Relation Algebra Reducts of Cylindric Algebras and Complete Representations

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Abstract

We show that the class $\mathfrak{Ra}\mathbf{CA}_\gamma$, for any ordinal $\gamma \geq 3$, is pseudo-elementary and has a recursively enumerable elementary theory. $\mathbf{S}_c K$ denotes the class of strong subalgebras of members of the class K. We show, for $\gamma \geq 5$ and any class K of relation algebras satisfying

$$\mathfrak{Ra}\mathbf{RCA}_\gamma \subseteq K \subseteq \mathbf{S}_c\mathfrak{Ra}\mathbf{CA}_5,$$

that K is not closed under subalgebras and is not elementary. For infinite γ, the inclusion $\mathfrak{Ra}\mathbf{CA}_\gamma \subset \mathbf{S}_c\mathfrak{Ra}\mathbf{CA}_\gamma$ is strict.

For infinite γ and for a countable relation algebra $\mathcal{A}$ (or any atomic relation algebra with countably many atoms) we show that $\mathcal{A} \in \mathbf{S}_c\mathfrak{Ra}\mathbf{CA}_\gamma$ if and only if $\mathcal{A}$ has a complete representation.

1 Introduction

There are two kinds of algebras of relations, largely due to Alfred Tarski: *relation algebra* (although the history of relation algebra goes much further back [9, 3]) and n-dimensional *cylindric algebra*, for various n. Relation algebras are closely related to fields of binary relations and n-dimensional cylindric algebras are based on fields of n-ary relations. Both types of algebra have been studied intensively and are widely used.

For any n-dimensional cylindric algebra $\mathcal{C}$ ($n \geq 3$) the relation algebra reduct $\mathfrak{Ra}(\mathcal{C})$ can be defined by taking the two dimensional elements of $\mathcal{C}$ and using the third dimension to define converse and composition. The *relation algebra reduct* is the key tool for connecting cylindric algebras to relation algebras. Quite a lot is known about the class of subalgebras of relation algebra reducts of n-dimensional cylindric algebras: the class is a canonical variety, for example. The neat embedding theorem [5, 5.3.13, 5.3.16] says that a relation algebra is representable if and only if it is a subalgebra of a relation algebra reduct of an ω-dimensional cylindric algebra.

Much less is known about the class of relation algebra reducts of n-dimensional cylindric algebras (here we do not take subalgebras). Maddux and Németi independently proved that $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under subalgebras for $n \geq 4$

[7, 10] and Simon proved the same for $n = 3$ [13]. It is also known that the related classes $\mathfrak{Nr}_m\mathbf{CA}_n$ of neat reducts of cylindric algebras, for $1 < m < n$, are not closed under subalgebras [12]. In this article we will show, for $n \geq 5$, that $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under elementary equivalence. [A corresponding result was already established for neat reducts of cylindric algebras [11], but the construction used there does not seem to work for relation algebra reducts.] On the other hand, the class is pseudo-elementary, even for infinite n, and hence the elementary theory of this class is at least recursively enumerable.

The class **CRA** of *completely representable relation algebras* is known to be pseudo-elementary but not closed under elementary equivalence [6, theorem 17.6]. So the class cannot be defined by any first-order theory. A transfinite game can be defined that does characterise this class, but not everyone likes transfinite games.

In some ways these classes $\mathfrak{Ra}\mathbf{CA}_n$ $(n \geq 5)$ are similar to the class **CRA**, although the former is defined algebraicly and the latter is defined semantically. In each case the class is pseudo-elementary, but not pseudo-universal. None of these classes is elementary. In this paper we will establish another connection between the classes $\mathfrak{Ra}\mathbf{CA}_n$ and the class **CRA**, at least for countable algebras, by proving a complete version of the neat embedding theorem: a countable relation algebra has a complete representation if and only if it is atomic and it is a strong subalgebra of the relation algebra reduct of an ω-dimensional cylindric algebra. This can be thought of as an algebraic characterisation of **CRA**, at least for the countable case. Whether this characterisation works for uncountable algebras remains unknown.

2 Prerequisites

${}^{\gamma}X$ denotes the set of all functions from the ordinal γ to the set X. Equivalently, we may consider $\bar{x} \in {}^{\gamma}X$ as a sequence $(x_0, x_1, \ldots) = (x_i : i < \gamma)$, it will be implicit that the i'th element of $\bar{x}$ (or equivalently $\bar{x}(i)$) is x_i. If $\theta : X \to Y$ is any function and $\bar{x} \in {}^{\gamma}X$ then $\theta(\bar{x}) \in {}^{\gamma}Y$ is defined by $(\theta(\bar{x}))(i) = \theta(\bar{x}(i))$. We write ${}^{<\omega}X$ for $\bigcup_{n<\omega} {}^{n}X$.

We assume some knowledge about cylindric algebras and relation algebras [8, 4, 5, 6]. For any ordinal γ, $\mathbf{CA}_\gamma$ denotes the class of γ-dimensional cylindric algebras and $\mathbf{RCA}_\gamma$ denotes the class of algebras isomorphic to cylindric set algebras of dimension γ (the class of representable γ-dimensional cylindric algebras) [4, definition 1.1.1 & page 171]. **RA** is the class of all relation algebras. A relation algebra $\mathcal{A}$ is *representable* if there is a structure $\mathcal{M}$ in which each element $a \in \mathcal{A}$ is interpretted as a binary relation $a^{\mathcal{M}}$ so as to preserve all the relation algebra operators, e.g. $(a+b)^{\mathcal{M}} = a^{\mathcal{M}} \cup b^{\mathcal{M}}$ and $(a;b)^{\mathcal{M}} = a^{\mathcal{M}}|b^{\mathcal{M}}$, etc.

An atom of a boolean algebra with operators is a minimal non-zero element. A boolean algebra with operators is atomic if every non-zero element is above some atom. If $\mathcal{A}$ is an atomic relation algebra, the *relation algebra atom structure* $\mathsf{At}(\mathcal{A}) = (A, Id, \smile, C)$ consists of the set A of atoms of $\mathcal{A}$, the set Id of atoms below the identity of $\mathcal{A}$, the function $\smile$ that takes an atom to its converse, and the list C of consistent triples of atoms (a, b, c) — those where $a; b \geq c$. Since the relation algebra operators are completely additive, the atom structure suffices to define the operators over arbitrary elements of $\mathcal{A}$. The following properties always hold in a relation algebra atom structure [6, lemma 3.24]. For all $x, y, z, t \in A$,

- $x = y$ iff there is $e \in Id$ such that $(x, e, y) \in C$.
- If $(x, y, z) \in C$ then $(\breve{x}, z, y), (\breve{y}, \breve{x}, \breve{z}) \in C$.
- $(\exists u \in A\ ((x, y, u), (u, z, t) \in C)) \Leftrightarrow (\exists v \in A\ ((y, z, v), (x, v, t) \in C))$.

Conversely, if $\alpha = (A, Id, \smile, C)$ has the type of a relation algebra atom structure we can define the *complex algebra* of α, which has the type of a relation algebra, by $\mathfrak{Cm}(\alpha) = (\wp(A), \emptyset, A, \cup, \setminus, Id, \smile, ;)$, where the converse operator $\smile$ is extended from atoms to sets of atoms by $S^\smile = \{s^\smile : s \in S\}$, and composition of sets of atoms is defined by $S; T = \{a \in \alpha : \exists s \in S,\ \exists t \in T,\ (s, t, a) \in C\}$, where $S, T \subseteq A$.

For later use, we recall some facts about cylindric algebras and the substitution operators. Let $\mathcal{C} \in \mathbf{CA}_n$, $i, j < n$ and $x \in \mathcal{C}$. Define

$$s^i_j x = \begin{cases} x & \text{if } i = j \\ c_i(d_{ij} \cdot x) & \text{otherwise} \end{cases}$$

FACT 1 *Let $\mathcal{C} \in \mathbf{CA}_n$ (some $n \geq 3$), $x, y \in \mathcal{C}$, $i, j, k, l < n$.*

1. *s^i_j is a completely additive endomorphism of $\mathcal{C}$ [4, 1.5.3].*
2. *If $x \cdot c_i y = 0$ then $y \cdot c_i x = 0$ [4, 1.2.5].*
3. *If $k \notin \{i, j\}$ then $c_k s^i_j x = s^i_j c_k x$ [4, 1.5.8(ii)].*
4. *If $i \neq j$ then $c_i s^i_j x = s^i_j x$ [4, 1.5.9(ii)]*
5. *$c_j s^i_j x = c_i s^j_i x$ [4, 1.5.9(i)]*
6. *$s^i_j s^j_i x = s^i_j x$ [4, 1.5.10(v)]*

DEFINITION 2 *Let $\lambda \leq \mu$ be ordinals and let $\mathcal{C} \in \mathbf{CA}_\mu$. The* neat reduct *$\mathfrak{Nr}_\lambda(\mathcal{C}) \in \mathbf{CA}_\lambda$ has as its domain $\{x \in \mathcal{C} : \lambda \leq i < \beta \to c_i x = x\}$ and all the operators are inherited from $\mathcal{C}$. $\mathfrak{Nr}_\lambda \mathbf{CA}_\mu$ denotes the class $\{\mathfrak{Nr}(\mathcal{C}) : \mathcal{C} \in \mathbf{CA}_\mu\}$.*

Let $\lambda \geq 3$ and let $\mathcal{C} \in \mathbf{CA}_\lambda$. The relation algebra reduct *$\mathfrak{Ra}(\mathcal{C})$ is the algebra of the type of relation algebras whose domain is $\mathfrak{Nr}_2(\mathcal{C})$ with boolean operators inherited from $\mathcal{C}$ and with the relation algebra operators defined by*

$$\begin{aligned} 1' &= d_{01} \\ a^\smile &= s^2_0 s^0_1 s^1_2 a \\ a; b &= c_2(s^1_2 a \cdot s^0_2 b) \end{aligned}$$

for $a, b \in \mathfrak{Nr}_2(\mathcal{C})$. For $\lambda \geq 4$, $\mathfrak{Ra}(\mathcal{C})$ is a relation algebra [5, 5.3.8]. $\mathfrak{Ra}\mathbf{CA}_\lambda$ denotes the class $\{\mathfrak{Ra}(\mathcal{C}) : \mathcal{C} \in \mathbf{CA}_\lambda\}$.

LEMMA 3 *If* $3 \leq \lambda \leq \mu$ *then* $\mathfrak{Ra}(\mathbf{CA}_\mu) \subseteq \mathfrak{Ra}(\mathbf{CA}_\lambda)$.

PROPOSITION 4 ([6, 13.31]) *Let* $4 \leq \gamma$, $\mathcal{C} \in \mathbf{CA}_\gamma$, $i, j, k < \gamma$, $k \notin \{i, j\}$ *and* $\alpha, \beta, \gamma \in \mathfrak{Ra}(\mathcal{C})$.

$$s_i^0 s_j^1(\alpha ; \beta) = c_k(s_i^0 s_k^1 \alpha \,.\, s_k^0 s_j^1 \beta)$$

DEFINITION 5 *A* representation $\mathcal{M}$ *with base* D *of a boolean algebra* $\mathcal{B}$ *interprets each element* $b \in \mathcal{B}$ *as a subset of* D *such that* $0^{\mathcal{M}} = \emptyset$, $1^{\mathcal{M}} = D$, $(-b)^{\mathcal{M}} = D \setminus b^{\mathcal{M}}$ *and* $(b + b')^{\mathcal{M}} = b^{\mathcal{M}} \cup b'^{\mathcal{M}}$, *for all* $b, b' \in \mathcal{B}$.

A complete representation $\mathcal{M}$ *of* $\mathcal{B}$ *has the additional property that, for any subset* X *of the universe of* $\mathcal{B}$, *if the supremum* $\sum^{\mathcal{B}} X$ *exists in* $\mathcal{B}$ *then* $(\sum^{\mathcal{B}} X)^{\mathcal{M}} = \bigcup_{b \in X} b^{\mathcal{M}}$.

For any boolean algebras with operators $\mathcal{A} \subseteq \mathcal{B}$ *we say "*$\mathcal{A}$ *is a strong subalgebra of* $\mathcal{B}$*" and we write* $\mathcal{A} \subseteq_c \mathcal{B}$ *if whenever the supremum* $\sum^{\mathcal{A}} X$ *exists in* $\mathcal{A}$ *then the supremum exists in* $\mathcal{B}$ *and* $\sum^{\mathcal{A}} X = \sum^{\mathcal{B}} X$.

For any class K *of similar boolean algebras with operators, we write* $\mathbf{P}K, \mathbf{S}K$ *(respectively) for the class of direct products of members of* K *and the class of subalgebras of members of* K. *We write* $\mathbf{S_c}K$ *for* $\{\mathcal{A} : \exists \mathcal{B} \in K,\ \mathcal{A} \subseteq_c \mathcal{B}\}$.

It is easy to show, using the De Morgan laws, that if $\mathcal{M}$ is a complete representation of $\mathcal{B}$ then infima are also preserved in $\mathcal{M}$: if $\prod^{\mathcal{B}} X$ exists then $(\prod^{\mathcal{B}} X)^{\mathcal{M}} = \bigcap_{b \in X} b^{\mathcal{M}}$. Similarly, if $\mathcal{A} \subseteq_c \mathcal{B}$ then whenever $\prod^{\mathcal{A}} X$ exists then $\prod^{\mathcal{B}} X = \prod^{\mathcal{A}} X$ also exists.

LEMMA 6 ([6, theorem 2.21]) *Let* $\mathcal{B}$ *be a boolean algebra and let* $\mathcal{M}$ *be a representation of* $\mathcal{B}$. *The following are equivalent.*

- $\mathcal{M}$ *is a* complete *representation of* $\mathcal{B}$.
- $\mathcal{M}$ *is an* atomic *representation of* $\mathcal{B}$ *i.e.* $1^{\mathcal{M}} = \bigcup\{\beta^{\mathcal{M}} : \beta \in \mathsf{At}(\mathcal{B})\}$.

LEMMA 7 ([6, lemma 2.16]) *If* $\mathcal{B}$ *is an atomic boolean algebra and* $\mathcal{A} \subseteq_c \mathcal{B}$ *then* $\mathcal{A}$ *is atomic too.*

PROOF:

Suppose that $\mathcal{B}$ is atomic but $\mathcal{A}$ is not. Then there is $a \in \mathcal{A}$ with $a \neq 0$ with no atom of $\mathcal{A}$ below a. But $\mathcal{B} \supseteq \mathcal{A}$ is atomic so there is $\beta \in \mathsf{At}(\mathcal{B})$ with $\beta \leq a$. Let $F = \{r \in \mathcal{A} : \beta \leq r\}$. We have $a \in F$. Then $\prod^{\mathcal{A}} F = 0$ but $\beta \leq \prod^{\mathcal{B}} F$. Hence $\mathcal{A} \not\subseteq_c \mathcal{B}$. □

LEMMA 8 *If* $\mathcal{M}$ *is a complete representation of* $\mathcal{B}$ *and* $\mathcal{A} \subseteq_c \mathcal{B}$ *then* $\mathcal{M}$ *induces a complete representation of* $\mathcal{A}$.

PROOF:

Suppose for contradiction that $\mathcal{M}$ is a complete representation of $\mathcal{B}$ but it does not induce a complete representation of $\mathcal{A}$. By lemma 6 there are points $m, n \in \mathcal{M}$ with $(m, n) \in 1^{\mathcal{M}}$ but $(m, n) \notin \alpha^{\mathcal{M}}$ for all $\alpha \in \mathsf{At}(\mathcal{A})$. By the same lemma, $(m, n) \in \beta^{\mathcal{M}}$ for some $\beta \in \mathsf{At}(\mathcal{B})$. Let $F = \{a \in \mathcal{A} : \beta \leq a\}$. Then $\prod^{\mathcal{A}} F = 0$ but $\prod^{\mathcal{B}} F \geq \beta$. This contradicts $\mathcal{A} \subseteq_c \mathcal{B}$.

□

LEMMA 9 *Let $\mathcal{A} \subseteq \mathcal{B}$ be relation algebras and let $\mathcal{A}$ be atomic. $\mathcal{A} \subseteq_c \mathcal{B}$ if and only if for all $b \in \mathcal{B} \setminus \{0\}$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $a.b \neq 0$.*

PROOF:

If $b \in \mathcal{B} \setminus \{0\}$ and for all $a \in \mathsf{At}(\mathcal{A})$ $a.b = 0$ then $\sum^{\mathcal{A}} \mathsf{At}(\mathcal{A}) = 1$ but $\sum^{\mathcal{B}} \mathsf{At}\mathcal{A} \leq 1 - b$ if it exists, so $\mathcal{A} \not\subseteq_c \mathcal{B}$.

Conversely, if $\mathcal{A} \not\subseteq_c \mathcal{B}$ then there is a set $S \subseteq \mathcal{A}$ such that $\sum^{\mathcal{A}} S$ exists but there is $b \in \mathcal{B}$ with $b \not\geq \sum^{\mathcal{A}} S$ and b is an upper bound for S. But then, $b' = \sum^{\mathcal{A}} S - b \neq 0$ must be disjoint from all atoms of $\mathcal{A}$. □

LEMMA 10 *Let $n \geq 3$ and let $\mathcal{A}$ be an atomic relation algebra, $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$ for some $\mathcal{C} \in \mathbf{CA}_n$. For all $x \in \mathcal{C} \setminus \{0\}$ and all $i, j < n$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $s_i^0 s_j^1 a \,.\, x \neq 0$.*

PROOF:

Recall fact 1.1, that s_j^i is a completely additive operator. Hence $\sum\{s_j^i a : a \in \mathsf{At}(\mathcal{A})\} = s_j^i \sum \mathsf{At}(\mathcal{A}) = s_j^i 1 = 1$, for any $i, j < n$. Let $x \in \mathcal{C} \setminus \{0\}$. It is impossible that $s_i^0 s_j^1 a \,.\, x = 0$ for all $a \in \mathsf{At}(\mathcal{A})$ because this would imply that $1 - x$ was an upper bound for $\{s_i^0 s_j^1 a : a \in \mathsf{At}(\mathcal{A})\}$, contradicting $\sum\{s_i^0 s_j^1 a : a \in \mathsf{At}(\mathcal{A})\} = 1$. □

CRA denotes the class of completely representable relation algebras and **FRA** denotes the class of full relation algebras — the closure under isomorphism of the class of relation algebras of the form $\mathcal{F}(D) = (\wp(D \times D), D \times D, \emptyset, \cup, \setminus, Id_D, ^{\smile}, ;)$ for some domain D.

THEOREM 11 $\mathbf{CRA} = \mathbf{S_c P(FRA)}$.

PROOF:

Let $\mathcal{A} \in \mathbf{CRA}$ and let $\mathcal{M}$ be a complete representation of $\mathcal{A}$. $1^{\mathcal{M}}$ is an equivalence relation over the base of $\mathcal{M}$. Let E be the set of equivalence classes. For each $D \in E$ let $\mathcal{F}(D)$ be the full relation algebra over D.

The map $f : a \mapsto (a^{\mathcal{M}} \cap (D \times D) : D \in E)$ is an embedding of $\mathcal{A}$ into $\prod_{D \in E} \mathcal{F}(D)$. If $\emptyset \neq x = (b_D : D \in E) \in \prod_{D \in E} \mathcal{F}(D)$ then there is $D \in E$ and $d, e \in D$ such that $(d, e) \in x$. Since $\mathcal{M}$

is a complete representation of $\mathcal{A}$ and $(d,e) \in 1^{\mathcal{M}}$, by lemma 6 there is $a \in \mathsf{At}(\mathcal{A})$ such that $(d,e) \in a^{\mathcal{M}}$. Hence, for all non-zero $x \in \prod_{D \in E} \mathcal{F}(D)$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $x.a \neq 0$. By lemma 9, $\mathcal{A} \subseteq_c \prod_{D \in E} \mathcal{F}(D) \in \mathbf{P}(\mathbf{FRA})$. Thus $\mathbf{CRA} \subseteq \mathbf{S_c P}(\mathbf{FRA})$.

Conversely, let $\mathcal{A} \in \mathbf{S_c P}(\mathbf{FRA})$, say $\mathcal{A} \subseteq_c \prod_{D \in \Delta} \mathcal{F}(D)$, for some Δ. Now $\prod_{D \in \Delta} \mathcal{F}(D)$ is completely representable — the identity map is a complete representation of it. **CRA** is closed under strong subalgebras (lemma 8) so $\mathcal{A} \in \mathbf{CRA}$. Hence $\mathbf{S_c P}(\mathbf{FRA}) \subseteq \mathbf{CRA}$.

□

3 $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary

DEFINITION 12 *Let K be a class of structures in a signature L. We say that K is* pseudo-elementary *if there is a many-sorted signature L^s, where the signature L_1 of the first sort contains L, and some L-theory U such that $K = \{M^1\restriction_L : M \models U\}$. Here $M^1\restriction_L$ is the L-structure obtained from M by (a) restricting the domain to the first-sorted elements only and (b) restricting the language to L.*

THEOREM 13 *For any ordinal $\gamma \geq 3$ the class $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary.*

PROOF:

For finite γ it is quite easy to define $\mathfrak{Ra}\mathbf{CA}_\gamma$ in a two-sorted language. The first sort is for relation algebra elements and the second sort is for cylindric algebra elements. The defining theory includes sentences requiring the first-sorted elements to form a relation algebra and the second-sorted elements to form an γ-dimensional cylindric algebra. The signature of the defining theory also includes a function I from sort one to sort two and the defining theory includes a sentence requiring that I respects the operators (e.g. $I(1') = d_{01}$) and is injective. Finally, there is a sentence that says for any cylindric algebra element y such that $\bigwedge_{2 \leq i < \gamma} c_i y = y$ that there is a relation algebra element x such that $y = I(x)$. This ensures that I is a surjection onto the relation algebra reduct of the cylindric algebra.

For infinite γ this method won't work because the conjunction $\bigwedge_{2 \leq i \leq \gamma} c_i y = y$ is infinitary. Instead, we use a three sorted defining theory, with one sort for a relation algebra (r), the second sort for the boolean part of a cylindric algebra (b) and the third sort for a set of dimensions (δ). We will use superscripts r, b, δ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the relation algebra, the boolean part of the cylindric algebra or the dimension set, respectively. Our signature includes dimension sort constants i^δ, for each $i < \gamma$ to represent the dimensions. It also includes the relation algebra operators for the first sort, a function d^b taking two dimension sort arguments and returning a boolean sort element, and a function c^b taking one argument of sort δ and a second argument of sort b and returning an element of sort b. The defining theory for $\mathfrak{RaCA}_\gamma$ includes sentences demanding that the first sort is a relation algebra, that all constants i^δ for $i < \gamma$ are distinct, and that the last two sorts define a cylindric algebra of dimension at least γ. For example, in place of the cylindric algebra axiom $d_{ij} = c_k(d_{ik}.d_{kj})$ (all $i, j, k < \gamma$) we have the sentence

$$\forall x^\delta, y^\delta, z^\delta \ (d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta).^b d^b(z^\delta, y^\delta)))$$

(here $x^\delta, y^\delta, z^\delta$ are variables of sort δ, $.^b$ is the boolean intersection operator for cylindric algebras, henceforth we drop sort superscripts for boolean operators) with similar translations of the other cylindric algebra axioms. We also have a function I^b from sort r to sort b and sentences requiring I^b to be injective and to respect the relation algebra operations as follows: for all x^r, y^r,

$$\begin{aligned} I^b(1') &= d^b(0^\delta, 1^\delta) \\ I^b(x^r) &= s_0^2 s_1^0 s_2^1 I^b(x^{\smile r}) \\ I^b(x^r; y^r) &= c_2^b(s_2^1 I^b(x) \ . \ s_2^0 I^b(y)) \end{aligned}$$

where s_j^i the substitution operator from sort b to sort b. More precisely, an equation $x^b = s_j^i y^b$ abbreviates the formula

$$[(i^\delta = j^\delta) \to (x^b = y^b)] \wedge [(i^\delta \neq j^\delta) \to (x^b = c^b(d^b(i^\delta, j^\delta).y^b))]$$

Finally, we require that I^b maps *onto* the set of two dimensional elements:

$$\forall y^b \ ((\forall z^\delta (z^\delta \neq 0^\delta, 1^\delta \to c^b(z^\delta, y^b) = y^b)) \to \exists x^r \ (y^b = I^b(x^r)))$$

Clearly, any relation algebra $\mathcal{A} \in \mathfrak{RaCA}_\gamma$ is the first sort of a model of this theory. Conversely, a model of this theory will consist of a relation algebra (sort r) and a cylindric algebra whose dimension is the cardinality of the set of δ-sorted elements. This cardinality is at least γ since we required that all the constants $\{i^\delta : i < \gamma\}$ are distinct. So the first sort of a model will be the relation algebra reduct of a cylindric algebra of dimension $\gamma' \geq \gamma$. By lemma 3 this implies that the first sort of a model must belong to $\mathfrak{RaCA}_\gamma$. Hence this three sorted theory does define $\mathfrak{RaCA}_\gamma$. □

COROLLARY 14 *For $\gamma \geq 3$ the elementary theory of $\mathfrak{RaCA}_\gamma$ is recursively enumerable.*

PROOF:

The defining three-sorted theory in the proof of the previous theorem is recursive. Use [6, theorem 9.37]. □

Since $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary and the defining theory is recursive, it is possible to devise a two-player game $\Gamma(\mathcal{A})$ to test if a relation algebra $\mathcal{A}$ belongs to this class [6, definition 9.32, proposition 9.33]. The number of rounds in a play of $\Gamma(\mathcal{A})$ is the cardinal $|\mathcal{A}|+\omega$. In each of these rounds the first player, $\forall$, makes a move and the second player, $\exists$, has to respond. There are rules which stipulate which responses by $\exists$ are legal and which are not. If $\exists$ makes an illegal response in any round then $\forall$ wins the play, otherwise $\exists$ makes a legal response in every round and $\exists$ wins the play. $\exists$ has a winning strategy in $\Gamma(\mathcal{A})$ if and only if $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\gamma$.

For $n < \omega$, a shortened version of this game, $\Gamma_n(\mathcal{A})$, can be defined. This is very similar, but play stops after n rounds. If $\exists$ responds legally in each of the n rounds she wins the play, otherwise $\forall$ wins. [6, Propositions 9.34, 9.36] state (in the more general setting of arbitrary pseudo-elementary classes) that for each $n < \omega$ there is a first-order formula η_n in the signature of relation algebras such that $\exists$ has a winning strategy in $\Gamma_n(\mathcal{A})$ if and only if $\mathcal{A} \models \eta_n$, and that if $\exists$ has a winning strategy in $\Gamma_n(\mathcal{A})$ for all $n < \omega$ then $\mathcal{A}$ is elementarily equivalent to a member of $\mathfrak{Ra}\mathbf{CA}_\gamma$. Thus $\{\eta_n : n < \omega\}$ axiomatises the elementary theory of $\mathfrak{Ra}\mathbf{CA}_\gamma$.

However, the game $\Gamma(\mathcal{A})$ is not very easy to use in practice — it seems that games that use the atoms of an atomic boolean algebra with operators are easier to use then these more general games. Furthermore, we want to prove not only that $\mathfrak{Ra}\mathbf{CA}_\gamma$ is not elementary, but various other classes also fail to be elementary (see theorem 32). We also want to draw out the connection between relation algebra reducts and complete representations. For these reasons, we omit details of the game $\Gamma(\mathcal{A})$ and define three other games $F^n(\alpha), G(\alpha), H(\alpha)$ played on the *atom structure* of a relation algebra. The games are increasingly difficult for $\exists$ to win (and increasingly easy for $\forall$ to win). For countable α, a winning strategy for $\exists$ in $F^\omega(\alpha)$ is equivalent to $\alpha \in \mathsf{At}(\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega)$, a winning strategy for $\exists$ in $G(\alpha)$ implies that $\alpha \in \mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega)$ (we do not know if the converse is true) and a winning strategy for $\exists$ in $H(\alpha)$ is equivalent to $\alpha \in \mathsf{At}(\mathfrak{Ra}\mathbf{RCA}_\omega)$.

4 Games

DEFINITION 15 (Networks and Hypernetworks) *Let α be a relation algebra atom structure. A* network *over α (sometimes called an atomic network) is a complete labelled graph N whose nodes are $\mathsf{nodes}(N)$ with each edge labelled by an atom from α such that*

- $N(i,i) \leq 1'$,
- $N(j,i) = N(i,j)^\smile$,
- $N(i,j); N(j,k) \geq N(i,k)$

for all nodes $i,j,k \in \mathsf{nodes}(N)$.

A network N is strict *if $N(i,j) \leq 1' \iff i = j$.*

Define an equivalence relation $\sim$ over the set of all finite sequences over $\mathsf{nodes}(N)$ *by* $\bar{x} \sim \bar{y}$ *iff* $|\bar{x}| = |\bar{y}|$ *and* $N(x_i, y_i) \leq 1'$ *for all* $i < |\bar{x}|$.

A hypernetwork $N = (N^a, N^h)$ *consists of a network* N^a *together with a labelling function for hyperlabels* $N^h : {}^{<\omega}\mathsf{nodes}(N) \to \Lambda$ *(some set of labels* Λ*) such that for* $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(N)$ *if* $\bar{x} \sim \bar{y}$ *then* $N^h(\bar{x}) = N^h(\bar{y})$*. If* $|\bar{x}| = k \in \mathbb{N}$ *and* $N^h(\bar{x}) = \lambda$ *then we say that* λ *is a* k*-ary hyperlabel. [It is possible, though not very desirable, that* λ *could be a* k*-ary hyperlabel and an* l*-ary hyperlabel for* $k \neq l$*.]*

When there is no risk of ambiguity we may drop the superscripts a, h*.*

The following notation is defined for hypernetworks, but applies equally to networks. If N *is a hypernetwork and* S *is any set then* $N{\restriction_S}$ *is the* n*-dimensional hypernetwork defined by restricting* N *to the set of nodes* $S \cap \mathsf{nodes}(N)$*. For hypernetworks* M, N *if there is a set* S *such that* $M = N{\restriction_S}$ *then we write* $M \subseteq N$*. If* $N_0 \subseteq N_1 \subseteq \ldots$ *is a nested sequence of hypernetworks then we define the* limit $N = \bigcup_{i<\omega} N_i$ *to be the hypernetwork defined by* $\mathsf{nodes}(N) = \bigcup_{i<\omega} \mathsf{nodes}(N_i)$*,* $N^a(x, y) = N_i(x, y)$ *if* $x, y \in \mathsf{nodes}(N_i)$*, and* $N^h(\bar{x}) = N_i(\bar{x})$ *if* $\mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(N_i)$*. This is well-defined since the hypernetworks are nested and since hyperedges* $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ *are only finitely long.*

For hypernetworks M, N *and any set* S*, we write* $M \equiv^S N$ *if* $N{\restriction_S} = M{\restriction_S}$*. For hypernetworks* M, N *with the same set of nodes, and any set* S*, we write* $M \equiv_S N$ *if* $M \equiv^{\mathsf{nodes}(M) \setminus S} N$*. We write* $M \equiv_k N$ *for* $M \equiv_{\{k\}} N$*.*

Let N *be a network and let* θ *be any function. The network* $N\theta$ *is a complete labelled graph with nodes* $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$*, and labelling defined by* $(N\theta)(i, j) = N(\theta(i), \theta(j))$*, for* $i, j \in \theta^{-1}(\mathsf{nodes}(N))$*. Similarly, for a hypernetwork* $N = (N^a, N^h)$*, we define* $N\theta$ *to be the hypernetwork* $(N^a\theta, N^h\theta)$ *with hyperlabelling defined by* $N^h\theta(x_0, x_1, \ldots) = N^h(\theta(x_0), \theta(x_1), \ldots)$ *for* $(x_0, x_1, \ldots) \in {}^{<\omega}\theta^{-1}(\mathsf{nodes}(N))$*.*

Let M, N *be hypernetworks. A* partial isomorphism $\theta : M \to N$ *is a partial map* $\theta : \mathsf{nodes}(M) \to \mathsf{nodes}(N)$ *such that for any* $i, j \in \mathsf{dom}(\theta) \subseteq \mathsf{nodes}(M)$ *we have* $M^a(i, j) = N^a(\theta(i), \theta(j))$ *and for any finite sequence* $\bar{x} \in {}^{<\omega}\mathsf{dom}(\theta)$ *we have* $M^h(\bar{x}) = N^h\theta(\bar{x})$*. If* $M = N$ *we may call* θ *a partial isomorphism of* N*.*

A hyperedge $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ *of* N *is called* short *if there are* $y_0, y_1 \in \mathsf{nodes}(N)$ *and for all* $i < |\bar{x}|$ *either* $N(x_i, y_0) \leq 1'$ *or* $N(x_i, y_1) \leq 1'$*. Other hyperedges are called* long*. A hypernetwork* N *is called* λ-neat *if* $N(\bar{x}) = \lambda$*, for all short hyperedges* $\bar{x}$ *of* N*. If* N *is a* λ*-neat hypernetwork then* $N\theta$ *is a* λ*-neat hypernetwork.*

REMARK 16 *We will use* λ_0*-neat hypernetworks extensively in what follows. The idea is to keep a constant label (*λ_0*) on short hyperedges of the hypernetworks we use. These hypernetworks can be used to form the atoms of a cylindric algebra (at least in the finite dimensional case). The fact that short hyperlabels are constant means that the atoms of the relation algebra reduct of this cylindric algebra should be no smaller than the atoms of the original relation algebra. This will help us prove that the relation algebra is a relation algebra reduct of a cylindric algebra.*

DEFINITION 17 *For $n \geq 3$ and $\mathcal{C} \in \mathbf{CA}_n$, if $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ is an atomic relation algebra and N is an $\mathcal{A}$-network then we define $\widehat{N} \in \mathcal{C}$ by*

$$\widehat{N} = \bigwedge_{i,j \in \mathsf{nodes}(N)} s_i^0 s_j^1 N(i,j)$$

$\widehat{N} \in \mathcal{C}$ depends implicitly on $\mathcal{C}$. Where there might be confusion we will write $\widehat{N}^{\mathcal{C}}$ to make this explicit.

LEMMA 18 *Let $n \geq 3$, $\mathcal{C} \in \mathbf{CA}_n$ and let $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ be atomic. For any network N over $\mathcal{A}$ and $k \in n \setminus \mathsf{nodes}(N)$ we have $c_k\widehat{N} = \widehat{N}$.*

PROOF:

By facts 1.1 and 1.3. □

LEMMA 19 *Let $3 \leq n$, $\mathcal{C} \in \mathbf{CA}_n$ and let $\mathcal{A} \subseteq_c \mathfrak{Ra}\mathcal{C}$ be an atomic relation algebra.*

1. *For any $x \in \mathcal{C} \setminus \{0\}$ and any finite set $I \subseteq n$ there is a network N such that $\mathsf{nodes}(N) = I$ and $x \,.\, \widehat{N} \neq 0$.*
2. *For any networks M, N if $\widehat{M} \,.\, \widehat{N} \neq 0$ then $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$.*

PROOF:

The proof of the first part is based on repeated use of lemma 10. We define the edge labelling of N one edge at a time. Initially no edges are labelled. Suppose $E \subseteq \mathsf{nodes}(N) \times \mathsf{nodes}(N)$ is the set of labelled edges of N (initially $E = \emptyset$) and $x \,.\, \prod_{(i,j)\in E} s_i^0 s_j^1 N(i,j) \neq 0$. Pick an edge $(k,l) \in (I \times I) \setminus E$. By lemma 10 there is $a \in \mathsf{At}(\mathcal{A})$ such that $x \,.\, \prod_{(i,j)\in E} s_i^0 s_j^1 N(i,j) \,.\, s_k^0 s_l^1 a \neq 0$. Extend the labelling of N so that the edge (k,l) is now included in E. Eventually, all edges will be labelled, thus proving the first part of the lemma.

For the second part, if it is not true that $M \equiv^{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} N$ then there are $i,j \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ such that $M(i,j) \neq N(i,j)$. Since edges are labelled by atoms we have $M(i,j).N(i,j) = 0$ so $0 = s_i^0 s_j^1 0 = s_i^0 s_j^1 M(i,j) \,.\, s_i^0 s_j^1 N(i,j) \geq \widehat{M} \,.\, \widehat{N}$.
□

DEFINITION 20 (Games) *For any relation algebra atom structure α and $3 \leq n \leq \omega$, we define two-player games $F^n(\alpha)$, $G(\alpha)$ and $H(\alpha)$, each with ω rounds, and for $n < \omega$ we define $H_n(\alpha)$ with n rounds.*

- *In a play of $F^n(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \ldots$ where $\mathsf{nodes}(N_i)$ is a finite subset of $n = \{j : j < n\}$, for each i. In the initial round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$*

must play a network N_0 *with* $\mathsf{nodes}(N_0) \subseteq \{0,1\}$, *such that* $N_0(i,j) = a$ *for some* $i, j \in \mathsf{nodes}(N_0)$.

In a subsequent round of a play of $F^n(\alpha)$ $\forall$ *can pick a previously played network* N *and* $i, j \in \mathsf{nodes}(N)$, $k \in n \setminus \{i,j\}$, *and atoms* $a, b \in \alpha$ *such that* $a;b \geq N(i,j)$. *This move is called a* triangle move *and is denoted* (N,i,j,k,a,b). *In order to make a legal response,* $\exists$ *must play a network* $M \equiv_k N$ *such that* $M(i,k) = a$ *and* $M(k,j) = b$ *and* $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$.

$\exists$ *wins* $F^n(\alpha)$ *if she responds with a legal move in each of the* ω *rounds. If she fails to make a legal response in any round then* $\forall$ *wins.*

- $G(\alpha)$ *is similar to* $F^\omega(\alpha)$. *The initial round in a play of* $G(\alpha)$ *is the same as in a play of* $F^\omega(\alpha)$. *In any subsequent round* $\forall$ *can play a triangle move, as in* $F^\omega(\alpha)$ *and the rules for* $\exists$*'s response are the same. In* $G(\alpha)$, $\forall$ *has the option of playing a* permutation move (N,θ) *by picking a previously played network* N *and a partial finite surjection* $\theta : \omega \to \mathsf{nodes}(N)$. $\exists$ *must respond with* $N\theta$. *Also,* $\forall$ *can play an* amalgamation move (M,N) *by picking previously played networks* M, N *such that* $0 < |\mathsf{nodes}(M) \cap \mathsf{nodes}(N)| \leq 2$ *and* $M \equiv^{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} N$. *To make a legal response,* $\exists$ *must respond with some network* L *extending* M *and* N. *If she fails to make a legal response in any of the* ω *rounds of the play,* $\forall$ *wins. If she succeeds in each round, she wins.*

- *Fix some hyperlabel* λ_0. $H(\alpha)$ *is similar to* $G(\alpha)$ *and* $F^\omega(\alpha)$, *but in this game the play consists of a sequence of* λ_0*-neat hypernetworks* $N_0, N_1, \ldots$ *where* $\mathsf{nodes}(N_i)$ *is a finite subset of* ω, *for each* $i < \omega$. *In the initial round* $\forall$ *picks* $a \in \alpha$ *and* $\exists$ *must play a* λ_0*-neat hypernetwork* N_0 *with nodes contained in* $\{0,1\}$ *and* $N_0(i,j) = a$ *for some nodes* i, j. *At a later stage* $\forall$ *can make any triangle move* (N,i,j,k,a,b) *by picking a previously played hypernetwork* N *and* $i, j \in \mathsf{nodes}(N)$, $k \in \omega \setminus \mathsf{nodes}(N)$ *and* $a;b \geq N(i,j)$. *[In* H *we require that* $\forall$ *chooses* k *as a 'new node', i.e. not in* $\mathsf{nodes}(N)$, *whereas in* F^n *for finite* n *it was necessary to allow* $\forall$ *to 'reuse old nodes'.] For a legal response,* $\exists$ *must play a* λ_0*-neat hypernetwork* $M \equiv_k N$ *where* $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$ *and* $M(i,k) = a$ *and* $M(k,j) = b$. *Alternatively,* $\forall$ *can play a* permutation move *by picking a previously played hypernetwork* N *and a partial, finite surjection* $\theta : \omega \to \mathsf{nodes}(N)$, *this move is denoted* (N,θ). $\exists$ *must respond with* $N\theta$. *Finally,* $\forall$ *can play an* amalgamation move *by picking previously played hypernetworks* M, N *such that* $M \equiv^{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} N$ *and* $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. *This move is denoted* (M,N). *To make a legal response,* $\exists$ *must play a* λ_0*-neat hypernetwork* L *extending* M *and* N, *where* $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.

  Again, $\exists$ *wins* $H(\alpha)$ *if she responds legally in each of the* ω *rounds, otherwise* $\forall$ *wins.*

- *For* $n < \omega$ *the game* $H_n(\alpha)$ *is similar to* $H(\alpha)$ *but play ends after* n *rounds,* *so a play of $H_n(\alpha)$ could be*

$$N_0, N_1, \dots, N_n$$

If $\exists$ responds legally in each of these n rounds she wins, otherwise $\forall$ wins.

REMARK 21 *It will simplify things a bit if we alter the rules of the game $H(\alpha)$ slightly so that only strict hypernetworks are played. In the initial round if $\forall$ plays a then $\exists$ plays a hypernetwork N_0 where $\mathsf{nodes}(N_0) = \{0\}$ if $a \leq 1'$ and $\mathsf{nodes}(N_0) = \{0, 1\}$ otherwise. In the former case $N_0(0,0) = a$ and in the latter case the edge labelling is completely determined by $N_0(0,1) = a$.*

$\forall$ is only allowed to play permutation moves (N, θ) if θ is injective.

$\forall$ is only allowed to play a triangle move (N, i, j, k, a, b) if there does not exist $l \in \mathsf{nodes}(N)$ such that $N(i,l) = a$ and $N(l,j) = b$.

$\forall$ is only allowed to play an amalgamation move (M, N) if for all $m \in \mathsf{nodes}(M) \setminus \mathsf{nodes}(N)$ and all $n \in \mathsf{nodes}(N) \setminus \mathsf{nodes}(M)$ the map $\{(m,n)\} \cup \{(x,x) : x \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)\}$ is not a partial isomorphism. I.e. he can only play (M, N) if the amalgamated part is 'as large as possible'. If, as a result of these restrictions, $\forall$ cannot move at some stage then he loses and the game halts.

It is easy to check that $\forall$ has a winning strategy in $H(\alpha)$ iff he has a winning strategy with these restrictions to his moves. Also, if $\forall$ plays with these restrictions to his moves, if $\exists$ has a winning strategy then she has a winning strategy which only directs her to play strict hypernetworks. The same holds when we consider $H^n(\alpha)$. We will assume that $\forall$ plays according to these restrictions and $\exists$ only plays strict hypernetworks.

THEOREM 22 *Let $\gamma \geq \omega$. Let $\mathcal{A}$ be a relation algebra. With reference to the four conditions below, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $\mathcal{A}$ is atomic with countably many atoms then (4) $\Rightarrow$ (1) and all conditions are equivalent.*

1. *$\mathcal{A}$ has a complete representation.*
2. *There is an atomic representable cylindric algebra $\mathcal{C} \in \mathbf{RCA}_\gamma$ such that $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$.*
3. *$\mathcal{A}$ is atomic and for all $n < \omega$, $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$.*
4. *$\exists$ has a winning strategy in $F^\omega(\mathsf{At}(\mathcal{A}))$.*

PROOF:

(1) $\Rightarrow$ (2) Let $\mathcal{M}$ be a complete representation of $\mathcal{A}$. $1^{\mathcal{M}}$ must be an equivalence relation over the domain of $\mathcal{M}$. Let E be the set of equivalence classes of $1^{\mathcal{M}}$. For each equivalence class $D \in E$ pick an arbitrary sequence $f_D \in {}^\gamma D$. Let $W_D = \{f \in {}^\gamma D : \{i < \gamma : f(i) \neq f_D(i)\}$ is finite$\}$ and let $W = \bigcup_{D \in E} W_D$.

Note, for all $f \in W$, that there is $D \in E$ such that $f \in {}^{\gamma}D$. Let $\mathcal{C} = (\wp(W), \emptyset, W, \cup, \setminus, d_{ij}, c_i : i, j < \gamma) \in \mathbf{RCA}_\gamma$, where $d_{ij} = \{f \in W : f(i) = f(j)\}$ and $c_i x = \{f \in W : \exists m \in \mathcal{M},\ f[i/m] \in x\}$ (here $f[i/m]$ is the function identical to f on all values other than i and $f[i/m](i) = m$). $\mathcal{C}$ is an atomic cylindric set algebra — the atoms are the singleton sets.

If $f, g \in W$ and $g(0) = f(0)$ then there is $D \in E$ (the equivalence class containing $f(0)$) with $f, g \in W_D$. Hence, by definition of W_D, the set $\{i < \gamma : f(i) \neq g(i)\}$ is finite.

Let $x \in \mathfrak{Ra}(\mathcal{C})$, i.e. $c_i x = x$ for $2 \leq i < \gamma$. If $f \in x$ and $g \in W$ satisfies $g(0) = f(0),\ g(1) = f(1)$ then $g \in x$ since, as we just saw, $\{2 \leq i < \gamma : f(i) \neq g(i)\}$ is finite. It follows that for any $m, n \in \mathcal{M}$ we have $\{g \in W : g(0) = m,\ g(1) = n\} \in \mathsf{At}(\mathfrak{Ra}(\mathcal{C}))$ and $\mathfrak{Ra}(\mathcal{C})$ is atomic. $\mathcal{A}$ embeds into $\mathfrak{Ra}(\mathcal{C})$ by $I : a \mapsto \{f \in W : \mathcal{M} \models a(f(0), f(1))\}$. Now, for any atom $\{g \in W : g(0) = m,\ g(1) = n\} \in \mathsf{At}(\mathfrak{Ra}(\mathcal{C}))$, m, n belong to the same equivalence class of $1^{\mathcal{M}}$ so $(m, n) \in 1^{\mathcal{M}}$. By lemma 6 $\mathcal{A}$ is atomic and there is $a \in \mathsf{At}(\mathcal{A})$ such that $\mathcal{M} \models a(m, n)$. Hence $\{g \in W : g(0) = m,\ g(1) = n\} \subseteq I(a)$. By lemma 9, $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$.

(2) ⇒ (3) Trivial (use lemma 7 for atomicity of $\mathcal{A}$, use lemma 3 for $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_n$).

(3) ⇒ (4) For each $n < \omega$ let $\mathcal{A} \subseteq_c \mathfrak{Ra}\mathcal{C}_n$ for some $\mathcal{C}_n \in \mathbf{CA}_n$. We have to show that $\exists$ has a winning strategy in $F^\omega(\mathsf{At}(\mathcal{A}))$.

$\exists$'s strategy, roughly, is to always play networks N such that $\widehat{N} \neq 0$. But note that $\widehat{N}$ cannot be evaluated in an n-dimensional cylindric algebra $\mathcal{C}_n$ unless $\mathsf{nodes}(N) \subseteq n$. Since the evaluation of $\widehat{N}$ depends on the cylindric algebra we are working with, we will write $\widehat{N}^{\mathcal{C}}$ for the element of the cylindric algebra $\mathcal{C}$ determined by $\widehat{N}$, for this section only. So, more accurately, her strategy is to play networks N such that $\widehat{N}^{\mathcal{C}_n} \neq 0$ for infinitely many (we could in fact prove this for cofinitely many) $n < \omega$ where $\mathsf{nodes}(N) \subseteq n$.

In the initial round, let $\forall$ play $a \in \mathsf{At}(\mathcal{A})$. $\exists$ plays the network N_a with nodes $\{0, 1\}$ and labelling determined by $N_a(0, 1) = a$. Then $\widehat{N_a}^{\mathcal{C}_n} = a \neq 0$, for all $n \geq 3$.

At a later stage suppose $\forall$ plays the triangle move (N, i, j, k, a, a'), where $k \neq i, j$, $a; a' \geq N(i, j)$ and N was previously played so $\widehat{N}^{\mathcal{C}_n} \neq 0$ for infinitely many $n < \omega$. By proposition 4, $c_k(s_i^0 s_k^1 a \ .\ s_k^0 s_j^1 a') = s_i^0 s_j^1 (a; a') \geq s_i^0 s_j^1 N(i, j) \geq \widehat{N}^{\mathcal{C}_n}$ in $\mathcal{C}_n$, for all such n. By lemma 18, $c_k \widehat{N}^{\mathcal{C}_n} = \widehat{N}^{\mathcal{C}_n}$, provided $k < n$. Therefore $c_k(s_i^0 s_k^1 a \ .\ s_k^0 s_j^1 a') \geq c_k \widehat{N}^{\mathcal{C}_n}$ holds for infinitely many n, and hence $x = s_i^0 s_k^1 a \ .\ s_k^0 s_j^1 a' \ .\ c_k \widehat{N}^{\mathcal{C}_n} \neq 0$, for infinitely many n, by fact 1.2.

By lemma 19, for each of this infinite set of finite ordinals n, there is a network M where $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$ such that $\widehat{M}^{\mathcal{C}_n} \,.\, c_k \widehat{N}^{\mathcal{C}_n} \,.\, s^{01}_{ik} a \,.\, s^{01}_{kj} a' \neq 0$. It follows (use lemma 19) that $M \equiv_k N$, $M(i,k) = a$ and $M(k,j) = a'$. $\exists$ plays such a network M. Thus $\exists$ can preserve the condition $\widehat{M}^{\mathcal{C}_n} \neq 0$ for infinitely many n. This condition suffices to prove that M is a consistent network, so M is a legal move.

Now suppose $\mathcal{A}$ is atomic with countably many atoms. The implication (4) $\Rightarrow$ (1) is [6, theorem 11.7(2)], or see lemma 26 for a very similar proof.

□

PROBLEM 23 *If $\mathcal{A}$ is atomic (but possibly has uncountably many atoms) and $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ (some $\gamma \geq \omega$) does it follow that $\mathcal{A}$ has a complete representation? It is known that the implication (4) $\Rightarrow$ (1) theorem 22 can fail for uncountable algebras (e.g. let $\mathcal{A}$ be the rainbow algebra $\mathcal{A}_{\omega_1,\omega}$, then show that $\exists$ has a winning strategy in $F(\mathsf{At}(\mathcal{A}))$ but $\mathcal{A}$ is not completely representable [6, theorem 17.25], but we do not know if $\mathcal{A} \in \mathbf{S_c}\mathbf{CA}_\omega$).*

For finite $n < \omega$ an n-dimensional version of this theorem can also be obtained, but instead of classical representations we have to use 'n-square representations' [6, definition 5.7]. But we do not have to follow that particular deviation, we only need the n-dimensional version of part of the preceding theorem.

THEOREM 24 *Let $3 \leq n < \omega$ and let $\mathcal{A}$ be an atomic relation algebra. If $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ then $\exists$ has a winning strategy in $F^n(\mathcal{A})$.*

The proof is very similar to the implication (3) $\Rightarrow$ (4) of theorem 22. If $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_n$ then $\exists$ always plays hypernetworks N with $\mathsf{nodes}(N) \subseteq n$ such that $\widehat{N}^{\mathcal{C}} \neq 0$. We omit the details.

The theorems above help us determine if an atomic relation algebra is a strong subalgebra of a member of $\mathfrak{Ra}\mathbf{CA}_\gamma$. The next theorem uses the game G and can be used to prove that an atom structure is not in $\mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega)$. We do not have much use for this game (or the following theorem) except to prove that the inclusion $\mathfrak{Ra}\mathbf{CA}_\gamma \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ is strict, for infinite γ.

THEOREM 25 *Let α be a relation algebra atom structure. If $\alpha \in \mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega)$ then $\exists$ has a winning strategy in $G(\alpha)$.*

PROOF:

Assume $\alpha = \mathsf{At}(\mathfrak{Ra}\mathcal{C})$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. For all $a \in \alpha$ and $x \in \mathfrak{Ra}\mathcal{C}$ if $a.x \neq 0$ then $a \leq x$, by atomicity of a. By considering $x = s^j_1 s^i_0 y$ it follows, using facts 1.1, 1.3, 1.5, 1.6, that for all $i < j < \omega$, $a \in \alpha$ and $y \in \mathcal{C}$,

and some other facts

$$[(\forall k \in \omega \setminus \{i,j\} c_k y = y) \wedge y \,.\, s^0_i s^1_j a \neq 0] \rightarrow s^0_i s^1_j a \leq y \qquad (1)$$

$\exists$'s strategy is to always play networks N such that $\widehat{N} \neq 0$.

As in the proof of theorem 22(3) $\Rightarrow$ (4), $\exists$ can always play N such that $\widehat{N} \neq 0$ in the initial round and in response to any triangle move by $\forall$. If $\forall$ plays the permutation move (N, θ) and $\exists$ responds with $N\theta$ then it can be shown, using the fact that θ is not surjective onto ω and by [6, lemma 13.31], that there is a string σ_θ of substitutions and cylindrifications such that $\widehat{N\theta} = \sigma_\theta \widehat{N} \neq 0$.

If $\forall$ plays an amalgamation move (M, N) where $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) = \{i, j\}$ (some i, j, possibly equal) then $M(i,j) = N(i,j)$. First we suppose $i \neq j$, without loss $i < j$. Let $i_0, \ldots, i_k$ enumerate $\mathsf{nodes}(M) \setminus \{i,j\}$ and let $j_0, \ldots, j_l$ enumerate $\mathsf{nodes}(N) \setminus \{i,j\}$. By lemma 18,

$$\begin{aligned} c_{j_0} \ldots c_{j_l} \widehat{M} &= \widehat{M} \\ c_{i_0} \ldots c_{i_k} \widehat{N} &= \widehat{N} \end{aligned}$$

and by facts 1.3 and 1.4,

$$\begin{aligned} 0 \neq c_{i_0} \ldots c_{i_k} \widehat{M} &\leq c_{i_0} \ldots c_{i_k} s_i^0 s_j^1 M(i,j) \\ &= s_i^0 s_j^1 M(i,j) \end{aligned}$$

Therefore, by (1) applied to $M(i,j)$, $s_i^0 s_j^1 M(i,j) \leq c_{i_0} \ldots c_{i_k} \widehat{M}$, so

$$c_{i_0} \ldots c_{i_k} \widehat{M} = s_i^0 s_j^1 M(i,j) = s_i^0 s_j^1 N(i,j) = c_{j_0} \ldots c_{j_l} \widehat{N}$$

Hence

$$c_{j_0} \ldots c_{j_l} \widehat{M} = \widehat{M} \leq c_{i_0} \ldots c_{i_k} \widehat{M} = c_{j_0} \ldots c_{j_l} \widehat{N}$$

By fact 1.2, it follows that

$$x = \widehat{M} \,.\, \widehat{N} \neq 0$$

If $i = j$ then $\widehat{M} \,.\, \widehat{N} \neq 0$ still holds, by fact 1.??.

By lemma 19 there is a network L with $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N) \neq 0$ and $\widehat{L} \,.\, x \neq 0$. We have $\widehat{L} \,.\, \widehat{M} = \widehat{M}$ so by lemma 19 $\widehat{L} \equiv^{\mathsf{nodes}(M)} \widehat{M}$. It follows that $L \supseteq M$ and similarly $L \supseteq N$, so L is a legal response to the amalgamation move.

□

LEMMA 26 *If α is a countable relation algebra atom structure and $\exists$ has a winning strategy in $G(\alpha)$ then $\mathfrak{Cm}(\alpha)$ has a complete representation in which for any partial isomorphism ι of size two or less and any finite subset X of the domain of the representation there is a partial isomorphism θ extending ι with X contained within its range.*

PROOF:

Suppose $\forall$ plays $a \in \alpha$ in the initial round, $\exists$ uses her winning strategy and $\forall$ plays all possible moves in the course of the play. Since α is countable, this can be done. Observe that if N occurs in the play, $i, j, i', j' \in \mathsf{nodes}(N)$ and $N(i,j) = N(i',j')$, then at some point $\forall$ plays the permutation move (N, θ) where θ is a finite surjection onto $\mathsf{nodes}(N)$, extending $\{(i',i),(j',j)\}$ and $\mathsf{dom}(\theta) \cap \mathsf{nodes}(N) = \{i', j'\}$. So $N\theta$ occurs in the play. At some point $\forall$ plays the amalgamation move $(N, N\theta)$ and $\exists$ responds with M, say. Note that the partial isomorphism $\{(i',i),(j',j)\}$ of M extends to a partial isomorphism θ where $\mathsf{rng}(\theta) = \mathsf{nodes}(N)$.

Now we define a nested subsequence of the play. Start with N_0, the network $\exists$ plays in the initial round in response to $\forall$'s move a. Repeatedly extend the network so as to witness all triangle moves and so that if $\iota = \{(i',i),(j',j)\}$ is a partial isomorphism of one of the networks in the sequence (note that since the networks are nested, it is a partial isomorphism of all the networks that follow) then eventually we take the last played network N and extend it to a network M such that there is a partial isomorphism θ extending ι and $\mathsf{rng}(\theta) = \mathsf{nodes}(N)$. Let N_a be the limit of this sequence (see definition 15, this is well-defined since the sequence is nested). Observe that if $\iota = \{(i',i),(j',j)\}$ is any partial isomorphism of N_a and X is any finite subset of $\mathsf{nodes}(N_a)$ then

$$\text{there is a partial isomorphism } \theta \supseteq \iota,\ \mathsf{rng}(\theta) \supseteq X \qquad (2)$$

Rename the nodes, if necessary, so that $a \neq b \in \alpha$ implies $\mathsf{nodes}(N_a) \cap \mathsf{nodes}(N_b) = \emptyset$.

Now define a representation $\mathcal{N}$ of $\mathfrak{Cm}(\alpha)$ with base $\bigcup_{a\in\alpha} \mathsf{nodes}(N_a)$, by

$$S^{\mathcal{N}} = \{(i,j) : \exists a \in \alpha,\ s \in S,\ N_a(i,j) = s\}$$

for any subset S of α. By lemma 6, $\mathcal{N}$ is a complete representation of $\mathfrak{Cm}\alpha$. By (2), for every partial isomorphism ι of size two or less and every finite subset X of the domain of $\mathcal{N}$ there is a partial isomorphism $\theta \supseteq \iota$ with $\mathsf{rng}(\theta) \supseteq X$. $\Box$

These theorems provide techniques to prove that a relation algebra is not in $\mathfrak{Ra}\mathbf{CA}_n$. The next theorem will be useful to prove that an atomic relation algebra is in $\mathfrak{Ra}\mathbf{CA}_\gamma$ for $\gamma \geq \omega$.

Recall that $H(\alpha)$ is the hypernetwork game of definition 20 where the nodes of any hypernetwork played form a finite subset of ω. Also recall remark 21 which imposes restrictions on $\forall$'s moves and requires $\exists$ to always play strict hypernetworks.

THEOREM 27 *Let $\gamma \geq \omega$ and let α be a countable relation algebra atom structure. If $\exists$ has a winning strategy in $H(\alpha)$ then there is a cylindric set algebra $\mathcal{M}$ of dimension γ such that $\mathfrak{Ra}(\mathcal{M})$ is atomic and $\mathsf{At}(\mathfrak{Ra}(\mathcal{M})) \cong \alpha$.*

PROOF:

A minor complication arises due to the fact that α might be the atom structure of a non-simple relation algebra. Let C the the set of consistent triples of α. Define a binary relation $\sim$ over α by $a \sim b \iff [\exists c, d, f \in \alpha,\ (c, a, d), (d, f, b) \in C]$. The properties of relation algebra atom structures (see section 2) prove that $\sim$ is an equivalence relation. Let $A \subseteq \alpha$ contain exactly one atom from each $\sim$-equivalence class.

Suppose $\exists$ has a winning strategy in $H(\alpha)$. For $a \in A$, consider a play of $H(\alpha)$ in which $\forall$ plays a in the initial round, where $\forall$ plays all possible triangle moves eventually, i.e. if (N, i, j, k, b, b') is a legal $\forall$-move in round r then $\forall$ plays (N, i, j, k, b, b') is some round $s \geq r$, if (N_r, θ) is a legal permutation move in round r then $\forall$ plays (N_s, θ') for some $\theta' \supseteq \theta$ eventually, and if (M, N) is a legal amalgamation move in round r then $\forall$ plays (M, N) eventually. All these $\forall$-moves are subject to the restrictions imposed in remark 21. Since α is countable it is possible for $\forall$ to schedule all these moves in. Suppose that $\exists$ uses her winning strategy in the play. By remark 21 we can assume that she only plays strict hypernetworks.

Suppose that N occurs in the play and θ is a partial isomorphism from N to N. Also suppose that θ is maximal, in the sense that there is no partial isomorphism θ' of N, properly extending θ. Let θ^+ be any finite injective map from a subset of ω onto $\mathsf{nodes}(N)$, such that θ^+ extends θ, $\mathsf{rng}(\theta^+) = \mathsf{nodes}(N)$ and $\mathsf{dom}(\theta^+) \cap \mathsf{nodes}(N) = \mathsf{dom}(\theta)$. Since $\forall$ plays all permutation moves, eventually $N\theta^+$ occurs in the play. Now $N \equiv^{\mathsf{nodes}(N) \cap \mathsf{nodes}(N\theta^+)} N\theta^+$, so eventually $\forall$ plays the amalgamation move $(N, N\theta^+)$. Thus, if N occurs in the play and θ is a partial isomorphism of N then a network extending N and $N\theta^+$ also occurs in the play, for some injection $\theta^+ \supseteq \theta$ with $\mathsf{rng}(\theta^+) = \mathsf{dom}(N)$. Even if θ is a partial isomorphism, but not a maximal one, for N then by considering a maximal extension of θ we see that this property still holds.

The play of $H(\alpha)$ is unlikely to form a nested sequence of hypernetworks, because $\forall$ plays permutation moves, so we consider instead a nested subsequence of the play. Start with N_0. Repeatedly extend the hypernetwork so as to witness all triangle moves and so that if N occurs in the nested sequence and θ is a partial isomorphism of N then there is a bijection $\theta^+ \supseteq \theta$ from a finite subset of ω onto $\mathsf{nodes}(N)$ and a hypernetwork extending N and $N\theta^+$ also occurs in the sequence. By the previous observation, such extensions can be made. Let the nested sequence be $N_0 \subseteq N_1 \subseteq \ldots$ and let $N_a = \bigcup_{i<\omega} N_i$. This limit is well-defined since the hypernetworks are nested. Note, for $b \in \alpha$, that

$$(\exists i, j \in \mathsf{nodes}(N_a),\ N_a(i, j) = b) \iff b \sim a \tag{3}$$

Let θ be any finite partial isomorphism of N_a and let X be any finite subset of $\mathsf{nodes}(N_a)$. Since X is finite, there is $i < \omega$ such that $\mathsf{nodes}(N_i) \supseteq X \cup \mathsf{dom}(\theta)$. There is a bijection $\theta^+ \supseteq \theta$ onto $\mathsf{nodes}(N_i)$ and $j \geq i$ such that $N_j \supseteq N_i, N_i\theta^+$. Then θ^+ is a partial isomorphism of N_j and $\mathsf{rng}(\theta^+) = \mathsf{nodes}(N_i) \supseteq X$. Hence, if θ is any finite partial isomorphism of N_a and X is any finite subset of $\mathsf{nodes}(N_a)$ then

$$\exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_a \text{ where } \mathsf{rng}(\theta^+) \supseteq X \tag{4}$$

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of $\mathsf{nodes}(N_a)$ within its domain.

Let L be the signature with one binary predicate symbol (a) for each $a \in \alpha$, and one k-ary predicate symbol (λ) for each k-ary hyperlabel λ. [Notational point: if λ is k-ary and l-ary for $k \neq l$ then make one k-ary predicate symbol λ and one l-ary predicate symbol λ', so that every predicate symbol has a unique arity.] The set of variables for L-formulas is $\{x_i : i < \gamma\}$. Rename the nodes of the hypernetworks so that $a \neq b \in A \Rightarrow \mathsf{nodes}(N_a) \cap \mathsf{nodes}(N_b) = \emptyset$. For each $a \in A$ pick $f_a \in {}^{\gamma}\mathsf{nodes}(N_a)$. Let $U_a = \{f \in {}^{\gamma}\mathsf{nodes}(N_a) : \{i < \omega : g(i) \neq f_a(i)\}$ is finite$\}$.

We can make U_a into the base of an L-structure $\mathcal{N}_a$ and evaluate L-formulas at $f \in U_a$ as follow. For $b \in \alpha$, $i, j, i_0 \ldots, i_{k-1} < \gamma$, k-ary hyperlabels λ, and all L-formulas ϕ, ψ, let

$$\begin{aligned}
\mathcal{N}_a, f \models a(x_i, x_j) &\iff N_a(f(i), f(j)) = b \\
\mathcal{N}_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff N_a(f(i_0), \ldots, f(i_{k-1})) = \lambda \\
\mathcal{N}_a, f \models \neg\phi &\iff \mathcal{N}_a, f \not\models \phi \\
\mathcal{N}_a, f \models (\phi \vee \psi) &\iff \mathcal{N}_a, f \models \phi \text{ or } \mathcal{N}_a, f \models \psi \\
\mathcal{N}_a, f \models \exists x_i \phi &\iff \mathcal{N}_a, f[i/m] \models \phi \text{ some } m \in \mathsf{nodes}(N_a)
\end{aligned}$$

Let $\phi(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ be an arbitrary L-formula using only variables belonging to $\{x_{i_0}, \ldots, x_{i_k}\}$. Let $f, g \in \mathcal{N}_a$ (some $a \in A$) and suppose
$\{(f(i_0), g(i_0)), (f(i_1), g(i_1)), \ldots, (f(i_k), g(i_k))\}$ is a partial isomorphism of N_a. We can prove by induction over the quantifier depth of ϕ and using (4), that

$$\mathcal{N}_a, f \models \phi \iff \mathcal{N}_a, g \models \phi \tag{5}$$

Now define an L-structure $\mathcal{N}$ as the disjoint union of the structures $\mathcal{N}_a$ just defined. So $\mathcal{N}, f \models \phi$ if and only if there is $a \in A$ such that $f \in U_a$ and $\mathcal{N}_a, f \models \phi$. For each L-formula ϕ let $\phi^{\mathcal{N}} = \bigcup_{a \in A}\{f \in U_a : \mathcal{N}_a, f \models \phi\}$. Let $Form^{\mathcal{N}} = \{\phi^{\mathcal{N}} : \phi$ is an L-formula$\}$. We define a γ-dimensional cylindric set algebra

$$\mathcal{M} = (Form^{\mathcal{N}}, \bot, \top, \vee, \neg, 1'(x_i, x_j), \exists x_i : i, j < \gamma)$$

Note that $\mathcal{M}$ is not atomic (unless α is trivial).

Let $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{N}} \in \mathfrak{Ra}(\mathcal{M})$. Pick $f \in \phi^{\mathcal{N}}$ (so $f \in U_a$ for some $a \in A$) and let $b \in \alpha$ satisfy $N_a(f(0), f(1)) = b$. We will show that $b(x_0, x_1)^{\mathcal{N}} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{N}}$. For this, take any $g \in b(x_0, x_1)^{\mathcal{N}}$, so $N_a(g(0), g(1)) = b$. By (3), $b \sim a$ and $g \in U_a$. The map $\{(f(0), g(0)), (f(1), g(1))\}$ is a partial isomorphism of N_a — here it is crucial that short hyperedges have constant label λ_0. By (4) this extends to a finite partial isomorphism θ of N_a whose domain includes $f(i_0), \ldots, f(i_k)$. Let $g' \in U_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathsf{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

By (5), $\mathcal{N}_a, g' \models \phi(x_{i_0}, \ldots, x_{i_k})$. Observe that $g'(0) = \theta(0) = g(0)$ and similarly $g'(1) = g(1)$, so g is identical to g' over $\{0, 1\}$ and it differs from g' on only a finite set of coordinates. Since $\phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{N}} \in \mathfrak{Ra}(\mathcal{M})$ we deduce $\mathcal{N}_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{N}}$. This proves that $b(x_0, x_1)^{\mathcal{N}} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{N}}$.

Hence every non-zero element of $\mathfrak{Ra}(\mathcal{M})$ is above an element $a(x_0, x_1)^{\mathcal{N}}$ (some $a \in \alpha$) and these latter elements are atomic. It follows that $\mathfrak{Ra}(\mathcal{M})$ is atomic and $\alpha \cong \mathsf{At}(\mathfrak{Ra}(\mathcal{M}))$ — the isomorphism is $a \mapsto a(x_0, x_1)^{\mathcal{N}}$. $\square$

5 Rainbow algebra

DEFINITION 28 *We define a* rainbow algebra atom structure α *(in the terminology of [6,] it is very similar, though not identical, to* $\mathsf{At}(\mathcal{A}_{\mathbb{Z},\mathbb{N}})$*).*

Let F be the set of partial, order preserving functions $f : \mathbb{Z} \to \mathbb{N}$ where $|\mathsf{dom}(f)| \leq 2$. The atoms of α are $\{1', \mathsf{y}, \mathsf{b}, \mathsf{w}\} \cup \{\mathsf{g}_i : i \in \mathbb{Z}\} \cup \{\mathsf{w}_f : f \in F\} \cup \{\mathsf{r}_{ij} : i, j \in \mathbb{N}\}$. Non-identity atoms have colours: y is yellow, b is black, w, w_f are white, g_i is green and r_{ij} is red. All atoms are self-converse except the red atoms, for these $\mathsf{r}_{ij}^{\smile} = \mathsf{r}_{ji}$. Composition of atoms is defined by listing the forbidden triples of atoms *(the set of consistent triples of atoms is the complement in $\alpha \times \alpha \times \alpha$ of the set of forbidden triples). The forbidden triples (a, b, c) are those where $a, b, c \in \alpha$ and $a; b \not\geq c$. If (a, b, c) is a triple of atoms, its* Peircean transforms *are $(a, b, c), (b, c^{\smile}, a^{\smile}), (c^{\smile}, a, b^{\smile}), (b^{\smile}, a^{\smile}, c^{\smile}), (a^{\smile}, c, b), (c, b^{\smile}, a)$. If a triple of atoms is forbidden, all of its Peircean transforms are also forbidden.*

The forbidden triples of atoms of α are the Peircean transform of the following.

$$
\begin{array}{rll}
(1', x, y) & \textit{unless } x = y & (6) \\
(\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{g}_{i^*}), (\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{w}) & \textit{any } i, i', i^* \in \mathbb{Z} & (7) \\
(\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{w}_f) & \textit{any } i, i' \in \mathbb{Z}, \ \textit{any } f \in F & (8) \\
(\mathsf{y}, \mathsf{y}, \mathsf{y}), (\mathsf{y}, \mathsf{y}, \mathsf{b}) & & (9) \\
(\mathsf{g}_i, \mathsf{y}, \mathsf{w}_f) & \textit{unless } i \in \mathsf{dom}(f) & (10) \\
(\mathsf{g}_i, \mathsf{g}_j, \mathsf{r}_{kl}) & \textit{unless } \{(i,k),(j,l)\} \textit{ is an} & (11) \\
 & \quad \textit{order-preserving function} & \\
(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) & \textit{unless } i = i^*, \ j = j' \textit{ and } k' = k^* & (12)
\end{array}
$$

and no other triple of atoms is forbidden.

Let $\mathcal{A}$ be the complex algebra over α (so the domain of $\mathcal{A}$ consists of arbitrary sets of atoms).

We will show that $\mathcal{A} \notin \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$, but an elementary extension $\mathcal{A}'$ of $\mathcal{A}$ belongs to $\mathfrak{Ra}\mathbf{RCA}_\gamma$, for each $\gamma \geq \omega$.

LEMMA 29 *If $\mathsf{At}(\mathcal{A}) = \alpha$ then $\mathcal{A} \notin \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$.*

PROOF:

We prove that $\forall$ has a winning strategy in $F^5(\alpha)$, see figure 1. In the initial round $\forall$ plays w and $\exists$ must play a network N_0 with $N_0(0,1) = \mathsf{w}$. In the next round $\forall$ plays the triangle move $(N_0, 0, 1, 2, \mathsf{g}_0, \mathsf{y})$ and $\exists$ must play a network $N_1 \equiv_2 N_0$ with $N_1(0,2) = \mathsf{g}_0, \ N_1(2,1) = \mathsf{y}$. In the following round $\forall$ plays the triangle move $(N_1, 0, 1, 3, \mathsf{g}_1, \mathsf{y})$ and $\exists$ must play $N_2 \equiv_3 N_1$ with $N_2(0,3) = \mathsf{g}_1, \ N_2(3,1) = \mathsf{y}$. $\exists$ must choose an atomic label for the edge $(2,3)$ of N_2. By considering the triangle $(2,3,0)$ we see that the identity, a green atom or a white atom are impossible. From the triangle $(2,3,1)$ we see that the yellow atom or the black atom are impossible. So $\exists$ must let $N_2(2,3)$ be a red atom, say r_{mn} (some $m, n \in \mathbb{N}$) and since $0 < 1$ we must have $m < n$. In the next move $\forall$ plays the triangle move $(N_3, 0, 1, 4, \mathsf{g}_{-1}, \mathsf{y})$ and $\exists$ must play $N_3 \equiv_4 N_2$ such that $N_3(0,4) = \mathsf{g}_{-1}, \ N_3(4,1) = \mathsf{y}$. As before we must have $N_3(4,3)$ and $N_3(4,2)$ both being red atoms and from the triangle $(2,3,4)$ we see that the indices of these red atoms must match, so we have $N_3(4,3) = \mathsf{r}_{ln}, \ N_3(4,2) = \mathsf{r}_{lm}$, for some $l < m \in \mathbb{N}$.

In the next round $\forall$ plays $(N_3, 0, 1, 3, \mathsf{g}_{-2}, \mathsf{y})$ and $\exists$ must play $N_4 \equiv_3 N_3$ with $N_4(0,3) = \mathsf{g}_{-2}, \ N_4(3,1) = \mathsf{y}$. In figure 1, node 3 of N_4 is marked $3'$ to distinguish it from node 3 of N_3. This time we get $N_4(3,2) = \mathsf{r}_{jl}$ for some $j < l \in \mathbb{N}$. In this way $\forall$ can force an infinite descending sequence of natural numbers $n > m > l > j > \ldots$. This is impossible. Hence $\exists$ has no winning strategy.

By theorem 24, $\alpha \notin \mathsf{At}(\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5)$. $\square$

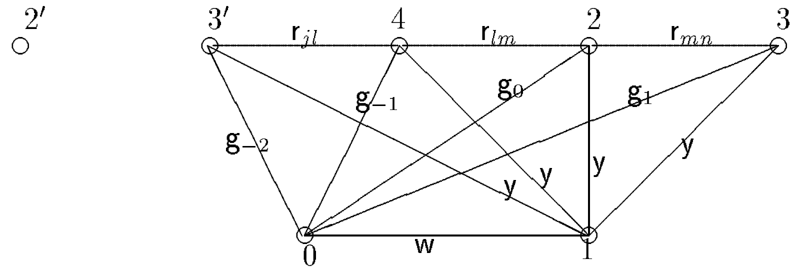

Figure 1: How $\forall$ can win $F^5(\alpha)$

Recall from definition 20 that $H_n(\alpha)$ is the hypernetwork game with n rounds.

LEMMA 30 *$\exists$ has a winning strategy in $H_n(\alpha)$, for any $n < \omega$.*

PROOF:

We need some more notation and terminology. Every irreflexive edge of any hypernetwork played in the game has an owner, $\forall$ or $\exists$. We call such edges $\forall$-edges or $\exists$-edges, as appropriate. And a long hyperedge $\bar{x}$ in a hypernetwork N occurring in the play has an *envelope* $\nu_N(\bar{x}) \subseteq \mathsf{nodes}(N)$.

In the initial round, if $\forall$ plays $a \in \alpha$ and $\exists$ plays N_0 then all irreflexive edges of N_0 belong to $\forall$. There are no long hyperedges in N_0. If, in a later round, $\forall$ plays the permutation move (N, θ) and $\exists$ responds with $N\theta$ then owners and envelopes are inherited in the obvious way: $(\theta(m), \theta(n))$ is a $\forall$-edge of N iff (m, n) is a $\forall$-edge of $N\theta$ (any $m \neq n \in \mathsf{dom}(\theta)$), and $\nu_N(\theta(\bar{x})) = \nu_{N\theta}(\bar{x})$ (any long hyperedge $\bar{x}$ of $N\theta$). If $\forall$ plays a triangle move (N, i, j, k, a, b) and $\exists$ responds with M then the owner in M of an edge not incident with the new node k is the same as it was in N and the envelope in M of a long hyperedge not incident with k is the same as it was in N. By remark 21 we know that $a \neq 1'$ and $b \neq 1'$. The edges $(i, k), (k, i), (j, k), (k, j)$ belong to $\forall$ in M, all edges $(l, k), (k, l)$ for $l \in \mathsf{nodes}(N) \setminus \{i, j\}$ belong to $\exists$ in M. If $\bar{x}$ is any long hyperedge of M with $k \in \mathsf{rng}(\bar{x})$ then $\nu_M(\bar{x}) = M$.

If $\forall$ plays the amalgamation move (M, N) and $\exists$ responds with L then, for $m \neq n \in \mathsf{nodes}(L)$, the owner of an edge (m, n) is $\exists$ in L if either (i) $m, n \in \mathsf{nodes}(M)$ and the owner of (m, n) is $\exists$ in M, (ii) $m, n, \in \mathsf{nodes}(N)$ and the owner of (m, n) is $\exists$ in N, or (iii) either $m \in \mathsf{nodes}(M) \setminus \mathsf{nodes}(N)$ and $n \in \mathsf{nodes}(N) \setminus \mathsf{nodes}(M)$ or the other way round. All other irreflexive edges of L belong to $\forall$. If $\bar{x}$ is a long hyperedge of L then

$$\nu_L(\bar{x}) = \begin{cases} \nu_M(\bar{x}) & \text{if } \mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(M) \\ \nu_N(\bar{x}) & \text{if } \mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(N),\ \mathsf{rng}(\bar{x}) \not\subseteq \mathsf{nodes}(M) \\ \mathsf{nodes}(M) & \text{otherwise} \end{cases}$$

In fact the first two parts of the following claim show that if $\bar{x} \subseteq \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ then $\nu_M(\bar{x}) = \nu_N(\bar{x})$. This completes the definition of owners and envelopes.

Claim: Let M, N occur in a play of $H(\alpha)$. Let $\bar{x}$ be a long hyperedge of M and let $\bar{y}$ be a long hyperedge of N.

1. For any hyperedge $\bar{x}'$ with $\mathsf{rng}(\bar{x}') \subseteq \nu_M(\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$.
2. If $\bar{x}$ is a long hyperedge of M and $\bar{y}$ is a long hyperedge of N and $M(\bar{x}) = N(\bar{y})$ then there is a local isomorphism $\theta : \nu_M(\bar{x}) \to \nu_N(\bar{y})$ such that $\theta(x_i) = y_i$, for $i < |\bar{x}|$.
3. For any $x \in \mathsf{nodes}(M) \setminus \nu_M(\bar{x})$ and $i, j, k \in \nu_M(\bar{x})$, if $(x, i), (x, j), (x, k)$ belong to $\forall$ in M then either $i = j$, $i = k$ or $j = k$.

The claim can be proved by a simple induction over the number of rounds taken before M and N are played.

$\exists$ uses the *default strategy* for choosing hyperlabels for long hyperedges, as follows. In response to a triangle move (N, i, j, k, a, b), all long hyperedges not incident with k necessarily keep the hyperlabel they had in N. We are assuming $a \neq 1'$ and $b \neq 1'$. All long hyperedges incident with k in M have unique hyperlabels, not occurring as the hyperlabel of any previously played hypernetwork and not occurring as the hyperlabel of any other hyperedge in M. We assume we have an infinite supply of hyperlabels of all finite arities, so this is possible. In response to an amalgamation move (M, N) all long hyperedges whose range is contained in $\mathsf{nodes}(M)$ have hyperlabel determined by M, and those whose range is contained in $\mathsf{nodes}(N)$ have hyperlabel determined by N. If $\bar{x}$ is a long hyperedge of $\exists$'s response L where $\mathsf{rng}(\bar{x}) \not\subseteq \mathsf{nodes}(M), \mathsf{nodes}(N)$ then $\bar{x}$ are given a new hyperlabel, not used in any previously played hypernetwork and not used within L as the label of any hyperedges other than $\bar{x}$.

In order to define $\exists$'s strategy it remains to explain how she chooses labels for edges in response to $\forall$-moves. Let $N_0, N_1, \ldots, N_r$ be the start of a play of $H_n(\alpha)$ just before round $r+1$ (where $r < n$). $\exists$ computes partial functions $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$. Inductively, for each $s \leq r$, suppose:

I. $\rho_0 \subseteq \ldots \subseteq \rho_r$,

II. $\mathsf{dom}(\rho_s) = \{i \in \mathbb{Z} : \exists t \leq s,\ x, y \in \mathsf{nodes}(N_t),\ N_t(x, y) = \mathsf{g}_i\}$.

III. ρ_s is order preserving: if $i < j \in \mathsf{dom}(\rho_s)$ then $\rho_s(i) < \rho_s(j)$. The range of ρ_s is 'widely spaced': if $i < j \in \mathsf{dom}(\rho_s)$ then $\rho_s(i), (\rho_s(j) - \rho_s(i)) \geq 3^{n-r}$ ($n - r$ is the number of rounds remaining in the game).

IV. If $u, v \in \mathsf{nodes}(N_s)$, $N_s(u, v) = \mathsf{r}_{\gamma,\delta}$, $N_s(x, u) = \mathsf{g}_i$, $N_s(x, v) = \mathsf{g}_j$, $N_s(y, u) = N_s(y, v) = \mathsf{y}$ and

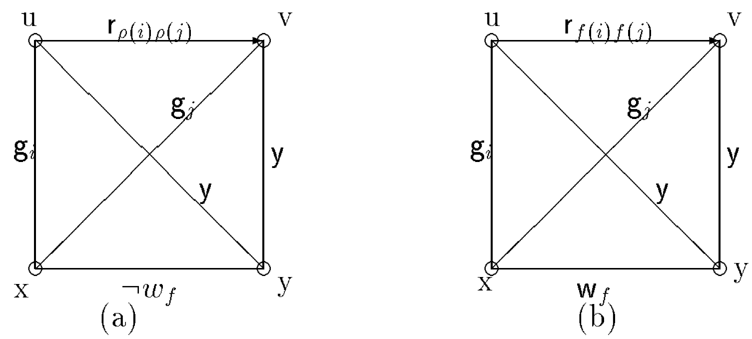

Figure 2: Property IV and red indices

(a) if $N_s(x, y) \neq \mathsf{w}_f$ (any $f \in F$) then $\rho_s(i) = \gamma$, $\rho_s(j) = \delta$,

(b) if $N_s(x, y) = \mathsf{w}_f$ (some $f \in F$) then $\gamma = f(i)$, $\delta = f(j)$.

See figure 2.

V. If $N_s(x, y)$ is green or yellow then (x, y) belongs to $\forall$ in N_s.

VI. N_s is a λ_0-neat hypernetwork.

To start with if $\forall$ plays $a \neq 1'$ in the initial round then $\mathsf{nodes}(N_0) = \{0, 1\}$, the edge labelling of N_0 is determined by $N_0(0, 1) = a$. If $\forall$ plays $1'$ then $\mathsf{nodes}(N_0) = \{0\}$ and $N_0(0, 0) = 1'$. If $a = \mathsf{g}_p$ (some $p \in \mathbb{Z}$) let $\rho_0 = \{(p, 3^n)\}$, otherwise let $\rho_0 = \emptyset$. All properties hold when $r = 0$.

Suppose the properties hold after round r (some $r < n$). We'll define how $\exists$ chooses atoms for new edges and maintains the properties above in response to a $\forall$-move in round $r + 1$. In response to a permutation move (N, θ) $\exists$ has nothing to do: her response, $N_{r+1} = N\theta$, is forced. There are no new edge labels, so she lets $\rho_{r+1} = \rho_r$.

In response to a triangle move $(N_s, i, j, k, \mathsf{g}_p, \mathsf{g}_q)$ by $\forall$ (some $s \leq r$ and some $p, q \in \mathbb{Z}$), $\exists$ must extend ρ_r to ρ_{r+1} so that $p, q \in \mathsf{dom}(\rho_{r+1})$ (property II) and the gap between elements of its range is at least 3^{n-r-1} (property III). Inductively, ρ_r is order-preserving and the gap between elements of its range is at least 3^{n-r}, so this can be maintained. If $\forall$ chooses non-green atoms, green atoms with the same suffix, or green atoms whose suffices already belong to $\mathsf{dom}(\rho_r)$, there would be fewer elements to add to the domain of ρ_{r+1} so it only makes it easier for $\exists$ to define ρ_{r+1}. This establishes properties (I–III) for round $r + 1$.

To choose edge labels in response to a triangle move by $\forall$, $\exists$ uses her normal strategy for rainbow algebras: she chooses a white atom if possible, else the black atom, and if neither of these are consistent then she chooses a red atom. The case where she has to choose a red atoms is the tricky one, the functions ρ_s and the suffix f in a label w_f will help her choose the suffices of red atoms for this case.

Let $\forall$ play the triangle move (N_s, i, j, k, a, b) in round $r+1$. $\exists$ has to choose labels for the edges $\{(x,k),(k,x) : x \in \mathsf{nodes}(N_s) \setminus \{i,j\}\}$. She chooses the label for the edges (x,k) one at a time, this then determines the label of the reverse edge (k,x) uniquely. She selects the first permissible option below.

1. Suppose it is not the case that $N_s(x,i)$ and a are both green, and it is not the case that $N_s(x,j)$ and b are both green. Let $S = \{p \in \mathbb{Z} : (N_s(x,i) = \mathsf{g}_p \wedge a = \mathsf{y}) \vee (N_s(x,i) = \mathsf{y} \wedge a = \mathsf{g}_p) \vee (N_s(x,j) = \mathsf{g}_p \wedge b = \mathsf{y}) \vee (N_s(x,j) = \mathsf{y} \wedge b = \mathsf{g}_p)\}$. Clearly $|S| \leq 2$.

   Suppose $N_s(x,i) = \mathsf{g}_p$, $N_s(x,j) = \mathsf{g}_q$, $a = b = \mathsf{y}$ (so $S = \{p,q\}$) and $N_s(i,j) = \mathsf{r}_{\beta,\gamma}$ (some $p,q \in \mathbb{Z}$, some $\beta,\gamma \in \mathbb{N}$). By property VI, and definition 28(11) for N_s, p and q must have the same order relation as β and γ. Suppose $p < q$ and $\beta < \gamma$, or $p > q$ and $\beta > \gamma$. $\exists$ lets $f = \{(p,\beta),(q,\gamma)\}$. If $p = q$ and $\beta = \gamma$ she lets $f = \{(p,\beta)\}$. If $N_s(x,i) = N_s(x,j) = \mathsf{y}$, $a = \mathsf{g}_p$, $b = \mathsf{g}_q$ and $N_s(i,j) = \mathsf{r}_{\beta,\gamma}$ then f is calculated in the same way. Otherwise (neither case above holds) then f can be an arbitrary order preserving function from S into $\mathbb{N}$ (e.g. if $S = \{p,q\}$ and $p < q$ let $f(p) = 0$, $f(q) = 1$).

   Having defined f $\exists$ lets $N_{r+1}(x,k) = \mathsf{w}_f$. This maintains property IV for round $r+1$. Property V is clear since w_f is neither green nor yellow.

   The only forbidden triples of atoms involving w_f are (8) and (10) of definition 28. Since $\exists$ does not choose green or yellow atoms to label new edges and $N_{r+1}(x,k) = \mathsf{w}_f$, all triangles involving the new edge (x,k) are consistent in N_{r+1}, so property VI holds after round $r+1$.

2. Else, if it is not the case that $N_s(x,i) = a = \mathsf{y}$ and it is not the case that $N_s(x,j) = b = \mathsf{y}$, $\exists$ lets $N_{r+1}(x,k) = \mathsf{b}$. Property V is clear (since b is not green or yellow) and all triangles (x,y,k) are consistent in N_{r+1}, so property VI holds after round $r+1$.

3. If neither case above apply, then either $N_s(x,i) = \mathsf{g}_p$, $a = \mathsf{g}_q$ (some p,q) and $N_s(x,j) = b = \mathsf{y}$ or $N_s(x,i) = a = \mathsf{y}$ and $N_s(x,j) = \mathsf{g}_p$, $b = \mathsf{g}_q$. Assume the first alternative. $\exists$ lets $N_{r+1}(x,k) = \mathsf{r}_{\gamma,\delta}$, where γ,δ remain to be specified. There are two subcases.

   (a) $N_s(i,j) = \mathsf{w}_f$ (some $f \in F$). By consistency of N_s we have $p \in \mathsf{dom}(f)$ and since $\forall$'s move was legal $a;b = \mathsf{g}_q;\mathsf{y} \geq N_s(i,j) = \mathsf{w}_f$ so $q \in \mathsf{dom}(f)$. $\exists$ lets $\gamma = f(p), \delta = f(q)$, maintaining property IVb for round $r+1$. The only forbidden triples of atoms involving $\mathsf{r}_{\gamma,\delta}$ are (11) and (12) of definition 28. Since f is order preserving, triangles involving the new edge (x, k) cannot give a forbidden triple of the form (11). For forbidden triple (12), let $y \in \mathsf{nodes}(N_s)$ and suppose $N_{r+1}(x, y), N_{r+1}(y, k)$ are both red. We have $y \notin \{i, j\}$ so $\exists$ chose the red label (y, k). By her strategy, we must have $N_s(y, i) = \mathsf{g}_t$ for some t (else she would have chosen a white label) and $N_s(y, j) = \mathsf{y}$ (else she would have chosen black). By consistency of N_s we have $t \in \mathsf{dom}(f)$ and by the current part of her strategy she let $N_{r+1}(y, k) = \mathsf{r}_{f(t),f(q)}$. By property IVb for N_s we have $N_{r+1}(x, y) = \mathsf{r}_{f(p),f(t)}$. So the triple of atoms from the triangle (x, y, k) is not forbidden by (12). This establishes property VI for N_{r+1}.

(b) $N_s(i, j) \neq \mathsf{w}_f$ (any $f \in F$). $\exists$ lets $\gamma = \rho_{r+1}(p)$, $\delta = \rho_{r+1}(q)$, maintaining property IVa. As above, the triple of atoms from a triangle (x, y, k) will not be forbidden by (11), so suppose $N_s(x, y), N_{r+1}(y, k)$ are both red. Since $\exists$ chooses a red atom for $N_{r+1}(y, k)$ and by her strategy, we must have $N_s(i, y) = \mathsf{g}_t$ (some t, else she would have chosen a white atom) and $N_s(j, y) = \mathsf{y}$ (else she would have chosen the black atom). By property (IVa) for N_{r+1} we have $N_{r+1}(x, y) = \mathsf{r}_{\rho_{r+1}(p),\rho_{r+1}(t)}$ and by her strategy $N_{r+1}(y, k) = \mathsf{r}_{\rho_{r+1}(t),\rho_{r+1}(q)}$, hence the triple of atoms from the triangle (x, y, k) is not forbidden by (12). Thus property VI holds for N_{r+1}.

Thus $\exists$ can maintain all the properties in round $r + 1$ in response to a triangle move by $\forall$.

Finally we consider an amalgamation move (N_s, N_t) by $\forall$ in round $r + 1$. Essentially, the claim (above) reduces this case to a case very similar to the triangle move case. $\exists$ has to choose a labels for each edge (i, j) where $i \in \mathsf{nodes}(N_s) \setminus \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \setminus \mathsf{nodes}(N_s)$ (this then determines the label for the reverse edge (j, i)).

Let $\bar{x}$ enumerate $\mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$. If $\bar{x}$ is long in N_s then by the claim there is a partial isomorphism $\theta : \nu_{N_s}(\bar{x}) \to \nu_{N_t}(\bar{x})$ fixing $\bar{x}$. By remark 21, since we are assuming that $\forall$ only plays 'maximal amalgamations', we see that $\nu_{N_s}(\bar{x}) = \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t) = \mathsf{rng}(\bar{x}) = \nu_{N_t}(\bar{x})$. It remains to label the edges (i, j) in N_{r+1} where $i \in \mathsf{nodes}(N_s) \setminus \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \setminus \mathsf{nodes}(N_s)$. Her strategy for labelling these edges is similar to her strategy for dealing with triangle moves. She chooses the labels for edges (i, j) one at a time. As before she chooses a white atom if possible, else the black atom if possible, otherwise a red atom. Since she never chooses a green atom, she lets $\rho_{r+1} = \rho_r$ and properties I, II and III remain true after round $r + 1$. She uses the first possible of the cases below.

1. There is no $x \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x)$ and $N_t(x,j)$ are both green. If there are $u, v \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(u,v) = \mathsf{r}_{\beta,\gamma}$, $N_s(i,u) = \mathsf{g}_p$, $N_s(i,v) = \mathsf{g}_q$, $N_t(u,j) = N_t(v,j) = \mathsf{y}$ (some $\beta, \gamma \in \mathbb{N}$, some $p, q \in \mathbb{Z}$) or the roles of i and j are swapped, she lets $f = \{(p,\beta),(q,\gamma)\}$ and sets $N_{r+1}(i,j) = \mathsf{w}_f$. Since all the edges labelled by green or yellow atoms belong to $\forall$ (property V), we can apply the claim to show that the points u, v are unique, so f is well-defined.

   If there are no such points u, v as just described then let $S = \{p \in \mathbb{Z} : \exists y \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t),\ (N_s(i,y) = \mathsf{g}_p \wedge N_t(y,j) = \mathsf{y}) \vee (N_s(i,y) = \mathsf{y} \wedge N_t(y,j) = \mathsf{g}_p)\}$. By the claim, $|S| \leq 2$. Let f be any order preserving function from S into $\mathbb{N}$. $\exists$ lets $N_{r+1}(i,j) = \mathsf{w}_f$. Property VI holds for N_{r+1}, as for triangle moves.

2. Otherwise, if there is no $x \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x) = N_t(x,j) = \mathsf{y}$, then she lets $N_r(i,j) = \mathsf{b}$. As with triangle moves, all properties are maintained.

3. Otherwise, there are $x, y \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x) = \mathsf{g}_k$, $N_t(x,j) = \mathsf{g}_l$ (some $k, l \in \mathbb{N}$) and $N_s(i,y) = N_t(y,j) = \mathsf{y}$. She labels (i,j) in N_r with a red atom $\mathsf{r}_{\beta,\gamma}$ where:

   (a) If $N_s(x,y) = \mathsf{w}_f$ (some $f \in F$). Then $\beta = f(k)$, $\gamma = f(l)$. This maintains property IVb.

   (b) Otherwise ($N_s(x,y) \neq \mathsf{w}_f$, all $f \in F$). Then $\beta = \rho_{r+1}(k)$, $\gamma = \rho_{r+1}(l)$. This maintains property IVa.

   In either case, we can show that property VI holds for N_{r+1}, as in the case of triangle moves.

□

6 Non-elementary classes

LEMMA 31 *There is a countable relation algebra $\mathcal{A}'$ such that $\mathcal{A}' \equiv \mathcal{A}$ and $\exists$ has a winning strategy in $H(\mathcal{A}')$.*

PROOF:

We have seen that for $n < \omega$ $\exists$ has a winning strategy σ_n in $H_n(\mathcal{A})$. We can assume that σ_n is deterministic. Let $\mathcal{B}$ be a non-principal ultrapower of $\mathcal{A}$. We can show that $\exists$ has a winning strategy σ in $H(\mathcal{B})$ — essentially she uses σ_n in the n'th component of the ultraproduct so that at each round of $H(\mathcal{B})$ $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. Now use an elementary chain argument to construct countable elementary subalgebras $\mathcal{A} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \mathcal{B}$. For this, let $\mathcal{A}_{i+1}$ be a countable elementary subalgebra of $\mathcal{B}$ containing $\mathcal{A}_i$

and all elements of $\mathcal{B}$ that σ selects in a play of $H_\omega(\mathcal{B})$ in which $\forall$ only chooses elements from $\mathcal{A}_i$. Now let $\mathcal{A}' = \bigcup_{i<\omega} \mathcal{A}_i$. This is a countable elementary subalgebra of $\mathcal{B}$ and $\exists$ has a winning strategy in $H(\mathcal{A}')$.

$\square$

THEOREM 32 *Let $\gamma \geq \omega$ and let K be any class of relation algebras with $\mathfrak{Ra}(\mathbf{RCA}_\gamma) \subseteq K \subseteq \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$. Then K is not closed under subalgebra or elementary equivalence, hence K is not an elementary class.*

PROOF:

The countable relation algebra $\mathcal{A}'$ of the previous lemma must belong to $\mathfrak{Ra}(\mathbf{RCA}_\gamma)$, by lemma 31 and theorem 27, hence $\mathcal{A}' \in K$. But $\mathcal{A} \notin K$ (lemma 29) and $\mathcal{A} \preceq \mathcal{A}'$. $\square$

THEOREM 33 *Let $\gamma \geq \omega$. The inclusion $\mathfrak{Ra}\mathbf{CA}_\gamma \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ is strict.*

PROOF:

A relation algebra is integral if its identity is an atom. A permutational representation of an integral relation algebra is one in which, for any pair of points x, y, there is an automorphism of the representation taking x to y (in model theory this kind of representation is sometimes called *transitive*). An integral relation algebra is called non-permutational if none of its representations is permutational. In [1] a finite, integral, representable, non-permuational relation algebra $\mathcal{A}$ is defined and it is shown that the representations of $\mathcal{A}$ are all finite (they have size 45). Since $\mathcal{A}$ is finite and representable it is completely representable, so by theorem 22 it belongs to $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$.

Since all representations of $\mathcal{A}$ are finite and not permutational, in any representation of $\mathcal{A}$ there is a partial isomorphism of size one that does not extend to an automorphism of the representation. Hence, by lemma 26, $\forall$ has a winning strategy in $G(\mathsf{At}(\mathcal{A}_n))$, so by theorem 27 it does not belong to $\mathfrak{Ra}\mathbf{CA}_\gamma$. This proves that the inclusion in the theorem is strict. $\square$

PROBLEM 34 *In fact [1] define a whole sequence $\mathcal{A}_n$ of finite, non-permutational relation algebras and prove that a non-principal ultraproduct $\mathcal{B}$ of the $\mathcal{A}_n$ has a permutational representation. If it could be shown that $\mathcal{B}$ has a homogeneous representation, where arbitrary finite partial isomorphism extend to full automorphisms, then it would follow that $\exists$ has a winning strategy in $H(\mathsf{At}(\mathcal{B}))$ so a countable elementary subalgebra of $\mathcal{B}$ belongs to $\mathfrak{Ra}\mathbf{CA}_\gamma$, by theorem 27. This would show that $\mathfrak{Ra}\mathbf{CA}_\gamma$ cannot be defined by finitely many axioms over $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$, for infinite γ.*

PROBLEM 35 *For which finite values n is it the case that the inclusion $\mathfrak{Ra}\mathbf{CA}_n \subseteq \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ is strict?*

PROBLEM 36 *It is easy to check that* $\mathfrak{Ra}\mathbf{CA}_\gamma$ *and* $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ *are closed under direct products. We have seen that neither class is closed under subalgebras (for* $\gamma \geq 5$*). Is either class closed under homomorphic images?*

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