Lecture Notes on
C^*-Algebras and K-Theory

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Abstract: The aim of these lectures is to explain the basics of the theory of C^*-algebras and their associated K-groups in the light of noncommutative geometry. Part I is an introduction to C*-algebras, covering the philosophy of noncommutative geometry, Banach algebras and C*-algebras, commutative C*-algebras, the original Gelfand-Naimark duality theorem, categories and functors, natural transformations and equivalence of categories, the categorical version of the Gelfand-Naimark duality theorem, the GNS-construction, the Gelfand-Naimark representation theorem, ideals and compact operators, the structure of finite-dimensional C^*-algebras, and applications to groupoids. Part II is an introduction to the K-theory of C*-algebras, covering projections in C*-algebras, the definition of K^0, vector bundles and topological K-theory, K^0 for nonunital C*-algebras, some explicit computations, properties of C*-algebraic K-theory (half-exactness, excision, etc.), higher K-groups, the index map, and Bott periodicity.

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1 Introduction

Noncommutative geometry was created by Alain Connes around 1980 in an effort to relate the theory of algebras of operators on Hilbert spaces to differential geometry and algebraic topology. For example, one of his goals was to generalize the famous index theorems of Atiyah and Singer to the setting of foliated manifolds. More generally, noncommutative geometry turned out to be a powerful tool for the study of singular spaces. This program has eventually led to a huge edifice, described in [3]. Connes's book is very advanced, partly because the underlying theory is difficult, and also because his examples typically come from the frontiers of mathematics. Thus many parts of [3] are difficult even for experts in noncommutative geometry. As an example of a good introductory text, we mention [8].

Noncommutative geometry relies on a certain technical apparatus, which includes homological algebra and the theory of C^*-algebras, which are certain algebras of operators on a Hilbert space. These notes are concerned with the latter theory, which is an important field of mathematics and mathematical physics worth studying also independently of noncommutative geometry. To see at what point C^*-algebras enter the game, we now introduce the 'philosophy' of noncommutative geometry in a simplified way.

A modern view of mathematics, going back to Hilbert and Bourbaki, is that the subject is concerned with structures defined on sets. The three basic examples of structures are topology, algebra, and (partial) order. As you know, a topology on a set X is a choice of a collection of so-called open subsets of X, an algebraic structure on a set A in the simplest case is a map from $A \times A$ to A, whereas a partial ordering on a set describes a certain relation $\leq$ between its elements. Higher structures are defined in terms of these basic ones; for example, differential geometry starts from the assumption that certain open sets defining the topology of X are homeomorphic to $\mathbb{R}^n$ for some n. Alternatively, the theory of groups arises by putting certain axioms on an algebraic structure $A \times A \to A$, whereas an algebra over a commutative ring k in the usual sense involves two such maps on A as well as two maps $k \times k \to k$ on a second set k, along with a map $k \times A \to A$.

A key guiding thought is that whenever a set carries more than one structure, these structures should be related by certain compatibility conditions. Almost every decent definition in mathematics illustrates this point, the definition of a C^*-algebra given below being a particularly rich example.

Algebraic topology tries to describe the topology of X in terms of certain abelian groups $A(X)$, which are defined by X in a functorial way.[1] That is, in the so-called covariant case a map[2] $\varphi : X \to Y$ defines a homomorphism $\varphi_* : A(X) \to A(Y)$, whereas in the contravariant case one has a homomorphism $\varphi^* : A(Y) \to A(X)$. These are supposed to satisfy the obvious composition rules: given another map $\psi : Y \to Z$ with associated homomorphism $\psi_* : A(Y) \to A(Z)$ (in the covariant case), one should have $(\psi \circ \varphi)_* = \psi_* \circ \varphi_* : A(X) \to A(Z)$. Similarly, in the contravariant case one requires $(\psi \circ \varphi)^* = \varphi^* \circ \psi^* : A(Z) \to A(X)$, where $\psi^* : A(Z) \to A(Y)$. Finally, in both cases the identity map $\mathrm{id}_X : X \to X$ induces the identity map from $A(X)$ to $A(X)$.

Consequently, if $X \cong Y$ (i.e, X is homeomorphic to Y), then $A(X) \cong A(Y)$ (i.e., $A(X)$ and $A(Y)$ are isomorphic as groups). This conclusion is most powerfully used in a negative way: if $A(X)$ and $A(Y)$ are not isomorphic, then X and Y cannot be homeomorphic. Thus the groups $A(X)$ capture some aspects of the topology of X. In any case, as the name suggests, in algebraic topology one trades a topological structure for an algebraic one, viz. that of an abelian group.

The first step in the program of noncommutative geometry is somewhat similar: again the topology of X is used to define an abelian group $A(X)$, but this time $A(X)$ has the richer structure of a commutative algebra over $k = \mathbb{C}$. Namely, one simply takes $A(X) = C(X) := C(X, \mathbb{C})$, the collection of all continuous complex-valued functions on X. The association $X \mapsto C(X)$ is contravariant: a map $\varphi : X \to Y$ defines a homomorphism of commutative algebras $\varphi^* : C(Y) \to C(X)$ as the pullback, that is, for $f \in C(Y)$ one puts $\varphi^*(f) = f \circ \varphi \in C(X)$. The question then arises to what extent the commutative algebra $C(X)$ captures X and its topology. For example, the worst possible scenario arises when X has the coarse topology,[3] so that $C(X) \cong \mathbb{C}$, which

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[1] The $A(X)$ are typically (co)homology groups or homotopy groups (though the latter are not necessarily abelian).

[2] All maps between topological spaces are assumed to be continuous, unless the contrary is explicitly stated.

[3] I.e., the only open sets are X and $\emptyset$.

means that $C(X)$ does not contain any information whatsoever about the space X.

On the other hand, in the most favourable situation, when X is a compact Hausdorff space, it turns out that X can be reconstructed as a topological space from $C(X)$. This reconstruction is done as follows. One defines a character of a commutative algebra A (over $\mathbb{C}$) as a nonzero homomorphism $\omega : A \to \mathbb{C}$ of algebras (that is, $\omega(ab) = \omega(a)\omega(b)$), and turns the set $\Delta(A)$ of all characters of A into a topological space by saying that $\omega_n \to \omega$ when $\omega_n(a) \to \omega(a)$ in $\mathbb{C}$ for all $a \in A$. For $A = C(X)$, one has an obvious map $X \to \Delta(C(X))$, written as $x \mapsto \omega_x$, defined by $\omega_x(f) = f(x)$. When X is a compact Hausdorff space, it turns out that this map is a homeomorphism, so that $X \cong \Delta(C(X))$.

This results shows that every possible further structure on X that is defined in terms of its topology, could equally well be defined in terms of the commutative algebra $C(X)$. The best example is given by the notion of a vector bundle over X; roughly speaking, this is a collection $E = \cup_{x\in X} E_x$ of vector spaces parametrized by X, each of which isomorphic to $\mathbb{C}^n$ for some fixed n. More precisely, a vector bundle over X is an open surjection $\pi : E \to X$ with the property that each fiber $E_x := \pi^{-1}(x)$ is a vector space isomorphic to $\mathbb{C}^n$, and each $x \in X$ has a neighbourhood U_x such that $\pi^{-1}(U_x) \cong U_x \times \mathbb{C}^n$, the isomorphism $\pi^{-1}(U_x) \to U_x \times \mathbb{C}^n$ being linear on each fiber. The simplest example is $E = X \times \mathbb{C}^n$ with $\pi(x, v) = x$, but there might be other possibilities.[4]

Now, similar to the passage from X to $C(X)$, one may describe E algebraically by its space of sections

$$\Gamma(E) = \{s : X \to E \mid \pi \circ s = \mathrm{id}\}.$$

The key point is that $\Gamma(E)$ is a module over $C(X)$, the action $C(X) \times \Gamma(E) \to \Gamma(E)$ being given by $f \cdot s : x \mapsto f(x)s(x)$. Furthermore, E may be reconstructed as a vector bundle over X from $\Gamma(E)$ as a $C(X)$ module: E is isomorphic to $\cup_{x\in X} \tilde{E}_x$, where $\tilde{E}_x := \Gamma(E)/(C(X, x) \cdot \Gamma(E))$. Here $C(X, x) := \{f \in C(X) \mid f(x) = 0\}$.

Similar to the notion of a direct sum $V \oplus W$ of two vector spaces, one may form the direct sum $E \oplus F$ of two vector bundles over a given space X. This leads to the definition of $K^0(X)$, which is the abelian group with one generator for each isomorphism class $[E]$ of vector bundles over X, and relations $[E] + [F] = [E \oplus F]$. The association $X \mapsto K^0(X)$ is contravariantly functorial (like $X \mapsto C(X)$), by a construction known as a pullback, too: given a vector bundle $\pi : E \to Y$ and a map $\varphi : X \to Y$ one defines the pullback bundle φ^*E over X by

$$\varphi^*E = E \times_Y X := \{(v, x) \in E \times X \mid \pi(v) = \varphi(x)\},$$

with projection $(v, x) \mapsto x$. Thus a generator $[E]$ of $K^0(Y)$ is mapped into a generator $[\varphi^*E]$ of $K^0(X)$, and since the pullback construction preserves direct sums, this induces a homomorphism $\varphi^* : K^0(Y) \to K^0(X)$. For compact Hausdorff spaces, the so-called topological K-theory $K^0(X)$ (due to Atiyah and Hirzebruch) is an important example of the general strategy of algebraic topology.

Here again, it should be possible to define $K^0(X)$ in terms of $C(X)$. This may indeed be done. First, define $M_\infty(C(X))$ as the collection of all infinite matrices with a finite number of entries in $C(X)$. This is an algebra through the usual matrix multiplication rule, where one now multiplies functions within $C(X)$ (i.e., pointwise). Define an idempotent in $M_\infty(C(X))$ as an element e satisfying $e^2 = e$. Two idempotents can be added, like any two elements of $M_\infty(C(X))$, but in addition there is an operation of direct sum, viz.

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

We now define an equivalence relation among all idempotents in $M_\infty(C(X))$. Each element of $M_\infty(C(X))$ may also be seen as a continuous function $f : X \to M_n(\mathbb{C})$ for some n, where $M_n(\mathbb{C})$ is the space of complex $n \times n$ matrices. This induces a norm[5] on $M_\infty(C(X))$ by $\|f\| := \sup_{x\in X} \|f(x)\|_n$, where $\|\cdot\|_n$ is the usual norm on $M_n(\mathbb{C})$:

$$\|a\|_n = \sup\{\|az\|, z \in \mathbb{C}^n, \|z\| = 1\}, \tag{1}$$

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[4] Unless X is contractible.

[5] Note that $M_\infty(C(X))$ is not complete in this norm.

where $\|z\|^2 = \sum_{k=1}^n \overline{z}_k z_k$ is the usual norm on $\mathbb{C}^n$. This leads to a notion of homotopy: two idempotents e, f in $M_\infty(C(X))$ are said to be homotopic or homotopy equivalent when there is a norm-continuous path $f : [0,1] \rightarrow M_\infty(C(X))$ of idempotents (i.e., $f(t)^2 = f(t)$ for all t) with $f(0) = e$ and $f(1) = f$. Finally, we define $K_0(C(X))$ as the abelian group with one generator for each homotopy class $[e]$ of idempotents in $M_\infty(C(X))$, and relations $[e] + [f] = [e \oplus f]$. As promised, it then turns out that $K_0(C(X)) \cong K^0(X)$ in a natural way.

These results do not yet give a completely algebraic reformulation of the topological concept of a compact Hausdorff space. The second step in the program of noncommutative geometry is to give an abstract characterization of those commutative algebras that are isomorphic to $C(X)$, for some compact Hausdorff space X. This problem was solved, ahead of its time, by Gelfand and Naimark in 1943 (see [6]). They noted that the space $C(X)$ has the following additional structure beyond just being a commutative algebra over $\mathbb{C}$ (see Exercises). Firstly, it has a norm, given by

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\},$$

in which it is a Banach space. This Banach space structure of $C(X)$ is compatible with its structure as commutative algebra by the property

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty. \tag{2}$$

Secondly, $C(X)$ has an involution $f \mapsto f^*$, given by $f^*(x) = \overline{f(x)}$. This involution is related to the norm as well as to the algebraic structure by the property

$$\|f^* f\| = \|f\|^2. \tag{3}$$

We summarize these properties by saying that $C(X)$ is a commutative C^*-algebra with unit; abstractly, a commutative C^*-algebra is defined as a Banach space that at the same time is a commutative algebra with involution, such that (2) and (3) hold.

The first theorem of Gelfand and Naimark then reads as follows: *Every commutative C^*-algebra A with unit is isomorphic to $C(X)$ for some compact Hausdorff space X.* The isomorphism is constructed as explained above: one takes $X := \Delta(A)$, which turns out to be a compact Hausdorff space, and the map $A \rightarrow C(X)$ is the so-called Gelfand transform $a \mapsto \hat{a}$, where $\hat{a}(\omega) := \omega(a)$. It is typical of commutative C^*-algebras, as opposed to general commutative algebras over $\mathbb{C}$, that the Gelfand transform is an isomorphism.

This theorem can be expanded into a categorical statement: the category of compact Hausdorff spaces as objects and continuous maps as arrows is dual (or anti-equivalent) to the category with unital commutative C^*-algebras as objects and unital homomorphisms as arrows. This means, roughly speaking, the following. As we have seen, the association $C : X \mapsto C(X)$ can be contravariantly extended to $\varphi \mapsto \varphi^* : C(Y) \rightarrow C(X)$ for any $\varphi : X \rightarrow Y$. Similarly, a unital homomorphism $\psi : A \rightarrow B$ defines a map $\psi^* : \Delta(B) \rightarrow \Delta(A)$ by basically the same contravariant pullback construction: a character $\omega : B \rightarrow \mathbb{C}$ on B is mapped to the character $\psi^*\omega := \omega \circ \psi : A \rightarrow \mathbb{C}$ on A. Moreover, C and Δ are inverses to each other up to isomorphism, that is, $\Delta(C(X)) \cong X$ (as we have seen), and $C(\Delta(A)) \cong A$ for any compact Hausdorff space X and any unital commutative C^*-algebra A.

We now come to the third and decisive step of noncommutative geometry: *whenever a definition or construction works for commutative C^*-algebras, try to extend it to noncommutative C^*-algebras.* The first thing to do here is to make sense of the notion of a noncommutative C^*-algebra itself: this simply consists of omitting the word ‘commutative’ in the definition of a commutative C^*-algebra! Thus a C^*-algebra is defined as a Banach space that at the same time is an algebra with involution, such that (2) and (3) hold (see Appendix).

A key example of a noncommutative C^*-algebra is the algebra $A = M_n(\mathbb{C})$ of complex $n \times n$ matrices, with norm (1) and involution $a^*_{ij} = \overline{a_{ji}}$ (see Exercises). This is a special case of $A = B(H)$, the algebra of all bounded operators on a Hilbert space H (cf. the Appendix below), which is a C^*-algebra in the usual operator norm (83) and the usual adjoint or Hermitian conjugate as the involution, i.e., $(z, a^*w) = (az, w)$.[6] Furthermore, any subalgebra of $B(H)$ that is closed in the norm-topology and closed under Hermitian conjugation is obviously a C^*-algebra as well.

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[6] Here (,) is the inner product on H (taken linear in the second entry).

Similar to their characterization of commutative C^*-algebras, Gelfand and Naimark completely clarified the nature of general C^*-algebras. Their second theorem, contained in the same paper as their first, reads: *Every C^*-algebra A is isomorphic to a norm-closed and *-closed subalgebra of $B(H)$ for some Hilbert space H.* The proof of this theorem is based on the so-called GNS-construction (after Gelfand, Naimark, and Segal), which is of central importance to the theory of C^*-algebras, and which basically explains why C^*-algebras are naturally related to Hilbert spaces.

This construction starts with the concept of state on a C^*-algebra A, which for simplicity we assume to contain a unit. A state is a linear functional ω that is positive, in the sense that $\omega(a^*a) \geq 0$ for all $a \in A$, and normalized, in that $\omega(1) = 1$.[7] The characters of a commutative C^*-algebra are examples of states. Let us first suppose that $\omega(a^*a) > 0$ for all a, and that A has a unit. In that case, A is a pre-Hilbert space in the inner product $(a, b)_\omega := \omega(a^*b)$, which may be completed into a Hilbert space H_ω. Then A acts on H_ω by means of $\pi_\omega : A \to B(H_\omega)$, given by $\pi_\omega(a)b := ab$.[8] It is easy to see that π_ω is injective: if $\pi_\omega(a) = 0$ then, taking $b = 1$, one infers that $a = 0$ as an element of H_ω, but then $(a, a)_\omega = \omega(a^*a) = 0$, contradicting the assumption that $\omega(a^*a) > 0$. Moreover, one checks that π_ω is a homomorphism of C^*-algebras, for example,

$$\pi_\omega(a)\pi_\omega(b)c = \pi_\omega(a)bc = abc = \pi_\omega(ab)c$$

for all c, which implies $\pi_\omega(a)\pi_\omega(b) = \pi_\omega(ab)$. Thus A is isomorphic to $\pi_\omega(A) \subset B(H_\omega)$.[9] In general A may not possess such strictly positive functionals, but it always has sufficiently many states. For an arbitrary state ω the Hilbert space H_ω is constructed by first dividing A by the kernel of $(\cdot, \cdot)_\omega$, and proceeding in the same way. The representation π_ω may then fail to be injective, but by taking the direct sum of enough such representations one always arrives at an injective one.

In conclusion, the theory of (locally)[10] compact Hausdorff spaces forms the commutative corner of the world of C^*-algebras, whereas in general one deals with operators on a Hilbert space. This unifies part of point-set topology with functional analysis in a nontrivial way. Connes's main idea was to first reformulate the constructions of algebraic topology and differential geometry in terms of commutative C^*-algebras, and then to try and generalize these constructions to arbitrary C^*-algebras. Since the latter are typically noncommutative, this explains the word 'noncommutative' in noncommutative geometry.

We have, in fact, already seen one such generalization, since the definition of $K_0(C(X))$ may be generalized to any (unital) C^*-algebra practically without any modification.[11] This leads to the subject of K-theory for C^*-algebras, which is one of the main techniques in noncommutative geometry.

From a historical perspective, it is interesting to remark that the ideology of noncommutative geometry is closely related to that of quantum mechanics, and this analogy actually played an important role in shaping the field.[12] From 1900 onwards, physicists began to recognize that the classical physics of Newton, Maxwell, and Lorentz could not describe all of Nature. The fascinating era thus initiated by Planck, to be continued mainly by Einstein and Bohr, ended in 1927 with the final formulation of quantum mechanics by von Neumann.[13] This theory replaced classical mechanics, and was initially discovered in two halves.

One half of quantum theory was discovered in 1925 by Heisenberg by a principle of 'reinterpretation' (Umdeutung) of the observables of classical mechanics. The latter were typically functions

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[7] Positive functionals on a C^*-algebra are automatically continuous, with norm $\|\omega\| = \omega(1)$.

[8] This is initially defined on the dense subspace $A \subset H_\omega$, and subsequently extended by continuity.

[9] A more technical argument shows that π_ω is isometric, so that its image is closed.

[10] If on extends the first theorem of Gelfand and Naimark to commutative C^*-algebras without a unit, one ends up with locally compact spaces.

[11] Only the definition of the norm on $M_\infty(A)$ requires some discussion.

[12] More generally, the theory of operator algebras was decisively influenced by quantum mechanics and quantum field theory.

[13] John von Neumann (1903–1957) was a Hungarian prodigy; he wrote his first mathematical paper at the age of seventeen. Except for this first paper, his early work was in set theory and the foundations of mathematics. In the Fall of 1926, he moved to Göttingen to work with Hilbert, the most prominent mathematician of his time. Around 1920, Hilbert had initiated his *Beweistheory*, an approach to the axiomatization of mathematics that was doomed to fail in view of Gödel's later work. However, at the time that von Neumann arrived, Hilbert was mainly interested in quantum mechanics.

on some phase space, forming a commutative algebra under pointwise multiplication, similar to $C(X)$ above.[14] According to Heisenberg, the passage from classical mechanics to quantum mechanics is achieved by replacing such classical observables by quantum observables. Heisenberg introduced a mathematical structure for the latter, including a multiplication rule which he immediately recognized as being noncommutative in nature. His boss in Göttingen was Born, who was one of the few physicists of his day to be familiar with the concept of a matrix, and saw that Heisenberg's quantum observables were infinite matrices. Thus Heisenberg's version of quantum mechanics was initially known as 'matrix mechanics.' Born turned to his former teacher Hilbert for mathematical advice.[15] Aided by his assistants Nordheim and von Neumann, Hilbert thus ran a seminar on the mathematical structure of quantum mechanics, and the three wrote a joint paper on the subject (now obsolete). In any case, the fact that quantum mechanics involved some noncommutative algebra was seen as a crucial feature of the theory.

The second half of quantum mechanics, discovered in 1926 by Schrödinger, was called 'wave mechanics,' in which the famous symbol Ψ, denoting a 'wave function,' played an important role. The possible relationship between these alternative formulations of quantum mechanics, which at first sight looked completely different, was much discussed at the time. It was von Neumann who, in 1927 at the age of 23, found the mathematical structure of quantum mechanics, and clarified the relationship between the work of Heisenberg and that of Schrödinger.

In this process, he defined the abstract concept of a Hilbert space, which previously had only appeared in some examples. These examples went back to the work of Hilbert and his pupils in Göttingen on integral equations, spectral theory, and infinite-dimensional quadratic forms. Hilbert's famous memoirs on integral equations had appeared between 1904 and 1906. In 1908, his student E. Schmidt had defined the space ℓ^2 in the modern sense, and F. Riesz had studied the space of all continuous linear maps on ℓ^2 in 1912. Various examples of L^2-spaces had emerged around the same time. However, the abstract notion of a Hilbert space was missing until von Neumann provided it.

Von Neumann saw that Schrödinger's wave functions were unit vectors in a certain Hilbert space,[16] and that Heisenberg's quantum observables were linear operators on a different Hilbert space.[17] A unitary transformation between these spaces then provided the the mathematical equivalence between wave mechanics and matrix mechanics.[18] In a series of papers that appeared between 1927–1929, von Neumann defined Hilbert space, formulated quantum mechanics in this language, and developed the spectral theory of bounded as well as unbounded normal operators on a Hilbert space. This work culminated in his book [13], which to this day remains the definitive account of the mathematical structure of elementary quantum mechanics.[19] Since certain crucial aspects of von Neumann's formulation of quantum mechanics have found their way into the theory of C^*-algebras, it is of interest to review this formulation.

The quantum observables of a given physical system are the selfadjoint linear operators a on a Hilbert space H. The states of the system are the so-called density operators ρ on H, that is, the positive trace-class operators on H with unit trace. The expectation value of an observable a in a state ρ is given by $\mathrm{Tr}\,(\rho a)$. For example, if ρ is the (orthogonal) projection $[v]$ on a unit vector v (expressing the statement that the system is in the state v), and a is the projection $[w]$ on a unit vector w (seen as the elementary observable corresponding to the yes-no question: 'is the system in the state w?'), then the number $\mathrm{Tr}\,([v][w]) = |(v,w)|^2$ is what physicists call the transition probability between v and w. This gives the probability that a system that has been prepared in the state $[v]$ is actually found to be in the state $[w]$.

Let $B(H)$ be the space of all bounded operators on H (whose selfadjoint elements are the

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[14] In classical mechanics one usually works with smooth functions on a manifold.

[15] Hilbert had been interested in the mathematical structure of physical theories for a long time; his Sixth Problem (1900) called for the mathematical axiomatization of physics.

[16] Which in the simplest case was $L^2(\mathbb{R}^3)$.

[17] Typically ℓ^2.

[18] Similar, mathematically incomplete insights had been reached by Pauli, Schrödinger, and Dirac.

[19] Von Neumann's book was preceded by Dirac's *The Principles of Quantum Mechanics* (1930), which contains another brilliant, but mathematically questionable account of quantum mechanics in terms of linear spaces and operators.

bounded observables), with unit operator 1. We have already defined the notion of a state on a general (unital) C^*-algebra A in an abstract way; this definition is due to von Neumann in the special case $A = B(H)$. The relation with density operators is that any such operator ρ defines a state ω_ρ on $B(H)$ by $\omega_\rho(a) = \mathrm{Tr}\,(\rho a)$. What about the converse? Von Neumann implicitly assumed a certain continuity condition on states, which in modern terminology singles out the so-called normal states on $B(H)$. When H is finite-dimensional, any state is normal. In general, von Neumann showed that a state ω on $B(H)$ is normal iff it is of the form $\omega = \omega_\rho$ for some density matrix ρ on H.

Von Neumann was familiar with Minkowski's convexity theory, and he recognized that his and Schrödinger's notion of a quantum-mechanical state could be reformulated in those terms. The set of all states on $B(H)$ is obviously convex, as is its subset of all normal states on $B(H)$. Now Minkowski had defined the extreme boundary of a convex set K as the set of all $\omega \in K$ that are indecomposable, in the sense that if $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ for some $0 < \lambda < 1$ and $\omega_1, \omega_2 \in K$ then $\omega_1 = \omega_2 = \omega$. (In some examples of compact convex sets in $\mathbb{R}^n$, the extreme boundary of K coincides with its geometric boundary; cf. the closed unit ball. However, the extreme boundary of an equilateral triangle consists only of its corners. If K fails to be compact, its extreme boundary may be empty, as illustrated by the open unit ball in any dimension.)

Von Neumann saw that Schrödinger's wave functions precisely correspond to the points in the extreme boundary of the normal state space of $B(H)$. More precisely, ρ is an extreme point in this convex set iff it is a one-dimensional projection $\rho = [v]$ for some unit vector (or 'wave function') $v \in H$.[20] In view of this, points in the extreme boundary of some state space are nowadays called pure states, whereas all other states are said to be mixed.

Having defined Hilbert space and linear operators, von Neumann got bored with single operators, and turned to algebras thereof. In one of his papers on Hilbert space theory (1929), von Neumann defined a ring of operators M (nowadays called a von Neumann algebra) as a *-subalgebra of the algebra $B(H)$ of all bounded operators on a Hilbert space H that contains the unit 1 and is closed (i.e., sequentially complete) in the weak operator topology. This means that M is a unital subalgebra of $B(H)$ under operator mulitplication, that M is closed under the involution $a \mapsto a^*$, and that if for some sequence $\{a_n\} \subset M$ one has $|(v, (a_n - a)w)| \to 0$ for all $v, w \in H$, then $a \in M$. For example, $B(H)$ is itself a von Neumann algebra. When H is finite-dimensional, any direct sum of matrix algebras containing 1 is a von Neumann algebra.

Such a ring of operators is automatically closed also in the norm-topology, so that a von Neumann algebra is a C^*-algebra when it is studied with respect to this topology. However, the natural topology on a von Neumann algebra is a different one, so that the theory of such operator algebras is quite different from the theory of C^*-algebras. Both remain closely related to quantum theory. In the Bibliography we list some books on C^*-algebras and von Neumann algebras.

2 Commutative C^*-algebras

Our first goal is to prove a slight generalization of the first theorem of Gelfand and Naimark quoted above:

Theorem 2.1 *Every commutative C^*-algebra A is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. Here $C_0(X)$ is seen as a C^*-algebra under pointwise operations, the sup-norm*

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\},$$

and the involution $f^*(x) = \overline{f(x)}$.

To explain the formulation of this theorem, we should clarify two issues. Firstly:

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[20] The passage from the unit vector v to the projection $[v]$ loses the phase information in v, but this information drops out of the expectation values (v, av) defined by v anyway.

Definition 2.2 *A* ***morphism*** *between C^*-algebras A, B is a (complex-) linear map $\varphi : A \to B$ such that*

$$\begin{aligned} \varphi(ab) &= \varphi(a)\varphi(b); && (4) \\ \varphi(a^*) &= \varphi(a)^* && (5) \end{aligned}$$

for all $a, b \in A$. An ***isomorphism*** *is a bijective morphism. Two C^*-algebras are* ***isomorphic*** *when there exists an isomorphism between them.*

A morphism between C^*-algebras is usually called a ***-homomorphism**. One immediately checks that the inverse of a bijective morphism is a morphism. It is remarkable, however, that a the image of a morphism between C^*-algebras is automatically closed (and hence a C^*-algebra), while an injective morphism (and hence an isomorphism) is automatically isometric. We will prove this at a later stage; see section 10. For this reason the condition that an isomorphism be isometric is not included in the definition. As a trivial example of this definition, note that the C^*-algebra given by all 2×2 matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{C},$$

is isomorphic to $\mathbb{C}$.

Secondly, recall that a space X is called locally compact when each point has a compact neighbourhood.

Definition 2.3 *Let X be a locally compact Hausdorff space X. The space $C_0(X)$ consists of all continuous functions on X that* ***vanish at infinity*** *in the sense that for each $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for all x outside K.*

So when X is compact one trivially has $C_0(X) = C(X)$. When X is not compact, the sup-norm can still be defined, as $C_0(X) \subset C_b(X)$, and just as for $C(X)$ one easily checks that $C_0(X)$ is a commutative C^*-algebra in this norm. For $X = \mathbb{R}$, this definition just means that $f \in C_0(\mathbb{R})$ when $\lim_{x\to\pm\infty} f(x) = 0$.

The proof of Theorem 2.1 relies on the theory of commutative Banach algebras (cf. Definition 11.17). This class of algebras is of some interest in itself, as the following example shows. Consider $L^1(\mathbb{R})$, with the usual linear structure, and norm

$$\|f\|_1 := \int_{\mathbb{R}} dx\, |f(x)|.$$

The associative product $*$ defining the Banach algebra structure is convolution, that is,

$$f * g(x) := \int_{\mathbb{R}} dy\, f(x-y)g(y).$$

Using Fubini's theorem on product integrals, one easily estimates

$$\|f * g\|_1 \leq \|f\|_1\, \|g\|_1,$$

which proves that $L^1(\mathbb{R})$ is a commutative Banach algebra. Furthermore, $L^1(\mathbb{R})$ admits an involution given by $f^*(x) := \overline{f(-x)}$, but note that this norm does not satisfy (94), so that $L^1(\mathbb{R})$ is *not* a C^*-algebra with respect to these operations. There is no unit in $L^1(\mathbb{R})$; it would have been Dirac's delta-function (i.e., the measure on $\mathbb{R}$ which assigns 1 to $x = 0$ and 0 to all other x), but this does not lie in $L^1(\mathbb{R})$.

Each $p \in \mathbb{R}$ defines a character ω_p on $L^1(\mathbb{R})$ by means of the Fourier transform

$$\omega_p(f) := \hat{f}(p) = \int_{\mathbb{R}} dx\, f(x)e^{ipx};$$

since it is a well-known property of the Fourier transform that it diagonalizes the convolution product, in that $\widehat{fg} = \hat{f}\hat{g}$ (the product on the right-hand side being pointwise multiplication). In particular, the Gelfand transform $f \mapsto \hat{f}$, which we recall to be defined by $\hat{f}(\omega) := \omega(f)$, is nothing but the Fourier transform on characters of the form ω_p. In fact, all characters turn out to be of this form, so that $\Delta(L^1(\mathbb{R})) \cong \mathbb{R}$.

By the Riemann–Lebesgue lemma, one has $\hat{f} \in C_0(\mathbb{R})$. Also, as is easily shown, $\|\hat{f}\|_\infty \leq \|f\|_1$. These are typical properties of the Gelfand transform defined on commutative Banach algebras. For a C^*-algebra, the Gelfand transform should be isometric. This is not the case for the norm $\|\cdot\|_1$ in which, as we have already remarked $L^1(\mathbb{R})$ fails to be a C^*-algebra. However, we can define a new norm on $L^1(\mathbb{R})$ in such a way that the Gelfand transform is isometric by construction. In other words, we put $\|f\| := \|\hat{f}\|_\infty$. It is not difficult to show that this is indeed a norm, and that both (93) and (94) are now satisfied. An appeal to the Stone–Weierstrass theorem shows that the image of $L^1(\mathbb{R})$ under the Gelfand transform is dense in $C_0(\mathbb{R})$. Hence, if we define $C^*(\mathbb{R})$ as the closure of $L^1(\mathbb{R})$ in the new norm $\|\cdot\|$, it follows that $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$. This is consistent with Theorem 2.1, and the given argument is basically an outline of its proof in general.

Let us now turn to the proof of Theorem 2.1. The first few steps are valid for general commutative Banach algebras A. Recall

Definition 2.4 *A* ***character*** *of a commutative (Banach) algebra A (over $\mathbb{C}$) is a nonzero homomorphism $\omega : A \to \mathbb{C}$ of algebras,*[21] *in particular,*

$$\omega(ab) = \omega(a)\omega(b). \tag{6}$$

The set of all characters of A is called the ***structure space*** *$\Delta(A)$ of A.*[22] *This is a topological space in the so-called* ***Gelfand topology****, defined as the weakest topology making all functions $\omega \mapsto \omega(a)$, $a \in A$, continuous. In other words, the Gelfand topology is generated by all sets of the form $\hat{a}^{-1}(\mathcal{O})$, $a \in A$, $\mathcal{O} \subset \mathbb{C}$ open.*

It is easily shown that a neighbourhood base of the Gelfand topology is given by sets of the form

$$\mathcal{O}^{\epsilon}_{a_1,\ldots,a_n}(\omega) := \{\rho \in \Delta(A) \mid |\rho(a_i) - \omega(a_i)| < \epsilon \,\forall i\}, \tag{7}$$

where $\omega \in \Delta(A)$, $\epsilon > 0$, $n < \infty$, and $a_i \in A$.

The relevance of the following result to the proof of Theorem 2.1 should be clear.

Proposition 2.5 *If a commutative Banach algebra A has a unit 1, then $\Delta(A)$ is a compact Hausdorff space in the Gelfand topology.*[23]

The Hausdorff property is an easy exercise. The compactness property is much harder. It relies on the following lemma, which will be proved later on.

Lemma 2.6 *Let A be a commutative Banach algebra with unit. Then each $\omega \in \Delta(A)$ is continuous, with norm*

$$\|\omega\| = \omega(1) = 1, \tag{8}$$

so that $\Delta(A) \subset A^*$.

Let us note that, according to this lemma, $\Delta(A)$ may not only be equipped with the Gelfand topology, but also with the relative norm topology on A^*. The latter topology will, however, only be used in proofs. More importantly, recall from functional analysis that the **weak*-topology** on A^* is the topology of pointwise convergence, in which $\omega_n \to \omega$ when $\omega_n(a) \to \omega(a)$ in $\mathbb{C}$ for all $a \in A$. This is the weakest topology in which all functions $\rho \mapsto \rho(a)$, $a \in A$, $\rho \in A^*$, are continuous; a neighbourhood base is given by (7), where one now puts $\omega, \rho \in A^*$. It follows that the Gelfand topology on $\Delta(A)$ coincides with the relative[24] weak*-topology on A^*.

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[21] Since $\omega \in \Delta(A)$ is a homomorphism, it is linear by definition.

[22] It is not obvious that A has any character. We will show in Lemma 2.9 below that $\Delta(A)$ is not empty when A is a C^*-algebra.

[23] If A has no unit, $\Delta(A)$ will merely be locally compact, as we shall see.

[24] I.e., restricted from A^* to $\Delta(A)$.

Now, it is an easy exercise to show that if A has a unit, then $\Delta(A)$ is a *closed* subspace of the unit ball[25] of A^* in the weak*-topology (and also in the norm topology). Combining this with the basic Banach-Alaoglu theorem of functional analysis,[26] Proposition 2.5 follows. ■

Apart from such technicalities, the fundamental idea of the proof of Theorem 2.1 is very simple. As mentioned before, it consists of the Gelfand transform.

Proposition 2.7 *Let A be a commutative Banach algebra with unit, and let $\Delta(A)$ carry the Gelfand topology.*[27] *The **Gelfand transform** $a \mapsto \hat{a}$, defined by*

$$\hat{a}(\omega) := \omega(a), a \in A, \omega \in \Delta(A), \tag{9}$$

is an algebra homomorphism from A to $C(\Delta(A))$ (equipped with pointwise operations).

The homomorphism property is trivial, since

$$(\hat{a}\hat{b})(\omega) := \hat{a}(\omega)\hat{b}(\omega) = \omega(a)\omega(b) = \omega(ab),$$

by definition of a character. Hence the only nontrivial claim is that $\hat{a}$ is a continuous function on $\Delta(A)$. Recall that any Banach space A is naturally embedded in its double dual $A^{**} := (A^*)^*$, so $A \subset A^{**}$, by the very same formula $a \mapsto \hat{a}$, with $\hat{a}(\omega) := \omega(a)$. Hence the Gelfand transform is simply the restriction of this embedding to $\Delta(A) \subset A^*$.

Once again, we evoke a basic result of functional analysis without proof (cf. [15]). Namely, under this embedding, *A is precisely the subset of elements of A^{**} that are continuous functionals on A^* with respect to the weak*-topology.*[28] Of course, elements of $A \subset A^{**}$ will still be continuous when restricted to $\Delta(A) \subset A^*$, so Proposition 2.7 follows. ■

We can sharpen Proposition 2.7 when A is a C^*-algebra.

Lemma 2.8 *Let A be a commutative C^*-algebra with unit, and equip $C(\Delta(A))$ with the usual structure of $C(X)$ as a commutative C^*-algebra, $X = \Delta(A)$. Then the Gelfand transform is a *-homomorphism (i.e., a morphism of C^*-algebras).*

In view of Proposition 2.7 and Definition 2.2, we only need to show that $\widehat{a^*} = \overline{\hat{a}}$, and for this purpose it suffices that $\omega(a) \in \mathbb{R}$ whenever $a^* = a$, $a \in A$. To do so, we suppose that $\omega(a) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. Note that $b := a - \alpha 1$ is selfadjoint. Since $\omega(1) = 1$, one has $\omega(b) = i\beta$. Hence for $t \in \mathbb{R}$ one computes

$$|\omega(b + it1)|^2 = \beta^2 + 2t\beta + t^2.$$

On the other hand, using $\|\omega\| = 1$ (so that $|\omega(c)| \leq \|c\|$) and (94), we estimate

$$|\omega(b + it1)|^2 \leq \|b + it1\|^2 = \|(b + it1)^*(b + it1)\| = \|b^2 + t^2\| \leq \|b\|^2 + t^2.$$

Combining the displayed equations yields $\beta^2 + t\beta \leq \|b\|^2$ for all $t \in \mathbb{R}$. This is impossible unless $\beta = 0$, so that $\omega(a)$ is real when $a = a^*$. Consequently, by (9) the function $\hat{A}$ is real-valued. ■

To prove Theorem 2.1, it remains to be shown that the Gelfand transform is a bijection on a commutative C^*-algebra (given this, it is trivial that its inverse is again a *-homomorphism). We will prove both injectivity and surjectivity from the most difficult part of the entire proof:[29]

Lemma 2.9 *Let A be a commutative C^*-algebra with unit. Then the Gelfand transform is isometric, that is,*

$$\|\hat{a}\|_\infty = \sup_{\omega \in \Delta(A)} |\omega(a)| = \|a\|.$$

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[25] The unit ball is the set of all functionals $\varphi \in A^*$ for which $\|\varphi\| \leq 1$.

[26] This states that for any Banach space A the unit ball in A^* is compact in the weak*-topology (but not in the norm topology, except when A is finite-dimensional!). See, e.g., [15].

[27] Clearly, we can now say that this is the weakest topology making all $\hat{a}$, $a \in A$, continuous.

[28] Whereas A^{**} by definition consists of all functionals on on A^* that are continuous with respect to the norm-topology on A^*.

[29] Don't get confused: an isomorphism between two C^*-algebras is automatically isometric, but in the present proof we prove the bijectivity property of an isomorphism by showing that the Gelfand transform is isometric.

Given this lemma, injectivity is obvious. Surjectivity follows from the Stone-Weierstrass theorem, which we now recall.[30]

Let X be a compact Hausdorff space. A subalgebra $\hat{A}$ of $C(X)$ (regarded as a commutative C^*-algebra in the usual way) that

1. separates points on X (i.e., if $x \neq y$ there is $f \in \hat{A}$ such that $f(x) \neq f(y)$);
2. is closed under complex conjugation (i.e., if $f \in \hat{A}$ then $\overline{f} \in \hat{A}$);
3. contains the unit function 1_X (where $1_X(x) = 1$ for all $x \in X$);

is dense in $C(X)$ in the sup-norm.

Applying this to $X = C(\Delta)$ and $\hat{A}$ as the image of the Gelfand transform on A, it is trivial that property 1 holds; property 2 follows from Lemma 2.8, and property 3 is a consequence of (8). Furthermore, if the Gelfand transform is isometric, $\hat{A}$ must be closed in $C(\Delta(A))$, so that it coincides with $C(\Delta(A))$ by the Stone-Weierstrass theorem. This proves surjectivity.

To prove Lemmas 2.6 and 2.9, and also as the technical basis for much of the theory of general C^*-algebras, we need to develop some spectral theory. The basics are valid for general Banach algebras, so we briefly return to this setting, specializing to the C^*-case as appropriate.

3 Spectral theory for Banach algebras

Until further notice, let A be a Banach algebra with unit.

Definition 3.1 *The* ***resolvent*** $\rho(a)$ *of* $a \in A$ *is the set of all* $z \in \mathbb{C}$ *for which* $a - z1$ *has a (two-sided) inverse in* A.

The ***spectrum*** $\sigma(a)$ *of* $a \in A$ *is the complement of* $\rho(a)$ *in* $\mathbb{C}$*; in other words,* $\sigma(a)$ *is the set of all* $z \in \mathbb{C}$ *for which* $a - z1$ *has no (two-sided) inverse in* A.

When A is the algebra of $n \times n$ matrices, the spectrum is just the set of eigenvalues. For $A = B(H)$, Definition 3.1 reproduces the usual notion of the spectrum of an operator on a Hilbert space. The third example, which we present as a lemma, is an easy exercise.

Lemma 3.2 *For* $f \in A = C(X)$*, one has* $\sigma(f) = \{f(x), x \in X\}$.

These examples have a physical interpretation (also cf. the Introduction). In classical physics the observables are real-valued functions on some phase space X, so the spectrum of an observable is just the set of values it can assume. In quantum physics one assumes that the observables are selfadjoint operators on a Hilbert space H, and one assumes that the values such an observable can assume lie in its spectrum. Hence Definition 3.1 unifies these to cases to some extent.

Theorem 3.3 *The spectrum* $\sigma(a)$ *of any element* $a \in A$ *is*

1. *contained in the set* $\{z \in \mathbb{C} \mid |z| \leq \|a\|\}$*;*
2. *compact;*
3. *not empty.*

The proof uses two lemmas. We assume that A is unital.

Lemma 3.4 *When* $\|a\| < 1$ *the sum* $\sum_{k=0}^{n} a^k$ *converges to* $(1-a)^{-1}$.

Hence $(a - z1)^{-1}$ *always exists when* $|z| > \|a\|$.

---

[30] This theorem is contained in elementary analysis courses for $X = [0,1]$; for the general case see [15].

We first show that the sum is a Cauchy sequence. Indeed, for $n > m$ one has

$$\| \sum_{k=0}^{n} a^k - \sum_{k=0}^{m} a^k \| = \| \sum_{k=m+1}^{n} a^k \| \leq \sum_{k=m+1}^{n} \|a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k.$$

For $n, m \to \infty$ this goes to 0 by the theory of the geometric series. Since A is complete, the Cauchy sequence $\sum_{k=0}^{n} a^k$ converges for $n \to \infty$. Now compute

$$\sum_{k=0}^{n} a^k(1-a) = \sum_{k=0}^{n} (a^k - a^{k+1}) = 1 - a^{n+1}.$$

Hence

$$\|1 - \sum_{k=0}^{n} a^k(1-a)\| = \|a^{n+1}\| \leq \|a\|^{n+1},$$

which $\to 0$ for $n \to \infty$, as $\|a\| < 1$ by assumption. Thus

$$\lim_{n\to\infty} \sum_{k=0}^{n} a^k(1-a) = 1.$$

By a similar argument,

$$\lim_{n\to\infty} (1-a) \sum_{k=0}^{n} a^k = 1.$$

so that, by continuity of multiplication in a Banach algebra, one finally has

$$\lim_{n\to\infty} \sum_{k=0}^{n} a^k = (1-a)^{-1}.$$

The second claim of the lemma follows because $(a-z)^{-1} = -z^{-1}(1-a/z)^{-1}$, which exists because $\|a/z\| < 1$ when $|z| > \|a\|$. ■

To prove that $\sigma(a)$ is compact, it remains to be shown that it is closed.

Lemma 3.5 *The set*

$$G(A) := \{a \in A | \, a^{-1} \text{ exists}\}$$

of invertible elements in A is open in A.

Given $a \in G(A)$, take a $b \in A$ for which $\|b\| < \|a^{-1}\|^{-1}$. By (88) this implies

$$\|a^{-1}b\| \leq \|a^{-1}\| \, \|b\| < 1. \tag{10}$$

Hence $a+b = a(1+a^{-1}b)$ has an inverse, namely $(1+a^{-1}b)^{-1}a^{-1}$, which exists by (10) and Lemma 3.4. It follows that all $c \in A$ for which $\|a-c\| < \epsilon$ lie in $G(A)$, for $\epsilon \leq \|a^{-1}\|^{-1}$. ■

To resume the proof of Theorem 3.3, given $a \in A$ we now define a function $f : \mathbb{C} \to A$ by $f(z) := z - a$. Since $\|f(z+\delta) - f(z)\| = \delta$, we see that f is continuous (take $\delta = \epsilon$ in the definition of continuity). Because $G(A)$ is open in A by Lemma 3.5, it follows from the topological definition of a continuous function that $f^{-1}(G(A))$ is open in $\mathbb{C}$. But $f^{-1}(G(A))$ is the set of all $z \in \mathbb{C}$ where $z-a$ has an inverse, so that $f^{-1}(G(A)) = \rho(a)$. This set being open, its complement $\sigma(a)$ is closed.

Finally, define $g : \rho(a) \to A$ by $g(z) := (z-a)^{-1}$. For fixed $z_0 \in \rho(a)$, choose $z \in \mathbb{C}$ such that $|z - z_0| < \|(a-z_0)^{-1}\|^{-1}$. From the proof of Lemma 3.5, with a replaced by $a - z_0$ and c replaced by $a - z$, we see that $z \in \rho(a)$, as $\|a - z_0 - (a-z)\| = |z - z_0|$. Moreover, the power series

$$\frac{1}{z_0 - a} \sum_{k=0}^{n} \left(\frac{z_0 - z}{z_0 - a} \right)^k$$

converges for $n \to \infty$ by Lemma 3.4, because

$$\|(z_0 - z)(z_0 - a)^{-1}\| = |z_0 - z| \, \|(z_0 - a)^{-1}\| < 1.$$

By Lemma 3.4, the limit $n \to \infty$ of this power series is

$$\frac{1}{z_0 - a} \sum_{k=0}^{\infty} \left(\frac{z_0 - z}{z_0 - a} \right)^k = \frac{1}{z_0 - a} \left(1 - \left(\frac{z_0 - z}{z_0 - a} \right) \right)^{-1} = \frac{1}{z - a} = g(z).$$

Hence

$$g(z) = \sum_{k=0}^{\infty} (z_0 - z)^k (z_0 - a)^{-k-1} \tag{11}$$

is a norm-convergent power series in z. For $z \neq 0$ we write $\|g(z)\| = |z|^{-1} \|(1 - a/z)^{-1}\|$ and observe that $\lim_{z \to \infty} 1 - a/z = 1$, since $\lim_{z \to \infty} \|a/z\| = 0$ by Definition 11.9.3. Hence $\lim_{z \to \infty} (1 - a/z)^{-1} = 1$, and

$$\lim_{z \to \infty} \|g(z)\| = 0. \tag{12}$$

Let $\rho \in A^*$ be a functional on A; since ρ is bounded, (11) implies that the function $g_\rho : z \to \rho(g(z))$ is given by a convergent power series, and (12) implies that

$$\lim_{z \to \infty} g_\rho(z) = 0. \tag{13}$$

Now suppose that $\sigma(a) = \emptyset$, so that $\rho(a) = \mathbb{C}$. The function g, and hence g_ρ, is then defined on $\mathbb{C}$, where it is analytic and vanishes at infinity. In particular, g_ρ is bounded, so that by Liouville's theorem it must be constant. By (13) this constant is zero, so that $g = 0$ by Corollary 11.16. This is absurd, so that $\rho(a) \neq \mathbb{C}$ hence $\sigma(a) \neq \emptyset$. This finishes the proof of Theorem 3.3. ■

For one thing, we are now in a position to prove Lemma 2.6. Indeed, we now know that $(a - z)^{-1}$ exists when $|z| > \|a\|$. Since a character $\omega \in \Delta(A)$ is a homomorphism of algebras, also $\omega(a - z)$ must be invertible, so that $\omega(a - z) \neq 0$, i.e., $\omega(a) \neq z$. It follows that $|\omega(a)| \leq \|a\|$, so that $\|\omega\| \leq 1$. But the property $\omega(1) = 1$ was an easy exercise, so that finally $\|\omega\| = 1$. ■

We now return to Theorem 2.1. Define the **spectral radius** $r(a)$ of $a \in A$ by

$$r(a) := \sup\{|z|, z \in \sigma(a)\}. \tag{14}$$

From Theorem 3.3.1 one immediately infers

$$r(a) \leq \|a\|. \tag{15}$$

The proof of Theorem 2.1 relies on two properties, which are valid in any commutative Banach algebra A with unit.

Proposition 3.6 *For each $a \in A$ one has*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}; \tag{16}$$

$$\sigma(a) = \sigma(\hat{a}), \tag{17}$$

where (cf. Lemma 3.2) $\sigma(\hat{a}) = \{\omega(a) \mid \omega \in \Delta(A)\}$.

Before proving this proposition, we show how it implies Lemma 2.9, and therefore Theorem 2.1, at least in the unital case. Let A be a commutative C^*-algebra with unit, and first take $a \in A$ sucht that $a^* = a$. Then (94) implies $\|a^2\| = \|a\|^2$, so $\|a\| = r(a)$ from (16). On the other hand, (17) implies $r(a) = \|\hat{a}\|_\infty$, so that $\|a\| = \|\hat{a}\|_\infty$. It is a very easy exercise to show that this implies the same equality for general $a \in A$. This completes the proof of Theorem 2.1 in the unital case, up to the proof of Proposition 3.6.

The proof of (16) is relatively easy (at least compared to the second part!). By Lemma 3.4, for $|z| > \|a\|$ the function g in the proof of Lemma 3.5 has the norm-convergent power series expansion

$$g(z) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k . \tag{18}$$

On the other hand, we have seen that for any $z \in \rho(a)$ one may find a $z_0 \in \rho(a)$ such that the power series (11) converges. If $|z| > r(a)$ then $z \in \rho(a)$, so (11) converges for $|z| > r(a)$. At this point the proof relies on the theory of analytic functions with values in a Banach space, which says that, accordingly, (18) is norm-convergent for $|z| > r(a)$, uniformly in z. Comparing with (15), this sharpens what we know from Lemma 3.4. The same theory says that (18) cannot norm-converge uniformly in z unless $\|a^n\|/|z|^n < 1$ for large enough n. This is true for all z for which $|z| > r(a)$, so that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a). \tag{19}$$

To derive a second inequality we use the following **polynomial spectral mapping property**.

Lemma 3.7 *For a polynomial p on $\mathbb{C}$, define $p(\sigma(a))$ as $\{p(z) | z \in \sigma(a)\}$. Then*

$$p(\sigma(a)) = \sigma(p(a)). \tag{20}$$

To prove this equality, choose $z, \alpha \in \mathbb{C}$ and compare the factorizations

$$\begin{aligned} p(z) - \alpha &= c\prod_{i=1}^{n}(z - \beta_i(\alpha)); \\ p(a) - \alpha 1 &= c\prod_{i=1}^{n}(a - \beta_i(\alpha)1). \end{aligned} \tag{21}$$

Here the coefficients c and $\beta_i(\alpha)$ are determined by p and α. When $\alpha \in \rho(p(a))$ then $p(a) - \alpha 1$ is invertible, which implies that all $a - \beta_i(\alpha)1$ must be invertible. Hence $\alpha \in \sigma(p(a))$ implies that at least one of the $a - \beta_i(\alpha)1$ is not invertible, so that $\beta_i(\alpha) \in \sigma(a)$ for at least one i. Hence $\alpha \in p(\beta_i(\alpha)) - \alpha = 0$, i.e., $\alpha \in p(\sigma(a))$. This proves the inclusion $\sigma(p(a)) \subseteq p(\sigma(a))$.

Conversely, when $\alpha \in p(\sigma(a))$ then $\alpha = p(z)$ for some $z \in \sigma(a)$, so that for some i one must have $\beta_i(\alpha) = z$ for this particular z. Hence $\beta_i(\alpha) \in \sigma(a)$, so that $a - \beta_i(\alpha)$ is not invertible, implying that $p(a) - \alpha 1$ is not invertible, so that $\in \sigma(p(a))$. This shows that $p(\sigma(a)) \subseteq \sigma(p(a))$, and (20) follows. ■

To conclude the proof of (16), we note that since $\sigma(a)$ is closed there is an $\alpha \in \sigma(a)$ for which $|\alpha| = r(a)$. Since $\alpha^n \in \sigma(a^n)$ by Lemma 3.7, one has $|\alpha^n| \leq \|a^n\|$ by (15). Hence $\|a^n\|^{1/n} \geq |\alpha| = r(a)$. Combining this with (19) yields

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a) \leq \|a^n\|^{1/n}.$$

Hence the limit must exist, and

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n} = r(a).$$ ■

To prove (17), we need to develop the theory of ideals in commutative C^*-algebras. We will do this directly also for the general noncommutative case, specializing to the commutative situation as appropriate.

4 Ideals in Banach algebras and C^*-algebras

We proceed in two stages, first discussing ideals in general Banach algebras, and subsequently passing to the special case of C^*-algebras.

Definition 4.1 *Let A be a Banach algebra.*

*A **left ideal (right ideal)** in A is a closed linear subalgebra J for which $a \in J$ implies $ba \in J$ ($ab \in J$) for all $b \in A$.*

*An **ideal** in A is a subspace that is both a left and a right ideal (i.e., a two-sided ideal).*

*A **maximal ideal** is a proper ideal*[31] *$J \subset A$ for which $J \subset \tilde{J} \subset A$ for some proper ideal $\tilde{J}$ implies $\tilde{J} = J$.*[32]

Note that an ideal is by definition two-sided. Compared to the purely algebraic definition, an ideal in a Banach algebra has to be closed by definition. This turns out to make an enormous difference, restricting the possible ideals. In particular, an ideal in a Banach algebra is itself a Banach algebra. An ideal J that contains an invertible element a must coincide with A, since $a^{-1}a = 1$ must lie in J, so that all $B = B1$ must lie in J. This shows the need for considering Banach algebras with and without unit; it is usually harmless to add a unit to a Banach algebra A, but a given proper ideal $J \neq A$ does not contain 1, and one cannot add 1 to J without ruining the property that it is a proper ideal.

Definition 4.1 literally applies to C^*-algebras. One would expect that an ideal in a C^*-algebra is required to be selfadjoint, but this is turns out to be a consequence of the given definition.

Proposition 4.2 *Let J be an ideal in a C^*-algebra A. If $a \in J$ then $a^* \in J$; in other words, every ideal in a C^*-algebra is selfadjoint.*

The proof of this theorem is surprisingly difficult, using techniques we have not yet discussed. Hence we postpone it until section 10.

In the commutative case, left and right ideals are the same as ideals. For example, if $A = C(X)$ for a compact space X, then each closed subspace $Y \subset X$ defines an ideal

$$C(X,Y) := \{f \in C(X) \mid f(x) = 0 \; \forall x \in Y\}. \tag{22}$$

Note that $C(X,Y)$ is indeed closed by definition of the sup-norm, and that $C(X,Y) \cong C_0(X\backslash Y)$. In fact, it is an easy exercise that all ideals in $C(X)$ are of this form. It is not necessary to assume that Y is closed, but this assumption entails no loss of generality, since $C(X,Y) = C(X,\overline{Y})$, where $\overline{Y}$ is the closure of Y. We will see shortly that $C(X,Y)$ is maximal iff Y is a point, and that all maximal ideal in $C(X)$ are of this form.

Proposition 4.3 *If J is an ideal in a Banach algebra A then the quotient A/J is a Banach algebra in the norm*

$$\|\tau(a)\| := \inf_{j \in J} \|a + j\| \tag{23}$$

and the multiplication

$$\tau(a)\tau(b) := \tau(ab). \tag{24}$$

Here $\tau : A \to A/J$ is the canonical projection. If A is unital and J is proper, then A/J is unital, with unit $\tau(1)$.

We omit the standard proof that A/J is a Banach space in the norm (23). As far as the Banach algebra structure is concerned, first note that (24) is well defined: when $j_1, j_2 \in J$ one has

$$\tau(a + j_1)\tau(b + j_2) = \tau(ab + aj_2 + j_1 b + j_1 j_2) = \tau(ab) = \tau(a)\tau(b),$$

since $aj_2 + j_1 b + j_1 j_2 \in J$ by definition of an ideal, and $\tau(j) = 0$ for all $j \in J$. To prove (88), observe that, by definition of the infimum, for given $a \in A$, for each $\epsilon > 0$ there exists a $j \in J$ such that

$$\|\tau(a)\| + \epsilon \geq \|a + j\|. \tag{25}$$

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[31] A **proper ideal** of A is an ideal that is neither A nor 0.

[32] Note that a maximal ideal $J \subset A$ defined in the purely algebraic sense is automatically closed. To prove this, note that the closure $\overline{J}$ of J cannot be A, since J does not contain any invertible element of A (otherwise it would coincide with A), and the set $G(A)$ of all invertible elements in A is open (see Lemma 3.5). Since $J \subseteq \overline{J} \subset A$ and J is maximal, $\overline{J} = J$.

For if such a j would not exist, then $\|\tau(a)\| \leq \|a+j\| - \epsilon$ for all $j \in J$, violating (23). On the other hand, for any $j \in J$ it is clear from (23) that

$$\|\tau(a)\| = \|\tau(a+j)\| \leq \|a+j\|. \tag{26}$$

For $a, b \in A$ choose $\epsilon > 0$ and $j_1, j_2 \in J$ such that (25) holds for a, b, and estimate

$$\begin{aligned} \|\tau(a)\tau(b)\| &= \|\tau(a+j_1)\tau(b+j_2)\| = \|\tau((a+j_1)(b+j_2))\| \\ &\leq \|(a+j_1)(b+j_2)\| \leq \|a+j_1\|\,\|b+j_2\| \\ &\leq (\|\tau(a)\|+\epsilon)(\|\tau(b)\|+\epsilon). \end{aligned} \tag{27}$$

Letting $\epsilon \rightarrow 0$ yields

$$\|\tau(a)\tau(b)\| \leq \|\tau(a)\|\,\|\tau(b)\|. \tag{28}$$

When A has a unit, it is obvious from (24) that $\tau(1)$ is a unit in A/J. By (26) with $a=1$ and $j=0$ one has $\|\tau(1)\| \leq \|1\| = 1$. On the other hand, from (28) and (24) with $b=1$ and $a \in A\backslash J$ one derives $\|\tau(1)\| \geq 1$. Hence $\|\tau(1)\| = 1$. ■

Like Proposition 4.2, the proof of the following C^*-algebraic analogue of Proposition 4.3 is difficult, so we postpone it until section 10 as well.

Proposition 4.4 *Let J be an ideal in a C^*-algebra A. The quotient A/J is a C^*-algebra with respect to the norm (23), the multiplication (24), and the involution*

$$\tau(a)^* = \tau(a^*). \tag{29}$$

In the commutative case, the basic example (with X and Y compact, as above), is

$$C(X)/C(X,Y) \cong C(Y), \tag{30}$$

as two elements f, g of $C(X)$ are identified in $C(X)/C(X,Y)$ when $f-g \in C(X,Y)$, i.e., when they coincide on Y. If one looks at $C(X,Y)$ as the kernel of the restriction map $r_Y : C(X) \rightarrow C(Y)$, then elementary linear analysis shows that $\mathrm{ran}(r_Y) \cong C(X)/\ker(r_Y)$, which is just (30).

We now return to the proof of Theorem 2.1, which was complete up to the proof of (17). This hinges on the following key result.

Theorem 4.5 *Let A be a unital commutative Banach algebra with structure space $\Delta(A)$.*

1. *If $\omega \in \Delta(A)$ then $J_\omega := \ker(\omega)$ is a maximal ideal in A;*
2. *Every maximal ideal is of this form;*
3. *$\omega_1 = \omega_2$ iff $J_{\omega_1} = J_{\omega_2}$.*

Hence there is a bijective correspondence between $\Delta(A)$ and the set of all maximal ideals in A.

The first and the third claim are easy exercises. Before demonstrating the second claim, let us show how Theorem 4.5 implies (17). Namely, precisely the difficult second claim enters the proof of a crucial lemma.

Lemma 4.6 *One has $\omega(a) \neq 0$ for all $\omega \in \Delta(A)$ iff $a \in G(A)$ (i.e., a is invertible).*

If $a \in G(A)$ then $\omega(a)\omega(a^{-1}) = \omega(aa^{-1}) = \omega(1) = 1$, so that $\omega(a) \neq 0$. Conversely, assume there exists $a \notin G(A)$ such that $\omega(a) \neq 0$ for all $\omega \in \Delta(A)$. The assumption $a \notin G(A)$ implies that the ideal

$$J_a = \{ab \mid b \in A\}$$

generated by a does not contain 1, so that $J_a \neq A$. By abstract nonsense[33] there must be a maximal ideal $J \subset A$ containing J_a. By Theorem 4.5.2, it must be that $J = J_\omega$ for some $\omega \in \Delta(A)$, so that $J_a \subseteq \ker(\omega)$. Hence $\omega(ab) = \omega(a)\omega(b) = 0$ for all b, and since $\omega(a) \neq 0$ by assumption, it follows that $\omega(b) = 0$ for all $b \in A$, hence $\omega = 0$. This contradicts the definition of $\Delta(A)$. ■

Eq. (17) easily follows from this lemma: it implies that $z \in \rho(a)$, that is, $a - z \in G(A)$, iff $\omega(a - z) \neq 0$ for all $\omega \in \Delta(A)$, so that $z \in \rho(a)$ iff $\omega(a) \neq z$ for all ω. Taking the complement $\sigma(a) = \mathbb{C}\backslash\rho(a)$, we obtain

$$\sigma(a) = \{\omega(a) \mid \omega \in \Delta(A)\}.$$

But $\omega(a) = \hat{a}(\omega)$ by definition of the Gelfand transform, so that (17) follows from Lemma 3.2. This concludes the proof of Theorem 2.1 in the unital case.

We now prove Theorem 4.5.2. Let J be a maximal ideal. Since $J \neq A$, there is a nonzero $b \in A$ which is not in J. Form

$$J_b := \{ba + j|, a \in A, j \in J\}.$$

Since A is commutative, J_b is an ideal. Taking $a = 0$ we see $J \subseteq J_b$. Taking $a = 1$ and $j = 0$ we see that $b \in J_b$, so that $J_b \neq J$. Hence $J_b = A$, as J is maximal. In particular, $1 \in J_b$, so that $1 = ba + j$ for some $a \in A, j \in J$. Applying the canonical projection $\tau : A \to A/J$ to this equation gives

$$\tau(1) = 1 = \tau(ba) = \tau(b)\tau(a),$$

because of (24) and $\tau(J) = 0$. hence $\tau(a) = \tau(b)^{-1}$ in A/J. Since b was arbitrary (though nonzero), this shows that every nonzero element of A/J is invertible, since every such element is of the form $\pi(b)$ for some $b \notin J$.

The structure of commutative Banach algebras in which every nonzero element is invertible is described by the **Gelfand–Mazur theorem**.

Theorem 4.7 *If every nonzero element of a unital Banach algebra A is invertible, then $A \cong \mathbb{C}$ as Banach algebras.*

By Theorem 3.3.3 one has $\sigma(a) \neq \emptyset$ for each $a \neq 0$, hence there is a $z \in \mathbb{C}$ for which $a - z$ is not invertible. Hence $a - z = 0$ or $a = z1$ by assumption, and the map $a \to z$ is the desired algebra isomorphism. ■

Continuing the proof of Theorem 4.5.2, we have $A/J \cong \mathbb{C}$, so that there is a homomorphism $\psi : A/J \to \mathbb{C}$. Now define a map $\omega : A \to \mathbb{C}$ by $\omega(a) := \psi(\tau(a))$. This map is clearly linear, since τ and ψ are. Also,

$$\omega(a)\omega(b) = \psi(\tau(a))\psi(\tau(b)) = \psi(\tau(a)\tau(b)) = \psi(\tau(ab)) = \omega(ab),$$

because of (24) and the fact that ψ is a homomorphism.

Therefore, ω is multiplicative; it is nonzero because $\omega(b) \neq 0$, or because $\omega(1) = 1$. Hence $\omega \in \Delta(A)$. Finally, trivially

$$J = \ker(\tau) \subseteq \ker(\psi \circ \tau) = \ker(\omega).$$

But if $b \notin J$ then $\tau(b) \neq 0$, hence $\omega(b) = \psi \circ \tau(b) \neq 0$, since ψ is an isomoprhism. So $J^c \subseteq (\ker(\omega))^c$, or $\ker(\omega) \subseteq J$. Finally, $J = \ker(\omega)$. ■

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[33] Let C be the collection of all proper ideals containing J_a. This is a partially ordered set by inclusion. By Hausdorff's maximality theorem, C has a maximal totally ordered subcollection T. Then the union J of all members of T is an ideal in A. It is clear that $J_a \subset J$, whereas $J \neq A$ because none of the elements of C contains 1, while A does. Since C is maximal, J is a maximal ideal. Note that this argument, and thereby the whole of Gelfand's theory of commutative Banach algebra's depends on the axioms of choice (which is equivalent to Hausdorff's maximality theorem).

5 C^*-algebras without unit

We still need to prove Theorem 2.1 for the nonunital case, and will use this opportunity to introduce a general technique for handling C^*-algebras without a unit. As mentioned above, such C^*-algebras are quite important whenever ideals play a role.

The discussion of general (not necessarily commutative) Banach algebras without unit is easier than that of C^*-algebras. When a Banach algebra A does not contain a unit, we can always add one, as follows. Form the vector space $A \oplus \mathbb{C}$, and turn this into an algebra by means of

$$(a + \lambda 1)(b + \mu 1) := ab + \lambda b + \mu a + \lambda\mu 1, \tag{31}$$

where we have written $a + \lambda 1$ for (a, λ), etc. In other words, the number 1 in $\mathbb{C}$ is identified with the unit 1. Furthermore, define a norm on $A \oplus \mathbb{C}$ by

$$\|a + \lambda 1\| := \|a\| + |\lambda|. \tag{32}$$

In particular, $\|1\| = 1$. Using (93) in A, as well as 11.9.3, one sees from (31) and (32) that

$$\|(a + \lambda 1)(b + \mu 1)\| \le \|a\| \, \|b\| + |\lambda| \, \|b\| + |\mu| \, \|a\| + |\lambda| \, |\mu| = \|a + \lambda 1\| \, \|b + \mu 1\|,$$

so that $A \oplus \mathbb{C}$ is a Banach algebra with unit. Since by (32) the norm of $a \in A$ in A coincides with the norm of $a + 01$ in $A \oplus \mathbb{C}$, we have shown the following.

Proposition 5.1 *For every Banach algebra without unit there exists a unital Banach algebra $\tilde{A}$, the* ***unitization*** *of A, and an isometric (hence injective) morphism $A \to \tilde{A}$, such that $\tilde{A}/A \cong \mathbb{C}$.*

The unitization of A with the given properties is not unique. Indeed, when A is a C^*-algebra it is natural to equip $A \oplus \mathbb{C}$ with the involution

$$(a + \lambda 1)^* := a^* + \overline{\lambda} 1. \tag{33}$$

However, if $A \oplus \mathbb{C}$ is normed through (32) it fails to be a C^*-algebra, since (94) is not satisfied.[34] To guarantee that $A \oplus \mathbb{C}$ is a unital C^*-algebra one needs a different norm, constructed as follows. Recall Definition 11.12.

Proposition 5.2 *Let A be a C^*-algebra.*

1. *The map $\rho : A \to L(A)$ given by*
$$\rho(a)b := ab \tag{34}$$
establishes an isomorphism between A and $\rho(A) \subset L(A)$.
2. *When A has no unit, define a norm on $A \oplus \mathbb{C}$ by*
$$\|a + \lambda 1\| := \|\rho(a) + \lambda 1\|, \tag{35}$$
where the norm on the right-hand side is the operator norm (82) in $L(A)$, and 1 on the right-hand side is the unit operator in $L(A)$. With the operations (31) and (33), the norm (35) turns $A \oplus \mathbb{C}$ into a C^*-algebra with unit, called $\dot{A}$.

By (93) we have $\|\rho(a)b\| = \|ab\| \le \|a\| \, \|b\|$ for all b, so that $\|\rho(a)\| \le \|a\|$ by (82). On the other hand, using (94) and (95) we can write

$$\|a\| = \|aa^*\|/\|a\| = \|\rho(a)\frac{a^*}{\|a\|}\| \le \|\rho(a)\|;$$

in the last step we used (84) and $\|(a^*/\|a\|)\| = 1$. Hence

$$\|\rho(a)\| = \|a\|. \tag{36}$$

---

[34] This may already be seen in the simplest of all examples $A = \mathbb{C}$.

Being isometric, the map ρ must be injective; it is clearly a homomorphism, so that we have proved the first claim of the proposition.

It is clear from (31) and (33) that the map $a + \lambda 1 \to \rho(a) + \lambda 1$ (where the symbol 1 on the left-hand side is defined below (31), and the 1 on the right-hand side is the unit in $L(A)$) is a morphism. Hence the norm (35) satisfies (93), because (88) is satisfied in the Banach algebra $L(A)$. Moreover, in order to prove that the norm (35) satisfies (94), by Lemma 11.20 it suffices to prove that

$$\|\rho(a) + \lambda 1\|^2 \leq \|(\rho(a) + \lambda 1)^*(\rho(a) + \lambda 1)\| \tag{37}$$

for all $a \in A$ and $\lambda \in \mathbb{C}$.

To do so, we use a trick similar to the one involving (25), but with inf replaced by sup. Namely, in view of (82), for given $c \in L(A)$ and $\epsilon > 0$ there exists a $b \in A$, with $\|b\| = 1$, such that $\|c\|^2 - \epsilon \leq \|c(b)\|^2$. Applying this with $c = \rho(a) + \lambda 1$, we infer that for every $\epsilon > 0$ one has

$$\|\rho(a) + \lambda 1\|^2 - \epsilon \leq \|(\rho(a) + \lambda 1)b\|^2 = \|ab + \lambda b\|^2 = \|(ab + \lambda b)^*(ab + \lambda b)\|.$$

Here we used (94) in A. Using (34), the right-hand side may be rearranged as

$$\|\rho(b^*)\rho(a^* + \overline{\lambda}1)\rho(a + \lambda 1)b\| \leq \|\rho(b^*)\| \, \|(\rho(a) + \lambda 1)^*(\rho(a) + \lambda 1)\| \, \|b\|.$$

Since $\|\rho(b^*)\| = \|b^*\| = \|b\| = 1$ by (36) and (95), and $\|b\| = 1$ also in the last term, the inequality (37) follows by letting $\epsilon \to 0$. ■

Hence the C^*-algebraic version of Theorem 5.1 is

Proposition 5.3 *For every C^*-algebra without unit there exists a unique unital C^*-algebra $\dot{A}$ and an isometric (hence injective) morphism $A \to \dot{A}$, such that $\dot{A}/A \cong \mathbb{C}$.*

The uniqueness of $\dot{A}$ will follow from Corollary 8.5 below.

In the commutative case, the unitization procedure has a simple topological meaning, which illustrates the general principle that the use of commutative C^*-algebras often allows one to trade topological properties for algebraic ones. Recall that the **one-point compactification** $\dot{X}$ of a non-compact topological space X is the set $\dot{X} = X \cup \infty$, whose open sets are the open sets in X plus those subsets of $X \cup \infty$ whose complement is compact in X. The injection $i : X \hookrightarrow \dot{X}$ is continuous, and any continuous function $f \in C_0(X)$ extends uniquely to a function $f \in C(\dot{X})$ satisfying $f(\infty) = 0$. The space $\dot{X}$ is the solution (unique up to homeomorphism) of a so-called universal problem by Alexandroff's theorem: If $\varphi : X \to Y$ is a map between locally compact Hausdorff spaces such that $Y \backslash f(X)$ is a point and f is a homeomorphism onto its image, then there is a unique homeomorphism $\psi : \dot{X} \to Y$ such that $\varphi = \psi \circ i$.

The proof of the following lemmas is an easy exercise.

Lemma 5.4 *Let $A = C_0(X)$ for some noncompact locally compact Hausdorff space X. Then $\dot{A} \cong C(\dot{X})$, where $1 \in \dot{A}$ is identified with $1_{\dot{X}} \in C(\dot{X})$. Conversely, removing $\mathbb{C}1_{\dot{X}}$ from $C(\dot{X})$ corresponds to removing $\mathbb{C}$ from $\dot{A} = A \oplus \mathbb{C}$ (as a vector space), leaving one with $C_0(X)$.*

Hence the unitization of $C_0(X)$ corresponds to the one-point compactification of X.

Lemma 5.5 *Let A be a commutative C^*-algebra without unit.*

1. *Each $\omega \in \Delta(A)$ extends to a character $\dot{\omega}$ on $\dot{A}$ by*

$$\dot{\omega}(a + \lambda 1) := \omega(a) + \lambda. \tag{38}$$

2. *The functional ω_∞ on $\dot{A}$, defined by*

$$\omega_\infty(a + \lambda 1) := \lambda, \tag{39}$$

is a character of $\dot{A}$.

3. *There are no other characters on* $\dot{A}$.
4. $\Delta(\dot{A})$ *is homeomorphic to the one-point compactification of* $\Delta(A)$.

In fact, this lemma is true for any commutative Banach algebra, with respect to any unitization.

We are now in a position to prove Theorem 2.1 also in the nonunital case. Applying the unital case of Theorem 2.1 to $\dot{A}$ and using Lemma 5.5, one finds $\dot{A} \cong C(\dot{X})$ with $X := \Delta(A)$. Removing $\mathbb{C}$ from $\dot{A} = A \oplus \mathbb{C}$ precisely leaves one with $C_0(X)$ by Lemma 5.4, so that finally $A \cong C_0(X)$. ■

Note that the Gelfand transform on a commutative C^*-algebra without unit indeed takes values in $C_0(\Delta(A))$, since by (39) one has $\hat{a}(\omega_\infty) = \omega_\infty(a + 0) = 0$ for the (unique continuous) extension of $\hat{a}$ from $\Delta(A)$ to its one-point compactification.

6 Categorical version of the first Gelfand–Naimark theorem

We have seen that any compact Hausdorff space X defines a commutative C^*-algebra with unit $C(X)$. As mentioned in the Introduction, the association $X \mapsto C(X)$ is contravariant in the following sense: a map[35] $\varphi : X \to Y$ defines a morphism $\varphi^* : C(Y) \to C(X)$ of C^*-algebras via the pullback, that is, for $f \in C(Y)$ one puts $\varphi^*(f) = f \circ \varphi \in C(X)$. This, of course, relies on the fact that the composition of two continuous functions is again continuous. This association has the property $(\psi \circ \varphi)^* = \varphi^* \circ \psi^* : C(Z) \to C(X)$, where $\psi^* : C(Z) \to C(Y)$ is the pullback of a map $\psi : Y \to Z$. Furthermore, the identity map $\mathrm{id}_X : X \to X$ induces the identity map on $C(X)$.

In the opposite direction, a commutative C^*-algebra with unit A defines a compact Hausdorff space $\Delta(A)$ in a similar contravariant way: since $\Delta(A) \subset A^* \subset C(A)$, one may equally well define the pullback of a unit-preserving morphism $\varphi : A \to B$ as $\varphi^* : \Delta(B) \to \Delta(A)$; this relies on the fact that the composition of a character with a morphism is again a character. Also here one has $(\psi \circ \varphi)^* = \varphi^* \circ \psi^* : \Delta(C) \to \Delta(A)$, where $\psi^* : \Delta(C) \to \Delta(B)$ is the pullback of a unital morphism $\psi : B \to C$.

What is the precise relationship between the maps C and Δ? They are not inverse to each other; as we shall see, one has $\Delta(C(X)) \cong X$ and $C(\Delta(A)) \cong A$, where the symbol $\cong$ stands for homeomorphism and isomorhism, respectively. Moreover, the specific homeomorphims $X \to \Delta(C(X))$ for each X turn out to be well-behaved with respect to maps $f : X \to Y$, ans similarly for the isomorphisms $A \to C(\Delta(A))$. To describe the situation in an optimal way, we use the formalism of category theory, which we briefly recall.

Definition 6.1 *A* ***category*** C *consists of:*

- *A class*[36] C^0 *of* ***objects;***
- *A class* C^1 *of* ***arrows;***
- *maps* $s : C^1 \to C^0$ *(the* ***source map****),* $t : C^1 \to C^0$ *(the* ***target map****),* $i : C^0 \to C^1$ *(the* ***inclusion map****), and* ***multiplication***

$$m : C^2 := C^1 \times_{C^0} C^1 = \{(\varphi, \psi) \in C^1 \times C^1 \mid s(\varphi) = t(\psi)\} \to C^1.$$

We write $x \xrightarrow{\varphi} y$ *when* $\varphi \in C^1$ *satisfies* $s(\varphi) = x$ *and* $t(\varphi) = y$*, and* $\varphi\psi$ *or* $\varphi \circ \psi$ *for* $m(\varphi, \psi)$.

These maps are subject to the following axioms:

1. $s(i(x)) = t(i(x)) = x$ *for all* $x \in C^0$;
2. *If* $\varphi, \psi \in C^2$ *then* $s(\varphi\psi) = s(\psi)$ *and* $t(\varphi\psi) = t(\varphi)$;
3. *If* $(\varphi, \psi) \in C^2$ *and* $(\psi, \rho) \in C^2$ *then* $(\varphi\psi)\rho = \varphi(\psi\rho)$;

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[35] Recall Footnote 2; a map is continuous by definition.

[36] We refrain from defining the concept of a class; this would lead us into logic and the foundations of mathematics. We just mention that a class may be "larger" than a set.

4. $i(t(\varphi))\circ\varphi=\varphi\circ i(s(\varphi))=\varphi$ *for all* $\varphi\in C^1$.

We interpret φ as an arrow from $s(\varphi)$ to $t(\varphi)$, so that $i(x)$ is an arrow from x to x. The composition $\varphi\circ\psi$ of arrows is defined whenever $s(\varphi)=t(\psi)$ (so that on paper the direction of an arrow is from right to left!). Arrow composition is associative whenever defined, and each $i(x)$ acts as an identity under this composition operation. The class of all arrows from x to y in a category C is sometimes written as $(x,y)_C$.

The "simplest" example of category is the category S of all sets, where S^0 consists of sets and S^1 consists of functions.[37] Of course, in the present context our basis examples are the category CH of compact Hausdorff spaces as objects and maps as arrows, and the category CC_1A of commutative C^*-algebras with unit as objects and unital morphisms as arrows. The former is a subcategory (as defined in the obvious way) of the category T of all topological spaces and maps, and the latter is a subcategory of the category CA of all C^*-algebras and morphisms.

An important feature of category theory is that the notion of isomorphism is intrinsically defined: one calls two objects $x,y\in C^0$ **isomorphic**, written $x\cong y$, when there exist $x\xrightarrow{\varphi}y$ and $y\xrightarrow{\psi}x$ such that $\varphi\psi=i(y)$ and $\psi\varphi=i(x)$. Thus two topological spaces are isomorphic in T when they are homeomorphic,[38] and two C^*-algebras are isomorphic in CA when they are isomorphic according to Definition 2.2.

Note that a **groupoid** is a category[39] in which all arrows are isomorphisms, i.e., are invertible. Groupoids are usually denoted by G, and elements of G^1 are typically called γ.Thus, further to the axioms for a category, in a groupoid one has a map $I:G^1\rightarrow G^1$, called the **inverse**, also written $I(f)=f^{-1}$, satsfying $s\circ I=t$, $t\circ I=s$, $\gamma^{-1}\gamma=i(s(\gamma))$, and $\gamma^{-1}\gamma=i(t(\gamma))$. Groupoids provide interesting examples of categories, and may help getting used to the latter concept.[40] Moreover, at a later stage we'll show how, as independently discovered by Connes and Renault, certain groupoids canonically define C^*-algebras. This construction is of paramount importance for noncommutative geometry. Here are some examples.

1. The simplest example of a groupoid comes from a set X without additional structure. One takes $G^1=G^0=X$, and $s=t=i=\text{id}$, with multiplication $xx=x$ for all $x\in X$ and inverse $x^{-1}=x$. Thus each point of X is the unique arrow to and from itself, and there are no other arrows. One says that as a groupoid X is a **space**. As we shall see, when X is a locally compact space, the C^*-algebra associated to this groupoid is $C^*(X)=C_0(X)$.

2. A set X defines another groupoid, called the **pair groupoid** over X. Here one takes $G^1=X\times X$ and $G^0=X$, with $s(x,y)=y$, $t(x,y)=x$, $i(x)=(x,x)$, $(x,y)\circ(y,z)=(x,z)$, and $(x,y)^{-1}=(y,x)$. When X is finite with cardinality n, the associated C^*-algebra turns out to be $C^*(X\times X)=M_n(\mathbb{C})$.

3. An equivalence relation $\sim$ on a set X defines a subgroupoid $(X,\sim)$ of the pair groupoid over X; here $G^0=X$, while G^1 consists of those pairs $(x,y)\in X\times X$ for which $x\sim y$. When $x\sim y$ for any (x,y) one recovers the pair groupoid over X; on the other hand, when $x\sim x$ and nothing else the groupoid defined by the equivalence relation is X as a space. It is a useful exercise that the axioms for an equivalence relation imply that the definition of a groupoid is satisfied.[41]

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[37] Here one sees that a class is larger than a set, since the set of all sets does not exist, whereas the category of all sets does.

[38] In algebraic topology one works in the homotopy category hT, in which the objects are topological spaces and the arrows are homotopy classes of continuous maps. In that case, isomorphism of objects comes down to homotopy equivalence of spaces.

[39] It is often required that that this category is "small," in that its class of arrows is a set.

[40] Note, though, that the categories CC_1A and CH fail to be groupoids by a long shot!

[41] The fundamental idea of Connes, which underlies about half of noncommutative geometry, is as follows. When X is a locally compact Hausdorff space, the quotient $X/\sim$ may be ill-behaved in being non-Hausdorff; it may even have no nontrivial open sets. In such cases, the space $C_0(X/\sim)$ carries practically no information about $X/\sim$, not even as a set. In such situations, Connes proposed to describe the quotient $X/\sim$ by the C^*-algebra $C^*(X,\sim)$ of the groupoid defined by $\sim$, which may have a rich structure. Note that a complication arises: the subset of $X\times X$ defined by $\sim$ often needs to be retopologized in order to define $C^*(X,\sim)$. Also, in foliation theory, one does not in fact work with $C^*(X,\sim)$ but with a closely related C^*-algebra.

4. A group G is a groupoid with $G^1 = G$ and $G^0 = \{e\}$. When G is finite, $C^*(G)$ is the group ring $\mathbb{C}G$. More generally, when G is locally compact $C^*(G)$ is the so-called convolution C^*-algebra of G, which lies at the basis of modern harmonic analysis.

We return to general categories. According to the well-known category theorist Peter Freyd, *Topology is the study of continuous maps; category theory is the study of functors.*

Definition 6.2 *Let G and H be categories. A* ***covariant functor*** *or simply* ***functor*** $\Phi : G \to H$ *is a pair of maps $\Phi^i : G^i \to H^i$, $i = 0, 1$, such that*

1. $i_H \circ \Phi^0 = \Phi^1 \circ i_G$;
2. $s_H \circ \Phi^1 = \Phi^0 \circ s_G$ *and* $t_H \circ \Phi^1 = \Phi^0 \circ t_G$;
3. *If* $(\varphi, \psi) \in G^2$ *then* $\Phi^1(\varphi\psi) = \Phi^1(\varphi)\Phi^1(\psi)$.

It follows that Φ^0 is in fact determined by Φ^1, since i is injective. Nonetheless, it is useful to keep them apart. In any case, functors are simply homomorphisms between categories in the obvious sense. Now, the map $C : CH \to CC_1A$ defined by $C^0(X) = C(X)$ and $C^1(\varphi) = \varphi^*$ fails to be a functor, because in fact one has $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$, as we have seen. Similarly, the map $\Delta : CC_1A \to CH$ defined by $\Delta^0(A) = \Delta(A)$ and $\Delta^1(\psi) = \psi^*$ is not a functor for the same reason. So in our application we rather need something like an anti-homomorphism.

Definition 6.3 *Let G and H be categories. A* ***contravariant functor*** $\Phi : G \to H$ *is a pair of maps $\Phi^i : G^i \to H^i$, $i = 0, 1$, such that*

1. $i_H \circ \Phi^0 = \Phi^1 \circ i_G$;
2. $s_H \circ \Phi^1 = \Phi^0 \circ t_G$ *and* $t_H \circ \Phi^1 = \Phi^0 \circ s_G$;
3. *If* $(\varphi, \psi) \in G^2$ *then* $\Phi^1(\varphi\psi) = \Phi^1(\psi)\Phi^1(\varphi)$.

Note that a (possibly contravariant) functor $\Phi : G \to H$ preserves isomorphism of objects: if $x \cong y$ in G then $\Phi^0(x) \cong \Phi^0(y)$ in H, since if $x \xrightarrow{\varphi} y$ is invertible then so is $\Phi^0(x) \xrightarrow{\Phi^1(\varphi)} \Phi^0(y)$ by the axioms.

It should be clear that $C : CH \to CC_1A$ and $\Delta : CC_1A \to CH$ are contravariant functors. To describe the relationship between the maps C and Δ, we need the idea of a natural transformation between two functors. Indeed, according to the founders of category theory Samuel Eilenberg and Saunders Mac Lane, *Categories are defined to define functors; functors are defined to define natural transformations.*

Definition 6.4 *A* ***natural transformation*** *between two functors $\Phi : G \to H$ and $\Psi : G \to H$ (that are either both covariant or both contravariant) is a map $\tau : G^0 \to H^1$ such that $s(\tau_x) = \Phi^0(x)$ and $t(\tau_x) = \Psi^0(x)$ (in other words, τ is a collection of maps $\{\Phi^0(x) \xrightarrow{\tau_x} \Psi^0(x)\}_{x \in G^0}$), such that the following diagram commutes for all arrows $x \xrightarrow{\varphi} y$:*

$$\begin{tikzcd}
\Phi^0(x) \arrow[r, "\tau_x"] \arrow[d, "\Phi^1(\varphi)"'] & \Psi^0(x) \arrow[d, "\Psi^1(\varphi)"] \\
\Phi^0(y) \arrow[r, "\tau_y"'] & \Psi^0(y)
\end{tikzcd}$$

The given functors are called ***naturally isomorphic,***[42] *written $\Phi \cong \Psi$, when there exists a natural transformation between them for which all arrows τ_x are invertible (i.e., are isomorphisms).*[43]

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[42] Or **equivalent**.

[43] Hence a natural transformation of functors between groupoids automatically defines a natural isomorphism.

It follows that if Φ and Ψ are naturally isomorphic, then $\Phi(x) \cong \Psi(x)$ for all $x \in G^0$, but this condition in itself is not sufficient to render Φ and Ψ naturally isomorphic; the isomorphisms between $\Phi(x)$ and $\Psi(x)$ must be compatible with the arrows, as expressed by the diagram in the definition.

The original motivation for this definition was to find a way of expressing the fact that the dual V^* of any finite-dimensional vector space V (over $\mathbb{C}$) is isomorphic to V, but in an "unnatural" way, in that an isomorphism depends on the coice of a basis, whereas the double dual V^{**} is isomorphic to V in a "natural" way, namely through the Gelfand transform $v \mapsto \hat{v}$ from V to V^{**}, where $\hat{v}(\theta) := \theta(v)$ for $\theta \in V^*$. This fact may now be expressed by saying that the functor $**$ from the category of all finite-dimensional vector spaces to itself (with linear maps as arrows) is equivalent to the identity functor.

Now the whole point is the following definition.

Definition 6.5 *Two categories G, H are called* ***equivalent,*** *written $G \simeq H$, when there exist functors $\alpha : G \to H$ and $\beta : H \to G$ such that $\alpha \circ \beta \cong \mathrm{id}_H$ and $\beta \circ \alpha \cong \mathrm{id}_G$. Similarly, G and H are called* ***anti-equivalent*** *or* ***dual*** *when there exist contravariant functors with the same properties.*[44]

Spelling out what this means, using Definition 6.4, yields the commutative diagram

$$\begin{tikzcd} x \arrow[r, "\tau_x"] \arrow[d, "\varphi"'] & \alpha^0\beta^0(x) \arrow[d, "\alpha\beta^1(\varphi)"] \\ y \arrow[r, "\tau_y"'] & \alpha^0\beta^0(y) \end{tikzcd} \tag{40}$$

for all $x \xrightarrow{\varphi} y \in H$, in which all arrows τ_x are invertible, and similarly for G.

Theorem 6.6 *The category CH of compact Hausdorff spaces and continuous maps is dual to the category CC_1A of unital commutative C^*-algebras and unital morphisms.*

We already have the contravariant functors $C : CH \to CC_1A$ and $\Delta : CC_1A \to CH$. According to the diagram (40), we need isomorphisms τ_A for any unital commutative C^*-algebra such that

$$\begin{tikzcd} A \arrow[r, "\tau_A"] \arrow[d, "\varphi"'] & C(\Delta(A)) \arrow[d, "\varphi^{**}"] \\ B \arrow[r, "\tau_B"'] & C(\Delta(B)) \end{tikzcd} \tag{41}$$

commutes for any unital morphism $\varphi : A \to B$, and isomorphisms ε_X for any compact Hausdorff space X such that

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "\psi"'] & \Delta(C(X)) \arrow[d, "\psi^{**}"] \\ Y \arrow[r, "\varepsilon_Y"'] & \Delta(C(Y)) \end{tikzcd} \tag{42}$$

commutes for any map $\psi : X \to Y$.

For τ_A we take the Gelfand transform $\tau_A(a) = \hat{a}$, with $\hat{a}(\omega) = \omega(a)$, which we already know to be an isomorphism by Theorem 2.1. Furthermore, it is an easy exercise that Diagram (41) commutes. For ε_X we take the evaluation map, i.e., $\varepsilon_X(x) = \omega_x$, where $\omega_x(f) = f(x)$. Again it is easy to show that Diagram (42) commutes. The only remaining step is to show that ε_X is an isomorphism in the category CH, which we now do. We first prove injectivity, then surjectivity, and finally continuity of ε_X (the latter also of its inverse).

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[44] Similarly, G and H are **isomorphic**, $G \cong H$, when $\alpha \circ \beta = \mathrm{id}_H$ and $\beta \circ \alpha = \mathrm{id}_G$. This notion is rarely useful, however. The "natural" notion of isomorphism in category theory is equivalence.

Since a compact Hausdorff space is normal,[45] Urysohn's lemma says that $C(X)$ separates points on X (i.e., for all $x \neq y$ there is an $f \in C(X)$ for which $f(x) \neq f(y)$). This immediately shows that ε_X is injective.

To prove surjectivity, suppose there is $\omega \in \Delta(C(X))$ that is not of the form $\omega = \omega_x$ for some $x \in X$. By Theorem 4.5, there then exists a maximal ideal J in $C(X)$ such that $J \neq \ker(\omega_x)$ for all x, in other words, for all x there is a $f^x \in J$ for which $f^x(x) \neq 0$. For, if for some x there would exist no such f^x, then $f(x) = 0$ for all $f \in J$, so that $J \subseteq \ker(\omega_x)$ and hence $J = \ker(\omega_x)$ by maximality of J, contradicting the assumption. For each x, the set $\mathcal{O}_x$ where f_x is nonzero is open, because f is continuous. This gives a covering $\{\mathcal{O}_x\}_{x\in X}$ of X. By compactness, there exists a finite subcovering $\{\mathcal{O}_{x_i}\}_{i=1,\ldots,N}$. Then form the function $g := \sum_{i=1}^{N} |f_{x_i}|^2$. This function is strictly positive by construction, so that it is invertible.[46] But J is an ideal, so that, with all $f_{x_i} \in J$ (since all $f_x \in J$) also $g \in J$. But an ideal containing an invertible element must coincide with $C(X)$, contradicting the assumption that J is maximal. Hence ε_X is surjective by *reductio ad absurdum.*

The space X may be equipped with the "new" topology induced by ε_X, in which the open sets are of the form $\varepsilon_X^{-1}(\mathcal{O})$, with $\mathcal{O}$ open in $\Delta(C(X))$ in the Gelfand topology. We claim that this new topology on X is weaker than the original one.[47] Namely, for $f \in C(X)$ one has $\hat{f} \circ \varepsilon_X = f$. Therefore, since the Gelfand topology on $\Delta(C(X))$ is the weakest topology for which all Gelfand transforms $\hat{f}$ are continuous,[48] the new topology on X is the weakest topology for which all f are continuous. But f was already continuous with respect to the given topology, so the claim follows.

Without proof we now state a result from topology.

Lemma 6.7 *Let a set X be Hausdorff in some topology T_1 and compact in a topology T_2. If T_1 is weaker than T_2 then $T_1 = T_2$.*

Since X is in fact compact and Hausdorff in both topologies, we conclude from this lemma that the new topology on X must coincide with the original one. In other words, ε_X is a homeomorphism. This finishes the proof of Theorem 6.6. ■

Corollary 6.8 *The space X in Theorem 2.1 is unique up to homeomorphism.*

Firstly, it follows from the comment after Definition 6.3 that if $A \cong C(X)$ in CC_1A for some $X \in CH^0$, then $\Delta(A) \cong \Delta(C(X))$ in CH. But we have just seen that $\Delta(C(X)) \cong X$ in CH, so that $\Delta(A) \cong X$, or $X \cong \Delta(A)$. ■

This conclusion would be invalid when X is not required to be Hausdorff, in which case $C(X) \cong C(Y)$ in CC_1A no longer implies $X \cong Y$ in CH.

We now examine the case of commutative C^*-algebras possibly without unit, versus locally compact Hausdorff spaces that are possibly noncompact. It is clear that a slight difficulty arises here, since a map $\varphi : X \to Y$ does not, in general, pull back to a morphism $\varphi^* : C_0(Y) \to C_0(X)$. For example, with Y equal to a point any $f \in C(Y) \cong \mathbb{C}$ pulls back to a constant function on X, which does not vanish at infinity. Hence some restriction is necessary on the class of allowed maps between locally compact Hausdorff spaces.

Definition 6.9 *A map $\varphi : X \to Y$ between locally compact Hausdorff spaces is* ***proper*** *when $\varphi^{-1}(K)$ is compact for any compact set $K \subset Y$. The category LCH consists of locally compact Hausdorff spaces as objects and proper maps as arrows.*[49]

Clearly, maps like $c : \mathbb{R} \to pt$ are not proper. We list some properties of proper maps $\varphi : X \to Y$.[50]

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[45] Recall that in a normal space any disjoint pair of closed sets is contained in a disjoint pair of open sets.

[46] Recall that $f \in C(X)$ is invertible iff $f(x) \neq 0$ for all $x \in X$, in which case $f^{-1}(x) = 1/f(x)$.

[47] A topology T_1 is called weaker than a topology T_2 on the same set if any open set of T_1 contains an open set of T_2. This includes the possibility $T_1 = T_2$.

[48] Cf. Footnote 27.

[49] This definition relies on the property that the composition of two proper maps is again proper.

[50] See Bourbaki, General Topology Ch. 1-4, , §10.

1. A proper map is automatically closed.[51]
2. φ is proper iff it is closed and $\varphi^{-1}(pt)$ is compact for any point $pt \in Y$.
3. If X is compact then any $\varphi : X \to Y$ is proper.
4. φ is proper iff its canonical extension $\dot{\varphi} : \dot{X} \to \dot{Y}$, given by $\dot{\varphi}(\infty_X) = \infty_Y$, is continuous.[52]
5. The pullback φ^* maps $C_0(Y)$ into $C_0(X)$.

The algebraic counterpart of a proper map is as follows.

Definition 6.10 *A morphism $\psi : A \to B$ between C^*-algebras is called **nondegenerate** when $\psi(A)B$ (i.e., the linear span of all expressions of the form $\psi(a)b$, $a \in A$, $b \in B$) is dense in B.*

Clearly, a unital morphism between unital C^*-algebras is nondegenerate. Conversely, a nondegenerate morphism $\psi : A \to B$ between unital C^*-algebras is automatically unital: it follows from the definition of a morphism that $p := \psi(1)$ is a projection in B, so that $\varphi(A)B \subset pB$. Since $B = pB \oplus (1-p)B$ as a vector space, $\varphi(A)B$ can only be dense in B when $p = 1$. Finally, a nondegenerate morphism $\psi : A \to B$ between nonunital C^*-algebras extends uniquely to a unital morphism $\dot{\psi} : \dot{A} \to \dot{B}$ between the unitizations in the obvious way.

Using one-point compactifications it is easy to check that the pullback $\varphi^* : C_0(Y) \to C_0(X)$ of a proper map $\varphi : X \to Y$ is nondegenerate. This eventually leads to the following generalization of Theorem 6.6:

Theorem 6.11 *The category LCH of locally compact Hausdorff spaces and proper continuous maps is dual to the category CCA of commutative C^*-algebras and nondegenerate morphisms.*

As an exercise, this can be derived from Theorem 6.6 and the above results. The only tricky point is the proof that LCH contains identities, i.e., that the map $\mathrm{id} : A \to A$ is nondegenerate. In other words, is $A^2 = AA$ dense in A? In fact:

Lemma 6.12 *In any C^*-algebra A one has $A^2 = A$.*

This will be proved in section 8.[53]

7 The structure of C^*-algebras

Having said everything about commutative C^*-algebras there is to say, we now turn to the general case. It is an exercise (also cf. the Appendix) to prove the following:

Proposition 7.1 *Let H be a Hilbert space. The algebra $B(H)$ of all bounded operators on H is a C^*-algebra under the operator adjoint $a \mapsto a^*$, (cf. (92)) and the operator norm (83). Moreover, each (operator) norm-closed *-algebra in $B(H)$ is a C^*-algebra.*

Our goal is to prove the converse, which is conveniently stated in terms of the following concept.

Definition 7.2 *A **representation** of a C^*-algebra A on a Hilbert space H is a morphism $\pi : A \to B(H)$. In particular, π is a linear map satisfying $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$.*

If we just say "representation of a C^*-algebra" without further comment, we mean a representation on a Hilbert space. It is not obvious from this definition, but true, that the image $\pi(A)$ is automatically norm-closed (and hence a C^*-algebra) in $B(H)$. This is a special case of the result that the image of a morphism between C^*-algebras is always closed (and hence a C^*-algebra); see section 10 below. As an example of a representation, consider $A = C_0(\mathbb{R})$ and $H = L^2(\mathbb{R})$, with

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[51] That is, the image of any closed set is closed.

[52] This is true even when X and/or Y are/is compact.

[53] The same proof shows that $A^n = A$ for any $n \in \mathbb{N}$.

π given by $\pi(f)\psi(x) = f(x)\psi(x)$.[54] This representation is injective; an example of a representation of $A = C_0(\mathbb{R})$ that isn't is given by $\pi(f) = f(x)$ for some $x \in \mathbb{R}$. As another example, we just mention that the unitary representation theory of locally compact groups is a special case of the representation theory of C^*-algebras, since for every such group G there exists a C^*-algebra $C^*(G)$ whose representations are in bijective correspondence with the unitary representations of G, preserving direct sums, etc.

For later use, we define the notion of equivalence of representations: two representations $\pi_1(A)$ and $\pi_2(A)$ of a C^*-algebra A on Hilbert spaces H_1 and H_2 are called **equivalent** or **unitarily equivalent** when there is a unitary map $u : H_1 \to H_2$ such that $\pi_2(a) = u\pi_1(a)u^*$ for all $a \in A$. In the previous example, this notion reduces to the unitary equivalence of group representations as it is usually defined.

We now state the structure of C^*-algebras.

Theorem 7.3 *Every C^*-algebra A admits an injective*[55] *representation $\pi : A \to B(H)$ on some Hilbert space H. In other words, every C^*-algebra is isomorphic to a norm-closed *-algebra in $B(H)$, for some Hilbert space H.*

Since an injective morphism (and hence an injective representation) is automatically isometric, this theorem yields the best characterization of C^*-algebras one could possibly hope for. Note that we now have two entirely different characterizations of commutative C^*-algebras, given by Theorems 2.1 and 7.3. These are, of course, related by applying Theorem 7.3 to $A = C_0(X)$.[56]

The proof of Theorem 7.3 uses a beautiful construction, which is important in its own right. This construction, in turn, relies on a notion from quantum mechanics. Let $A = B(H)$ and take $\Omega \in H$ with $\|\Omega\| = 1$. The functional $\omega_\Omega : a \mapsto (\Omega, a\Omega)$ has the properties $\omega_\Omega(a^*a) \geq 0$ for all $a \in A$ (positivity) and $\omega_\Omega(1) = 1$ (normalization). More generally, let π be a unital representation of a unital C^*-algebra A on some Hilbert space H.[57] Then any unit vector $\Omega \in H$ defines a functional ω_Ω on A by

$$\omega_\Omega(a) := (\Omega, \pi(a)\Omega), \tag{43}$$

which has the same two properties. This functional is called the **vector state** defined by Ω. This suggests the following concept.

Definition 7.4 *A **state** on a unital C^*-algebra A is a functional $\omega : A \to \mathbb{C}$ that is **positive**, in that $\omega(a^*a) \geq 0$ for all $a \in A$, and **normalized**, in that $\omega(1) = 1$.*

A state on a C^*-algebra algebra A without unit is a functional ω for which the canonical extension $\dot{\omega}$ to the unitization $\dot{A}$, defined by

$$\dot{\omega}(a + \lambda 1) := \omega(a) + \lambda, \tag{44}$$

is positive.

The main properties of states, which basically yield a number of equivalent definitions, are summarized in the following results, whose proof is a straightforward exercise.

Proposition 7.5

- *Any positive functional on a C^*-algebra is continuous.*
- *A continuous functional ω on a unital C^*-algebra is positive iff $\|\omega\| = \omega(1)$.*
- *A positive functional ω on a C^*-algebra is a state iff $\|\omega\| = 1$.*
- *A state ω on a C^*-algebra A without unit has a unique extension to a state $\dot{\omega}$ on the unitization $\dot{A}$, given by (44).*

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[54] This may be generalized to the case $A = C_0(X)$ and $H = L^2(X, \mu)$, where μ is a Borel measure on X.

[55] One could call an injective representation faithful in the usual sense of the word. However, this terminology is sometimes used for representations π for whch $\pi(a^*a) = 0$ implies $a = 0$, so we avoid it here.

[56] Theorem 7.3 does not imply that for $A = C_0(X)$ one may always choose $H = L^2(X, \mu)$, and in fact its proof does not lead to this choice at all. Nonetheless, except in pathological cases one may indeed find a Borel measure μ on X for which the representation of $C_0(X)$ on $L^2(X, \mu)$ by multiplication operators is injective.

[57] When A has no unit, π should be nondegenerate for ω_Ω to define a state.

- *Conversely, the restriction of a state on $\dot{A}$ to A is a state.*

We now turn to the GNS-construction, initially assuming that A is unital. First, we call a representation π **cyclic** if its carrier space H contains a **cyclic vector** Ω for π; this means that the closure of $\pi(A)\Omega$ coincides with H. Such representation are the building blocks of any representation.[58] The following theorem and its proof form the **GNS-construction**.[59]

Theorem 7.6 *Let ω be a state on a C^*-algebra A. There exists a cyclic representation π_ω of A on a Hilbert space H_ω with cyclic unit vector Ω_ω such that*

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega) \ \ \forall a \in A. \tag{45}$$

Using (43), we could somewhat cryptically write this as $\omega = \omega_{\Omega_\omega}$.

The idea of the proof is simple; the devil is in the details of step 4.

1. Given a state ω on A, define a bilinear form $(-,-)_0$ on A by

$$(a,b)_0 := \omega(a^*b). \tag{46}$$

It is easily shown from the positivity property of ω that this form is positive semi-definite[60] and sesquilinear.[61] Hence $(-,-)_0$ defines a pre-inner product on A. Its null space

$$N_\omega = \{a \in A \,|\, \omega(a^*a) = 0\} \tag{47}$$

is a left-ideal in A.

2. The form $(-,-)_0$ projects to an inner product[62] $(-,-)_\omega$ on the quotient A/N_ω. If $p_\omega : A \to A/N_\omega$ is the canonical projection, then by definition

$$(p_\omega a, p_\omega b)_\omega := (a,b)_0. \tag{48}$$

The Hilbert space H_ω is the completion of A/N_ω in this inner product.

3. The representation $\pi_\omega(A)$ is initially defined on $A/N_\omega \subset H_\omega$, that is, on all vectors of the form $p_\omega b$, $b \in A$, by

$$\pi_\omega(a)p_\omega b := p_\omega ab. \tag{49}$$

This is well defined because N_ω is a left ideal in A. It is trivial to show that π_ω is linear and satisfies $\pi_\omega(ab) = \pi_\omega(a)\pi_\omega(b)$ and $\pi_\omega(a)^* = \pi_\omega(a^*)$; the latter is proved by taking inner products with arbitrary vectors pB and pc in A/N_ω.

4. We will show that each $\pi_\omega(a)$ is bounded on A/N_ω. Hence $\pi_\omega(A)$ may be extended to all of H_ω by continuity.[63] It is easily shown that this extension still has the defining properties of a representation.

5. The vector defined by $\Omega_\omega := p_\omega 1$ is cyclic, since $\pi_\omega(a)\Omega_\omega = p_\omega a$, hence $\pi_\omega(A)\Omega_\omega = A/N_\omega$, which by construction is dense in H_ω.

6. Finally, (45) follows by a simple computation.

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[58] It may be shown that any non-degenerate representation π is a direct sum of cyclic representations. Here one says that a representation $\pi(A)$ is **non-degenerate** if $\pi(a)v = 0$ for all $a \in A$ implies $v = 0$.

[59] For Gelfand-Naimark-Segal. This construction is very important in quantum field theory and quantum statistical mechanics.

[60] This means that $(a,a)_0 \geq 0$ for all $a \in A$.

[61] That is, $\overline{(a,b)_0} = (b,a)_0$, which is equivalent to the property $\omega(a^*) = \overline{\omega(a)}$.

[62] Which by definition is a positive definite hermitian (=sequilinear) form.

[63] This relies on a simple but important fact in functional analysis: if B_0 is a dense subspace of a Banach space B, and a linear map $a : B_0 \to B_0$ is bounded, then a extends to a bounded operator $a : B \to B$ by continuity. Namely, if $v_\lambda \to v$ in B with all $v_\lambda \in B_0$, then $\{av_\lambda\}$ is a Cauchy sequence in B, so that one may put $av := \lim av_\lambda$.

When A has no unit, we simply define $\pi_\omega(A)$ as the restriction of $\pi_{\dot{\omega}}(\dot{A})$ to A.

To prove the claim in step 4, we compute $\|\pi_\omega(a)p_\omega b\|^2$, where $a, b \in A$. By (48) and (49) one has $\|\pi_\omega(a)p_\omega b\|^2 = \omega(b^*a^*ab)$. The key inequality is now

$$\omega(b^*a^*ab) \leq \|a\|^2\omega(b^*b), \tag{50}$$

whose proof we postpone until the end of section 9. But $\omega(b^*b) = \|p_\omega b\|^2$, so that $\|\pi_\omega(a)p_\omega b\| \leq \|a\|\,\|p_\omega b\|$, so that $\|\pi_\omega(a)\| \leq \|a\|$ from (83).

We now take up the proof of Theorem 7.3. Given a state ω on A, the GNS-construction provides us with a representation $\pi_\omega : A \to B(H_\omega)$. If this representation were injective, we would be finished, but firstly we do not know that any state exists on A at all, and secondly we have no guarantee that π_ω is indeed injective. The first problem is resolved by the following lemma, which we will prove once we have developed the theory of states.

Lemma 7.7 *For any selfadjoint element $a \in A$ (i.e., $a^* = a$), there exists a state ω_a on A such that $|\omega_a(a)| = \|a\|$.*

In fact, it will be shown in section 8 that this lemma follows from the more powerful

Proposition 7.8 *For any $a \in A$ and $\alpha \in \sigma(a)$ there is a state $\omega_\alpha \in S(A)$ for which $\omega_\alpha(a) = \alpha$.*

To prove this, assume that A has a unit; if not, pass to the unitization and use Proposition 7.5. Given $a \in A$ and $\alpha \in \sigma(a)$, define a linear map $\tilde{\omega}_\alpha : \mathbb{C}a + \mathbb{C}1 \to \mathbb{C}$ by $\tilde{\omega}_\alpha(\lambda a + \mu 1) := \lambda\alpha + \mu$. Since $\alpha \in \sigma(a)$, one has $\lambda\alpha + \mu \in \sigma(\lambda a + \mu 1)$; this easily follows from the definition of σ. Hence (15) with a replaced by $\lambda a + \mu 1$ implies $|\tilde{\omega}_\alpha(\lambda a + \mu 1)| \leq \|(\lambda a + \mu 1)\|$. Since $\tilde{\omega}_\alpha(1) = 1$, it follows from Proposition 7.5 that $\|\tilde{\omega}\| = 1$. By the Hahn-Banach Theorem 11.15, there exists an extension ω_α of $\tilde{\omega}_\alpha$ to A of norm 1. By Proposition 7.5, ω_α is a state on A, which clearly satisfies $\omega_\alpha(a) = \tilde{\omega}_\alpha(a) = \alpha$. ■

To solve the second problem of the possible lack of injectivity of π_ω, we proceed as follows. First, note that if π_1 and π_2 are representations of A on Hilbert spaces H_1 and H_2, respectively, then there is an obvious notion of the direct sum $\pi_1 \oplus \pi_2$, which is a representation of A on $H_1 \oplus H_2$; one simply puts

$$(\pi_1 \oplus \pi_2)(a) := \begin{pmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{pmatrix}, a \in A.$$

This immediately extends to the direct sum of a finite number of Hilbert spaces and representations. The direct sum of a countable number of Hilbert spaces H_i is defined by

$$\oplus_i H_i := \left\{ v \in \prod_i H_i \mid \sum_i \|v_i\|^2_{H_i} < \infty \right\};$$

Similar to the proof of completeness of ℓ^2, it is a simple exercise to prove that this space is complete in the inner product

$$(v, w) := \sum_i (v_i, w_i)_{H_i}.$$

However, for the proof of Theorem 7.3 we unfortunately need uncountable direct sums, since we wish to define the direct sum of all GNS-representations of A, which is a huge number. To detail, let $S(A)$ be the **state space** of A, that is, the set of all states on A. We will need to make sense of constructions such as $\oplus_{\omega \in S(A)} H_\omega$ and $\oplus_{\omega \in S(A)} \pi_\omega$. To see what this means, let us say a few words about $S(A)$.

This set has the structure of a topological space in two ways, since by Proposition 7.5 one has $S(A) \subset A^*$, so that $S(A)$ inherits both the norm-topology and the weak*-topology of A^*. In what follows, we always regard $S(A)$ as a topological space with respect to the latter.[64] Furthermore, $S(A)$ has the structure of a **convex set** in the vector space A^*. That is, if $\rho, \sigma \in S(A)$ and

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[64] Recall that for finite-dimensional A there is no difference between these topologies.

$\lambda \in [0,1]$ then $\lambda\rho + (1-\lambda)\sigma$, initially defined in A^*, actually belongs to $S(A)$. This is immediate from Proposition 7.5. Repeating this process, it follows that $\sum_i p_i\omega_i$ belongs to $S(A)$ when all $p_i \geq 0$ and $\sum_i p_i = 1$, and all $\omega_i \in S(A)$. Combining the two structures, it follows from the Banach-Alaoglu theorem that $S(A)$ is a compact convex set when A is unital.[65]

For example, for $A = M_2(\mathbb{C})$ one has $S(A) \cong B^3$, as compact convex sets, where the three-ball is defined as $B^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$. Even this space is already uncountable! For $A = C(X)$ it follows from the Riesz theorem of measure theory that $S(A)$ consists of all probability measures on X.

Now, to define $\oplus_{\omega\in S(A)} H_\omega$ and the like, let S be any set, and let $f : S \to \mathbb{R}^+$ be a positive function on S. Define

$$\sum_{\omega\in S} f(\omega) := \sup_{F\subset S} \sum_{\omega\in F} f(\omega),$$

where the supremum is taken over all finite subsets F of S. It is easy to show that if the left-hand side is finite, then it can only have a countable number of nonzero terms. This gives meaning to

$$H_u := \oplus_{\omega\in S(A)} H_\omega := \left\{ v \in \prod_{\omega\in S(A)} H_\omega \mid \sum_{\omega\in S(A)} \|v_\omega\|^2_{H_\omega} < \infty \right\}. \tag{51}$$

One can show that H_u is complete in the inner product

$$(v,w) := \sum_{\omega\in S(A)} (v_\omega, w_\omega)_{H_\omega};$$

note that this sum only has a countable number of terms. We then define the **universal representation** $\pi_u(A)$ by

$$\pi_u(A) := \oplus_{\omega\in S(A)} \pi_\omega(A); \tag{52}$$

this means, of course, that $\pi_u(a)v$ for $v \in H_u$ is defined by its components $(\pi_u(a)v)_\omega := \pi_\omega(a)v_\omega$, which again is a countable family. It is easy to see that this indeed is a representation.

The Hilbert space H_u will be the H in the statement of Theorem 7.3. To prove that $\pi = \pi_u$ is injective, take some fixed $\rho \in S(A)$, and define the vector Ω as having components $\Omega_\omega = 0$ for $\omega \neq \rho$, and Ω_ρ equal to the cyclic vector of the GNS-construction for ρ. This vector clearly lies in H_u. Then $(\pi_u(a)\Omega)\omega = \pi_\omega(a)\Omega_\rho$ for $\omega = \rho$, and zero otherwise. Now suppose that $\pi_u(a) = 0$ for some $a \in A$. Then $\pi_u(a)\Omega = 0$, hence $\pi_\rho(a)\Omega_\rho = 0$, hence

$$\|\pi_\rho(a)\Omega_\rho\|^2 = \rho(a^*a) = 0$$

by the GNS-construction. Taking $\rho = \omega_{a^*a}$, Lemma 7.7 implies

$$\|a\|^2 = \|a^*a\| = \omega_{a^*a}(a^*a) = 0,$$

hence $a = 0$, so that π_u is injective. This concludes the proof of Theorem 7.3, up to the proof of (50) and Lemma 7.7. ■

It should be noted that this proof relies on incredible overkill, in that H_u is far larger than necessary. For example, for $A = M_n(\mathbb{C})$ any vector state in $\mathbb{C}^n$ gives rise to an injective representation, namely the defining one.

To prove the inequality (50) and Lemma 7.7 we need to develop the theory of states, which in turn relies on the notion of positivity in C^*-algebra. This notion, in turn, depends on the the functional calculus in C^*-algebras. All these topics are of great interest in their own right as well, and no course in C^*-algebras would be complete without covering these seemingly technical developments.

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[65] When A has no unit, the space of all positive functionals ω with $\|\omega\| \leq 1$ is a compact convex set, but $S(A)$ isn't.

8 Spectrum and functional calculus

Throughout this section, A is supposed to be unital. In the nonunital case, the same considerations apply after passing to the unitization $\dot{A}$.

For each element a of a unital C^*-algebra A there is a smallest C^*-subalgebra $C^*(a)$ of A that contains a and 1, namely the closure of the linear span of 1 and all operators of the type $a_1 \ldots a_n$, where a_i is a or a^*. Following the terminology for operators on a Hilbert space, an element $a \in A$ is called **normal** when $aa^* = a^*a$. The crucial property of a normal operator is that $C^*(a)$ is commutative. In particular, when a is selfadjoint, $C^*(a)$ is simply the closure of the space of all polynomials in a.

Theorem 8.1 *Let a be a normal element of a unital C^*-algebra. Then the spectrum $\sigma_A(a)$ of a in A coincides with the spectrum $\sigma_{C^*(a)}(a)$ of a in $C^*(a)$ (so that we may unambiguously speak of the spectrum $\sigma(a)$).*

Recall that $G(A)$ consists of the invertible elements in A. Let $a \in G(A)$ be normal, and consider the C^*-algebra $C^*(a, a^{-1}, 1)$ generated by a, a^{-1}, and 1. One has $(a^{-1})^* = (a^*)^{-1}$, and a, a^*, a^{-1}, $(a^*)^{-1}$ and 1 all commute with each other. Hence $C^*(a, a^{-1}, 1)$ is commutative; it is the closure of the space of all polynomials in a, a^*, a^{-1}, $(a^*)^{-1}$, and 1. By Theorem 2.1 we have $C^*(a, a^{-1}, 1) \cong C(X)$ for some compact Hausdorff space X. Since a is invertible and the Gelfand transform (9) is an isomorphism, $\hat{a}$ is invertible in $C(X)$. That is, $\hat{a}(x) \neq 0$ for all $x \in X$. However, for any $f \in C(X)$ that is nonzero throughout X we have $0 < \|f\|_\infty^{-2} f f^* \leq 1$ pointwise, so that $0 \leq 1_X - \|f\|_\infty^{-2} f f* < 1$ pointwise, hence

$$\|1_X - f f * / \|f\|_\infty^2\|_\infty < 1.$$

Here $f^*(x) = \overline{f(x)}$. Using Lemma 3.4, in terms of $1 = 1_X$ we may therefore write

$$\frac{1}{f} = \frac{f^*}{\|f\|_\infty^2} \sum_{k=0}^{\infty} \left(1 - \frac{f f^*}{\|f\|_\infty^2}\right)^k . \tag{53}$$

Hence $\hat{a}^{-1}$ is a norm-convergent limit of a sequence of polynomials in $\hat{a}$ and $\hat{a}^*$. Gelfand transforming this result back to $C^*(a, a^{-1}, 1)$, we infer that a^{-1} is a norm-convergent limit of a sequence of polynomials in a and a^*. Hence a^{-1} lies in $C^*(a)$, and $C^*(a, a^{-1}, 1) = C^*(a)$.

Now replace a by $a - z$, where $z \in \mathbb{C}$. If a is normal, then $a - z$ is normal. So if we assume that $a - z \in G(A)$ the argument above applies, leading to the conclusion that the resolvent $\rho_A(a)$ in A coincides with the resolvent $\rho_{C^*(a)}(a)$ in $C^*(a)$. By Definition 3.1 we then conclude that $\sigma_A(a) = \sigma_{C^*(a)}(a)$. ■

Corollary 8.2 *If $a^* = a$ then the spectrum $\sigma(a)$ is a subset of $\mathbb{R}$.*

According to Theorem 2.1, the function $\hat{a}$ is real-valued when $a = a^*$. Hence by (17) the spectrum $\sigma_{C^*(a)}(a)$ is real, so that by the previous result $\sigma(a)$ is real. ■

The commutative C^*-algebra $C^*(a)$ is clearly important, so it is gratifying that its structure can be described explicitly.

Proposition 8.3 *Let a be normal. The structure space $\Delta(C^*(a))$ is homeomorphic with $\sigma(a)$, so that $C^*(a)$ is isomorphic to $C(\sigma(a))$. Under this isomorphism the Gelfand transform $\hat{a} : \sigma(a) \to \mathbb{C}$ is the identity function $\mathrm{id}_{\sigma(a)} : t \to t$.*

Given the isomorphism $C^*(a) \cong C(X)$ of Theorem 2.1 (where $X = \Delta(C^*(a))$), according to (17) the function $\hat{a}$ is a surjective map from X to $\sigma(a)$. We now prove injectivity. When $\omega_1, \omega_2 \in X$ and $\omega_1(a) = \omega_2(a)$, then, for all $n \in \mathbb{N}$, we have

$$\omega_1(a^n) = \omega_1(a)^n = \omega_2(a)^n = \omega_2(a^n)$$

by iterating (6) with $b = a$. Since also $\omega_1(1) = \omega_2(1) = 1$ by (8), we conclude by linearity that $\omega_1 = \omega_2$ on all polynomials in a. By continuity (cf. Lemma 2.6) this implies that $\omega_1 = \omega_2$ on $C^*(a)$, since the linear span of all polynomials is dense in $C^*(a)$. Using (9), we have proved that $\hat{a}(\omega_1) = \hat{a}(\omega_2)$ implies $\omega_1 = \omega_2$.

Since $\hat{a} \in C(X)$, $\hat{a}$ is continuous. To prove continuity of the inverse, one checks that for $z \in \sigma(a)$ the functional $\hat{a}^{-1}(z) \in \Delta(C^*(a))$ maps a to z (and hence a^n to z^n, etc.). Looking at the definition of the Gelfand topology in Definition 2.4, one then sees that $\hat{a}^{-1}$ is continuous. In conclusion, $\hat{a}$ is a homeomorphism. The final claim of the theorem is then obvious. ■

An immediate consequence is the **continuous functional calculus**.[66]

Theorem 8.4 *For each selfadjoint element $a \in A$ and each $f \in C(\sigma(a))$ there is an operator $f(a) \in A$, which is the obvious expression when f is a polynomial (and in general is given via the uniform approximation of f by polynomials), such that*

$$\begin{aligned} \|f(a)\| &= \|f\|_\infty; && (54) \\ \sigma(f(a)) &= f(\sigma(a)). && (55) \end{aligned}$$

In particular, the norm of $f(a)$ in $C^*(a)$ coincides with its norm in A.

Still writing $X = \Delta(C^*(a))$, there are maps $a \mapsto \hat{a}$ from $C^*(a)$ to $C(X)$ (i.e., the Gelfand transform GT), $\hat{a}^* : C(\sigma(a)) \to C(X)$, where $\hat{a}$ is seen as a map from X to $\sigma(a)$, and finally the continuous functional calculus CFC as a map from $C(\sigma(a))$ to $C^*(a)$. These maps form a commutative diagram:

$$\begin{tikzcd} C^*(\sigma(a)) \arrow[r, "\text{CFC}"] \arrow[dr, "\hat{a}^*"'] & C^*(a) \arrow[d, "\text{GT}"] \\ & C(X) \end{tikzcd}$$

By Proposition 8.3, $\hat{a}^*$ is an isomorphism, and so is GT by Theorem 2.1. Hence CFC is an isomorphism by commutativity of the diagram. Furthermore, the first two maps are isometric; the first by direct computation (any pullback $\varphi^* : C(Y) \to C(X)$ of a homeomorphism $\varphi : X \to Y$ is isometric), and the second by Lemma 2.9. Hence the continuous functional calculus is isometric, too. This yields (54).

Since $f(\sigma(a))$ is the set of values of f on $\sigma(a)$, (55) follows from (17), with a replaced by $f(a)$. ■

This leads us to a central result in the theory of C^*-algebras.

Corollary 8.5 *The norm in a C^*-algebra is unique (that is, given a C^*-algebra A there is no other norm in which A is a C^*-algebra).*

First assume $a = a^*$, and apply (54) with $f = \mathrm{id}_{\sigma(a)}$. By definition, the sup-norm of $\mathrm{id}_{\sigma(a)}$ is $r(a)$, so that

$$\|a\| = r(a) \quad (a = a^*). \tag{56}$$

Since a^*a is selfadjoint for any a, for general $a \in A$ we have, using (94),

$$\|a\| = \sqrt{r(a^*a)}. \tag{57}$$

Since the spectrum is determined by the algebraic structure alone, (57) shows that the norm is determined by the algebraic structure as well. ■

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[66] For operators on a Hilbert space there exist holomorphic and measurable functional calculi as well.

Note that Corollary 8.5 does not imply that a given *-algebra can be normed only in one way so as to be completed into a C^*-algebra. In Corollary 8.5 the completeness of A is assumed from the outset.

As an immediate application of (56), let us derive Lemma 7.7 from Proposition 7.8, as promised. Since $\sigma(a)$ is closed by Theorem 3.3.2, there is an $\alpha \in \sigma(a)$ for which $r(a) = |\alpha|$. For this α one finds ω_α from Proposition 7.8, giving $|\omega_\alpha(a)| = |a| = r(A) = \|A\|$ by (56). ■

An important consequence of Corollary 8.5 is

Proposition 8.6 *An isomorphism between C^*-algebras is isometric.*

Let $\varphi : A \to B$ an isomorphism (cf. Definition 2.2). Define a map $\| \cdot \|_\varphi : A \to \mathbb{R}^+$ by $\|a\|_\varphi := \|\varphi(a)\|_B$. It easily follows that this is a norm on A, in which A is complete: a sequence $\{a_\lambda\}$ in A is a Cauchy sequence with respect to $\| \cdot \|_\varphi$ when $\{\varphi(a_\lambda)\}$ is a Cauchy sequence in B. The latter converges to some $b \in B$, so that $\{a_\lambda\}$ converges to $\varphi^{-1}(b)$. Furthermore, (4) and (5) guarantee that the norm $\| \cdot \|_\varphi$ satisfies (93) and (94). Thus A is a C^*-algebra in $\| \cdot \|_\varphi$, so that $\| \cdot \|_\varphi = \| \cdot \|_A$. ■

To close this section, we prove Lemma 6.12. It is enough to show that any selfadjoint $a \in A$ is of the form $a = a_1 a_2$, where A has no unit (otherwise the claim is trivial). Embed $A \subset \dot{A}$ and consider $C^*(a) \subset \dot{A}$. Now factorize the function $t \mapsto t$ on $\sigma(a) \subset \mathbb{R}$ as $t = f_1(t) f_2(t)$ for some $f_i \in C(\sigma(a))$, so that $a = a_1 a_2$ by Proposition 8.3 for $a_i := CFC(f_i) \in C^*(a)$. Now the constant function $1_{\sigma(a)}$ maps to $1 \in C^*(a) \subset \dot{A}$ under CFC, so if $f \in C(\sigma(a))$ then $CFC(f - f(0)1_{\sigma(a)}) \in C^*(a) \cap A$.[67] Imposing the additional condition $f_i(0) = 0$ therefore guarantees that in fact the a_i lie in A. ■

9 Positivity in C^*-algebras

A bounded operator a on a Hilbert space H is called positive when $(v, av) \geq 0$ for all $v \in H$.[68] This property is equivalent to $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}^+$, and clearly also applies to closed subalgebras of $B(H)$.

Classically, a function f on some space X is called positive when it is pointwise positive, that is, when $f(x) \geq 0$ for all $x \in X$. This applies, in particular, to elements of the commutative C^*-algebra $C_0(X)$ (where X is a locally compact Hausdorff space).

These examples are not as dissimilar as they might seem. Let $\mathbb{P}H$ be the **projective Hilbert space** defined by H; this space ist most easily defined as the quotient of the unit sphere $H_1 := \{v \in H \mid (v, v) = 1\}$ by the equivalence relation $v \sim w$ when $w = zv$ for some $z \in \mathbb{C}$ with $|z| = 1$. Now, an operator $a \in B(H)$ defines a function $\tilde{a} \in C(\mathbb{P}H)$ by means of $\tilde{a}([v]) := (v, av)$; the continuity of this function is an easy exercise. Hence we see that $\tilde{a}$ is real-valed iff a is self-adjoint as an operator on H, and that a is a positive operator precisely when $\tilde{a}$ is positive as a function on $\mathbb{P}H$.[69]

Hence we have a notion of positivity for certain concrete C^*-algebras, which we would like to generalize to arbitrary abstract C^*-algebras. In particular, one is interested in finding a number of equivalent characterizations of positivity.

Definition 9.1 *An element a of a C^*-algebra A is called **positive** when $a = a^*$ and its spectrum is positive; i.e., $\sigma(a) \subset \mathbb{R}^+$. We write $a \geq 0$ when a is positive, and A^+ for the set of all positive elements in A.*

It is immediate from Theorems (17) and Proposition 8.3 that a is positive iff its Gelfand transform $\hat{a}$ is positive in $C(\sigma(a))$. The basic structure of A^+ is captured by the following definition.

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[67] If A has no unit, then necessarily $0 \in \sigma_{\dot{A}}(a)$.

[68] In quantum mechanics this means that the expectation value of the observable a is always positive.

[69] This argument exhibits a crucial difference between classical and quantum mechanics: if one regards $\mathbb{P}H$ as the phase space of a quantum system, the only observables are functions of the form $\tilde{a}$ for some self-adjoint $a \in B(H)$. This class by no means exhausts $C(\mathbb{P}H)$; for example, for $H = \mathbb{C}^2$ one has $\mathbb{P}H \cong S^2$, the two-sphere, on which the space of all real-valued continuous (or even smooth) functions is clearly infinite-dimensional.

Definition 9.2 *A* ***convex cone*** *in a real vector space V is a subspace V^+ such that*

1. *when $v \in V^+$ and $t \in \mathbb{R}^+$ then $tv \in V^+$;*
2. *when $v, w \in V^+$ then $v + w \in V^+$;*
3. *$V^+ \cap -V^+ = 0$.*

A ***linear partial ordering*** *in V is a partial ordering $\leq$ in which $v \leq w$ implies $v + f \leq w + f$ for all $f \in V$ and $tv \leq tw$ for all $t \in \mathbb{R}^+$.*

These two structures are equivalent: given $V^+ \subset V$ one defines $\leq$ by putting $v \leq w$ if $w - v \in V^+$, and given $\leq$ one defines $V^+ = \{v \in V \mid 0 \leq v\}$.

Proposition 9.3 *The set A^+ of all positive elements of a C^*-algebra A is a convex cone in the real vector space $A_{\mathrm{sa}} := \{a \in A \mid a^* = a\}$.*

The first property follows from $\sigma(ta) = t\sigma(a)$, which is a special case of (55).

Since $\sigma(a) \subseteq [0, r(a)]$, we have $|c - t| \leq c$ for all $t \in \sigma(a)$ and all $c \geq r(a)$. Hence $\sup_{t\in\sigma(a)} |c1_{\sigma(a)} - \hat{a}| \leq c$ by (17) and Proposition 8.3, so that $\|c1_{\sigma(a)} - \hat{a}\|_\infty \leq c$. Gelfand transforming back to $C^*(a)$, this implies $\|c1 - a\| \leq c$ for all $c \geq \|a\|$ by Theorem 8.4. Inverting this argument, one sees that if $\|c1 - a\| \leq c$ for some $c \geq \|a\|$, then $\sigma(a) \subset \mathbb{R}^+$.

Use this with $a \rightarrow a + b$ and $c = \|a\| + \|b\|$; clearly $c \geq \|a + b\|$ by 11.9.4. Then

$$\|c1 - (a + b)\| \leq \|(\|a\| - a)\| + \|(\|b\| - b)\| \leq c,$$

where in the last step we used the previous paragraph for a and for b separately. As we have seen, this inequality implies $a + b \in A^+$.

Finally, when $a \in A^+$ and $a \in -A^+$ it must be that $\sigma(a) = 0$, hence $a = 0$ by (56) and (14). ■

For example, when $a = a^*$ one checks the validity of

$$-\|a\|\,1 \leq a \leq \|a\|\,1 \tag{58}$$

by taking the Gelfand transform of $C^*(a)$. The implication

$$-b \leq a \leq b \implies \|a\| \leq \|b\| \tag{59}$$

then follows, because $-b \leq a \leq b$ and (58) for $a \rightarrow b$ yield $-\|b\|\,1 \leq a \leq \|b\|\,1$, so that $\sigma(a) \subseteq [-\|b\|, \|b\|]$, hence $\|a\| \leq \|b\|$ by (56) and (14). For later use we also record

Lemma 9.4 *When $a, b \in A^+$ and $\|a + b\| \leq k$ then $\|a\| \leq k$.*

By (58) we have $a + b \leq k1$, hence $0 \leq a \leq k1 - b$ by the linearity of the partial ordering, which also implies that $k1 - b \leq k1$, as $0 \leq b$. Hence, using $-k1 \leq 0$ (since $k \geq 0$) we obtain $-k1 \leq a \leq k1$, from which the lemma follows by (59). ■

We now come to the central result in the theory of positivity in C^*-algebras, which generalizes the cases $A = B(H)$ and $A = C_0(X)$.

Theorem 9.5 *One has*

$$\begin{aligned} A^+ &= \{a^2 \mid a^* = a\} &(60)\\ &= \{a^*a \mid a \in A\}. &(61)\end{aligned}$$

When $\sigma(a) \subset \mathbb{R}^+$ and $a = a^*$ then $\sqrt{a} \in A$ is defined by the continuous functional calculus for $f = \sqrt{\cdot}$, and satisfies $\sqrt{a}^2 = a$. Hence $A^+ \subseteq \{a^2 \mid a^* = a\}$. The opposite inclusion follows from (55) and Corollary 8.2. This proves (60).

The inclusion $A^+ \subseteq \{a^*a \mid a \in A\}$ is is trivial from (60).

Lemma 9.6 *Every selfadjoint element a has a decomposition $a = a_+ - a_-$, where $a_+, a_- \in A^+$ and $a_+a_- = 0$. Moreover, $\|a_\pm\| \leq \|a\|$.*

Apply the continuous functional calculus with $f = \mathrm{id}_{\sigma(a)} = f_+ - f_-$, where $\mathrm{id}_{\sigma(a)}(t)$, $f_+(t) = \max\{t, 0\}$, and $f_-(t) = \max\{-t, 0\}$. Since $\|f_\pm\|_\infty \leq r(a) = \|a\|$ (where we used (56)), the bound follows from (54) with $a \to a_\pm$. ■

We use this lemma to prove that $\{a^*a | \, a \in A\} \subseteq A^+$. Apply the lemma to $a = b^*b$ (noting that a is selfadjoint). Then

$$(a_-)^3 = -a_-(a_+ - a_-)a_- = -a_-aa_- = -a_-b^*ba_- = -(ba_-)^*ba_-.$$

Since $\sigma(a_-) \subset \mathbb{R}^+$ because a_- is positive, we see from (55) with $f(t) = t^3$ that $(a_-)^3 \geq 0$. Hence $-(ba_-)^*ba_- \geq 0$.

Lemma 9.7 *If $-c^*c \in A^+$ for some $c \in A$ then $c = 0$.*

We can write $c = d + ie$, d and e selfadjoint, so that

$$c^*c = 2d^2 + 2e^2 - cc^*. \tag{62}$$

Now for any $a, b \in A$ one has

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}. \tag{63}$$

This is because for $z \neq 0$ the invertibility of $ab - z$ implies the invertibility of $ba - z$. Namely, one computes that $(ba - z)^{-1} = b(ab - z)^{-1}a - z^{-1}1$. Applying this with c instead of A and c^* for b, we see that the assumption $\sigma(c^*c) \subset \mathbb{R}^-$ implies $\sigma(cc^*) \subset \mathbb{R}^-$, hence $\sigma(-cc^*) \subset \mathbb{R}^+$. By (62), (60), and Proposition 9.3 (applied to Definition 9.2.2) we see that $c^*c \geq 0$, i.e., $\sigma(c^*c) \subset \mathbb{R}^+$, so that the assumption $-c^*c \in A^+$ now yields $\sigma(c^*c) = 0$. Hence $c = 0$ by Proposition 9.3 applied to Definition 9.2.3. ■

The last claim before the lemma therefore implies $ba_- = 0$. As $(a_-)^3 = -(ba_-)^*ba_- = 0$ we see that $(a_-)^3 = 0$, and finally $a_- = 0$ by the continuous functional calculus with $f(t) = t^{1/3}$. Hence $b^*b = a_+$, which lies in a^+. This completes the proof of Theorem 9.5. ■

An important consequence of (61) is the fact that inequalities of the type $a_1 \leq a_2$ for selfadjoint a_1, a_2 are stable under conjugation by arbitrary elements $b \in A$, so that $a_1 \leq a_2$ implies $b^*a_1b \leq b^*a_2b$. This is because $a_1 \leq a_2$ is the same as $a_2 - a_1 \geq 0$; by (61) there is an $a_3 \in A$ such that $a_2 - a_1 = a_3^*a_3$. But clearly $(a_3b)^*a_3b \geq 0$, and this is nothing but $b^*ab \leq b^*a_2b$. For example, replace a in (58) by a^*a, and use (94), yielding $a^*a \leq \|a\|^21$. Applying the above principle gives

$$b^*a^*ab \leq \|a\|^2b^*b \tag{64}$$

for all $a, b \in A$. The definition of a state easily implies that if $a \leq b$, then $\omega(a) \leq \omega(b)$. Thus (50) follows.

10 Approximate units

It remains to prove Propositions 4.2 and 4.4, as well as the comments after Definition 2.2. In fact, these claims are closely related. The appropriate technical tool is the theory of approximate units. For any noncompact space X, the C^*-algebra $C_0(X)$ has no unit (the unit would be 1_X, which does not vanish at infinity because it is constant). There is a certain substitute for the absent unit, however. Taking $X = \mathbb{R}$ for simplicity, and pick a sequence of functions 1_n, $n \in \mathbb{N}$, that take the value 1 on $[-n, n]$ and vanish for $|x| > n + 1$. It is clear that one does not have $1_n \to 1_\mathbb{R}$ in the sup-norm, but instead one has $\lim_{n\to\infty} \|1_nf - f\|_\infty = 0$ for all $f \in C_0(\mathbb{R})$. Such a construction may be performed in any C^*-algebra.

Definition 10.1 *An* ***approximate unit*** *in a non-unital C^*-algebra A indexed by some directed set Λ (i.e., a set with a partial order and a sense in which $\lambda \to \infty$) is a family $\{1_\lambda\}_{\lambda\in\Lambda}$ of selfadjoint elements of A, such that*

$$\|1_\lambda\| \leq 1, \tag{65}$$

and

$$\lim_{\lambda\to\infty} \|1_\lambda a - a\| = \lim_{\lambda\to\infty} \|a1_\lambda - a\| = 0 \tag{66}$$

for all $a \in A$.

The following fact is very useful, though its proof is awful.

Proposition 10.2 *Every non-unital C^*-algebra A has an approximate unit. When A is separable, one may choose Λ countable.*

One takes Λ to be the set of all finite subsets of A, partially ordered by inclusion. Hence $\lambda \in \Lambda$ is of the form $\lambda = \{a_1, \ldots, a_n\}$, from which we build the element $b_\lambda := \sum_i a_i^* a_i$. Clearly b_λ is selfadjoint, and according to Theorem 9.5 and Proposition 9.3 one has $\sigma(b_\lambda) \subset \mathbb{R}^+$, so that $n^{-1}1 + b_\lambda$ is invertible in the unitization $\dot{A}$ of A. Hence we may form

$$1_\lambda := b_\lambda(n^{-1}1 + b_\lambda)^{-1}. \tag{67}$$

Since b_λ is selfadjoint and b_λ commutes with functions of itself (such as $(n^{-1}1 + b_\lambda)^{-1}$), one has $1_\lambda^* = 1_\lambda$. Although $(n^{-1}1 + b_\lambda)^{-1}$ is computed in $\dot{A}$, so that it is of the form $c + \mu 1$ for some $c \in A$ and $\mu \in \mathbb{C}$, one has $I_\lambda = b_\lambda c + \mu b_\lambda$, which actually lies in A. Using the continuous functional calculus with $f(t) = t/(n+t)$ on b_λ, one sees from (55) and the positivity of b_λ that $\sigma(1_\lambda) \subset [0, 1]$. This implies (65) because of (56).

Putting $c_i := 1_\lambda a_i - a_i$, a simple computation shows that

$$\sum_i c_i c_i^* = n^{-2} b_\lambda (n^{-1}1 + b_\lambda)^{-2}. \tag{68}$$

We now apply (54) with a replaced by $\to b_\lambda$ and $f(t) = n^{-2}t(n^{-1} + t)^{-2}$. Since $f \geq 0$ and f assumes its maximum at $t = 1/n$, one has $\sup_{t\in\mathbb{R}^+} |f(t)| = 1/4n$. As $\sigma(b_\lambda) \subset \mathbb{R}^+$, it follows that $\|f\|_\infty \leq 1/4n$, hence $\|n^{-2}b_\lambda(n^{-1}1 + b_\lambda)^{-2}\| \leq 1/4n$ by (54), so that $\|\sum_i c_i c_i^*\| \leq 1/4n$ by (68). Lemma 9.4 then shows that $\|c_i c_i^*\| \leq 1/4n$ for each $i = 1, \ldots, n$. Since any $a \in A$ sits in some directed subset of Λ with $n \to \infty$, it follows from (94) that

$$\lim_{\lambda\to\infty} \|1_\lambda a - a\|^2 = \lim_{\lambda\to\infty} \|(1_\lambda a - a)^* 1_\lambda a - a\| = \lim_{\lambda\to\infty} \|c_i^* c_i\| = 0.$$

The other equality in (66) follows analogously.

Finally, when A is separable one may draw all A_i occurring as elements of $\lambda \in \Lambda$ from a countable dense subset, so that Λ is countable. ■

We are now in a position to prove Propositions 4.2. Let $J \subset A$ be the given ideal, and put $J^* := \{a^* | \, a \in J\}$. Note that $j \in J$ implies $j^*j \in J \cap J^*$: it lies in J because J is an ideal, hence a left-ideal, and it lies in J^* because J^* is an ideal, hence a right-ideal. Since J is an ideal, $J \cap J^*$ is a C^*-subalgebra of A. Hence by 10.2 it has an approximate unit $\{1_\lambda\}$. Take $j \in J$. Using (94) and $1_\lambda^* = 1_\lambda$, we estimate

$$\|j^* - j^*1_\lambda\|^2 = \|(j - 1_\lambda j)(j^* - j^*1_\lambda)\| = \|(jj^* - jj^*1_\lambda) - 1_\lambda(jj^* - jj^*1_\lambda)\|$$

$$\leq \|(jj^* - jj^*1_\lambda)\| + \|1_\lambda(jj^* - jj^*1_\lambda)\| \leq \|(j^*j - j^*j1_\lambda)\| + \|1_\lambda\| \, \|(jj^* - jj^*1_\lambda)\|.$$

As we have seen, $j^*j \in J \cap J^*$, so that, also using (65), both terms vanish for $\lambda \to \infty$. Hence $\lim_{\lambda\to\infty} \|j^* - j^*1_\lambda\| = 0$. But 1_λ lies in $J \cap J^*$, so certainly $1_\lambda \in J$, and since J is an ideal it must be that $j^*1_\lambda \in J$ for all λ. Hence j^* is a norm-limit of elements in J; since J is closed, it follows that $j^* \in J$. ■

In view of Proposition 4.3, all we need to prove to establish Proposition 4.4 is the property (94). This uses

Lemma 10.3 *Let $\{1_\lambda\}$ be an approximate unit in J, and let $a \in A$. Then*

$$\|\tau(a)\| = \lim_{\lambda\to\infty} \|a - a1_\lambda\|. \tag{69}$$

It is obvious from (23) that

$$\|a - a1_\lambda\| \geq \|\tau(a)\|. \tag{70}$$

To derive the opposite inequality, add a unit 1 to A if necessary, pick any $j \in J$, and write

$$\|a - a1_\lambda\| = \|(a+j)(1-1_\lambda) + j(1_\lambda - 1)\| \leq \|a+j\| \, \|1-1_\lambda\| + \|j1_\lambda - j\|.$$

Note that

$$\|1 - 1_\lambda\| \leq 1 \tag{71}$$

by Definition 10.1 and the proof of Proposition 9.3. The second term on the right-hand side goes to zero for $\lambda \to \infty$, since $j \in J$. Hence

$$\lim_{\lambda\to\infty} \|a - a1_\lambda\| \leq \|a+j\|. \tag{72}$$

For each $\epsilon > 0$ we can choose $j \in J$ so that (25) holds. For this specific j we combine (70), (72), and (25) to find

$$\lim_{\lambda\to\infty} \|a - a1_\lambda\| - \epsilon \leq \|\tau(a)\| \leq \|a - a1_\lambda\|.$$

Letting $\epsilon \to 0$ proves (69). ■

We now prove (94) in A/J. Successively using (69), (94) in $\dot{A}$, (71), (69), (24), and (29), we find

$$\|\tau(a)\|^2 = \lim_{\lambda\to\infty} \|a - a1_\lambda\|^2 = \lim_{\lambda\to\infty} \|(a-a1_\lambda)^*(a-a1_\lambda)\|$$
$$= \lim_{\lambda\to\infty} \|(1-1_\lambda)a^*a(1-1_\lambda)\| \leq \lim_{\lambda\to\infty} \|1-1_\lambda\| \, \|a^*a(1-1_\lambda)\| \leq \lim_{\lambda\to\infty} \|a^*a(1-1_\lambda)\|$$
$$= \|\tau(a^*a)\| = \|\tau(a)\tau(a^*)\| = \|\tau(a)\tau(a)^*\|.$$

Lemma 11.20 then implies (94). This finishes the proof of Proposition 4.4. ■

We now state and prove the key result about morphisms.

Theorem 10.4 *Let $\varphi : A \to B$ be a nonzero morphism between C^*-algebras.*

1. *φ is continuous, with norm 1.*
2. *$\ker(\varphi)$ is an ideal in A.*
3. *$\varphi(A)$ is a C^*-subalgebra of B; in particular, $\varphi(A)$ is closed in B.*
4. *If φ is injective, then it is isometric.*

To prove continuity of φ, we may assume that A and B have units, which are preserved by φ.[70] If $z \in \rho(a)$, so that $(a-z)^{-1}$ exists in A, then $\varphi(a-z)$ is certainly invertible in B, for (4) implies that $(\varphi(a-z))^{-1} = \varphi((a-z)^{-1})$. Hence $\rho(a) \subseteq \rho(\varphi(a))$, so that

$$\sigma(\varphi(a)) \subseteq \sigma(a). \tag{73}$$

Hence $r(\varphi(a)) \leq r(a)$, so that $\|\varphi\| \leq 1$ follows from (57). This proves continuity of φ, which also immediately implies the second claim of the theorem.[71]

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[70] If A has no unit, we first replace φ by the morphism $\varphi_0 : A \to \overline{\varphi(A)}$, where the bar on the right-hand side denotes the closure in the norm-topology of B. Noting that φ_0 is nondegenerate by Lemma 6.12, we subsequently extend φ_0 to a unital morphism between the unitizations of A and $\overline{\varphi(A)}$, if necessary (cf. the comments after Definition 6.10).

[71] For a different proof of continuity, assume that $a^* = a$ by using (94) in the standard way, and reduce the situation to the commutative case by working with $\varphi_a : C^*(a) \to \overline{\varphi(C^*(a))}$. Even though we do not know yet that φ is continuous, it follows that the right-hand side is commutative as the closure of a commutative subalgebra of B. Subsequently, use Theorem 2.1 to reduce to the case of a unital morphism $\varphi : C(X) \to C(Y)$. By Theorem 6.6, φ is the pullback of some $\psi : Y \to X$. Lemma 3.2 then immediately yields (73).

We now prove the last claim of the theorem. Assume there is an $b \in A$ for which $\|\varphi(b)\| \neq \|b\|$. By (94), (4), and (5) this implies $\|\varphi(b^*b)\| \neq \|b^*b\|$. Put $a := b^*b$, noting that $a^* = a$. By (57) and (14) we must have $\sigma(a) \neq \sigma(\varphi(a))$. Then (73) implies $\sigma(\varphi(a)) \subset \sigma(a)$. By Urysohn's lemma there is a nonzero $f \in C(\sigma(a))$ which vanishes on $\sigma(\varphi(a))$, so that $f(\varphi(a)) = 0$. By Lemma 10.5 below we have $\varphi(f(a)) = 0$. This is a contradiction when φ is injective, in which case φ must therefore be isometric.

Combining the second claim with Proposition 4.4, we see that $A/\ker(\varphi)$ is a C^*-algebra. This fits into a commutative triangle of C^*-algebras:

$$\begin{tikzcd}
A \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & A/\ker(\varphi) \arrow[d, "\psi"] \\
& B
\end{tikzcd}$$

Here τ is the canonical projection and ψ is defined by $\psi([a]) = \varphi(a)$ (where $[a]$ is the equivalence class in $a/\ker(\varphi)$ of $a \in A$). By the theory of vector spaces, ψ is a vector space isomorphism between $A/\ker(\varphi)$ and $\varphi(A)$. Since φ and τ are morphisms between C^*-algebras, so is ψ. Since ψ is injective, it is isometric, as we have just shown. Hence $\psi(A/\ker(\varphi))$ has closed range in B. But $\psi(A/\ker(\varphi)) = \varphi(A)$, so that φ has closed range in B. Since φ is a morphism, its image is a *-algebra in B, which by the preceding sentence is closed in the norm of B. Hence $\varphi(A)$, inheriting all operations in B, is a C^*-algebra.

Finally, one trivially has $\|\tau\| = 1$ and $\|\psi\| = 1$ (because ψ is an isometry). It then follows from $\varphi = \psi \circ \tau$ that $\|\psi\| = 1$. ■

We have used the following fact:

Lemma 10.5 *When $\varphi : A \to B$ is a morphism and $a = a^*$ then*

$$f(\varphi(a)) = \varphi(f(a)) \tag{74}$$

for all $f \in C(\sigma(a))$ (here $f(a)$ is defined by the continuous continuous functional calculus, and so is $f(\varphi(a))$ in view of (73)).

The property is true for polynomials by (4), since for those f has its naive meaning. For general f the result then follows by continuity. ■

Note that Theorem 10.4.2 has a converse: every ideal in a C^*-algebra is the kernel of some morphism. This follows from Theorem 4.2, since J is the kernel of the canonical projection $\tau : A \to A/J$, where A/J is a C^*-algebra, and τ is a morphism by (24), and (29).

11 The C^*-algebra of compact operators

It would appear that the appropriate generalization of the C^*-algebra $M_n(\mathbb{C})$ of $n \times n$ matrices to infinite-dimensional Hilbert spaces H is the C^*-algebra $B(H)$ of all bounded operators on H. This is not the case. Firstly, unlike $M_n(\mathbb{C})$ (which can be shown to possess only one irreducible representation up to equivalence), $B(H)$ has a huge number of inequivalent representations; even when H is separable, most of these are realized on non-separable Hilbert spaces. Secondly, unlike $M_n(\mathbb{C})$, $B(H)$ is not simple when H is infinite-dimensional (i.e., it has proper ideals). Thirdly, $B(H)$ is non-separable in the norm-topology even when H is separable.

The appropriate generalization of $M_n(\mathbb{C})$ to an infinite-dimensional Hilbert space H turns out to be the C^*-algebra $B_0(H)$ of compact operators on H.[72] In noncommutative geometry elements of $B_0(H)$ play the role of infinitesimals. In K-theory, $B_0(H)$ is indistinguishable from $\mathbb{C}$. In the theory of operator algebras, $B_0(H)$ is a basic building block for C^*-algebras.

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[72] In the literature, one often finds the notation $K(H)$ for $B_0(H)$, but this is awkward in connection to K-theory.

Definition 11.1 *Let H be a Hilbert space. A* ***finite-rank operator*** *on H is an operator $a \in B(H)$ for which $aH := \{av| \, v \in H\}$ is finite-dimensional. Equivalently, a is a finite linear combination finite-dimensional projections on H.*

A ***compact operator*** *on H is an operator that can be approximated in norm by finite-rank operators.*[73]

Lemma 11.2 *The collection of all finite-rank operators on H is a *-algebra in $B(H)$. Its norm-closure $B_0(H)$ is a C^*-algebra, which contains precisely all compact operators on H.*

In other words, $B_0(H)$ is the smallest C^*-algebra in $B(H)$ that contains all finite-rank operators. The norm in $B_0(H)$ is, of course, the operator norm (83). It is clear that $B_0(H_1) \cong B_0(H_2)$ iff $\dim(H_1) = \dim(H_2)$, and one often writes B_0 or K for the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space $H \cong \ell^2$.

To prove the lemma, note that by definition $p^* = p$ for any projection p, so that $B_f(H)$ is a *-algebra. Alternatively, it is obvious that if $a \in B_f(H)$, then $a^* \in B_f(H)$. The second claim is trivial as well. ■

Proposition 11.3 *1. The unit operator 1 lies in $B_0(H)$ iff H is finite-dimensional.*

2. The C^*-algebra $B_0(H)$ is an ideal in $B(H)$.[74]

Firstly, for any sequence (or net) $a_n \in B_f(H)$ we may choose a unit vector $v_n \in (a_n H)^\perp$. Then $(a_n - 1)v = -v$, so that $\|(a_n - 1)v\| = 1$. Hence $\sup_{\|v\|=1} \|(a_n - 1)v\| \geq 1$, so that $\|a_n - 1\| \to 0$ is impossible by definition of the norm (83) in $B(H)$.

Secondly, when $a \in B_f(H)$ and $b \in B(H)$ then $ab \in B_f(H)$, since $abH \subseteq aH$. But since $ba = (a^* b^*)^*$, and $B_f(H)$ is a *-algebra, one has $a^* b^* \in B_f(H)$ and hence $ba \in B_f(H)$. Hence $B_f(H)$ is an ideal in $B(H)$, save for the fact that it is not norm-closed (unless H has finite dimension). Now if $a_n \to a$ then $a_n b \to ab$ and $ba_n \to ba$ by continuity of multiplication in $B(H)$. Hence $B_0(H)$ is an ideal by virtue of its definition. ■

One has a complete characterization of compact operators.

Theorem 11.4 *A self-adjoint operator $a \in B(H)$ is compact iff $a = \sum_{i \in I} a_i p_i$ (norm-convergent sum), where I is countable, $a_i \in \mathbb{R}$, each projection p_i is finite-dimensional (so that each nonzero eigenvalue of a has finite multiplicity), and the set of all eigenvalues of a only has 0 as a possible accumulation point.*

For a proof see, e.g., [15].

Unlike $B(H)$, the state space of $B_0(H)$ can be explicitly computed. We just state the results, referring to [15] for details. This involves the study of a number of algebraic ideals of $B(H)$, which are not closed.

Definition 11.5 *The* ***Hilbert-Schmidt norm*** *$\|a\|_2$ of $a \in B(H)$ is defined by*

$$\|a\|_2^2 := \sum_i \|a\mathbf{e}_i\|^2, \tag{75}$$

where $\{\mathbf{e}_i\}_i$ is an arbitrary basis of H; the right-hand side is independent of the choice of the basis. The ***Hilbert-Schmidt class*** *$B_2(H)$ consists of all $a \in B(H)$ for which $\|a\|_2 < \infty$.*

The ***trace norm*** *$\|a\|_1$ of $a \in B(H)$ is defined by*

$$\|a\|_1 := \|(a^* a)^{\frac{1}{4}}\|_2^2, \tag{76}$$

where $(a^* a)^{\frac{1}{4}}$ is defined by the continuous functional calculus. The ***trace class*** *$B_1(H)$ consists of all $a \in B(H)$ for which $\|a\|_1 < \infty$.*

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[73] If a is a compact operator then aB_1 is compact in H (with the norm-topology). Here B_1 is the unit ball in H, i.e., the set of all $v \in H$ with $\|v\| \leq 1$.

[74] The quotient $C(H) := B(H)/B_0(H)$ turns out to be a very interesting C^*-algebra, known as the **Calkin algebra**.

Both $B_1(H)$ and $B_2(H)$ are complete in the pertinent norms, and are therefore Banach spaces. Moreover, $B_2(H)$ is a Hilbert space in the inner product

$$(a, b) := \mathrm{Tr}\, a^* b. \tag{77}$$

If $a \in B_1(H)$ then

$$\mathrm{Tr}\, a := \sum_i (\mathbf{e}_i, a\mathbf{e}_i) \tag{78}$$

is finite and independent of the basis.[75]

Proposition 11.6 *One has the inclusions*

$$B_f(H) \subseteq B_1(H) \subseteq B_2(H) \subseteq B_0(H) \subseteq B(H), \tag{79}$$

with equalities iff H is finite-dimensional.

This chain of inclusions is sometimes seen as the non-commutative analogue of

$$\ell_c(X) \subseteq \ell^1(X) \subseteq \ell^2(X) \subseteq \ell_0(X) \subseteq \ell^\infty(X),$$

where X is an discrete set, with equalities iff X is finite. Since $\ell_1(X) = \ell_0(X)^*$ and $\ell^\infty(X) = \ell_1(X)^* = \ell_0(X)^{**}$, this analogy is strengthened by the following result.

Theorem 11.7 *One has $B_0(H)^* = B_1(H)$ and $B_1(H)^* = B_0(H)^{**} = B(H)$ under the pairing*

$$\hat{\rho}(a) = \mathrm{Tr}\, \rho a = \hat{a}(\rho). \tag{80}$$

Here $\hat{\rho} \in B_0(H)^*$ is identified with $\rho \in B_1(H)$, and $\hat{a} \in B_1(H)^*$ is identified with $a \in B(H)$.

Corollary 11.8 *Any state on $B_0(H)$ is of the form $a \mapsto \mathrm{Tr}\, \rho a$, where ρ is a* ***density matrix****, that is, an element of $B_1(H)$ that is positive in $B_0(H)$ (i.e., $\rho \geq 0$) and has unit trace ($\mathrm{Tr}\, \rho = 1$).*

Density matrices were introduced by John von Neumann [13] as the most general possible states in elementary quantum mechanics. Using modern terminology, he defined a state ω on $B(H)$ to be **normal** when $\omega(p) = \sum_j \omega(p_j)$ for each pairwise orthogonal family of projections $\{p_j\}_j$ with $\sum_j p_j = p$.[76] He then proved that a normal state on $B(H)$ is necessarily of the form described in Corollary 11.8. Thus the normal states on $B(H)$ coincide with the states on $B_0(H)$. In fact, this statement turns out to be equivalent to the claim $B(H) = B_1(H)^*$ in Theorem 11.7.

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[75] When $a \notin B_1(H)$, it may happen that $\mathrm{Tr}\, a$ depends on the basis; it may even be finite in one basis and infinite in another.

[76] Here the sum is defined as the least upper bound of all finite sums of the p_j, with respect to the ordering $\leq$ of projections defined by $p \leq q$ when $pH \subseteq qH$.

Appendix: Basic definitions

All vector spaces will be defined over $\mathbb{C}$, and all functions will be $\mathbb{C}$-valued, unless we explicitly state otherwise. The abbreviation 'iff' means 'if and only if', which is the same as the symbol $\Leftrightarrow$. An equation of the type $a := b$ means that a is by definition equal to b.

Definition 11.9 *A* ***norm*** *on a vector space V is a map $\| \, \| : V \to \mathbb{R}$ such that*

1. $\|v\| \geq 0$ *for all* $v \in V$*;*
2. $\|v\| = 0$ *iff* $v = 0$*;*
3. $\|\lambda v\| = |\lambda| \, \|v\|$ *for all* $\lambda \in \mathbb{C}$ *and* $v \in V$*;*
4. $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality).*

A norm on V defines a metric d on V by $d(v, w) := \|v - w\|$. A vector space with a norm that is complete in the associated metric (in the sense that every Cauchy sequence converges) is called a ***Banach space****. We will denote a generic Banach space by the symbol B.*

The two main examples of Banach spaces we will encounter are Hilbert spaces and certain collections of operators on Hilbert spaces.

Definition 11.10 *A* ***pre-inner product*** *on a vector space V is a map $(\, , \,) : V \times V \to \mathbb{C}$ such that*

1. $(\lambda_1 v_1 + \lambda_2 v_2, \mu_1 w_1 + \mu_2 w_2) = \overline{\lambda_1}\mu_1(v_1, w_1) + \overline{\lambda_1}\mu_2(v_1, w_2) + \overline{\lambda_2}\mu_1(v_2, w_1) + \overline{\lambda_2}\mu_2(v_2, w_2)$ *for all* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ *and* $v_1, v_2, w_1, w_2 \in V$*;*
2. $(v, v) \geq 0$ *for all* $v \in V$*.*

An equivalent set of conditions is

1. $\overline{(v, w)} = (w, v)$ *for all* $v, w \in V$*;*
2. $(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1(v, w_1) + \lambda_2(v, w_2)$ *for all* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* $v, w_1, w_2 \in V$*;*
3. $(v, v) \geq 0$ *for all* $v \in V$*.*

A pre-inner product for which $(v, v) = 0$ iff $v = 0$ is called an ***inner product****.*

The equivalence between the two definitions of a pre-inner product is elementary; in fact, to derive the first axiom of the second characterization from the first set of conditions, it is enough to assume that $(v, v) \in \mathbb{R}$ for all v (use this reality with $v \to v + iw$). Either way, one derives the **Cauchy-Schwarz inequality**

$$|(v, w)|^2 \leq (v, v)(w, w), \tag{81}$$

for all $v, w \in V$. Note that this inequality is valid even when $(\, , \,)$ is not an inner product, but merely a pre-inner product.

It follows from these properties that an inner product on V defines a norm on V by $\|v\| := \sqrt{(v, v)}$; the triangle inequality is automatic.

Definition 11.11 *A* ***Hilbert space*** *is a vector space with inner product that is complete in the associated norm. We will usually denote Hilbert spaces by the symbol H.*

A Hilbert space is completely characterized by its dimension (i.e., by the cardinality of an arbitrary orthogonal basis). To obtain an interesting theory, one therefore studies operators on a Hilbert space, rather than the Hilbert space itself. Although a satisfactory mathematical theory of unbounded operators on a Hilbert space exists, we will restrict ourselves to bounded operators. We recall this concept in the more general context of arbitrary Banach spaces.

Definition 11.12 *A* ***bounded operator*** *on a Banach space B is a linear map $a : B \to B$ for which*

$$\|a\| := \sup\{\|av\| \mid v \in B,\, \|v\| = 1\} < \infty. \tag{82}$$

The space of all bounded operators on a Banach space B is called $L(B)$.

The number $\|a\|$ is the **operator norm**, or simply the norm, of a. This terminology is justified, as it follows almost immediately from its definition (and from the properties of the the norm on B) that the operator norm is indeed a norm.

(It is easily shown that a linear map on a Banach space is continuous iff it is a bounded operator, but we will never use this result. Indeed, in arguments involving continuous operators on a Banach space one almost always uses boundedness rather than continuity.)

When B is a Hilbert space H the expression (82) becomes

$$\|a\| := \sup\{(av, av)^{\frac{1}{2}} \mid v \in H,\, (v, v) = 1\}. \tag{83}$$

When a is bounded, it follows that

$$\|av\| \le \|a\|\, \|v\| \tag{84}$$

for all $v \in B$. Conversely, when for $a \le 0$ there is a $C > 0$ such that $\|av\| \le C\|v\|$ for all v, then a is bounded, with operator norm $\|a\|$ equal to the smallest possible C for which the above inequality holds.

Proposition 11.13 *The space $L(B)$ of all bounded operators on a Banach space B is itself a Banach space in the operator norm.*

In view of the comments following (83), it only remains to be shown that $L(B)$ is complete in the operator norm. Let $\{a_n\}$ be a Cauchy sequence in $L(B)$. In other words, for any $\epsilon > 0$ there is a natural number $N(\epsilon)$ such that $\|a_n - a_m\| < \epsilon$ when $n, m > N(\epsilon)$. For arbitrary $v \in B$, the sequence $\{a_n v\}$ is a Cauchy sequence in B, because

$$\|a_n v - a_m v\| \le \|a_n - a_m\|\, \|v\| \le \epsilon\|v\| \tag{85}$$

for $n, m > N(\epsilon)$. Since B is complete by assumption, the sequence $\{a_n v\}$ converges to some $w \in B$. Now define a map a on B by $av := w = \lim_n a_n v$. This map is obviously linear. Taking $n \to \infty$ in (85), we obtain

$$\|av - a_m v\| \le \epsilon\|v\| \tag{86}$$

for all $m > N(\epsilon)$ and all $v \in B$. It now follows from (82) that $a - a_m$ is bounded. Since $a = (a - a_m) + a_m$, and $L(B)$ is a linear space, we infer that a is bounded. Moreover, (86) and (82) imply that $\|a - a_m\| \le \epsilon$ for all $m > N(\epsilon)$, so that $\{a_n\}$ converges to a. Since we have just seen that $a \in L(B)$, this proves that $L(B)$ is complete. ■

We define a **functional** on a Banach space B as a linear map $\rho : B \to \mathbb{C}$ that is continuous, in that $\rho(v)| \le C\|v\|$ for some C, and all $v \in B$. The smallest such C is the norm of ρ:

$$\|\rho\| := \sup\{|\rho(v)|,\, v \in B,\, \|v\| = 1\}. \tag{87}$$

Definition 11.14 *The* ***dual*** *B^* of B is the space of all functionals on B.*

Similarly to the proof of 11.13, one shows that B^* is a Banach space. For later use, we quote, without proof, the fundamental **Hahn-Banach theorem** of functional anlysis and an important consequence of it (cf. [15]).

Theorem 11.15 *For a functional ρ_0 on a linear subspace B_0 of a Banach space B there exists a functional ρ on B such that $\rho = \rho_0$ on B_0 and $\|\rho\| = \|\rho_0\|$. In other words, each functional defined on a linear subspace of B has an extension to B with the same norm.*

Corollary 11.16 *When $\rho(v) = 0$ for all $\rho \in B^*$ then $v = 0$.*

Recall that an **algebra** is a vector space with an associative bilinear operation ('multiplication') $\cdot : A \times A \to A$; we usually write ab for $a \cdot b$. We call A **unital** if there is an element $1 \in A$, the **unit**, such that $1a = a1 = a$ for all $a \in A$. It is easy to see that a unit is unique when it exists.

It is clear that $L(B)$ is an algebra with unit under operator multiplication. One easily shows that $\|ab\| \leq \|a\| \, \|b\|$.

Definition 11.17 *A **Banach algebra** is a Banach space A which is at the same time an algebra, in which for all $a, b \in A$ one has*

$$\|ab\| \leq \|a\| \, \|b\| \, . \tag{88}$$

It follows that multiplication in a Banach algebra is separately continuous in each variable. In a Banach algebra one needs a sharper definition of a unit than for general algebras: we require that $\|1\| = 1$.

As we have just seen, for any Banach space B the space $L(B)$ of all bounded operators on B is a Banach algebra. In what follows, we will restrict ourselves to the case that B is a Hilbert space H; this leads to the Banach algebra $L(H)$. In the theory of C^*-algebras, one usually writes $B(H)$ for $L(H)$. This algebra has additional structure.

Definition 11.18 *An **involution** on an algebra A is a real-linear map $A \to A^*$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$ one has*

$$a^{**} = a; \tag{89}$$

$$(ab)^* = b^* a^*; \tag{90}$$

$$(\lambda a)^* = \overline{\lambda} a^*. \tag{91}$$

*A ***-algebra** is an algebra with an involution.*

The operator adjoint $a \to a^*$ on a Hilbert space, defined by the property

$$(v, a^* w) := (av, w), \tag{92}$$

defines an involution on $B(H)$. Hence $B(H)$ is a *-algebra. The crucial property $\|a^*a\| = \|a\|^2$ for all $a \in B(H)$. motivates the following definition.

Definition 11.19 *A C^***-algebra** is a complex Banach space A that is at the same time a *-algebra, such that for all $a, b \in A$ one has*

$$\|ab\| \leq \|a\| \; \|b\|; \tag{93}$$

$$\|a^*a\| = \|a\|^2. \tag{94}$$

For C^*-algebras, the property $\|1\| = 1$ for a unit follows from the axioms, and does not need to be imposed separately. It is clear that $B(H)$ is a C^*-algebra. Moreover, each (operator) norm-closed *-algebra in $B(H)$ is obviously a C^*-algebra.

We note a very useful lemma, whose proof is an easy exercise.

Lemma 11.20
1. *If an involution on a Banach algebra A satisfies $\|a\|^2 \leq \|a^*a\|$, then A is a C^*-algebra.*
2. *For any element a of a C^*-algebra one has*

$$\|a^*\| = \|a\|. \tag{95}$$

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