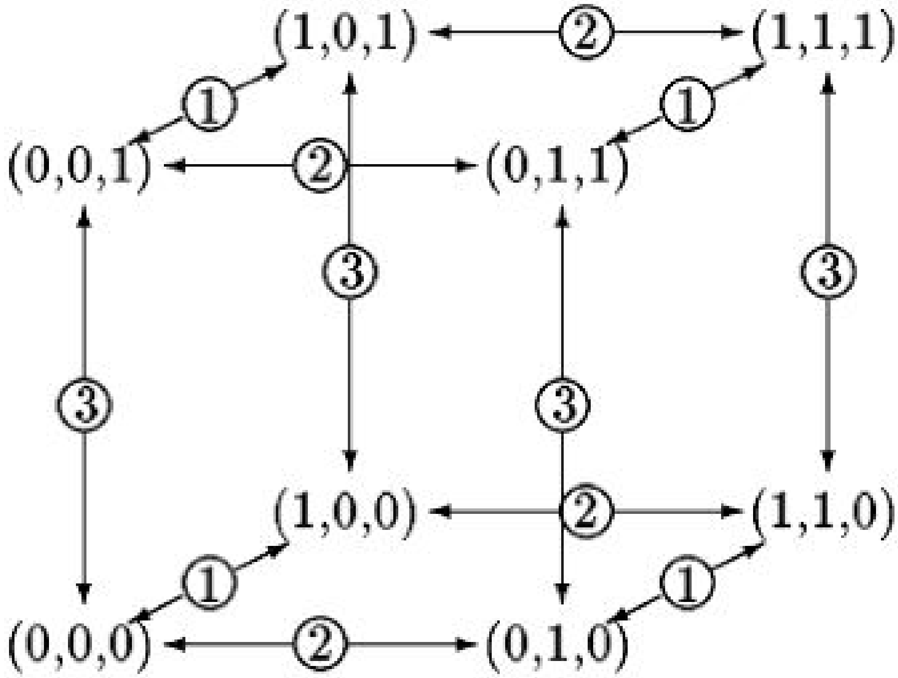
Muddy children

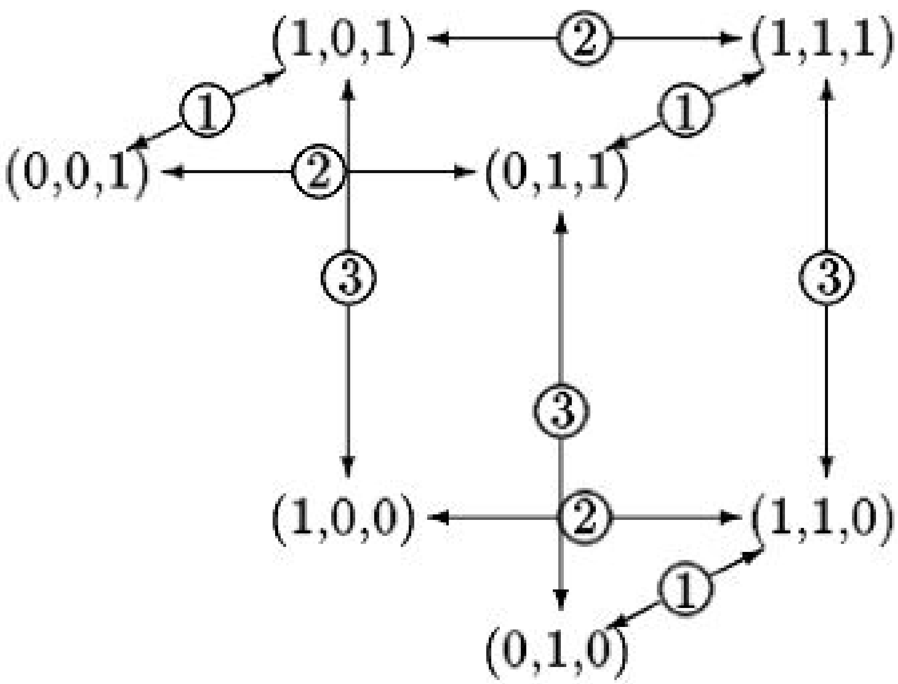
There are n children playing together. During their play some of the children, say k of them, get mud on their foreheads. Each can see the mud on others but not on his own forehead. Along comes a father, who says, "At least one of you has mud on your head". He then asks the following question, over and over: "Can any of you prove that you have mud on your head?" Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen?

There is a proof that the first $k-1$ times the father asks the question, the children will all say "no" but that the k-th time the children that are dirty will answer "yes".

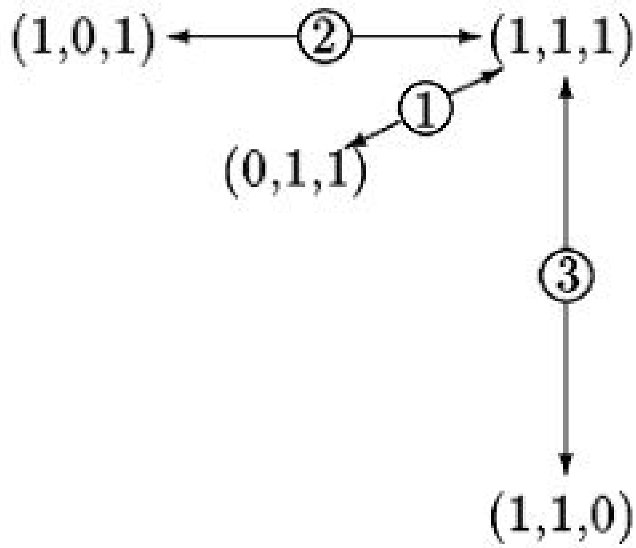
Kripke model with 3 children:



After father's announcement:



After all children say 'No':



Possibilities:

Let $\mathcal{A}$, a set of agents, and $\mathcal{P}$, a set of propositional variables, be given.

- A possibility w is a function that assigns to each propositional variable $p \in \mathcal{P}$ a truth value $w(p) \in \{0, 1\}$, and to each agent $a \in \mathcal{A}$ an information state $w(a)$.

- An information state σ is a set of possibilities.

Language of DES:

Let $\mathcal{A}$ be a non-empty set of agents, and let $\mathcal{P}$ be a set of propositional atoms. The language $\mathcal{L}_{\mathcal{P}}^{\mathcal{A}}$ of DES is the smallest set such that $\mathcal{P} \subseteq \mathcal{L}$, and if ϕ and ψ are in $\mathcal{L}_{\mathcal{P}}^{\mathcal{A}}$, and $a \in \mathcal{A}$, then $\neg\phi$, $\phi \wedge \psi$, $\Box_a\phi$ and $[\phi]_a\psi$ are in $\mathcal{L}_{\mathcal{P}}^{\mathcal{A}}$.

The part of the language that does not contain the $[.]_a$-operator is called the classical fragment of the language. This is just the language of classical multi-modal logic.

- $\Box_a\phi$: agent a has the information that ϕ.

- $[\phi]_a\psi$: an update of a's information with ϕ results in a situation where ψ is true, after a consciously updates his information with ϕ, ψ is true.

- The a's conscious update with ϕ in possibility w, $w[\![\phi]\!]_a$, only changes a's information state $w(a)$, i.e., we obtain a new possibility w' where
$w'(a) = \{v[\![\phi]\!]_a \mid v \in w(a) \ \& \ v \models \phi\}$

- group update: $[\phi]_{\mathcal{B}}\psi$, for $\mathcal{B} \subseteq \mathcal{A}$

Problem

We have to prove that this notion of conscious update is well-defined. That is, prove that for each agent and formula (1) it exists and (2) it is unique.

Proof (similar to proof in ch.8 Substitution):

- (non-wellfounded) sets as solution of a system of equations.
- Solution lemma (ch.6): Every system of equation (of a particular type) has a unique solution.
- Bisimulation/Strong extensionality (ch. 7): when are two (non-wellfounded) sets 'equal'.
  If there is a bisimulation B between w and v, i.e. if wBv then $w = v$.