Multiple View Geometry
in computer vision

Chapter 6: Camera Models

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Categorization

- Camera mapping: matrices with particular properties

$$\mathbf{x} = P\mathbf{X}$$

- Finite cameras
  - finite camera center
- Cameras at infinity
  - camera center is lying on plane at infinity

Categorization

- Finite cameras
  - Basic pinhole model
  - CCD camera
  - Finite projective camera
  - General projective camera
- Cameras at infinity
  - Affine cameras
    - Orthographic projection
    - Scaled orthographic projection
    - Weak perspective projection
    - General affine camera
  - General cameras at infinity

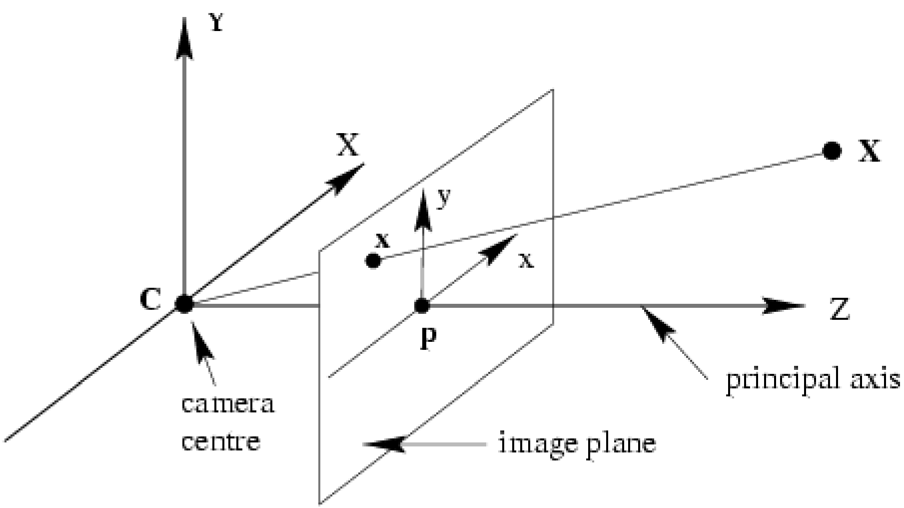
Finite cameras (pinhole model)

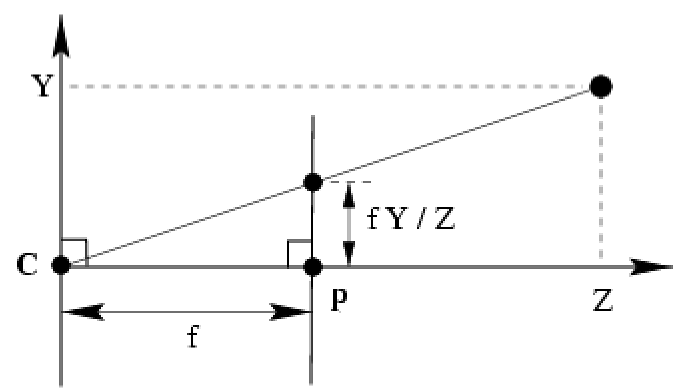
- Mapping from points in space onto a plane (Z = f)
- R^3 to R^2: $\mathbf{X} = (X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$
- Using homogeneous coordinates

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = P\mathbf{X}$$

$$P = diag(f, f, 1)[I \quad 0]$$





Finite cameras

- Principal point offset $\mathbf{X} = (X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$
  - Coordinate origin in image plane

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = K[I \quad 0]\mathbf{X}_{cam}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

- CCD camera/Finite projective camera
  - non-square pixels / skew parameter

$$K = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

$$\alpha_x = fm_x$$
$$\alpha_y = fm_y$$
$$x_0 = m_x p_x$$
$$y_0 = m_y p_y$$

Finite cameras

- Extrinsic camera parameters (rotation & translation)
  - transformation from world coordinate to camera coordinate frame

$$\tilde{\mathbf{X}}_{cam} = R(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$$ (camera orientation & camera center)

  - in homogeneous coordinates

$$\mathbf{X}_{cam} = \begin{bmatrix} R & -R\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{\mathbf{C}} \\ 0 & 0 \end{bmatrix} \mathbf{X}$$

  - projection

$$\mathbf{x} = KR\begin{bmatrix} I & -\tilde{\mathbf{C}} \end{bmatrix}\mathbf{X}$$

$$P = K\begin{bmatrix} R & t \end{bmatrix} \qquad t = -R\tilde{\mathbf{C}}$$

Finite cameras

- Finite projective camera

$$P = KR\begin{bmatrix} I & -\tilde{\mathbf{C}} \end{bmatrix} \qquad K = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

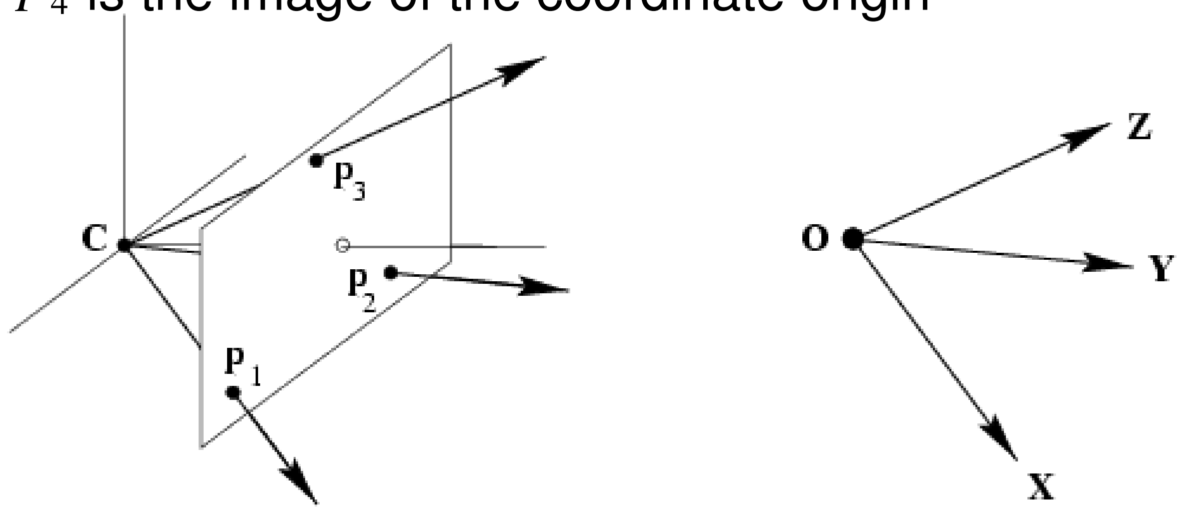
  - Left hand 3x3 sub-matrix (KR) is non-singular (RQ matrix decomposition: $M = KR$)
- General projective camera
  - remove non-singularity restriction
  - arbitrary homogeneous 3 x 4 matrix of rank 3
  - 11 degrees of freedom (5 intrinsic, 6 extrinsic)

Projective camera properties

- Camera center
  - P has 1-dim. null-space (rank 3, 4 columns), $P\mathbf{C}=0$
    C is camera center as homogeneous 4-vector

$$\mathbf{X}(\lambda)=\lambda\mathbf{A}+(1-\lambda)\mathbf{C} \quad \Rightarrow \quad \mathbf{x}=P\mathbf{X}(\lambda)=\lambda P\mathbf{A}+(1-\lambda)P\mathbf{C}=\lambda P\mathbf{A}$$

- Column points $P=\begin{pmatrix}p_1 & p_2 & p_3 & p_4\end{pmatrix}$
  - p_1, p_2, p_3 are the vanishing points of the WC axes X, Y, Z
  - p_4 is the image of the coordinate origin



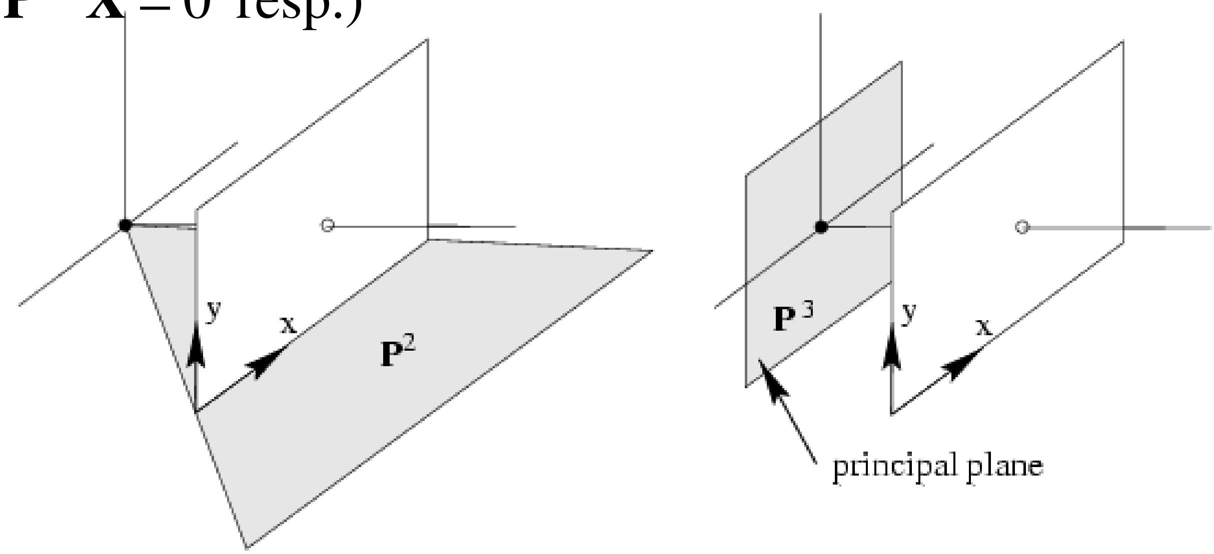
Projective camera properties

- Principal plane
  - $\mathbf{P}^3$ is plane through camera center parallel to image plane

    (set of points **X** which are imaged on line at infinity) $P\mathbf{X} = (x, y, 0)^T$ iff $\mathbf{P}^{3T}\mathbf{X} = 0$
- Axis planes
  - $\mathbf{P}^1, \mathbf{P}^2$ are planes through camera center, represent set of points that maps to image lines x=0 or y=0 resp.

    $(\mathbf{P}^{1T}\mathbf{X} = 0,\ \mathbf{P}^{2T}\mathbf{X} = 0\ \text{resp.})$

$$P = \begin{bmatrix} \mathbf{P}^{1T} \\ \mathbf{P}^{2T} \\ \mathbf{P}^{3T} \end{bmatrix}$$



Projective camera properties

- Principal point
  - intersection point of principal axis & image plane (project principal axis vector onto image plane)

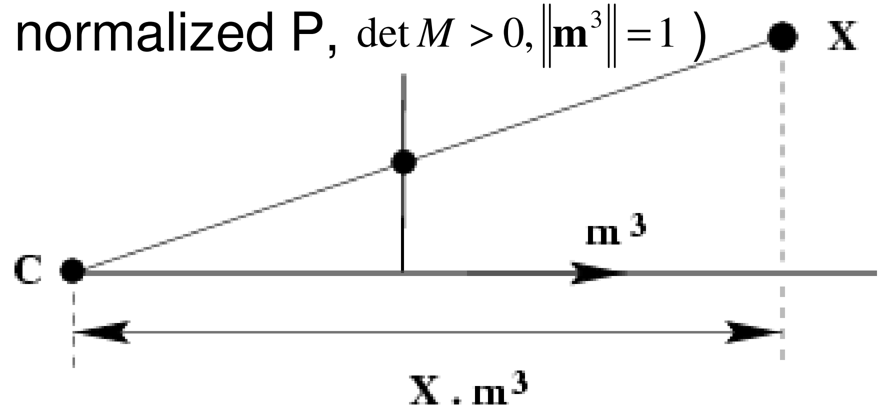
    $$P = \begin{bmatrix} M & \mathbf{p_4} \end{bmatrix} \quad \Rightarrow \quad x_0 = M\mathbf{m}^{3T}$$

- Principal axis vector
  - $\mathbf{m}^{3T}$ points in direction of principal axis, but: direction ambiguity!
  - $\mathbf{v} = \det(M)\mathbf{m}^{3T}$ is directed towards the front of the camera
    - This holds for $P_{cam} = K\begin{bmatrix} I & \mathbf{0} \end{bmatrix}$; unaffected by rotation, since det(R)>0

Action on points

- Back-projection to rays
  - Given a point **x** in the image plane, what 3D ray is it on?
  - Representation: join of two points
    - Camera center **C**
    - the point $P^{+}\mathbf{x}$ $\quad (P^{+} = P^{T}(\mathrm{PP}^{\mathrm{T}})^{-1})$, since $P(P^{+}\mathbf{x}) = I\mathbf{x} = \mathbf{x}$
  - Therefore: $\mathbf{X}(\lambda) = P^{+}\mathbf{x} + \lambda\mathbf{C}$
- Depth of points
  - $w = \mathbf{m}^{3T}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$ can be interpreted as the depth of **X** from **C** in direction of principal ray (for normalized P, $\det M > 0, \|\mathbf{m}^{3}\| = 1$)
  - Normalization:

$$d(X;P) = \frac{\mathrm{sgn}(\det M)w}{T\|\mathbf{m}^{3}\|}$$



Decomposition of P

- Camera center
  - point, for which $P\mathbf{C} = 0$ => SVD of P
  - Algebraically:

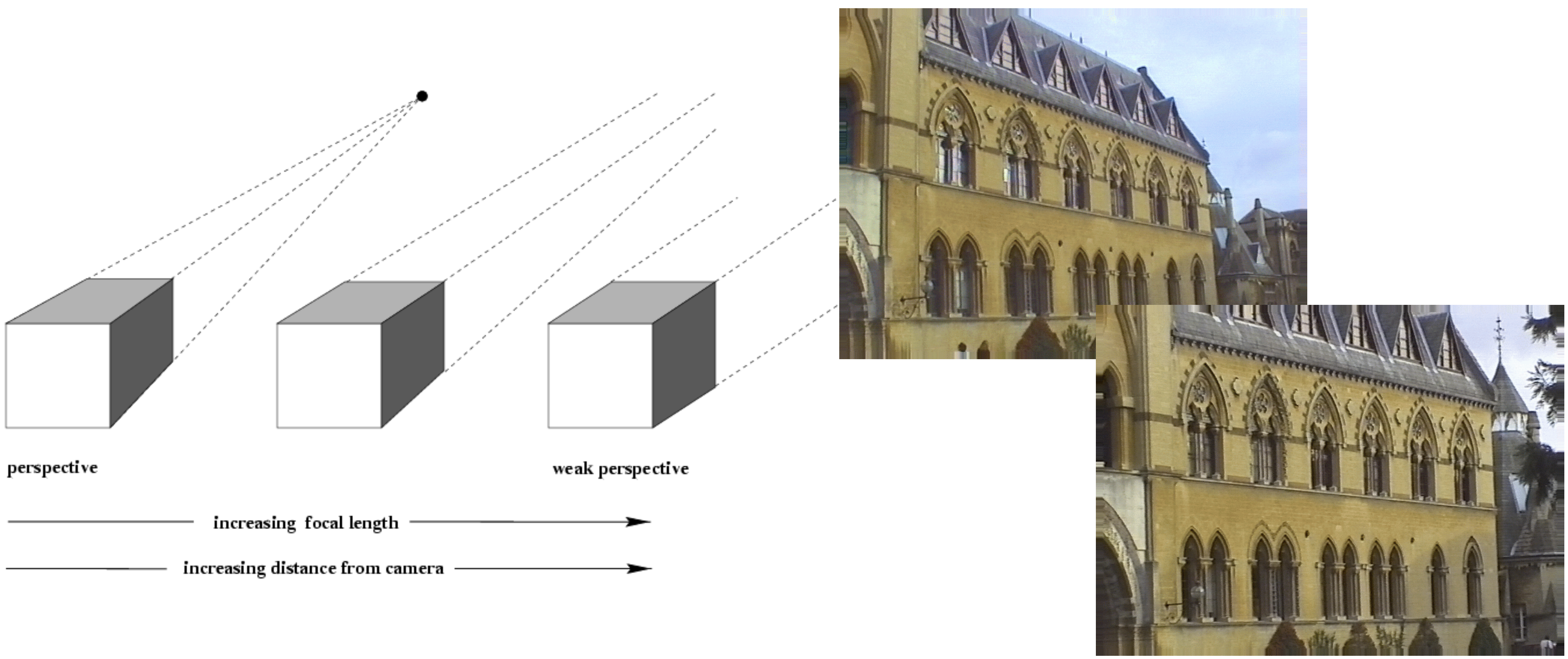
$$X = \det([\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4]) \quad Y = -\det([\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4])$$

$$Z = \det([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4]) \quad T = -\det([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3])$$

- Extrinsic/intrinsic parameters
  - $P = \begin{bmatrix} M & -M\tilde{\mathbf{C}} \end{bmatrix} = K\begin{bmatrix} R & -R\tilde{\mathbf{C}} \end{bmatrix}$
  - Decompose $M = KR$ using RQ-decomposition

Cameras at infinity

- Affine camera
  - An affine camera has a camera matrix P in which the last row is of the form (0,0,0,1).
  - Points at infinity are mapped to points at infinity.



Affine camera

- Tracking back (along the principal ray) while zooming in

$$P_t = KQ\begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T}(\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T}(\tilde{C} - t\mathbf{r}^3) \\ \mathbf{r}^{3T} & -\mathbf{r}^{3T}(\tilde{C} - t\mathbf{r}^3) \end{bmatrix} = K\begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix}\begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T}\tilde{C} \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T}\tilde{C} \\ \mathbf{r}^{3T} & d_t \end{bmatrix}$$

magnification factor
$k = d_t/d_0$

$$d_t = -\mathbf{r}^{3T}\tilde{\mathbf{C}} + t$$

depth of camera center
wrt. world origin

$$P_\infty = \lim_{t\to\infty} P_t = K\begin{bmatrix} \mathbf{r}^{1T} & -\mathbf{r}^{1T}\tilde{C}) \\ \mathbf{r}^{2T} & -\mathbf{r}^{2T}\tilde{C}) \\ \mathbf{0}^T & d_0 \end{bmatrix}$$

Affine camera

- Error
  - All points on plane through world origin perpendicular to principal axis direction $\mathbf{r}^3$ are unchanged.
  - Error of a 3D point at a distance Δ
    - image of the cameras $P_{t=0}$ and P_∞

$$X = \begin{pmatrix} \alpha\,\mathbf{r}^1 + \beta\,\mathbf{r}^2 + \Delta\,\mathbf{r}^3 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_{proj} = \begin{pmatrix} K_{2\times 2}\tilde{\mathbf{x}} + (d_0 + \Delta)\tilde{\mathbf{x}}_0 \\ d_0 + \Delta \end{pmatrix} \qquad \mathbf{x}_{affine} = \begin{pmatrix} K_{2\times 2}\tilde{\mathbf{x}} + d_0\tilde{\mathbf{x}}_0 \\ d_0 \end{pmatrix}$$

$$\Rightarrow \qquad \tilde{\mathbf{x}}_{affine} - \tilde{\mathbf{x}}_0 = \frac{d_0 + \Delta}{d_0}(\tilde{\mathbf{x}}_{proj} - \tilde{\mathbf{x}}_0)$$

(dehomogenized)

- The effect of the affine approximation to the true camera matrix is to move the image of a point **X** radially towards or away from the principal point by a factor equal to $d_0 + \Delta / d_0$

Hierarchy of affine cameras

- Orthographic projection

$$P = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$(X, Y, Z, 1)^T \mapsto (X, Y, 1)^T$

(dropping the Z-coordinate)

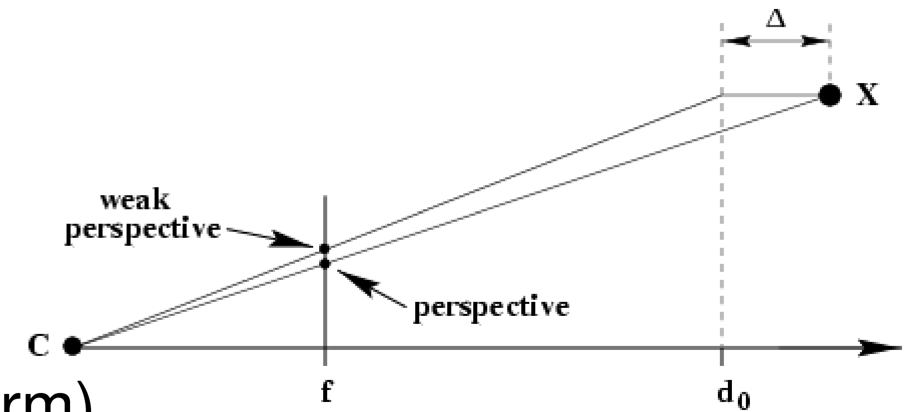
- Scaled orthographic projection

$$P = \begin{bmatrix} k & & \\ & k & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0}^T & 1/k \end{bmatrix}$$

Hierarchy of affine cameras

- Weak perspective projection

$$P = \begin{bmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0}^T & 1 \end{bmatrix}$$



- Affine camera (general form)

$$P = \begin{bmatrix} \alpha_x & s & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

($M_{2\times 3}$ rank 2)

Affine camera

- Properties
  - Principal plane is plane at infinity (also optical center, since it lies on principal plane)
  - Any camera matrix for which the principal plane is the plane at infinity is affine
  - Parallel world lines are mapped to parallel image lines
  - The vector $\mathbf{d}$ satisfying $M_{2\times 3}\mathbf{d} = \mathbf{0}$ is the direction of parallel projection, and $\left(\mathbf{d}^T, 0\right)^T$ the camera center, since

    $$P_A \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = \mathbf{0}$$