SES $0 \to J \to A \to B \to 0$ the C^*-algebras J and B are nuclear, then so is A. We will use this result below. Nuclear C^*-algebras have the pleasant property of being **exact**, in the sense that if C is nuclear, then $0 \to J\hat{\otimes}C \to A\hat{\otimes}C \to B\hat{\otimes}C \to 0$ is exact for any SES as above. Similarly, if J and B (and hence A) are exact, then the latter sequence is exact for any C^*-algebra C.

[55] Writing $B\dot{\otimes}A$ for the unitization of $B\hat{\otimes}A$, one by definition has $K_0(B\hat{\otimes}A) \subset K_0(B\dot{\otimes}A)$. However, the SES $0 \to B\dot{\otimes}A \to \dot{B}\hat{\otimes}A \to A \to 0$ is split by $a \mapsto 0 + a$, so that the ensuing map $K_0(B\dot{\otimes}A) \to K_0(\dot{B}\hat{\otimes}A)$ is injective. Hence $K_0(B\hat{\otimes}A) \subset K_0(B\dot{\otimes}A) \hookrightarrow K_0(\dot{B}\hat{\otimes}A)$.

[56] It will follow from the present proof that $K_0(C_0(\mathbb{R}^2)) \cong K_0(\mathbb{C})$; with $K_0(\mathbb{C}) \cong \mathbb{Z}$ this yields $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$, under which isomorphism b is mapped to $1 \in \mathbb{Z}$. Thus b is a generator of $K_0(C_0(\mathbb{R}^2))$.

$(\mathrm{id}_B \otimes \varphi)_* : K_0(B\hat{\otimes}A) \to K_0(B\hat{\otimes}C)$. It then easily follows that the following diagram commutes:

$$\begin{tikzcd}
K_0(B) \otimes_{\mathbb{Z}} K_0(A) \arrow[r, "\times"] \arrow[d, "\mathrm{id}_{K_0(B)} \otimes \varphi_*"'] & K_0(B\hat{\otimes}A) \arrow[d, "(\mathrm{id}_B \otimes \varphi)_*"] \\
K_0(B) \otimes_{\mathbb{Z}} K_0(C) \arrow[r, "\times"] & K_0(B\hat{\otimes}C).
\end{tikzcd}$$

Taking $B = S^2\mathbb{C}$, this immediately yields commutativity of

$$\begin{tikzcd}
K_0(A) \arrow[r, "b \times -"] \arrow[d, "\varphi_*"'] & K_2(A) \arrow[d, "(S^2\varphi)_*"] \\
K_0(C) \arrow[r, "b \times -"] & K_2(C)
\end{tikzcd}$$

for any $b \in K_0(S^2\mathbb{C})$, which in turn proves naturality of the map β_A.

We are now going to construct $\alpha_A : K_2(A) \to K_0(A)$, which will be the inverse of β_A. The definition of α_A is based on the theory of **Toeplitz operators**. These may be introduced in two equivalent ways. For the first, recall the shift operator $s : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined on the canonical basis by $se_n := e_{n+1}$. The Toeplitz C^*-algebra T' is defined as $T = C^*(s)$, i.e., the smallest C^*-algebra of operators on $\ell^2(\mathbb{N})$ that contains s.

The second approach is to start with $L^2(\mathbb{T})$, which of course is canonically isomorphic to $\ell^2(\mathbb{Z})$ through a Fourier transformation. The subspace $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$ then corresponds to a subspace $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ (where H stands for 'Hardy'), which consists of all L^2-functions on $\mathbb{T}$ that have nonnegative Fourier coefficients. Let $p : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection. Any $f \in C(\mathbb{T})$ defines a Toeplitz operator $T_f := p\hat{f}p$, where $\hat{f} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the multiplication operator defined by f. We then define T as the C^*-algebra of operators on $H^2(\mathbb{T})$ generated by all T_f, $f \in C(\mathbb{T})$.[57]

Approximating arbitrary $f \in C(\mathbb{T})$ by finite Fourier series, it is not difficult to show that T and T' are in fact isomorphic. This is convenient in proving a number of the claims below, which we leave as exercises. For the proof of Bott periodicity the main point is the SES

$$0 \to B_0(H^2(\mathbb{T})) \to T \to C(\mathbb{T}) \to 0, \tag{70}$$

which may be established as follows. First, one proves that T contains all compact operators on $H^2(\mathbb{T})$. This is most easily done for T'. Let p_n the projection onto the span of the first n basis vectors in $\ell^2(\mathbb{N})$. It easily follows that $\{p_n\}_{n\in\mathbb{N}}$ is an approximate unit for $B_0(\ell^2(\mathbb{N}))$, that $p_n a p_n \in T'$ for all $a \in B(\ell^2(\mathbb{N}))$, and finally that $B_0(\ell^2(\mathbb{N})) = \lim_n p_n B_0(\ell^2(\mathbb{N}))p_n$. This proves the claim. Simply defining the second arrow in (70) as inclusion, one then has exactness at $B_0(H^2(\mathbb{T}))$.

The third arrow in (70) is defined as the restriction of the canonical projection $\pi : B(H^2(\mathbb{T})) \to C(H^2(\mathbb{T}))$ to $T \subset B(H^2(\mathbb{T}))$, where $C(H) = B(H)/B_0(H)$ is the Calkin algebra on a Hilbert space H. Exactness of (70) at T is then an immediate consequence of the injectivity of the second arrow in (70) and the trivial fact that $\ker(\pi) = B_0(H^2(\mathbb{T}))$.

To proceed, one shows that $T_f T_g - T_{fg}$ is compact for all $f, g \in C(\mathbb{T})$, for example by first noting the bound

$$\|T_f\|_{B(H^2(\mathbb{T}))} = \|p\hat{f}p\|_{B(L^2(\mathbb{T}))} \leq \|\hat{f}\|_{B(L^2(\mathbb{T}))} = \|f\|_\infty,$$

since $\|p\| = 1$. This implies that if $f_n \to f$ in sup-norm then $T_{f_n} \to T_f$ in operator norm. Thus it suffices to prove the statement for finite Fourier series, which (once again) is most easily done by Fourier transforming the situation to $\ell^2(\mathbb{N})$. It immediately follows that $[T_f, T_g]$ is compact

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[57] In [6] the Toeplitz C^*-algebra is defined as the C^*-algebra generated by all T_f and all compact operators, but the latter are automatically included in the C^*-algebra generated by all T_f.

for all $f, g \in C(\mathbb{T})$. It may then be shown that $T/B_0(H^2(\mathbb{T})) \cong C(\mathbb{T})$ by considering the map $f \mapsto T_f \mapsto \pi(T_f)$ from $C(\mathbb{T})$ to $C(H^2(\mathbb{T}))$. Although $T_- : f \mapsto T_f$ fails to be a morphism, the compactness of all $T_f T_g - T_{fg}$ clearly makes $\pi \circ T_-$ a morphism. This morphism turns out to be injective, hence is an isomorphism onto its image in $C(H^2(\mathbb{T}))$. This yields exactness of (70) at $C(\mathbb{T})$.

We now, at last, come to the definition of α_A. It so happens that the SES (70) may be tensored with any C^*-algebra A to yield another SES[58]

$$0 \to B_0(H^2(\mathbb{T}))\hat{\otimes}A \to T\hat{\otimes}A \to C(\mathbb{T})\hat{\otimes}A \to 0. \tag{71}$$

This SES has an associated boundary map $\partial_1^A : K_1(C(\mathbb{T})\hat{\otimes}A) \to K_0(B_0(H^2(\mathbb{T}))\hat{\otimes}A)$. Using $C(\mathbb{T})\hat{\otimes}A \cong C(\mathbb{T}, A)$ and the stability of K-theory, we may reinterpret this as $\partial_1^A : K_1(C(\mathbb{T}, A)) \to K_0(A)$.

Now consider the SES

$$0 \to SA \to C(\mathbb{T}, A) \to A \to 0. \tag{72}$$

This is split by $a \to 1_{\mathbb{T}}a$, so that the functorially induced K-theory maps $K_n(SA) \to K_n(C(\mathbb{T}, A))$ are injective.[59] Hence $K_1(SA) = K_2(A)$ is a subgroup of $K_1(C(\mathbb{T}, A))$, and we define $\alpha_A : K_2(A) \to K_0(A)$ as the restriction of ∂_1^A to this subgroup.

We now show that

$$\alpha_A \circ \beta_A = \mathrm{id}, \tag{73}$$

which immediately proves injectivity of β_A as well. To do so, we use the commutativity of the following diagram:

$$\begin{tikzcd}
K_2(B) \otimes_{\mathbb{Z}} K_0(A) \arrow[r, "\times"] \arrow[d, "\alpha_B \otimes \mathrm{id}_{K_0(A)}"'] & K_2(B\hat{\otimes}A) \arrow[d, "\alpha_{B\hat{\otimes}A}"] \\
K_0(B) \otimes_{\mathbb{Z}} K_0(A) \arrow[r, "\times"] & K_0(B\hat{\otimes}A)
\end{tikzcd} \tag{74}$$

This follows from a simple computation (cf. [6]). Taking $B = \mathbb{C}$ in the above diagram yields

$$\alpha_A \circ \beta_A(y) = \alpha_A(b \times y) = \alpha_{\mathbb{C}}(b) \times y.$$

If we knew that

$$\alpha_{\mathbb{C}}(b) = 1, \tag{75}$$

then we could conclude the above computation using $1 \times y = y$ (in itself an easy exercise), which finally gives (73).

Eq. (75) turns out te be a special case of the **Gohberg–Krein index theorem**: if $f \in C(\mathbb{T})$ is nowhere zero, with winding number $w(f)$,[60] then

$$\mathrm{index}(T_f) = -w(f). \tag{76}$$

To make this meaningful, note that T_f is a Fredholm operator, since $T_f T_{f^{-1}} - 1$ and $T_{f^{-1}} T_f - 1$ are compact, as we have just seen (because $T_f T_{f^{-1}} = T_{ff^{-1}}$ plus a compact operator, and $T_{ff^{-1}} = T_{1_{\mathbb{T}}} = 1$).

To prove (76), one first argues from (65) in the special case $J = B_0(H)$, $A = B(H)$, $A/J = C(H)$ with $H = H^2(\mathbb{T})$ that index(T_f) only depends on $[\pi(T_f)] \in K_1'(C(H^2(\mathbb{T})))$. Hence it suffices to prove (76) for one function f per class $[\pi(T_f)]$. Now the LES of K-theory shows that

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[58] This is because T is nuclear, hence exact; cf. footnote 54. In particular, the minimal tensor product used in (71) is the only possible one.

[59] If an SES $0 \to J \to A \to B \to 0$ is split, then one has an induced SES of K-groups, (which is split as well). Hence $K_n(J) \to K_n(A)$ is injective.

[60] The homotopy class $[f]$ of $f : \mathbb{T} \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ defines an element $[f] \in \pi_1(\mathbb{C}^*) = \pi_1(\mathbb{T})$. There is an isomorphism $\pi_1(\mathbb{T}) \cong \mathbb{Z}$ that maps $[z \mapsto z]$ to 1. The winding number of f is by definition the image of $[f]$ under the ensuing isomorphism $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$.

$\partial_1' : K_1'(C(H)) \to \mathbb{Z}$ is an isomorphism for any infinite-dimensional H, so one merely needs to prove (76) for a family of functions $f_n \in C(\mathbb{T})$, $n \in \mathbb{Z}$, for which $\mathrm{index}(T_{f_n}) = n$. To start, we take $f_{-1}(z) = z$, for which (76) follows by direct computation. Since $T_f^* = T_{\overline{f}}$ and $\mathrm{index}(F^*) = -\mathrm{index}(F)$, this proves (76) also for $f_1(z) = \overline{z}$. We now use the multiplicativity property of the index, i.e., $\mathrm{index}(FG) = \mathrm{index}(F) + \mathrm{index}(G)$ for any two Fredholm operators F, G.[61] This implies

$$\mathrm{index}(T_{fg}) = \mathrm{index}(T_f T_g) = \mathrm{index}(T_f) + \mathrm{index}(T_g),$$

as $\mathrm{index}(T_{fg})$ only depends on $\pi(T_{fg}) = \pi(T_f T_g)$. Hence (76) follows for the family $f_n : z \mapsto z^{-n}$, and we are finished.

This result has the following reinterpretation. The SES (70) leads to a boundary map $\partial_1 : K_1(C(\mathbb{T})) \to K_0(H^2(\mathbb{T}))$,[62] and therefore also to $\partial_1' : K_1'(C(\mathbb{T})) \to K_0(H^2(\mathbb{T}))$. The conditions in the derivation of (65) are met, so applying this equation to the operator $F = T_f$, so that $\pi(T_f) = f$, yields, on the identification $K_0(H^2(\mathbb{T})) \cong \mathbb{Z}$,

$$\partial_1'([f]) = w(f). \tag{77}$$

Here $f : \mathbb{T} \to \mathbb{T}$ defines $[f] \in K_1'(C(\mathbb{T}))$.[63] As a special case, we apply (77) to the function $f(z) = \overline{z}$ to find

$$\partial_1'([z \mapsto \overline{z}]) = 1. \tag{78}$$

We now show that (78) is precisely the same as

$$\partial_1'(b') = 1, \tag{79}$$

where b' is the element of $K_1'(S\mathbb{C})$ corresponding to the Bott element $b \in K_1(S\mathbb{C}) = K_2(\mathbb{C})$ under the canonical isomorphism $K_1'(S\mathbb{C}) \cong K_1(S\mathbb{C})$. Since (79) clearly implies

$$\partial_1(b) = 1, \tag{80}$$

this yields (75) by definition of $\alpha_{\mathbb{C}}$. Indeed, the SES (72) for $A = \mathbb{C}$ gives an inclusion $K_1(S\mathbb{C}) \subset K_1(C(\mathbb{T}))$, and as a matter of fact one has equality,[64] so that also $K_1'(S\mathbb{C}) = K_1'(C(\mathbb{T})) = \pi_0(U_\infty(C(\mathbb{T})))$. The function $f(z) = \overline{z}$ defines an element of $C(\mathbb{T}, U_1(\mathbb{C})) = C(\mathbb{T}, \mathbb{T})$, and hence, under the isomorphism $U_n(C(\mathbb{T})) \cong C(\mathbb{T}, U_n(\mathbb{C}))$ for $n = 1$, of $U_1(C(\mathbb{T})) \subset U_\infty(C(\mathbb{T}))$. Thus f defines a class $[f] \in K_1'(C(\mathbb{T}))$, which we denote by $[z \mapsto \overline{z}]$. Checking the details of the isomorphism $K_1'(S\mathbb{C}) \cong K_1(S\mathbb{C})$, this class turns out to be precisely b', i.e.,[65]

$$b' = [z \mapsto \overline{z}] \in K_1'(S\mathbb{C}). \tag{81}$$

Given (78), this yields (79) and hence (80), (75), and ultimately (73). Having established (73) and therefore injectivity of β_A, it remains to prove surjectivity of β_A. This is based on a trick. Introduce the map

$$\tau : C_0(\mathbb{R}^2)\hat{\otimes}A\hat{\otimes}C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2)\hat{\otimes}A\hat{\otimes}C_0(\mathbb{R}^2)$$

given by $a \otimes b \otimes c \mapsto c \otimes b \otimes a$. It follows that τ is the composition

$$\begin{aligned} C_0(\mathbb{R}^2)\hat{\otimes}A\hat{\otimes}C_0(\mathbb{R}^2) &\xrightarrow{\mathrm{id}_{C_0(\mathbb{R}^2)}\otimes\sigma} C_0(\mathbb{R}^2)\hat{\otimes}C_0(\mathbb{R}^2)\hat{\otimes}A \to C_0(\mathbb{R}^4)\hat{\otimes}A \xrightarrow{\varphi^*\otimes\mathrm{id}_A} \\ C_0(\mathbb{R}^4)\hat{\otimes}A &\to C_0(\mathbb{R}^2)\hat{\otimes}C_0(\mathbb{R}^2)\hat{\otimes}A \xrightarrow{\mathrm{id}_{C_0(\mathbb{R}^2)}\otimes\sigma^{-1}} C_0(\mathbb{R}^2)\hat{\otimes}A\hat{\otimes}C_0(\mathbb{R}^2), \end{aligned}$$

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[61] This property again follows from (65) in the special case $J = B_0(H)$, $A = B(H)$, $A/J = C(H)$, since $\mathrm{index}(FG) = \partial_1'([\pi(FG)]) = \partial_1'([\pi(F)\pi(G)]) = \partial_1'([\pi(F)] \cdot [\pi(G)]) = \partial_1'([\pi(F)]) + \partial_1'([\pi(G)]) = \mathrm{index}(F) + \mathrm{index}(G)$, since ∂_1' is a group homomorphism from $K_1'(C(H))$ to $\mathbb{Z}$.

[62] In the previous paragraph ∂_1 was the boundary map for $J = B_0(H^2(\mathbb{T}))$ and $A = B(H^2(\mathbb{T}))$, whereas here it is the boundary map for the same J but $A = T$.

[63] This actually applies to any f that vanishes nowhere, see footnote 53.

[64] For any nonunital A one has $K_1(A) = K_1(\dot{A})$.

[65] This realization of b' also yields an explicit expression of $\beta_A' : K_0(A) \to K_1'(SA)$, which corresponds to $\beta_A' : K_0(A) \to K_2(A)$ under the isomorphism $K_1'(SA) \cong K_1(SA) = K_2(A)$. The pairing $\times' : K_1'(B) \otimes_{\mathbb{Z}} K_0(A) \to K_1'(B\hat{\otimes}A)$ turns out to be given by $[u] \times [p] = [u \otimes p + 1_m \otimes (1_n - p)]$, where $u \in U_m(B)$ and $p \in P_n(A)$. It is easily checked that $u \otimes p + 1_m \otimes (1_n - p) \in U_{mn}(B \otimes_{\mathbb{C}} A)$. With (81), this gives $\beta_A'([p]) := [z \mapsto p\overline{z} + 1_n - p]$. It is clear that if $p \overset{\mathrm{h}}{\sim} q$, then the corresponding loops lie in the same component of $U_\infty(\dot{S}A)$, so that β_A' is well defined. This expression is used in other proofs of Bott periodicity, see, for example, [9].

where $\sigma : A\hat{\otimes}C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2)\hat{\otimes}A$ simply permutes the two factors, i.e., $\sigma(a \otimes b) := b \otimes a$, the second map is the canonical isomorphism[66] $C_0(\mathbb{R}^2)\hat{\otimes}C_0(\mathbb{R}^2) \cong C_0(\mathbb{R}^4)$ tensored with the identity on A, and the map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ in the third map is the linear transformation defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

One easily shows directly from this definition that

$$\tau_*(b \times y) = \sigma_*(y) \times b \tag{82}$$

for all $b \in K_0(C_0(\mathbb{R}^2))$ and $y \in K_0(A\hat{\otimes}C_0(\mathbb{R}^2))$.

Now note that the above matrix has determinant one, so it lies in the connected component of the identity of $SO(4)$. We infer that $\varphi^* \overset{\mathrm{h}}{\sim} \mathrm{id}$. Hence the induced map $(\varphi^* \otimes \mathrm{id}_A)_*$ in K-theory is the identity, so that by functoriality of K-theory one has

$$\tau_* = \mathrm{id}. \tag{83}$$

In succession using (73) with $A\hat{\otimes}C_0(\mathbb{R}^2)$ instead of A, (69), (83), (82), and the commutativity of Diagram (74) with B and A replaced by A and $C_0(\mathbb{R}^2)$, respectively, we compute

$$\begin{aligned} y &= \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)} \circ \beta_{A\hat{\otimes}C_0(\mathbb{R}^2)}(y) \\ &= \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)}(b \times y) \\ &= \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)}(\tau_*(b \times y)) \\ &= \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)}(\sigma_*(y) \times b) \\ &= \alpha_A(\sigma_*(y)) \times b. \end{aligned}$$

Hence for any $y \in K_0(A\hat{\otimes}C_0(\mathbb{R}^2))$ one has $y = z \times b$ for some $z \in K_0(A)$. For any A, B the flip map $\theta : A\hat{\otimes}B \to B\hat{\otimes}A$ induces an isomorphism $\theta_* : K_0(A\hat{\otimes}B) \to K_0(B\hat{\otimes}A)$, so that also any $\theta(y) \in K_2(A) = K_0(C_0(\mathbb{R}^2)\hat{\otimes}A)$ is of the form $\theta(y) = b \times z = \beta_A(z)$. Hence any element of $K_2(A)$ is in the image of β_A, and we have finished the proof of Bott periodicity. ■

As an important corollary of Bott periodicity, we note that the LES

$$\cdots \to K_n(J) \to K_n(A) \to K_n(B) \overset{\partial_n}{\to} K_{n-1}(J) \to K_{n-1}(A) \to K_{n-1}(B) \to \cdots$$

induced by an SES $0 \to J \to A \to B \to 0$ (cf. (28)) now collapses to a cyclic exact sequence

$$\begin{tikzcd}
K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d, "\partial_0"] \\
K_1(B) \arrow[u, "\partial_1"] & K_1(A) \arrow[l] & K_1(J). \arrow[l]
\end{tikzcd}$$

Here ∂_0 is defined as the composition of $K_0(B) \overset{\beta_B}{\to} K_2(B) \overset{\partial_2}{\to} K_1(J)$. After some computations, this leads to the following explicit expression for $\partial_0' : K_0(B) \to K_1'(J)$ (assuming A and hence B are unital). Given a class $[p] \in K_0(B)$, with $p \in P_n(B)$, find $x \in M_n(A)$ with $\pi(x) = p$ and $x^* = x$. It follows that $e^{2\pi i x}$ is unitary in $M_n(A)$, and in fact $e^{2\pi i x} \in U_n(\dot{J})$, since $\pi(e^{2\pi i x}) = e^{2\pi i p} = 1_B$. One then has

$$\partial_0'([p]) = \left[e^{2\pi i x}\right]. \tag{84}$$

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[66] For any locally compact Hausdorff spaces X, Y one has $C_0(X)\hat{\otimes}C_0(Y) \cong C_0(X \times Y)$. The isomorphism is given by continuous extension of $f \otimes g \mapsto fg$, where $fg(x, y) := f(x)g(y)$. One may check that the image of this map is indeed $C_0(X \times Y)$ by using the Stone-Weierstrass theorem.

We close these lectures by mentioning that Bott periodicity also leads to a very nice overall picture of K-theory, namely

$$K_0(A) \cong \pi_1(U_\infty(A)); \tag{85}$$

$$K_1(A) \cong \pi_0(U_\infty(A)). \tag{86}$$

The second isomorphism is just $K_1(A) \cong K_1'(A)$. For the first,[67] recall (51), which we rewrite as

$$\dot{S}A = \{f \in C(\mathbb{T}, A) \mid f(1) \in \mathbb{C}1_A\}. \tag{87}$$

It follows that

$$U_\infty(\dot{S}A)) = \{f \in C(\mathbb{T}, U_\infty(A)) \mid f(1) \in U_\infty(\mathbb{C}1_A)\},$$

where $\mathbb{T}$ is parametrized by $z \in \mathbb{C}$ with $|z| = 1$. Since $U_\infty(\mathbb{C}1_A) \cong U_\infty(\mathbb{C})$ is connected, one may always move f so as to satisfy $f(0) = e$, where e is the identity in $U_\infty(\mathbb{C}1_A)$, and therefore also in $U_\infty(A)$. Consequently, using

$$K_0(A) \cong K_2(A) = K_1(SA) \cong K_1'(SA) = K_1'(\dot{S}A) = \pi_0(U_\infty(\dot{S}A)),$$

eq. (85) follows from the definition $\pi_1(G)$.

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[67] For any topological space X with base point x_0 the first homotopy group $\pi_1(X, x_0)$ is defined as $\pi_0(L_{x_0}X)$, where $L_{x_0}X := \{f \in C(\mathbb{T}, X) \mid f(1) = x_0\}$ is the space of based loops in X. For topological groups $X = G$ one takes $x_0 = e$ and simply writes $\pi_1(G) := \pi_1(G, e) = \pi_0(L_eG)$.

Bibliography

[1] B. Blackadar, *K-theory for Operator Algebras*, 2nd ed. (Cambridge University Press, Cambridge, 1999).

[2] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, 1994).

[3] K.R. Davidson, *C^*-algebras by Example* (AMS, Providence, 1996).

[4] D.E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras* (Oxford University Press, New York, 1998).

[5] J.M. Gracia-Bondía, J.C. Várilly, and H. Figueroa, *Elements of Noncommutative Geometry* (Birkhäuser, Boston, 2001).

[6] N. Higson and J. Roe, *Analytic K-Homology* (Oxford University Press, Oxford, 2000).

[7] R.V. Kadison (eds.), *Operator Algebras and Applications, Vols. 1&2* (AMS, Providence, 1982).

[8] B.V. Rajarama Bhat, G.A. Elliott, and P.A. Fillmore (eds.), *Lectures on Operator Theory* (AMS, Providence, 1999).

[9] M. Rørdam, F. Larsen, N. Laustsen, *An Introduction to K-theory for C*-algebras* (Cambridge University Press, Cambridge, 2000).

[10] R.M. Switzer, *Algebraic Topology - Homology and Homotopy* (Springer, Berlin, 1975).

[11] N.E. Wegge-Olsen, *K-theory and C^*-algebras* (Oxford University Press, Oxford, 1993).