TOPOLOGY AND MODAL LOGIC: SPACE FOR KNOWLEDGE

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1 Modal logic and topology

Let ***M*** be a topological space plus a valuation assigning sets to proposition letters:

$\boldsymbol{M}, s \models [\,]\phi$ iff s is in the topological interior of $[[\phi]]^{M}$ $\quad \exists O \in \mathbf{O}\ \forall t \in O$

Relevant valid *S4*-principles:

$[\,]\phi \rightarrow \phi, \quad [\,][\,]\phi \leftrightarrow [\,]\phi, \quad [\,](\phi \& \psi) \leftrightarrow [\,]\phi \,\&\, [\,]\psi, \quad [\,]\phi \vee [\,]\psi \leftrightarrow [\,]([\,]\phi \vee [\,]\psi)$

General Completeness Theorem A modal formula is valid iff it is provable in *S4*.

McKinsey and Tarski: The logic of any metric space without isolated points is *S4*.

In particular, *S4* is the complete logic of any space IR^n. Flurry of recent proofs.

2 Relational semantics as a special case

Standard graph-based semantics: $\boldsymbol{M}, s \models [\,]\phi$ iff for all t: if Rst, then $\boldsymbol{M}, t \models \phi$

This is the special case of Alexandroff topologies on pre-orders. Major difference: *[]* distributes over arbitrary infinite conjunctions in this case, but not in topology!

Interplay interesting: e.g., represent finite Kripke models with fractals on real line.

Still more general: modal neighbourhood semantics with point-to-set relations::

$\boldsymbol{M}, s \models [\,]\phi$ iff there is a set X with RsX, and for all $t \in X$: $\boldsymbol{M}, t \models \phi$

3 Model theory: topo-bisimulation and comparison games

Bisimulation games: Spoiler and Duplicator compare points across models:

> Spoiler chooses a current point *s* in one model plus an open neighbourhood *U*,
> Duplicator responds with an open neighbourhood *V* of the other current point *t*,
> Spoiler chooses a point *v* in *V*, Duplicator chooses a point *u* in *U*, making *u*–*v*
> the new match. Duplicator loses if the two points differ qua atomic properties.

Examples: comparing points on 'Spoons'. *Adequacy Theorem*: Spoiler's winning strategies in *k*-round game match modal 'difference formulas' of operator depth *k*.

Topo-bisimulations: the corresponding relations between topological spaces with a valuation, that encode winning strategies for Duplicator in the infinite game:

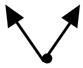
Zigzag:	whenever sEt, then, if $s \in U \in \mathbf{O}$, there exists a V
(and v.v.)	with (a) $t \in V \in \mathbf{O}'$ and (b) $\forall v \in V\ \exists u \in U$: uEv.

Coarse variant of homeomorphism, preserving truth of all modal formulas.

4 Axiom systems for special spaces

Landscape of stronger topological logics above S4? Restricted valuations: 'serial sets': finite unions of convex sets on the real line. A typical valid principle now:

$$\neg\phi \,\&\, {<>}\phi \rightarrow {<>}[\,]\phi$$

Complete logic is that of the *'2-fork frame'*  Extensions to IR^n.

5 Extended modal languages for topology

Add *universal/existential modalities* to define connectedness of topological spaces:

$U([\,]p \vee [\,]q) \,\&\, Ep \,\&\, Eq \rightarrow E(p\&q)$ (no non-empty open sets partition the space)

Example: *continuous maps* preserve all topological properties modally definable by means of *U, E,* $<>$, $\wedge$, $\vee$, *literals* $(\neg)p$. E.g., modal definition of *Compactness*.

Alternative : *first-order* preservation analysis of Chu spaces (van Benthem 2000).

6 Products 1, relational semantics

Combine modal languages in applications (time, space, knowledge): models $\boldsymbol{M} = (W, R_1, R_2, V)$ for component modalities *[1], [2]*. Simplest combination: *fusion* of component logics: union of axioms in the combined language.

Gabbay & Shehtman: special case of *product models* $\boldsymbol{F}x\boldsymbol{G}$:

$(x, y)\, R_1\, (u, v)$ iff $xR_1u \,\&\, y = v$; and ditto for R_2

With Horn-clause frame conditions, logics axiomatized by fusion plus extras

$[1][2]\phi \leftrightarrow [2][1]\phi$	Commutation
$<1>[2]\phi \leftrightarrow [2]<1>\phi$	Confluence

Behaviour of product logics very complex in general (Gabbay et al. 'Product Book'). *Grid pattern* in products often leads to *undecidability*: encoding Tiling Problems!

7 Products 2, topological semantics

Products of topological models $(W_i, \boldsymbol{O}_i, V_i)$ $(i = 1, 2)$: points are ordered pairs, horizontal and vertical topologies: copy component topologies on lines/columns:

$\boldsymbol{M}_1 \times \boldsymbol{M}_2, (s, t) \models [1]\phi$ iff there is an open neighbourhood
Y of s in $\boldsymbol{M}_1$ such that for all $y \in Y$: $\boldsymbol{M}_1 \times \boldsymbol{M}_2, (y, t) \models \phi$

Counterexamples to the two extra axioms: e.g., *refute Commutation* on *IR x IR*!

Theorem 1 The modal logic of products of topologies is just the fusion *S4+S4*.

Proof (van Benthem, Bezhanishvili, ten Cate & Sarenac), represent infinite binary trees with two successor relations up to bisimulation in the topo-product *Q x Q*. Details of the construction in the references below.

Topology *O2 commutes* with *O1* if *[1][2]p* $\rightarrow$ *[2][1]p* is valid in their product.

Simple: Alexandroff topologies *O2* commute with every topology *O1*.

Converse: If *O2* commutes with every topology, then it is Alexandroff!

Add *true product topology*: contained in intersection of horizontal and vertical topologies, *undefinable* in terms of these, modal logic of *[1x2]* axiomatizable as well.

8 Epistemic logic and topology

Intuitionism: open sets ~ evidence (Tarski, Vickers). Scott domains for information.

Epistemic logic in relational models: agents know what is true in all worlds they cannot distinguish from the actual one. *S4*: (a) veridicality, (b) positive introspection.

Special flavour: *iterated* knowledge: what you know about my knowledge, etc. ...

Common knowledge: what all members of a group know to every depth of mutual iteration ('Countable Iteration'). Coordinated behaviour (Aumann, economics).

Alternative: 'Equilibrium' as *greatest fixed-point*:

$$\nu p \bullet \phi \,\&\, [1]p \,\&\, [2]p \qquad \#$$

These two views are equivalent on relational models: fixed-points stop at stage ω!

Barwise's 'Three Theories of Common Understanding': these notions are *different*...

9 Common knowledge

Theorem 2 On topological models, greatest fixed-points for # are not always defined by a countable iteration of *[1], [2]*.

Proof Use Theorem 1 to find a sequence of squares converging to the origin of *QxQ* such that *(0, 0)* satisfies the countable iteration, but not its *[1]*-prefix version. Details are in the references (van Benthem & Sarenac 2004).

Compare the earlier lack of infinite distributivity.

Barwise's separation achieved for 2 of the 3 notions! Third ~ product topology?

10 Fixed-point logic

Approximation sequence for # on relational models always stops by stage ω by the

distributive syntactic format with *[], &* only.

On topological models, because of our truth definition, that syntactic format is really of the more complex form <>***[]***, *&*, which has no such distributivity.

Open: fixed-point logics like *dynamic logic*, *μ–calculus* in topological semantics?

11 Operations on topologies

Topologies as agents: new topologies are then like further (group) agents.

Intersection of horizontal and vertical topologies, union, products. Which operations ***O*** 'make sense'? Borrow idea from process logics. Safety for topo-bisimulation: if *A* has a bisimulation with *A'*, and *B* with *B'*, then so do ***O****(A, B)* and ***O****(A', B')*. Intersection and product are safe, but e.g., taking the union topology is not.

12 Conclusions and open questions

Viewpoint 1: topological semantics for modal languages very much alive.

And the approach can be extended, especially in the guise of modal neighbourhood semantics, to semantics of games, belief revision, and so on.

But to which extent is it reducible to iterated relational complexes **<>*[]***?

Viewpoint 2: modal languages for topological structure very much alive..

And the approach can be extended to *affine* and *metric* structure, and *vector spaces*.

Will modal 'fragment methodology' be as successful here as it has been elsewhere?

Pictures on blackboard!

References

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