Exercises on K-theory (2004)

Note: the set of exercises for Lecture x will (ideally!) be discussed on the same day as Lecture x at the given date. Those who wish to hand in the (attempted) solutions, should do do no later than the Tuesday morning of the week of Lecture x.

The exercises have a rating, ranging from no star (denoting an easy application of definitions), to five stars (denoting an open research problem). To do exercises with two stars or more, it is wise to consult the literature.

Lecture 1 (05-02-04)

No exercises

Lecture 2 (12-02-04)

1. Define direct sum $\oplus$ and isomorphism $\cong$ for complex vector bundles. Show that if $E \cong E'$ and $F \cong F'$, then $(E \oplus F) \cong (E' \cong F')$.
2. Given a vector bundle $\pi : E \to Y$ over Y and a map $\varphi : X \to Y$, one defines the pullback bundle $\varphi^* E$ over X by
$$\varphi^* E = E \times_Y X := \{(v, x) \in E \times X \mid \pi(v) = \varphi(x)\},$$
with projection $(v, x) \mapsto x$. Show that $\varphi^* E$ is a vector bundle over X.
3. Show that $\varphi^*(E \oplus F) \cong (\varphi^* E) \oplus (\varphi^* F)$.
4. Show that $K^0(\mathrm{pt}) \cong \mathbb{Z}$ and $K^1(\mathrm{pt}) \cong 0$.
5. Relate as many entries as possible in Table 1 in the Lecture Notes by the properties 1-4 of topological K-theory.

Lecture 3 (26-02-04)

1. Given an injective representation of a C^*-algebra A on some Hilbert space H, represent $M_n(A)$ on H^n in the obvious way. Prove the inequalities
$$\sup_{i,j} \|a_{ij}\|_A \leq \|a\|_{B(H^n)} \leq \sum_{i,j} \|a_{ij}\|_A$$
for any $a \in M_n(A)$, and conclude that $M_n(A)$ is complete in the norm inherited from $B(H^n)$.
2. Show that the one-dimensional projections in $M_2(\mathbb{C})$ are of the form
$$p(x, y, z) := \tfrac{1}{2} \begin{pmatrix} 1 + x & y + iz \\ y - iz & 1 - x \end{pmatrix},$$
where $(x, y, z) \in \mathbb{R}^3$ satisfies $x^2 + y^2 + z^2 = 1$.
3. Prove that $\overset{\mathrm{MvN}}{\sim}$ is an equivalence relation for projections in a C^*-algebra.
4. Prove the following claims:
   (a) * Any pair of projections in a C^*-algebra satisfies $\|p - q\| \leq 1$.
   (b) If $p \perp q$ (that is, $pq = qp = 0$), then $\|p - q\| = 1$.
   (c) ** If $\|p - q\| < 1$, then $p \overset{\mathrm{h}}{\sim} q$.
5. Let p, q be projections in a unital C^*-algebra B. Show that $p \overset{\mathrm{u}}{\sim} q$ iff $p \overset{\mathrm{MvN}}{\sim} q$ and $1 - p \overset{\mathrm{MvN}}{\sim} 1 - q$.

Lecture 4 (04-03-04)

In what follows A is a unital C^*-algebra.

1. Let $p \in P_n(A)$ and $q \in P_m(A)$. Show that $p \sim q$ in $P_\infty(A)$ iff pA^n and qA^m are isomorphic as right A modules.
2. For $p, q \in P_\infty(A)$, show that $p \oplus q \sim q \oplus p$.
3. For $p, q \in P_\infty(A)$ with $pq = qp = 0$, show that $[p] + [q] = [p + q]$ in $K_0(A)$.
4. For $p, q \in P_\infty(A)$, show that $[p] = [q]$ in $K_0(A)$ iff there is $m \in \mathbb{N}$ such that $p \oplus 1_m \overset{\text{MvN}}{\sim} q \oplus 1_m$.
5. Prove that the following definitions of $K_0(A)$ define isomorphic groups:

   (a) $K_0(A)$ is the abelian group with one generator for each equivalence class $[p]$ of projections $p \in P_\infty(A)$ under the relation $\sim$, and relations $[p]+[q] = [p \oplus q]$ between these generators.

   (b) $K_0(A)$ is the Grothendieck group of the abelian semigroup $V(A)$;

   Hint: use the universal property of the Grothendieck group.

Lecture 5 (11-03-04)

1. Let $\dot{A}$ be the unitization of a C^*-algebra A without unit, with associated *-homomorphisms $\pi : \dot{A} \to \mathbb{C}$, $\lambda : \mathbb{C} \to \dot{A}$, and $s = \lambda \circ \pi : \dot{A} \to \dot{A}$.

   (a) For $p, q \in P_\infty(\dot{A})$), show that
   $$[p] - [s(p)] = [q] - [s(q)]$$
   in $K_0(A)$ iff $p \oplus 1_k \sim q \oplus 1_l$ for some k, l.

   (b) Show that $[p] - [s(p)] = 0$ iff $p \oplus 1_k \sim s(p) \oplus 1_k$ for some k.

   (c) Show that a *-homomorphism $\varphi : A \to B$, with associated morphism $\dot{\varphi} : \dot{A} \to \dot{B}$, satisfies
   $$\dot{\varphi}_*([p] - [s(p)]) = [\dot{\varphi}_\infty(p)] - [s(\dot{\varphi}_\infty(p))].$$
2. Show that a projection p lies in $B_0(H)$ iff $\dim(pH) = \mathrm{Tr}\,(p) < \infty$.
3. Use the preceding exercises and the argument in the Lecture Notes to give a complete proof that $K_0(B_0(H)) \cong \mathbb{Z}$.

Lecture 6 (18-03-04)

1. Show that $\lim_n M_n(\mathbb{C}) \cong B_0(\ell^2)$, where the morphisms $M_n(\mathbb{C}) \xrightarrow{\varphi} M_{n+1}(\mathbb{C})$ are defined by
   $$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$
2. Prove that the category of abelian groups has direct limits.
3. * Compute the direct limit of
   $$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_3} \cdots$$
   in the category of abelian groups, where $\varphi_n(k) := nk$.
4. * Compute the direct limit of
   $$\mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_2} \cdots$$
   in the category of abelian groups.
5. ** Prove the continuity of K_0 under direct limits.

Lecture 7 (25-03-04)

1. Define the arrows in the sequence

$$0 \to C_0((0,1)) \to C([0,1]) \to \mathbb{C} \oplus \mathbb{C} \to 0,$$

show that it is exact, and show that the corresponding sequence of K_0-groups is not exact.

2. Let $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ be a SES of C^*-algebras, with corresponding sequence

$$0 \to M_n(\dot{J}) \xrightarrow{\dot{\iota}_n} M_n(\dot{A}) \xrightarrow{\dot{\pi}_n} M_n(\dot{B}) \to 0$$

of matrix-valued unitizations.

   (a) Show that $a \in \mathrm{ran}(\dot{\iota}_n)$ iff $\dot{\pi}_n(a) = s_n(\dot{\pi}(a))$.

   (b) * Suppose that $[\dot{\pi}_n(p)] - [s \circ \dot{\pi}_n(p)] = 0$ for some n and some $p \in K_0(\dot{A})$. Show that there exists $p' \sim p \oplus 1_k$ for some k (cf. exercise 5.1) such that $\dot{\pi}_n(p') = s \circ \dot{\pi}_n(p')$.

3. (a) ** Let X and Y be locally compact Hausdorff spaces. Show that any morphism $\varphi : C_0(Y) \to C_0(X)$ factors as $\varphi = i \circ \psi^*$, that is, as

$$C_0(Y) \xrightarrow{\psi^*} C_0(Z) \xrightarrow{i} C_0(X),$$

where $Z \subset X$ is open, $i(f) := f$ on Z and $i(f) := 0$ on $X \backslash Z$, and $\psi : Z \to Y$ is a continuous map.

   (b) When is φ nondegenerate?

   (c) Conclude that the restriction of C^*-algebraic K-theory to the commutative case has precisely the same functoriality and excision properties as topological K-theory.

4. Prove the stability of K_0:

   (a) $K_0(M_n(A)) \cong K_0(A)$, natural in A, for any $n \in \mathbb{N}$;

   (b) $K_0(B_0(A)) \cong K_0(A)$, where $B_0(A)$ is *defined* as the direct limit of $M_n(A)$ as $n \to \infty$.

Lecture 8 (01-04-04)

1. ** Show from the definition of $K_0(A, A/J)$ (and not from its isomorphism with $K_0(J)$) that the sequence $K_0(A, A/J) \to K_0(A) \to K_0(A/J)$ is exact (i.e., in the middle).

2. Suppose that the SES $0 \to J \to A \xrightarrow{\pi} A/J \to 0$ splits. Show that $K_0(A, A/J) \cong \ker(\pi_*)$.

3. * Show that the restriction of Proposition 8.3 to the commutative case yields Proposition 8.1. (Note: this includes the burden of outlining how the groups $K^0(X, Y)$ of topological K-theory arise as a special case of the groups $K_0(A, A/J)$.)

4. ** Give a complete proof of the excision property $K_0(J) \cong K_0(A, J)$.

Lecture 9 (08-04-04)

1. Prove that the suspension functor S is exact, in that in the sense that an SES $0 \to J \to A \to A/J \to 0$ is mapped into an SES $0 \to SJ \to SA \to S(A/J) \to 0$.

2. With $K_n(A) := K_0(S^n A)$, show that K_n is functorial, continuous, homotopy invariant, and stable.

3. * Let $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ be an SES in which B is contractible. Show that $\iota_* : K_0(J) \to K_0(A)$ is an isomorphism.

   Hint: to prove injectivity of ι_*, factor ι as $\iota = \iota_3 \circ \iota_2 \circ \iota_1$, where

   $$J \xrightarrow{\iota_1} AI' \xrightarrow{\iota_2} C_\pi(A, A/J) \xrightarrow{\iota_3} A,$$

   with $AI' := \{f : [0,1] \to A \mid f(0) \in J\}$, and the morphisms ι_i are given by $\iota_1(j) := j$ (i.e., the constant function taking the value j), $\iota_2(f) := (f(1), \pi \circ f)$, and $\iota_3(a, g) := a$. This gives rise to maps

   $$K_0(J) \xrightarrow{(\iota_1)_*} K_0(AI') \xrightarrow{(\iota_2)_*} K_0(C_\pi(A, A/J)) \xrightarrow{(\iota_3)_*} K_0(A).$$

   (a) Show that ι_1 is a homotopy equivalence, so that $K_0(J) \cong K_0(AI')$.

   (b) Construct maps making the sequence $0 \to C_0([0,1), J) \to AI' \to C_\pi(A, A/J) \to 0$ exact. Show that $C_0([0,1), J)$ is contractible, and conclude that $K_0(AI') \to K_0(C_\pi)$ is injective.

   (c) Show from the SES $0 \to SB \to C_\pi \to A \to 0$ and the assumed contractibility of B that $K_0(C_\pi) \to K_0(A)$ is injective.

Lecture 10 (22-04-04)

1. The aim of this exercise is to give an explicit expression for the connecting map $\partial_1 : K_1(A/J) \to K_0(J)$ (where A is unital). Recall that $K_1(A/J) := K_0(S(A/J))$, and that ∂_1 is defined as the composition $K_0(S(A/J)) \to K_0(C_\pi(A, A/J)) \to K_0(J)$, where the first map is induced by the map $S(A/J) \xrightarrow{i} C_\pi(A, A/J)$ with $i(f) := (0, f)$, and the second is the inverse of the isomorphism $K_0(J) \to K_0(C_\pi(A, A/J))$ induced by the map $j \mapsto (j, 0)$ from J to $C_\pi(A, A/J)$.

   (a) Show that $\dot{S}(A/J) = \{f \in C(I, A/J) \mid f(0) = f(1) \in \mathbb{C}1_{A/J}\}$.

   (b) Show that
   $$K_1(A/J) = \{[p]-[q] \mid p, q \in C(I, P_\infty(A/J)) : p(0) = p(1) = q(0) = q(1) \in P_\infty(\mathbb{C}1_{A/J})\}.$$

   (c) Show that $\dot{C}_\pi(A, A/J) = \{(a, f) \in A \oplus C(I, A/J) \mid f(1) = \pi(a)\}$.

   (d) Show that the map $K_1(A/J) \to K_0(C_\pi(A, A/J))$ is given by
   $$[p] - [q] \mapsto [(\hat{p}(1), p)] - [(\hat{q}(1), q)],$$
   where $\hat{p}(1)$ is defined by replacing $1_{A/J}$ in $p(1) \in P_\infty(\mathbb{C}1_{A/J})$ by 1_A, etc.

   (e) Show that the map $K_0(J) \to K_0(C_\pi(A, A/J))$ is the restriction of the map $K_0(\dot{J}) \to K_0(\dot{C}_\pi(A, A/J))$ given by
   $$[e] - [f] \mapsto [(e, \pi(e)] - [(f, \pi(f)],$$
   where $e, f \in P_\infty(\dot{J})$.

   (f) Show that that a loop $p \in C(I, P_\infty(A/J))$ with $p(0) = p(1) \in P_\infty(\mathbb{C}1_{A/J})$ has a lift (with respect to $\pi : A \to A/J$) to a curve $\hat{p} \in C(I, P_\infty(A))$ with $\hat{p}(1)$ as defined above.

   (g) Show that
   $$\mathrm{Twist}(p) := [\hat{p}(0)] - [\hat{p}(1)] \in K_0(J)$$
   is independent of the chosen lift $p \mapsto \hat{p}$.

   (h) Show that, under the map in (e),
   $$\mathrm{Twist}(p) - \mathrm{Twist}(q) \mapsto [\hat{p}(0), p(0)] - [\hat{q}(0), q(0)].$$

   (i) Show that $(\hat{p}(0), p(0)) \overset{h}{\sim} (\hat{p}(1), p)$ in $P_\infty(C_\pi)$. (Here $p(0)$ on the left-hand side is constant in $t \in I$, whereas p on the right-hand side is the given function on I.)

   (j) Conclude that the map $\partial_1 : K_1(A/J) \to K_0(J)$ is given by
   $$\partial_1([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q).$$

   There is another exercise on the next page!

2. The aim of this exercise is to derive an explicit expression for the connecting map $\partial_1' : K_1'(A/J) \to K_0(J)$ from the previous exercise.

   (a) Let $u \in U_n(A/J)$. Show that there exists a lift u of to $a \in M_n(A)$ (that is, $\pi_n(a) = u$) such that $\|a\| \leq 1$.

   (b) Sowh that the matrix

$$w = \begin{pmatrix} a & -(1-aa^*)^{1/2} \\ (1-a^*a)^{1/2} & a^* \end{pmatrix}:$$

      i. is unitary;

      ii. satisfies

$$\pi_{2n}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix};$$

      iii. is connected to the identity by a path of unitary matrices, which we call $w(t)$, with $w(0) = w$ and $w(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (c) Show that

$$p = \begin{pmatrix} aa^* & a(1-a^*a)^{1/2} \\ a^*(1-aa^*)^{1/2} & 1-a^*a \end{pmatrix}$$

   is in $P_{2n}(\dot{J})$, and that

$$[p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

   is in $K_0(J)$.

   (d) Show that the image of $[u]$ under the isomorphism $K_1'(A/J) \xrightarrow{\cong} K_1(A/J)$ discussed in the lecture notes is $[e] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$, where e is the projection-valued loop

$$e : t \mapsto u(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(t)^*,$$

   where $u(t) := \pi(w(t))$, and the second term is constant in t.

   (e) Show that

$$\operatorname{twist}(e) = \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^*\right] = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

   (f) Conclude that

$$\partial_1'([u]) = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Lecture 11 (29-04-04)

* The aim of this exercise is to study the so-called Toeplitz C^*-algebra. Recall the shift operator $s : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined on the canonical basis by $se_n := e_{n+1}$. The Toeplitz C^*-algebra T is defined as $T = C^*(s)$, i.e., the smallest C^*-algebra of operators on $\ell^2(\mathbb{N})$ that contains s.

Another approach is to start with $L^2(\mathbb{T})$, which is canonically isomorphic to $\ell^2(\mathbb{Z})$ through a Fourier transformation. The subspace $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$ then corresponds to a subspace $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ (where H stands for 'Hardy'), which consists of all L^2-functions on $\mathbb{T}$ that have nonnegative Fourier coefficients. Let $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection. Any $f \in C(\mathbb{T})$ defines a **Toeplitz operator** $T_f := P\hat{f}$, where $\hat{f} : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the restriction to $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ of the multiplication operator f on $L^2(\mathbb{T})$. We then define T' as the smallest C^*-algebra of operators on $H^2(\mathbb{T})$ that contains all T_f, $f \in C(\mathbb{T})$.

1. Show that T and T' are isomorphic.

   Hint: find the operator on $H^2(\mathbb{T})$ that corresponds to the shift operator s under Fourier transformation, and approximate arbitrary $f \in C(\mathbb{T})$ by finite Fourier series.

2. Show that $[T_f, T_g]$ is compact for all $f, g \in C(\mathbb{T})$.

   Hint: first show that $\|T_f\| \leq \|f\|_\infty$, which implies that if $f_n \to f$ in sup-norm then $T_{f_n} \to T_f$ in operator norm. Thus it suffices to prove the statement for finite Fourier series. The problem may then be transfered to $\ell^2(\mathbb{N})$.

3. Show that T and T' contain all compact operators on $\ell^2(\mathbb{N})$ and $H^2(\mathbb{T})$, respectively.

   Hint: It is enough to do this for T. Let p_n the projection onto the span of the first n basis vectors in $\ell^2(N)$.

   (a) Show that $\{p_n\}_{n\in\mathbb{N}}$ is an approximate unit for $B_0(\ell^2(\mathbb{N}))$.

   (b) Show that $p_n a p_n \in T$ for all $a \in B(\ell^2(\mathbb{N}))$. (Do this by expressing $p_n a p_n$ as a polynomial in s and s^*.)

   (c) Use $B_0(\ell^2(\mathbb{N})) = \lim_n p_n B_0(\ell^2(\mathbb{N})) p_n$.

4. Show that $T'/B_0(H^2(\mathbb{T})) \cong C(\mathbb{T})$.

   Hint: look at the map $f \mapsto T_f \mapsto \pi(T_f)$ from $C(\mathbb{T})$ to $C(H^2(\mathbb{T}))$, where $C(H) = B(H)/B_0(H)$ is the Calkin algebra and $\pi : B(H) \to C(H)$ is the canonical projection. Show that this map is an injective morphism.

5. Prove the Gohberg–Krein index theorem: if $f \in C(\mathbb{T})$ is nowhere zero, with winding number $w(f)$,[1] then

$$\text{index}(T_f) = -w(f).$$

   Hint:

   (a) First show that T_f is a Fredholm operator, using no. 2 above.

   (b) Check the theorem for $f(z) = z^n$; this is most easily done on $\ell^2(\mathbb{N})$.

   (c) Show that the invariance of the index under norm-homotopy implies that $\text{index}(T_f)$ only depends on the homotopy class of f.

   (d) Since the previous hint has verified the claim for any homotopy class, you are finished!

   Another way to prove the theorem is to use Bott periodicity and the 6-term sequence.

[1] The homotopy class $[f]$ of $f : \mathbb{T} \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ defines an element $[f] \in \pi_1(\mathbb{C}^*) \cong \pi_1(\mathbb{T}) \cong \mathbb{Z}$. The winding number of f is the image of $[f]$ under this isomorphism, which maps $[z \mapsto z]$ to 1.