Measuring Volatility

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Goal

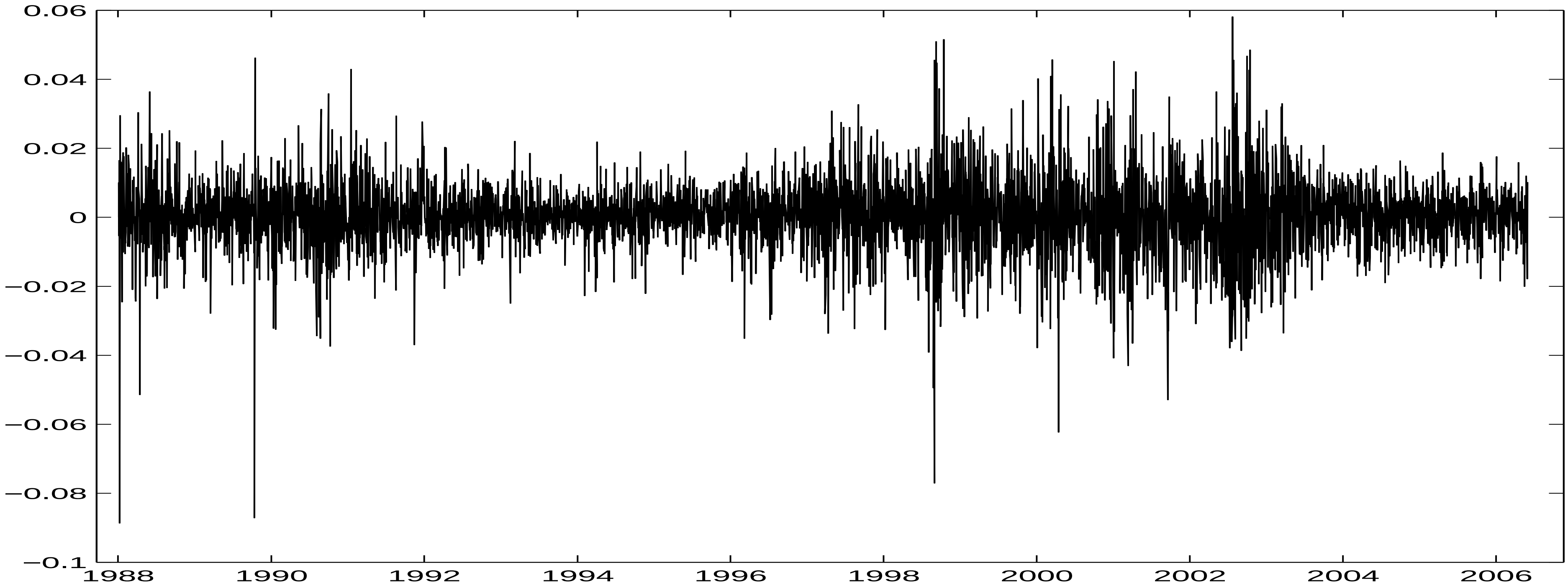
To find a good proxy for the daily scale factor s_n by using intraday data.

What does this mean?

Introduction

<u>DATA:</u> 18 years of S&P 500 tick data.

Daily returns r_n: <u>volatility clustering</u>



Introduction

Empirical stylized facts of daily asset returns:

- r_n roughly unpredictable
- r_n heavy tailed
- $|r_n|$ partly predictable (slowly decaying autocorrelations)

Introduction

Discrete time, close-to-close return models typically have a multiplicative form

$$r_n = s_n Z_n,$$

where

- s_n : the daily volatility (a scale factor >0)
- Z_n : the iid innovations

A particular example for the volatility dynamics is Garch(1,1):

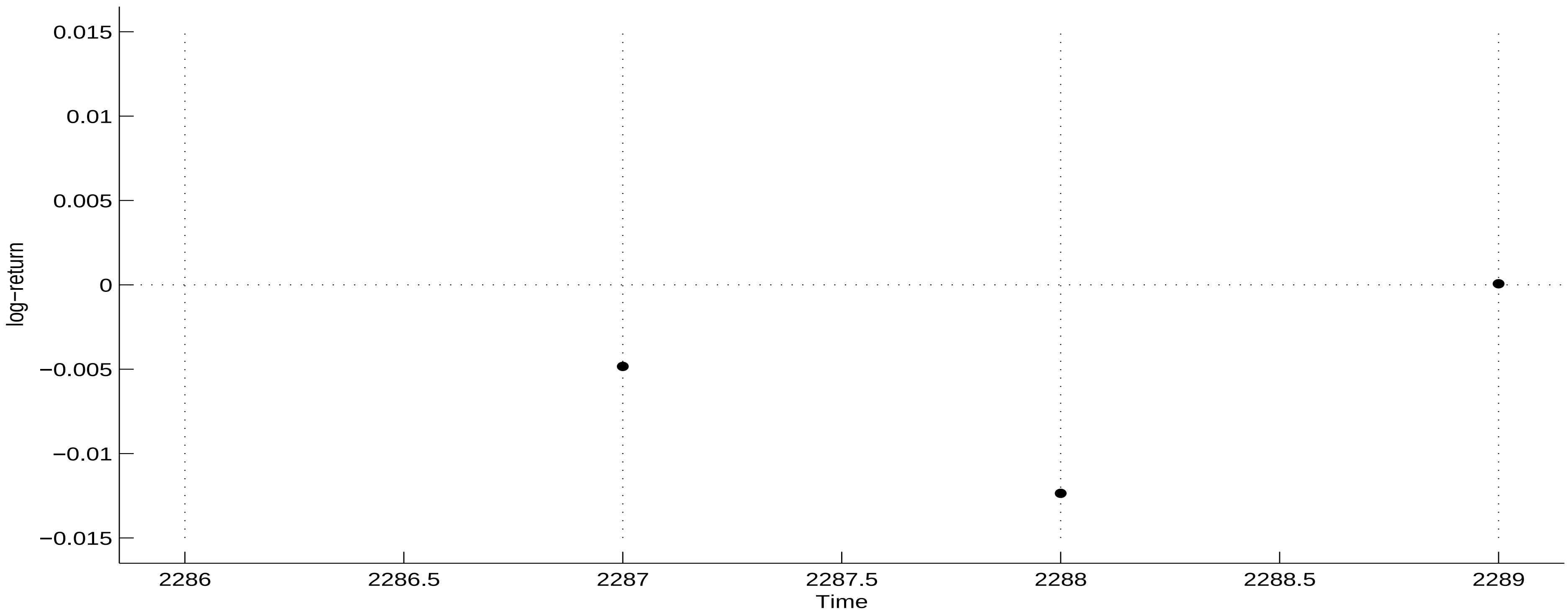
$$s_n^2 = \kappa + \alpha r_{n-1}^2 + \beta s_{n-1}^2.$$

Other models include:

stochastic volatility, long memory, Markov switching

Introduction

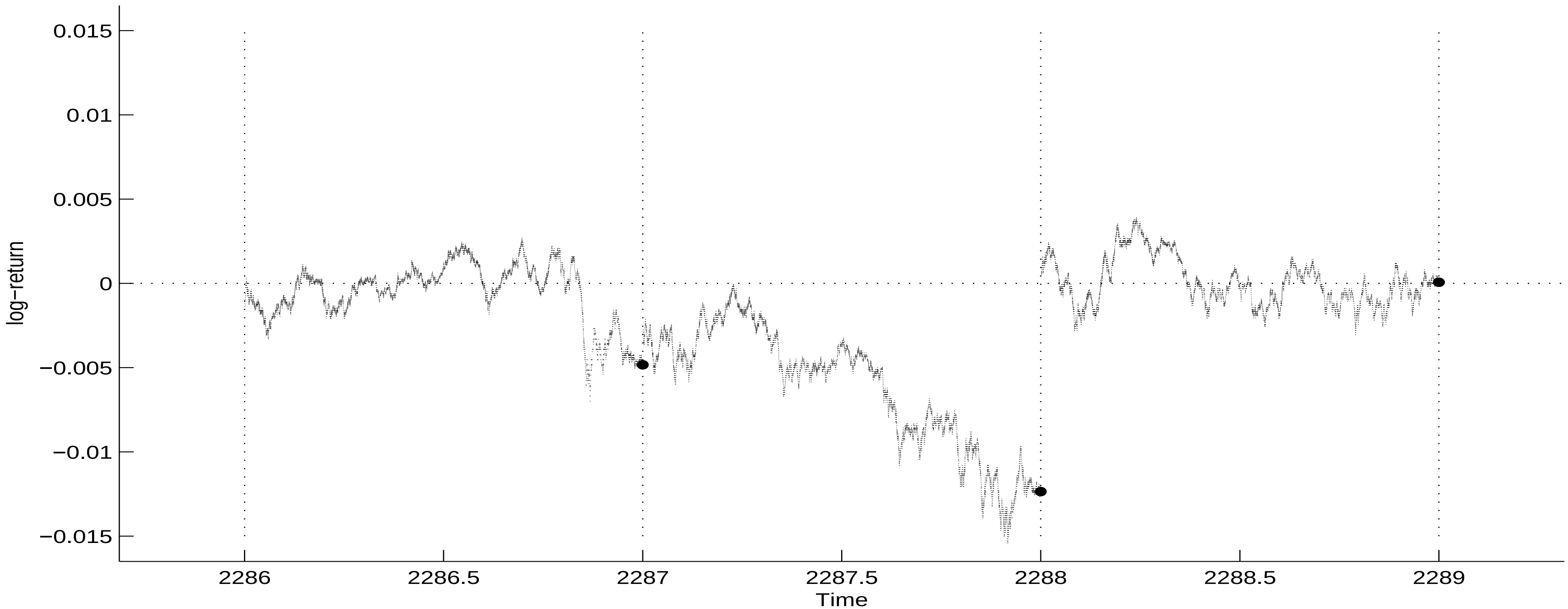
Three close-to-close returns ...



Introduction

Three daily returns ...

and the intraday log-return processes



How to use intraday information? (these 3 days: 14 thousand price ticks)

Introduction

We will use the intraday data to construct measurements (or proxies) for $s_n \dots$

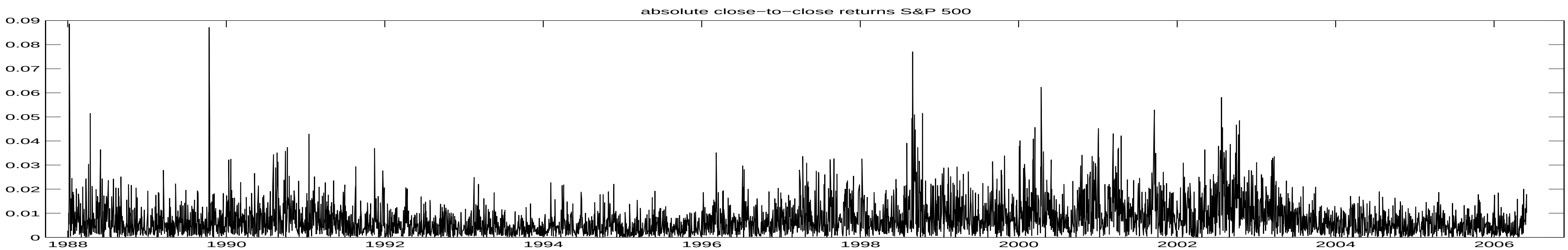
The discrete time model $r_n = s_n Z_n$ needs proxies for

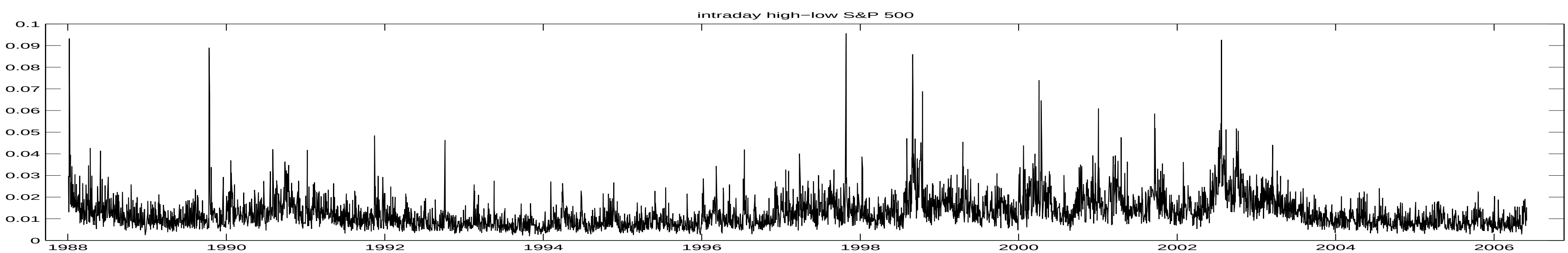
- estimation
- forecast evaluation

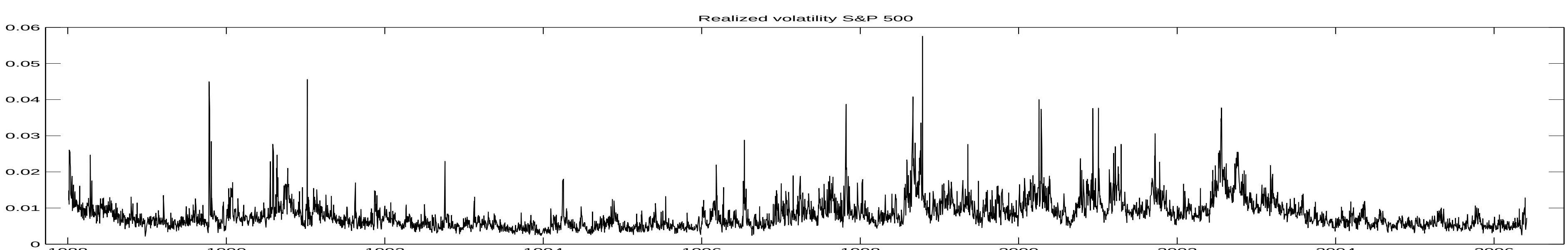
Which of the following three 'proxies' for s_n is best?

Three Proxies

absolute returns vs high-low vs realized volatility







Literature

Measuring Volatility:

Quadratic Variation	scale factor s_n
• realized variance (sum of squared intraday returns; ABDL, 1998, 2003; BNS, 2002)	• $\|r_n\|$
• realized (bi)power variation (BNS, 2004)	• high-low (under assumption of Brownian Motion: Parkinson, 1980; ABD, 2002)
• realized range (MD, 2007)	• More general: ?

Step 1/3: Intraday Extension of $r_n = s_n Z_n$

Let the intraday return process $R_n(\vartheta),\ \vartheta \in [0,1]$ be given by

$$R_n(\vartheta) = s_n \Psi_n(\vartheta),$$

where Ψ is an arbitrary cadlag process. So,

$$r_n = R_n(1) = s_n \Psi_n(1).$$

Moreover:

- (Ψ_n) iid.
- s_n and Ψ_n independent, for each n.

Step 2/3: Comparing Proxies

Definition. Let H be a positively homogeneous functional: $H(R_n) \geq 0$ and

$$H(\alpha R_n) = \alpha H(R_n) \qquad \alpha \geq 0$$

Then $H(R_n)$ is a proxy for s_n and

$$H(R_n) = s_n H(\Psi_n).$$

This covers the usual proxies:

high-low, realized volatility, realized range

Step 2/3: Comparing Proxies

Applying logarithms gives an additive measurement equation:

$$\log(H(R_n)) = \log(s_n) + U_n.$$

The errors $U_n = \log(H(\Psi_n))$ are iid.

A good proxy should add only little variation to the variation that is in $s_n \Rightarrow$

compare proxies by their <u>measurement</u> <u>variance:</u>

$$\lambda^2 = \mathrm{var}(U_n).$$

Step 2/3: Comparing Proxies

But: λ^2 cannot be estimated! However, by independence:

$$\mathrm{var}(\log(H^{(i)}(R_n))) = \mathrm{var}(\log(s_n)) + (\lambda^{(i)})^2.$$

So, for two functionals $H^{(1)}, H^{(2)}$:

$$(\lambda^{(1)})^2 \leq (\lambda^{(2)})^2$$

$$\Leftrightarrow$$

$$\mathrm{var}(\log(H^{(1)}(R_n))) \leq \mathrm{var}(\log(H^{(2)}(R_n)))$$

$\Rightarrow$ Rank proxies by the variance of their logarithm.

Step 3/3: Combining Proxies

Given $H^{(1)}, H^{(2)}$ consider the geometric combination

$$H^{(w)}(R_n) \equiv (H^{(1)}(R_n))^{w_1} \, (H^{(2)}(R_n))^{w_2} \qquad w_1 + w_2 = 1$$

$\Rightarrow$ Find $w = (w_1, w_2)$ such that $\mathrm{var}(\log(H^{(w)}(R_n)))$ is minimal.

Step 3/3: Combining Proxies

Define the log-proxy covariance Λ_p :

$$\Lambda_p \equiv \mathrm{Cov} \begin{pmatrix} \log(H^{(1)}(R_n)) \\ \log(H^{(2)}(R_n)) \end{pmatrix}$$

Then,

$$\arg\min_w \mathrm{var}(\log(H^{(w)}(R_n))) \qquad \text{s.t.} \sum w_i = 1.$$

$$\Leftrightarrow$$

$$\arg\min_w \; w' \Lambda_p w \qquad \text{s.t.} \sum w_i = 1.$$

Step 3/3: Combining Proxies

Similar to Markowitz minimal variance portfolio:

$$w^* = \frac{\Lambda_p^{-1}\iota}{\iota'\Lambda_p^{-1}\iota}$$

Λ_p can be estimated.

Empirical Analysis

S&P 500 tick data 1988–2006 (smaller PV corresponds to better proxy)

full sample

name	PV
RV5	0.064
RV10	0.080
RV15	0.089
RV20	0.100
RV30	0.117
abs-r	0.611
hl	0.161
maxar2	0.118
RAV5	0.058
RAV10	0.072
RVHL10	0.053
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Empirical Analysis

Same ranking for all subsamples

	full	1st	2nd	3rd	4th
name	ranking	ranking	ranking	ranking	ranking
RV5	4	4	4	4	4
RV10	6	6	6	6	6
RV15	7	7	7	7	7
RV20	8	8	8	8	8
RV30	9	9	10	9	9
abs-r	12	12	12	12	12
hl	11	11	11	11	11
maxar2	10	10	9	10	10
RAV5	3	3	3	3	3
RAV10	5	5	5	5	5
RVHL10	2	2	2	2	2
RAVHL10	1	1	1	1	1

Empirical Analysis

Optimal weights $\hat{w}$:

Proxy	weight
RAV10HIGH	1.04
RAV10LOW	0.72
RAV10	-0.76

$H^{(\hat{w})}$ has $PV = 0.038$

Empirical Analysis

Stability of optimized proxy $H^{(\hat{w})}$

	full	1st	2nd	3rd	4th
name	PV	PV	PV	PV	PV
$H^{(\hat{w})}$	0.038	0.039	0.043	0.043	0.028
$H^{(\hat{w},1)}$	0.038	0.039	0.043	0.043	0.028
$H^{(\hat{w},2)}$	0.039	0.039	0.043	0.043	0.029
$H^{(\hat{w},3)}$	0.039	0.040	0.044	0.042	0.029
$H^{(\hat{w},4)}$	0.039	0.040	0.045	0.044	0.027

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Empirical Analysis

Heuristic: forecast comparison

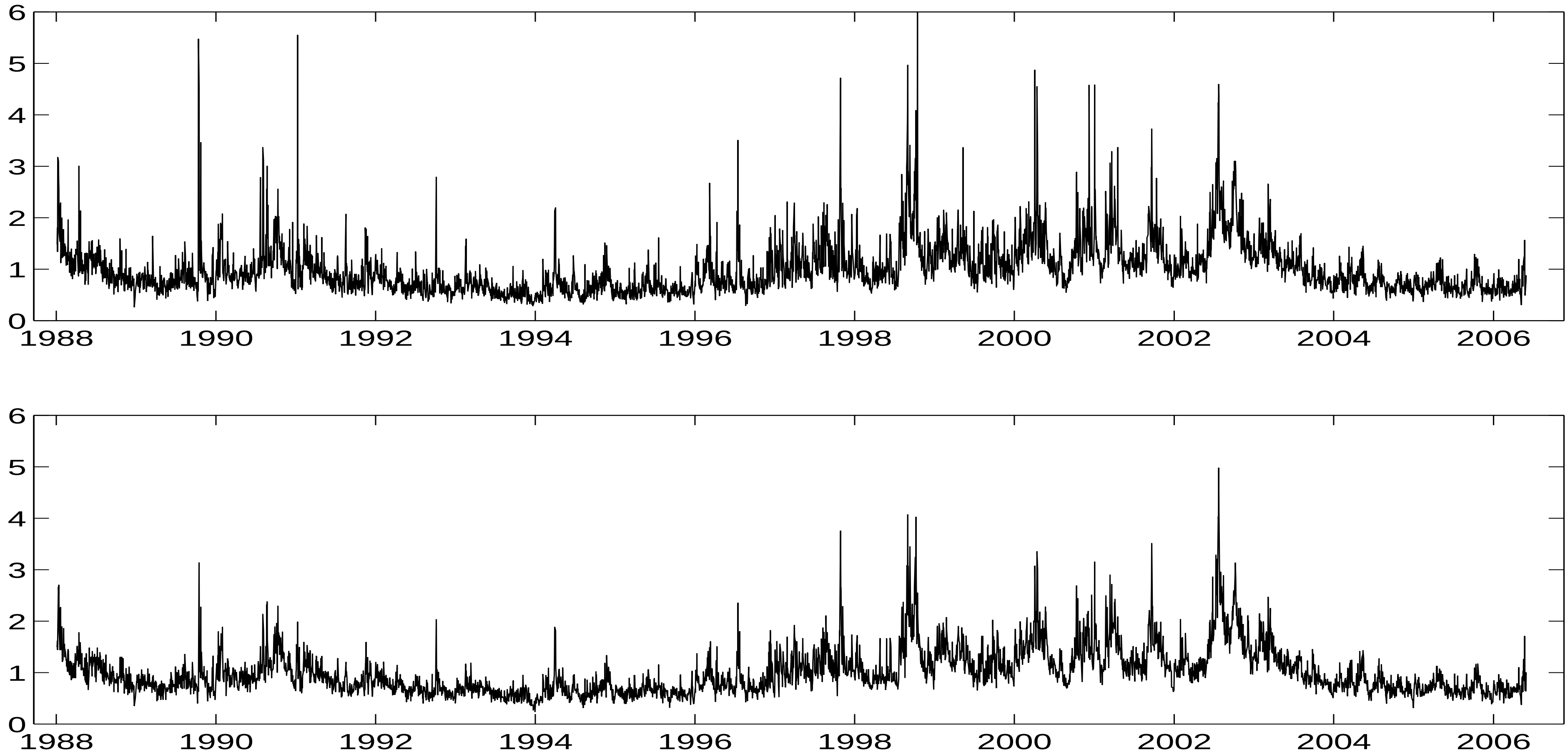
	RV30	RV20	RV15	RV10	RV5	$H^{(\hat{w})}$
RV30(-1)	0.35	0.39	0.42	0.46	0.50	0.58
RV20(-1)	0.38	0.42	0.45	0.49	0.54	0.61
RV15(-1)	0.39	0.44	0.47	0.50	0.55	0.63
RV10(-1)	0.41	0.45	0.48	0.52	0.57	0.66
RV5(-1)	0.43	0.48	0.51	0.55	0.60	0.69
$H^{(\hat{w})}(-1)$	0.44	0.48	0.51	0.54	0.60	0.71

R^2 of regression

$$\log(H^{(j)}(R_n)) = \alpha + \beta \log(H^{(i)}(R_{n-1})) + \varepsilon_n$$

Empirical Analysis

RV5 vs $H^{(\hat{w})}$

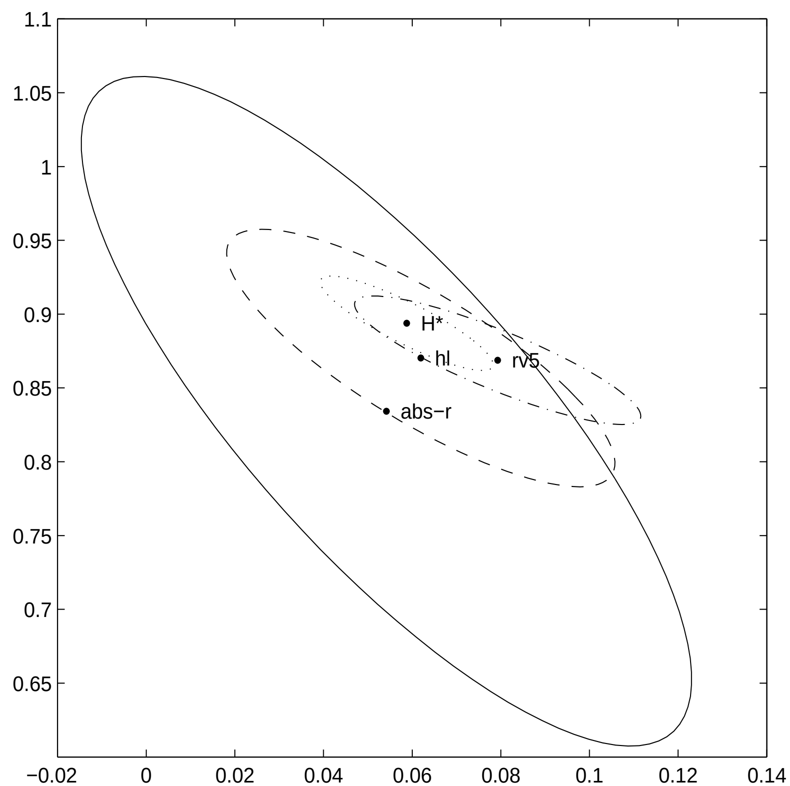


Future Research

Parameter estimation: consider the Garch(1,1) recursion

$$s_n^2 = \kappa + \alpha r_{n-1}^2 + \beta s_{n-1}^2$$

Decreasing the confidence region in the (α, β) plane:



Thank You!