Chapter 1

Objects in contact: boundary collisions as geometric wave propagation

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ABSTRACT *We provide a representation in geometric algebra for m-dimensional boundaries, to describe and analyze the geometry of wave propagation, which is the mathematical essence of collision computations. This representation of boundaries as direction-dependent rotors in an $(m+1,1)$-dimensional Minkowski space turns wave propagation into a geometric product of rotors; it is in fact a spectral decomposition of the propagation operation. Some differential-geometric properties of the propagation result are derived, such as: Gaussian curvatures add harmonically.*

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1.1 Introduction

1.1.1 Towards a 'systems theory' of collision

The motivation behind this work is the desire to make the computation of collision-free motions of robots efficiently computable. For translational motions, the boundary of permissible translations of a reference point is obtained from the obstacles and the robot by a kind of dilation, 'thickening' the obstacle (see below for details) to produce the forbidden states in configuration space of translations. The intuitive similarity of this operation to convolution suggests that we might be able to find a kind of Fourier transformation, in the sense that we might separate the shapes into independent 'spectral components' and combine those simply; after which the collision boundary would be obtained by the inverse transformation. This would enable the development of a 'systems theory' for collisions.

This was indeed done for two-dimensional boundaries [1], using a Legendre transformation and its coordinate-free counterpart (which is related to the polar curves of projective geometry). However, using classical differential geometry the generalization to the m-dimensional case was not straightforward. In this Chapter we do it using geometric algebra, which easily captures the geometric intuition in simple, computable expressions and allows compact derivations of fairly advanced results.

1.1.2 Collision is like wave propagation

This paper actually treats the mathematics of geometric wave propagation according to Huygens' principle, in which points on a wave front become secondary sources (also called propagators), of which the forward caustic generates the new wave front. The reason is that for arbitrary shapes of propagators, the generation of the secondary wave front is mathematically very closely related to the collision problem which we really want to treat (but the wave propagation formulation has some advantages).

This can be seen as follows, using Figure 1.1.

- *Huygens wave propagation*
  When we perform a Huygens wave propagation for a finite time interval, we place copies of a 'propagator' $\mathcal{A}$ at each position on a wave front $\mathcal{B}$ (see Figure 1.1a). By Huygens' principle, the forward caustic of these secondary wave fronts then forms the resulting propagated front, which we denote $\mathcal{A}\check{\oplus}\mathcal{B}$ (see Figure 1.1b). Each point P of $\mathcal{B}$ in this way locally 'causes' a point in the result (easiest to see when $\mathcal{A}$ is convex and $\mathcal{B}$ is differentiable). By performing a linear approximation to $\mathcal{B}$ at P, it is clear that at every point of the resulting caustic, the tangent is equal to that of the point which caused it, and equal to the tangent at the corresponding point of the 'propagator' $\mathcal{A}$.

- *Collision detection*
  Now consider Figure 1.1c, which depicts the collision of a movable object (the robot) $\mathcal{A}'$ with a fixed obstacle $\mathcal{B}$. The position of $\mathcal{A}'$ is indicated by that of its *reference point*, which is some fixed point that moves with it (of we would allow rotations, we would need to specify a reference frame, but for now a reference point is enough). This point is prevented from moving freely due to the collision at P. Computing such local contacts for all translational motions of $\mathcal{A}'$ generates the boundary of the 'free space' for the reference point of $\mathcal{A}'$. A linear approximation shows that the tangents at P of $\mathcal{B}$, $\mathcal{A}'$ and the reference point at the resulting boundary are proportional (with the outwardly directed normal vector of $\mathcal{A}'$ at P having an opposite sense).

This shows that the two operations of wave propagation and collision are mathematically closely related. Indeed, the boundary of free space in the collision problem is precisely $(-\mathcal{A}')\check{\oplus}\mathcal{B}$, i.e. the propagation of $\mathcal{B}$ with the

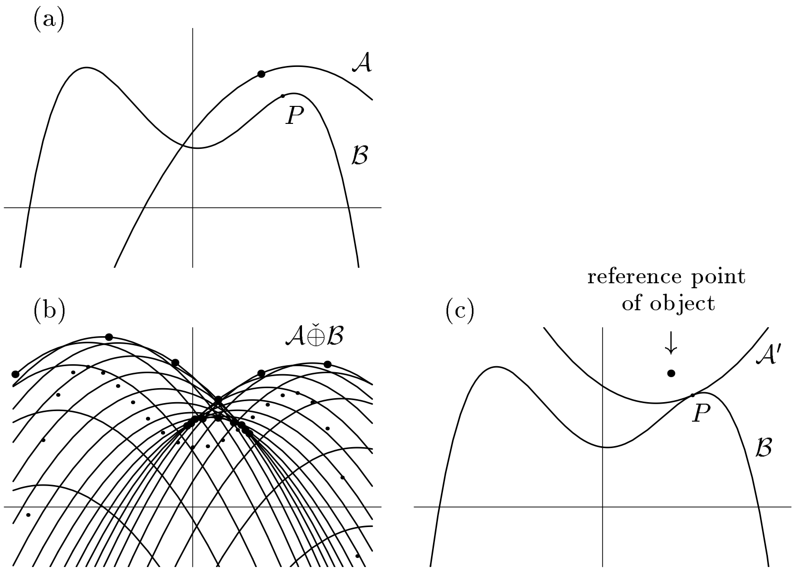


FIGURE 1.1. Wave propagation and collision are mathematically similar (see text).

propagator $-\mathcal{A}' \equiv \{-a \mid a \in \mathcal{A}'\}$. Thus an analysis of either is applicable to the other.

In this Chapter, we will use the wave propagation terminology, since it has the simplest relationship between directed tangents (no opposite orientations). We first come up with a representation for proper boundaries which unifies the 'hypersurface' aspects of a boundary with the assignment of a local 'inside'. Then we analyze the wave propagation in terms of this representation; it will turn out to be the 'Fourier-like' representation we were looking for, in which wave propagation becomes separable in a simple operation on the 'spectral components' (these are the tangent hypersurfaces of the objects involved).

1.1.3 Related problems

In fact, operations similar to wave propagation occur in many places in science and engineering. In *computer graphics*, 'growing' objects by thickening some specified skeleton shape involves a thickening by spheres (or other shapes), which is clearly equivalent to one step of Huygens wave propagation. In *image analysis*, the field of 'mathematical morphology' for object selection, originally designed to mimic the selective filtering of grains by sieves, involves extensive use of the 'dilation' operation – which is essentially 'growing' the object geometrically. The 'distance transform' and the related techniques of 'skeletonization', which involve computing the shortest distance to the nearest object, are also example of operations with a wave-propagation-like structure. In *robotics*, some have approached the 'path planning problem' in this manner, with the waves computing the distance function for the shortest path in the state space of the robot between initial state and goal state. In *milling* (carving away excess material to end up with a desired shape), the carving bit performs a destructive collision with the object to produce the result; the relationship between those three is again essentially Figure 1.1c. In *scanning tunneling microscopy*, an atomic probe (of unknown shape) is moved over an unknown atomic surface such that voltage between them is constant; this implies a contact operation on the equi-potential surfaces, which happens according to Figure 1.1c, in good approximation. Again, this gives the same relationship between surface, probe, and measured surface of positions.

Our analysis of the mathematics of geometric wave propagation is in principle applicable to all these fields. However, most of these applications contain an essential element which we will *not* treat here: in sharply concavely curved sections of $\mathcal{B}$, a locally tangent contact of $\mathcal{A}'$ can be precluded by an intersection of $\mathcal{A}'$ elsewhere along $\mathcal{B}$. This happens in the central section of Figure 1.1b, and part of the caustic then becomes excluded (e.g. you would not send your milling bit there, it would carve out unwanted sections). Mathematically, it is easier to treat these parts on a par with the rest, and in a genuine wave propagation they would be observed. And for the robotics collision application which motivated this work, these parts must be treated: if those concave parts of the obstacle $\mathcal{B}$ have been observed somehow, a path planning algorithm should already exclude the corresponding parts of $\mathcal{A}\check{\oplus}\mathcal{B}$ from consideration, even though later observations of other sections of $\mathcal{B}$ may exclude them eventually.

1.2 Boundary geometry

1.2.1 The oriented tangent space

In an m-dimensional *Euclidean* vector space $\mathcal{G}^1(\mathbf{I}_m)$, with pseudoscalar $\mathbf{I}_m$, we consider an object, noting specifically its boundary. This boundary is an $(m-1)$-dimensional hypersurface, with locally two 'sides': an *inside* and an *outside*. Assume the boundary to be smooth (we will not treat edges in this Chapter); then at every point $\mathbf{p}$ of the boundary, the boundary surface has a local tangent space with pseudoscalar $\mathbf{I}[\mathbf{p}]$ of grade $m-1$ (which we will mostly denote by $\mathbf{I}$, with $\mathbf{p}$ understood; throughout this Chapter we will use square brackets for non-linear arguments, round brackets for linear arguments). We may represent this tangent space by a dual vector $\mathbf{n}$ (with again $\mathbf{p}$ understood as parameter), defined by the geometric product with the inverse pseudoscalar:

$$\mathbf{n} = -\mathbf{I}\mathbf{I}_m^{-1}. \tag{1.1}$$

The '$-$' is introduced here to avoid awkward signs later on. We will call $\mathbf{n}$ the *normal vector* to $\mathbf{I}$. Since we are only interested in rigid body transformations, we can restrict our treatment to Euclidean geometry, in which the normal vector is well-behaved (the normal vector of a transformed object is the transformation of the normal vector), so using duals is permitted. We wish to denote the notion of 'inside' geometrically in the boundary representation, to make it more than merely the representation of the boundary surface. This involves orienting $\mathbf{n}$ (and hence $\mathbf{I}$). The usual convention for a circular blob in 2D (with the usual right-handed pseudoscalar $\mathbf{I}_2$) is 'when following the contour, the object is at the left-hand side'. So with a tangent $\mathbf{I}$ in the direction of motion, we have $\mathbf{I}\mathbf{I}_2$ as *inward* pointing direction. Therefore $\mathbf{n} = -\mathbf{I}\mathbf{I}_2^{-1} = \mathbf{I}\mathbf{I}_2$ is the *inward pointing normal vector*. We generalize this to m-dimensional space, deriving the sign of $\mathbf{I}$ using Equation (1.1) from the desire to have $\mathbf{n}$ be the locally *inward* pointing normal vector. Thus the tangent spaces have been oriented properly, whether represented by $\mathbf{I}$ or by $\mathbf{n}$.

1.2.2 Differential geometry of the boundary

The boundary surface at position $\mathbf{p}$ in $\mathcal{G}^1(\mathbf{I}_m)$ has directed tangent $\mathbf{I}[\mathbf{p}]$; when we move along the boundary surface, the tangent will change. The description of these changes can be found in [3](Chapters 4 & 5), and we briefly repeat and extend the elements relevant to our analysis.

So let $\mathbf{n}$ denote the inside pointing local unit normal vector, as a differentiable function of the position $\mathbf{p}$ on the boundary (sometimes we will write $\mathbf{n}[\mathbf{p}]$). Its direction can be derived from the first order differential structure of $\mathbf{p}$[1], but its sign must be explicitly determined by our notion of 'inside'. The second order differential structure of the boundary is obtained by differentiating such a properly oriented $\mathbf{n}$ using a vector derivative in some direction $\mathbf{a}$. We denote the resulting position-dependent linear function by $\underline{\mathbf{n}}$. So $\underline{\mathbf{n}} : \mathbf{I}_m[\mathbf{p}] \to \mathbf{I}[\mathbf{p}]$ is defined as the differential of $\mathbf{n}$:

$$\underline{\mathbf{n}}(\mathbf{a}) \equiv (\mathbf{a} \cdot \boldsymbol{\partial})\mathbf{n}, \tag{1.2}$$

and in our notation we will mostly let the dependence on $\mathbf{p}$ be understood implicitly, occasionally writing $\underline{\mathbf{n}}(\mathbf{a})[\mathbf{p}]$ if it needs to be specified. Since $\underline{\mathbf{n}}(\mathbf{a})$ is a linear function of the vector argument $\mathbf{a}$, we may extend it to arbitrary multivector arguments as an outermorphism (i.e. a $\wedge$-preserving linear operator). Specifically, we can form $\underline{\mathbf{n}}(\mathbf{a}_\wedge \mathbf{a}_2 \wedge \cdots \mathbf{a}_{m-1})$, with the

[1]For example, if the surface of the boundary is implicitly given by a scalar function ϕ as $\phi(\mathbf{p}) = 0$, then $\mathbf{n}$ is proportional to $\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})$.

$\mathbf{a}_i$ forming a basis for the tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$ at $\mathbf{p}$. The quantity $\underline{\mathbf{n}}(\mathbf{I})$ thus denotes the volumetric change of $\mathbf{n}$ as we move over a local tangent volume the size of the tangent pseudoscalar $\mathbf{I}$; the ratio with $\mathbf{I}$ (which is the determinant of the mapping $\underline{\mathbf{n}}$) is related to the *directed Gaussian curvature* $\boldsymbol{\kappa}$ of the boundary at $\mathbf{p}$ by:

$$\boldsymbol{\kappa} \equiv (-1)^{m-1}\underline{\mathbf{n}}(\mathbf{I})\mathbf{I}^{-1} = \underline{\mathbf{n}}(\widehat{\mathbf{I}})\mathbf{I}^{-1} \tag{1.3}$$

(the $\widehat{\cdot}$ denotes grade inversion), where we obtain an extra sign factor relative to the standard convention for the definition of $\boldsymbol{\kappa}$ which uses the *outward pointing normal vector*, due to the fact that $\mathbf{n} \to -\mathbf{n}$ gives $\underline{\mathbf{n}}(\mathbf{I}) \to \underline{\mathbf{n}}(\widehat{\mathbf{I}})$ by Equation (1.2).

The vector $\underline{\mathbf{n}}(\mathbf{a})$ gives the change in unit normal vector when moving in the $\mathbf{a}$-direction; this value is unique for the regular surfaces we treat. It is one of the properties of differentials that $\underline{\mathbf{n}}(\mathbf{P}(\mathbf{a})) = \underline{\mathbf{n}}(\mathbf{a})$ for all $\mathbf{a}$ (where $\mathbf{P}(\mathbf{a}) \equiv (\mathbf{a} \cdot \boldsymbol{\partial})\mathbf{p}$ is the projection onto the local tangent space) – and so a unique inverse to $\underline{\mathbf{n}}(\cdot)$ does not exist. Any $\mathbf{a}$ that projects to the same $\mathbf{P}(\mathbf{a})$ gives the same value for $\mathbf{n}$; but even when we limit the inverse to have values in the tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$, there may not be a unique solution. (An example is a cylinder with axis $\mathbf{z}$, where only the component of $\mathbf{P}(\mathbf{a})$ perpendicular to $\mathbf{z}$ determines the value of $\underline{\mathbf{n}}(\mathbf{a})$.) Therefore the inverse of $\underline{\mathbf{n}}(\cdot)$ usually produces a *set* of vectors in the space with pseudoscalar $\mathbf{I}_m$ based at $\mathbf{p}$ (we will denote this space by $\mathcal{G}^1(\mathbf{I}_m[\mathbf{p}])$). We prefer to limit the values to the local tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$, and so define:

$$\underline{\mathbf{n}}^{-1} : \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) \to \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) : \quad \underline{\mathbf{n}}^{-1}(\mathbf{m}) \equiv \{\mathbf{a} \mid \underline{\mathbf{n}}(\mathbf{a}) = \mathbf{m}\}. \tag{1.4}$$

(In words, $\underline{\mathbf{n}}^{-1}$ gives the set of 'tangent velocities' at the point $\mathbf{p}$ required to produce the change $\mathbf{m}$ in $\mathbf{n}$.) Such set-valued functions can be added, using the *Minkowski sum* (denoted by $\oplus$) as set addition:

$$A \oplus B = \{a + b \mid a \in A, b \in B\}, \tag{1.5}$$

where the '+' denotes is the vector addition. Note that if one of the arguments is $\emptyset$, then so is the result.

1.3 The boundary as a geometric object

In the representation of the boundary so far, we required a description of the position $\mathbf{p}$ (of which the differential structure gives us the direction of the local tangent space, characterizable by $\mathbf{I}$ or $\mathbf{n}$) and an orientation sign to specify 'inside' (which then gives the proper sign to $\mathbf{I}$ or $\mathbf{n}$). Thus the boundary is not yet a *single geometric object* in an embedding space. In [1], a single representation was found in a homogeneous embedding in projective $(m+1)$-space (for $m = 2$ only); it is actually structurally more clear to embed in the null-cone of the Minkowski space with Clifford algebra $\mathcal{C}\ell_{m+1,1}$, and we do so now.

1.3.1 Embedding in $\mathcal{C}\ell_{m+1,1}$

We embed the boundaries of the Euclidean space $\mathcal{G}^1(\mathbf{I}_m)$ through a conformal split in the higher dimensional space $\mathcal{G}^1(I_{m+2})$ with pseudoscalar $I_{m+2} \equiv E\mathbf{I}_m$. Here E is a bivector defined as

$$E \equiv e_0 \wedge e^0, \tag{1.6}$$

where e_0 and e^0 span the two extra dimensions. We choose them to be two reciprocal null vectors perpendicular to $\mathbf{I}_m$, so they satisfy

$$(e_0)^2 = (e^0)^2 = 0, \quad e_0 \cdot e^0 = 1, \quad e_0 \cdot \mathbf{I}_m = e^0 \cdot \mathbf{I}_m = 0. \tag{1.7}$$

We are therefore in the Clifford algebra $\mathcal{C}\ell_{m+1,1}$ of a Minkowski space (with e_0 and e^0 spanning the null cone), and $E^2 = 1$. As Chapter **??** has shown, this representation can be used to extend the more commonly used projective split (with e_0 as splitting vector) to an *isometric* embedding of Euclidean m-space onto the *horosphere* in $\mathcal{G}^1(I_{m+2})$. (The horosphere is the intersection of the null cone with the plane $x \cdot e^0 = 1$.) This is powerful embedding was introduced into geometric algebra in [2] and [5]; Chapter **??** uses the multiplicative split to produce this representation; we prefer the additive split form the original formulation in [5], though we use e^0 for their $-e$ since we want proper reciprocity between e_0 and e^0. In that split, a point $\mathbf{p}$ is represented as the null vector: $p' = e_0 + \mathbf{p} - e^0\mathbf{p}^2/2$ (rather than as the trivector $p'E$ used in Chapter **??**). We will use **bold** for elements of the algebra $\mathcal{G}(\mathbf{I}_m)$, and the usual *math* font for elements of the larger algebra $\mathcal{G}(I_{m+2})$.

Intuitively, e_0 is the representation of a point at the origin, and $-e^0$ is the representation of (the direction of) the point at infinity. In this Chapter, we will only need to embed *flats*, i.e. offset linear subspaces such as 0-dimensional points, 1-dimensional lines, etc., since those are the tangent spaces used to describe the boundaries. A flat with tangent $\mathbf{I}$ at position $\mathbf{p}$ is represented in the additive split as (see [5]):

$$e^0 \wedge p' \wedge \mathbf{I} = e^0 \wedge (e_0 + \mathbf{p}) \wedge \mathbf{I} \tag{1.8}$$

(leave off '$e^0\wedge$' to retrieve the usual homogeneous representation). For brevity, we will denote $e_0 + \mathbf{p}$, the homogeneous representation of the point at $\mathbf{p}$, by p.

Instead of with Equation (1.8), it is somewhat more convenient to work with its dual in our embedding space $\mathcal{G}(E\mathbf{I}_m)$:

$$\begin{aligned} (e^0 \wedge p \wedge \mathbf{I})\mathbf{I}_m^{-1}E &= -(e^0 \wedge p) \cdot \mathbf{n}E = p \cdot \big(e^0 \cdot (E\mathbf{n})\big) \\ &= p \cdot (e^0\mathbf{n}) = \mathbf{n} - e^0(\mathbf{p} \cdot \mathbf{n}) \end{aligned}$$

We define this as the representation $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ of the boundary at $\mathbf{p}$, so:

$$\mathcal{R}(\mathbf{n})[\mathbf{p}] \equiv p \cdot (e^0\mathbf{n}) = \mathbf{n} - e^0(\mathbf{p} \cdot \mathbf{n}). \tag{1.9}$$

This represents the flat by its normal vector and its scalar support $\mathbf{p}\cdot\mathbf{n}$ as an object in $\mathcal{C}\ell_{m+1,1}$. This representation is indexed by $\mathbf{n}$, and has an extra component in the e^0-direction; this scalar is the *support* of the flat. Over all $\mathbf{n}$ occurring in the boundary, this is therefore essentially the extended Gauss map: the sphere of directions, augmented by a scalar function specifying the directed support, see [6]. We will soon see that $(\mathcal{R}(\mathbf{n})[\mathbf{p}])^2 = 1$, so that the extended Gauss-sphere is geometrically embedded as an actual sphere in Minkowski space (rather than as a scalar-valued function on the tangent space sphere, which is the usual description).

1.3.2 Boundaries represented in $\mathcal{G}(E\mathbf{I}_m)$

We view a boundary as a collection of tangent flats, and assume throughout this Chapter that this collection is differentiable; so we limit ourselves to 'regular boundaries' in this sense. (This does not preclude the treatment of swallowtail catastrophes in the propagation result, as we will show in Section 1.4.5 – despite the characterization of such a curve in classical differential geometry as non-regular.)

Since the representation Equation (1.9) contains $\mathbf{n}$ explicitly, we will view the boundary as 'indexed by $\mathbf{n}$'. We are then required to view the positions $\mathbf{p}$ as a function of $\mathbf{n}$, so we write $\mathbf{p}[\mathbf{n}]$ – we use square brackets since $\mathbf{p}$ is not linear in $\mathbf{n}$. Also, this is not a single-valued function, since the same tangent may occur at different locations if the object is not convex. Once we have done this, the representation becomes a set-valued function of $\mathbf{n}$ only, which we denote by $\mathcal{R}(\mathbf{n}_\mathcal{A})$ for a boundary $\mathcal{A}$ – though we mostly omit the subscript if the context is clear.

The representation $\mathcal{R}(\mathbf{n})$ has some very nice properties:

- *representation commutes with differentiation*
  We have to be careful here, since it depends whether we differentiate $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ relative to variations in $\mathbf{p}$ (defining $\mathbf{n}$ as a function of $\mathbf{p}$) or in $\mathbf{n}$ (with $\mathbf{p}$ as a function of $\mathbf{n}$). Differentiating relative to $\mathbf{n}$, we obtain:

$$\begin{aligned}(\mathbf{m}\cdot\boldsymbol{\partial}_\mathbf{n})\mathcal{R}(\mathbf{n})[\mathbf{p}[\mathbf{n}]] &= (\mathbf{m}\cdot\boldsymbol{\partial}_\mathbf{n})\left(p[\mathbf{n}]\cdot(e^0\mathbf{n})\right)\\ &= \underline{\mathbf{p}}(\underline{\mathbf{n}}(\mathbf{m}))\cdot(e^0\mathbf{n}) + p[\mathbf{n}]\cdot(e^0\mathbf{m})\\ &= p[\mathbf{n}]\cdot(e^0\mathbf{m}) = \mathcal{R}(\mathbf{m})[\mathbf{p}[\mathbf{n}]], \qquad (1.10)\end{aligned}$$

  which gives the commutative relationship between differentiation and representation: $(\mathbf{m}\cdot\boldsymbol{\partial}_\mathbf{n})\mathcal{R}(\mathbf{n}) = \mathcal{R}((\mathbf{m}\cdot\boldsymbol{\partial}_\mathbf{n})\mathbf{n})$. It is convenient to use a shorthand for the differential, as in [3]: define $\underline{\mathcal{R}}(\mathbf{a}) \equiv (\mathbf{a}\cdot\boldsymbol{\partial}_\mathbf{n})\mathcal{R}(\mathbf{n})$, then

$$\underline{\mathcal{R}}(\mathbf{m})[\mathbf{p}] = \mathcal{R}(\mathbf{m})[\mathbf{p}].$$

  (Be careful to use both expressions at the same value of $\mathbf{p}[\mathbf{n}]$, *not* at $\mathbf{p}[\mathbf{m}]$!) Since we need to preserve the norm of $\mathbf{n}$ for the representation

($\mathbf{n}^2 = 1$), we will only use differentials for which $0 = (\mathbf{m} \cdot \boldsymbol{\partial}_{\mathbf{n}})\mathbf{n}^2 = \mathbf{m} \cdot \mathbf{n}$, i.e. $\mathbf{m}$ is perpendicular to $\mathbf{n}$.

On the other hand, taking the derivative of $\mathcal{R}(\mathbf{n})\mathbf{p}]$ to $\mathbf{p}$ yields

$$\begin{aligned}(\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathcal{R}(\mathbf{n})[\mathbf{p}] &= (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{p}})\left(p \cdot (e^0 \mathbf{n}[\mathbf{p}])\right) \\ &= \mathbf{P}(\mathbf{a}) \cdot (e^0 \mathbf{n}) + p \cdot \left(e^0 \underline{\mathbf{n}}(\mathbf{a})\right) \\ &= p \cdot \left(e^0 \underline{\mathbf{n}}(\mathbf{a})\right) = \mathcal{R}(\underline{\mathbf{n}}(\mathbf{a}))[\mathbf{p}],\end{aligned}$$

so that now $\mathcal{R}(\cdot)$ needs to be evaluated at $\underline{\mathbf{n}}(\mathbf{a})$ rather than at $\mathbf{a}$.

- *any tangent multivector u based at* $\mathbf{p}$ *is represented as* $\mathcal{R}(\underline{\mathbf{n}}(u))[\mathbf{p}]$
  The derivation above shows that a tangent vector $\mathbf{a}$ at $\mathbf{p}$ is represented as $\mathcal{R}(\underline{\mathbf{n}}(\mathbf{a}))[\mathbf{p}]$; since this is linear in $\mathbf{a}$ we can extend it as an outermorphism to any tangent blade at $\mathbf{p}$, and then by linearity to any tangent multivector. For a scalar α, this gives $\mathcal{R}(\underline{\mathbf{n}}(\alpha))[\mathbf{p}] = \mathcal{R}(\alpha)[\mathbf{p}] = \alpha$, as it should.

- *any multivector u from the differential space based at* $\mathbf{p}$ *is represented as* $\mathcal{R}(u)[\mathbf{p}]$
  This result is very useful, but a bit hard to formulate. By the *differential tangent space* at $\mathbf{p}$ we mean the space $\mathcal{G}^1(\underline{\mathbf{n}}(\mathbf{I}))$; the *differential space* at $\mathbf{p}$ is then the space spanned by $\mathbf{n}$ and the differential tangent space. Equation (1.10) shows that the vectors $\mathbf{m}$ from the differential tangent space are represented as $\mathcal{R}(\mathbf{m})[\mathbf{p}]$. This is a linear map, and can be extended by outermorphism to all of the differential tangent space. But since $\mathbf{n}$ is represented by $\mathcal{R}(\mathbf{n})[\mathbf{p}]$, which is of the same form, we can even extend the representation to *any* multivector of $\mathcal{G}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))$ based at $\mathbf{p}$, i.e. any multivector of the differential space at $\mathbf{p}$. So the representation of such a u at $\mathbf{p}$ is $p \cdot (e^0 u)$. For a scalar α, this gives $\mathcal{R}(\alpha)[\mathbf{p}] = \alpha$, as it should.

- *the representation at* $\mathbf{p}$ *commutes with the geometric product*
  For scalars, this holds by linearity. For vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ in the differential space at $\mathbf{p}$:

$$\begin{aligned}\mathcal{R}(\mathbf{m}_1)\,\mathcal{R}(\mathbf{m}_2) &= \left(p \cdot (e^0 \mathbf{m}_1)\right)\,\left(p \cdot (e^0 \mathbf{m}_2)\right) \\ &= \left(\mathbf{m}_1 - e^0 \mathbf{p} \cdot \mathbf{m}_1\right)\,\left(\mathbf{m}_2 - e^0 \mathbf{p} \cdot \mathbf{m}_2\right) \\ &= \mathbf{m}_1\mathbf{m}_2 - e^0\left((\mathbf{p} \cdot \mathbf{m}_1)\mathbf{m}_2 - \mathbf{m}_1(\mathbf{p} \cdot \mathbf{m}_2)\right) + 0 \\ &= (\mathbf{m}_1\mathbf{m}_2) - e^0\left(\mathbf{p} \cdot (\mathbf{m}_1\mathbf{m}_2)\right) = p \cdot \left(e^0(\mathbf{m}_1\mathbf{m}_2)\right) \\ &= \mathcal{R}(\mathbf{m}_1\mathbf{m}_2).\end{aligned}$$

  This result for vectors extends naturally to the whole geometric algebra of the differential space at $\mathbf{p}$. Note how this uses $e^0 e^0 = 0$; this is why we like the embedding in Minkowski space. This specifically means that the representation is isometric; for instance $\left(\mathcal{R}(\mathbf{n})\right)^2 =$

$\mathcal{R}(\mathbf{n}^2) = \mathcal{R}(1) = 1$, so all $\mathcal{R}(\mathbf{n})$ reside on a sphere in Minkowski space; this is the embedding of the extended Gaussian sphere of directions we referred to earlier.

We can derive a similar result for the tangent space at $\mathbf{p}$: the representation of elements of the tangent algebra also commutes with the geometric product; however, we will not need that in this Chapter.

- *the representation is invertible if* $\boldsymbol{\kappa} \neq 0$
  Observe that $p = e_0 + \mathbf{p}$ is perpendicular to the representation of any of the elements of the differential space at $\mathbf{p}$, since $p \cdot \mathcal{R}(u) = p \cdot \big(p \cdot (e^0 u)\big) = (p \wedge p) \cdot (e^0 u) = 0$. This gives m independent conditions, and thus determines p, if and only if the differential tangent space $\mathbf{n}(\mathbf{I})$ is $(m-1)$-dimensional (perpendicularity to $\mathbf{n}$ provides the one extra condition required). Since $\underline{\mathbf{n}}(\mathbf{I}) = \boldsymbol{\kappa}\widehat{\mathbf{I}}$ by Equation (1.3), and $\mathbf{I}$ is known to be $(m-1)$-dimensional for the regular surfaces we study, this requires that $\boldsymbol{\kappa} \neq 0$.

  So when $\boldsymbol{\kappa} \neq 0$, we have a proportional image of the full-rank tangent space at $\mathbf{p}$ present as the tangent space to our representation at $\mathcal{R}(\mathbf{n})[\mathbf{p}]$. The perpendicularity of p to this and to $\mathbf{n}$ gives m constraints, and is therefore sufficient to determine p by duality in the embedding space of $\mathcal{R}(\mathbf{n}) \wedge \mathcal{R}(\mathbf{I}) = \mathcal{R}(\mathbf{n} \wedge \mathbf{I}) = \mathcal{R}(-\mathbf{I}_m) = -\mathcal{R}(\mathbf{I}_m)$. And indeed:
  $$e^0 \wedge p = -\mathcal{R}(\mathbf{I}_m)\mathbf{I}_m^{-1} E. \tag{1.11}$$
  This is the flat of the point at $\mathbf{p}$, so that $\mathbf{p}$ is retrievable as: $\mathbf{p} = e_0 \cdot \big(\mathcal{R}(\mathbf{I}_m)\mathbf{I}_m^{-1} E\big) - e_0$, the usual formula in the additive split. The computation of Equation (1.11) is straightforward:
  $$\begin{aligned} -\mathcal{R}(\mathbf{I}_m)(E\mathbf{I}_m)^{-1} &= -\big(p \cdot (e^0 \mathbf{I}_m)\big)\,\mathbf{I}_m^{-1} E = -p \wedge \big(e^0 \mathbf{I}_m \mathbf{I}_m^{-1} E\big) \\ &= -p \wedge \big(e^0 E\big) = -(p \wedge e^0) = e^0 \wedge p. \end{aligned}$$
  (When $\boldsymbol{\kappa} \neq 0$, the dual of the largest possible tangent space of the representation produces a flat of equivalent positions with the same local shape as at $\mathbf{p}$; for instance, for a cylinder we obtain a line parallel to its axis. This need not be a disadvantage, since we will see that all such points behave similarly under wave propagation; this thus allows for effective lumping in propagation algorithms. We plan to investigate that.)

We thus have (in those non-flat cases) in $\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{p}[\mathbf{n}] \cdot \mathbf{n})$ an invertible representation of a boundary in terms of a '*spectrum*' of tangent directions indexed by $\mathbf{n}$ (or dually by $\mathbf{I}$), with '*amplitude*' $\mathbf{p}[\mathbf{n}] \cdot \mathbf{n}$ marked off in the e^0-direction. The advantage of the embedding space is that the original boundary and its dual representation reside in the same space $\mathcal{G}(E\mathbf{I}_m)$, so that the transition between them is purely algebraic, and geometric (through perpendicularity).

1.3.3 Example: a spherical boundary

Let us do an example: the representation of m-dimensional boundaries of which the surface is a sphere – we'll be interested in spherical blobs and in spherical holes.

The position of points on a sphere with center $\mathbf{c}$ and radius ρ can be defined by the scalar function equation $\phi(\mathbf{p}) = 0$, with $\phi(\mathbf{p}) = (\mathbf{p}-\mathbf{c})^2 - \rho^2$. Differentiation to $\mathbf{p}$ yields for the unit normal vector $\mathbf{n}$:

$$\mathbf{n} = \pm \frac{\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})}{|\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})|} = \pm \frac{\mathbf{p}-\mathbf{c}}{|\rho|}.$$

This needs to be oriented properly to point inwards. For a spherical blob, the inward pointing normal is positively proportional to $\mathbf{c}-\mathbf{p}$, for a spherical hole to $\mathbf{p}-\mathbf{c}$. We can therefore use the radius ρ of the sphere to indicate which it is; we prefer to denote blobs by a positive radius, holes by a negative radius, so we obtain

$$\mathbf{n} = \frac{\mathbf{c}-\mathbf{p}}{\rho}$$

as the inward pointing normal for a spherical boundary, whether hole or blob. Then to make $\mathcal{R}(\mathbf{n})$, we need to express $\mathbf{p}$ in terms of $\mathbf{n}$, which is simply $\mathbf{p} = \mathbf{c} - \rho\,\mathbf{n}$. This gives:

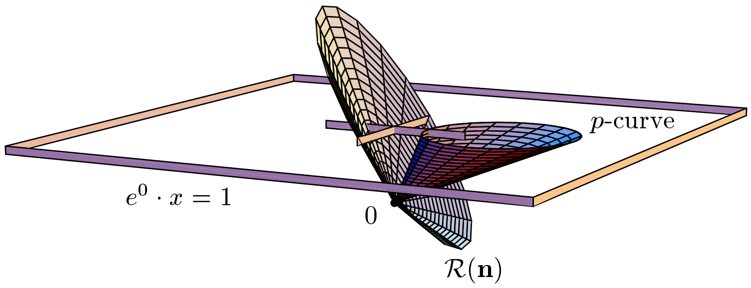
$$\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{p}\cdot\mathbf{n}) = \mathbf{n} - e^0(\mathbf{c}\cdot\mathbf{n} - \rho)$$

as the representation of the spherical boundary. This representation satisfies $\mathcal{R}(\mathbf{n})^2 = 1$ and $c \cdot \mathcal{R}(\mathbf{n}) = \rho$ (with $c = e_0 + \mathbf{c}$); so in the embedding space it is on the intersection of the Gaussian sphere in Minkowski space with the plane with normal vector c, at distance ρ. This is illustrated in Figure 1.2a for the 2-dimensional case of the circular blob. The Minkowski sphere looks like a cylinder to our Euclidean eyes; the intersection with the plane $c \cdot \mathcal{R}(\mathbf{n}) = \rho$ gives the tilted ellipse depicted. Figure 1.2b depicts the circular hole and its representation, which simply has the opposite sign. This cylinder is the Gauss sphere representation, but now in a space shared with the original curve, which makes the dual (i.e. perpendicular) nature of the representation explicit.

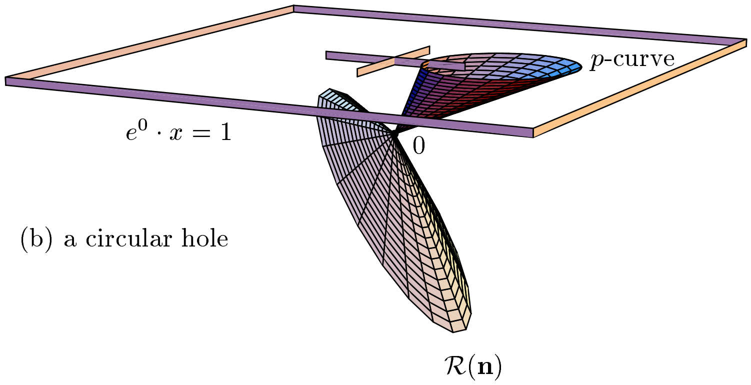
We may check the differential relationships:

$$\begin{aligned}
\underline{\mathcal{R}}(\mathbf{m}) &\equiv (\mathbf{m}\cdot\boldsymbol{\partial}_{\mathbf{n}})\big(\mathbf{n} - e^0(\mathbf{c}\cdot\mathbf{n} - \rho)\big) \\
&= \mathbf{m} - e^0(\mathbf{c}\cdot\mathbf{m}) = (e_0 + \mathbf{c} - \rho\,\mathbf{n})\cdot(e^0\mathbf{m}) = \mathcal{R}(\mathbf{m}),
\end{aligned}$$

since $\mathbf{m}\cdot\mathbf{n} = 0$. (Note that $\mathcal{R}(\mathbf{m})$ should *not* be interpreted as $\mathbf{m} - e^0(\mathbf{c}\cdot\mathbf{m} - \rho)$, simply substituting $\mathbf{m}$ for $\mathbf{n}$ in the expression for $\mathcal{R}(\mathbf{n})$; the latter is shorthand for $\mathcal{R}(\mathbf{n})[\mathbf{p}[\mathbf{n}]]$ and we need $\mathcal{R}(\mathbf{m})[\mathbf{p}[\mathbf{n}]]$, not $\mathcal{R}(\mathbf{m})[\mathbf{p}[\mathbf{m}]]$!)



(a) a circular blob



(b) a circular hole

FIGURE 1.2. The representation of a circular hole and a blob in $\mathcal{G}(\mathbf{I}_2)$. For the p-curve the vertical axis denotes e_0; since $p = e_0 + \mathbf{p}$, this curve resides in the plane $e^0 \cdot x = 1$ which is indicated. For the $\mathcal{R}(\mathbf{n})$ curves, the vertical axis is e^0, and the curve resides on the extended Gaussian sphere $\mathcal{R}(\mathbf{n})^2 = 1$, which due to the Minkowski geometry of $\mathcal{G}_{m+1,1}$ looks like a Euclidean cylinder in this projection. The curves have been made into cones to better indicate their spatial nature, and to help show that $p = e_0 + \mathbf{p}$ is everywhere perpendicular to $\mathcal{R}(\mathbf{I}_m) = \mathcal{R}(\mathbf{n}) \wedge \mathcal{R}(\mathbf{I})$. Since $e_0 \cdot e^0 = 1$, the axes e_0 and e^0 are parallel under the duality involved in the representation, which is why we can draw them this way.

To retrieve position and curvature from the representation, we take the derivative in the embedding space. With the above, we obtain through outermorphism:

$$\mathcal{R}(\mathbf{I}_m) = \mathcal{R}(\mathbf{n}) \wedge \mathcal{R}(-\mathbf{I}) = \mathbf{I}_m - e^0(\mathbf{c} \cdot \mathbf{I}_m + \rho \mathbf{I})$$

and then for the dual of this:

$$\begin{aligned} -\mathcal{R}(\mathbf{I}_m)\mathbf{I}_m^{-1}E &= -E + e^0(\mathbf{c} \cdot \mathbf{I}_m)\mathbf{I}_m^{-1}E + \rho\, e^0 \mathbf{I}\mathbf{I}_m^{-1}E \\ &= -E + e^0(\mathbf{c} \wedge 1)E - \rho\, e^0 \mathbf{n} \\ &= -E + e^0(\mathbf{c} - \rho\, \mathbf{n}) = e^0 \wedge (e_0 + \mathbf{c} - \rho\, \mathbf{n}). \end{aligned}$$

We retrieve $\mathbf{p}$ in terms of $\mathbf{n}$ from this by:

$$\mathbf{p} = e_0 \cdot \big(e^0 \wedge (e_0 + \mathbf{c} - \rho\, \mathbf{n})\big) - e_0 = \mathbf{c} - \rho\, \mathbf{n},$$

which is indeed the set of positions on the spherical boundary with inward pointing normal $\mathbf{n}$.

By the way, note that $\underline{\mathbf{n}}(\mathbf{a}) \equiv (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{n}[\mathbf{p}] = -\mathbf{a}/\rho$, by Equation (1.3.3). Therefore the Gaussian curvature is, by Equation (1.3), $\boldsymbol{\kappa} = \underline{\mathbf{n}}(\widehat{\mathbf{I}})\mathbf{I}^{-1} = 1/\rho^{m-1}$, as it should be.

1.3.4 Boundaries as direction-dependent rotors

The equation for $\mathcal{R}(\mathbf{n})[\mathbf{p}]$ can be written in an interesting alternative form:

$$\begin{aligned} \mathcal{R}(\mathbf{n})[\mathbf{p}] &= \mathbf{n} - e^0(\mathbf{p} \cdot \mathbf{n}) = (1 - e^0\mathbf{p}/2)\, \mathbf{n}\, (1 + e^0\mathbf{p}/2) \\ &= \exp(-e^0\mathbf{p}/2)\, \mathbf{n}\, \exp(e^0\mathbf{p}/2). \end{aligned}$$

Thus the $\mathbf{n}$-representation can be constructed from a normal vector $\mathbf{n}$ via the general rotor equation $\underline{U}(\mathbf{x}) = U\mathbf{x}\widehat{U}^{-1}$, using the $\mathbf{n}$-dependent rotor

$$U = T_{\mathbf{p}} \equiv \exp(-e^0\mathbf{p}[\mathbf{n}]/2) = 1 - e^0\mathbf{p}[\mathbf{n}]/2. \tag{1.12}$$

This is the versor $T_{\mathbf{p}}$ of a translation over $\mathbf{p}[\mathbf{n}]$ in the standard homogeneous model of a Euclidean space $\mathcal{G}^1(\mathbf{I}_m)$ in the Minkowski space $\mathcal{G}^1(E\mathbf{I}_m)$, see [5]. In this view, we can see an object boundary (as represented by $\mathcal{R}(\mathbf{n})$) as an $\mathbf{n}$-dependent translation $\mathbf{p}[\mathbf{n}]$ applied to the unit normal vector $\mathbf{n}$. Since the latter is the representation of a point blob at the origin as a (trivial) function of its orientation, this provides the view:

Any object boundary can be represented as a deformation by orientation-dependent translation of a point blob at the origin.

Non-convex objects may have a particular inward pointing normal vector $\mathbf{n}$ at different points $\mathbf{p}$, so for those the function $\mathbf{p}[\mathbf{n}]$ should be considered set-valued.

boundary operation	**boundary rotor (spectrum)**
null-boundary (a point blob)	1
arbitrary boundary creation	$T_{\mathbf{p}[\mathbf{n}]} = \exp(-e^0\mathbf{p}[\mathbf{n}]/2)$
translation over $\mathbf{t}$	$T_{\mathbf{t}}\, T_{\mathbf{p}[\mathbf{n}]} = \exp(-e^0\mathbf{t}/2)\, T_{\mathbf{p}[\mathbf{n}]}$
rotation (center $\mathbf{c}$, rotor $\mathbf{R}$)	$R_{\mathbf{c},\mathbf{R}}\, T_{\mathbf{p}[\mathbf{n}]} = (\mathbf{R} - e^0(\mathbf{c}\cdot\mathbf{R}))\, T_{\mathbf{p}[\mathbf{n}]}$
mirroring in hyperplane, support $\mathbf{d}$	$M_{\mathbf{d}}\, T_{\mathbf{p}[\mathbf{n}]} = (\mathbf{d} - e^0\mathbf{d}^2/2)\, T_{\mathbf{p}[\mathbf{n}]}$
point reflection in $\mathbf{c}$	$P_{\mathbf{c}}\, T_{\mathbf{p}[\mathbf{n}]} = (\mathbf{I}_m - e^0(\mathbf{c}\cdot\mathbf{I}_m))\, T_{\mathbf{p}[\mathbf{n}]}$
wave propagation by boundary $T_{\mathbf{q}}$	$T_{\mathbf{q}[\mathbf{n}]}\, T_{\mathbf{p}[\mathbf{n}]}$

FIGURE 1.3. The boundary rotor $T_{\mathbf{p}[\mathbf{n}]}$ associated with common operations on the boundary.

1.3.5 The effect of Euclidean transformations

When a boundary is subjected to a transformation, its representation must change. If we represent the boundary as a rotor, the transformed rotors under common Euclidean operations transform in a straightforward manner, according to the rules in Figure 1.3 (ignore the entry of wave propagation for now). These are easily proved by keeping track of what happens to the position and its differential (which gives $\mathbf{n}$). As an example we treat the rotation of a boundary.

When the boundary rotates around $\mathbf{c}$ over a bivector angle characterized by a rotor $\mathbf{R}$ (we use boldface since this rotor is in $\mathcal{G}(\mathbf{I}_m)$), then $\mathbf{p}$ becomes $(\mathbf{R}(\mathbf{p} - \mathbf{c})\mathbf{R}^{-1} + \mathbf{c})$. This is achieved on its versor $T_{\mathbf{p}}$ by: $T_{\mathbf{p}'} = T_{\mathbf{c}}(\mathbf{R}(T_{-\mathbf{c}}T_{\mathbf{p}})\mathbf{R}^{-1})$, as is easily verified. Differentiating yields that $\mathbf{n}$ is rotated as well, to $\mathbf{n}' = \mathbf{R}\mathbf{n}\mathbf{R}^{-1}$. The $\mathcal{R}(\mathbf{n})$-representation of the rotated boundary is achieved by applying $T_{\mathbf{p}'}$ to $\mathbf{n}'$ as a versor, which yields:

$$\begin{aligned}
\underline{T}_{\mathbf{p}'}(\mathbf{n}') &= T_{\mathbf{p}'}\mathbf{n}'T_{\mathbf{p}'}^{-1} \\
&= \left(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}}T_{\mathbf{p}}\mathbf{R}^{-1}\right)\mathbf{R}\mathbf{n}\mathbf{R}^{-1}\left(\mathbf{T}_{\mathbf{c}}\mathbf{R}\mathbf{T}_{-\mathbf{c}}\mathbf{T}_{\mathbf{p}}\mathbf{R}^{-1}\right)^{-1} \\
&= (T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})T_{\mathbf{p}}\,\mathbf{n}\,T_{\mathbf{p}}^{-1}(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})^{-1}.
\end{aligned}$$

Therefore the total result can be achieved by the application of a new versor to $\mathbf{n}$, i.e. we can bring the representation in a standard form which can immediately be used with the original, unrotated boundary, or other boundaries thus represented. This is a pleasant surprise! This new versor is $T_{\mathbf{p}}$ left-multiplied by:

$$T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}} = (1 - e^0\mathbf{c}/2)\mathbf{R}(1 + e^0\mathbf{c}/2) = \mathbf{R} - e^0(\mathbf{c}\cdot\mathbf{R}) \equiv R_{\mathbf{c},\mathbf{R}} \qquad (1.13)$$

and that is the entry in Figure 1.3 for rotation. The other entries are derived similarly.

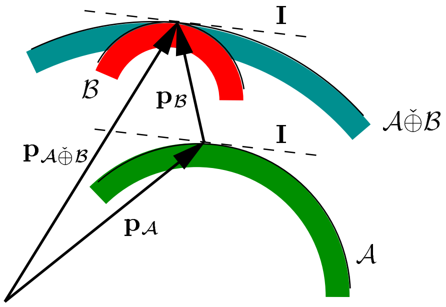


FIGURE 1.4. The definition of wave propagation.

1.4 Wave propagation of boundaries

1.4.1 Definition of propagation

Propagation combines two boundaries $\mathcal{A}$ and $\mathcal{B}$ to produce a boundary $\mathcal{A}\check{\oplus}\mathcal{B}$ according to the following rules (which can be taken as the definition of propagation, or alternatively derived from a formulaic definition as in [1]):

Propagation definition:

- The resulting position vector after combining a point $\mathbf{p}_{\mathcal{A}}$ on $\mathcal{A}$ and a point $\mathbf{p}_{\mathcal{B}}$ on $\mathcal{B}$ is the position $\mathbf{p}_{\mathcal{A}} + \mathbf{p}_{\mathcal{B}}$:

$$\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}} = \mathbf{p}_{\mathcal{A}} + \mathbf{p}_{\mathcal{B}} \tag{1.14}$$

- The points $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$ *must* have the same inward pointing normal vector (to $\mathcal{A}$ and $\mathcal{B}$, respectively), and this is also the inward pointing normal vector at the resulting position in the resulting boundary. Symbolically:

$$\mathbf{n}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}] = \mathbf{n}_{\mathcal{A}}[\mathbf{p}_{\mathcal{A}}] = \mathbf{n}_{\mathcal{B}}[\mathbf{p}_{\mathcal{B}}]. \tag{1.15}$$

These conditions together fully determine the propagation result and the dependence of its geometrical properties on the geometrical properties of $\mathcal{A}$ and $\mathcal{B}$.

1.4.2 Propagation in the embedded representations

Since $\mathcal{R}(\mathbf{n})$ is an invertible description of a boundary, if we can construct the representation of the wave propagation result then we know what the resulting boundary is. But this is extremely simple, since the representation lends itself to direct implementation of the definition of propagation of Equation (1.14) and Equation (1.15).

Let $\mathbf{p}_\mathcal{A}[\mathbf{n}]$ be defined as: $\mathbf{p}_\mathcal{A}[\mathbf{n}] \equiv \{\mathbf{x} \in \mathcal{A} \mid \mathbf{n}_\mathcal{A}[\mathbf{x}] = \mathbf{n}\}$, so as the set of all positions of the boundary where the inward normal vector is $\mathbf{n}$; and similarly for $\mathbf{p}_\mathcal{B}[\cdot]$. Then the propagation result of $\mathcal{R}(\mathbf{n}_\mathcal{A}) = \mathbf{n}_\mathcal{A} - e^0(\mathbf{p}_\mathcal{A}[\mathbf{n}_\mathcal{A}] \cdot \mathbf{n}_\mathcal{A})$ and $\mathcal{R}(\mathbf{n}_\mathcal{B}) = \mathbf{n}_\mathcal{B} - e^0(\mathbf{p}_\mathcal{B}[\mathbf{n}_\mathcal{B}] \cdot \mathbf{n}_\mathcal{B})$ is by Equation (1.15) indexed by the same normal vector $\mathbf{n}$, and

$$\begin{aligned}
\mathcal{R}(\mathbf{n}_{\mathcal{A}\check{\oplus}\mathcal{B}}) &= \mathbf{n} - e^0 \left(\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{n}] \cdot \mathbf{n}\right) \\
&= \mathbf{n} - e^0 \left((\mathbf{p}_\mathcal{A}[\mathbf{n}] \oplus \mathbf{p}_\mathcal{B}[\mathbf{n}]) \cdot \mathbf{n}\right) \\
&= \mathbf{n} - e^0 \left((\mathbf{p}_\mathcal{A}[\mathbf{n}] \cdot \mathbf{n}) \oplus (\mathbf{p}_\mathcal{B}[\mathbf{n}] \cdot \mathbf{n})\right). \qquad (1.16)
\end{aligned}$$

So basically, *the e^0 components add up at the same* $\mathbf{n}$ (we must use $\oplus$ since $\mathbf{p}[\mathbf{n}]$ is a set-valued function).

In the direction-dependent rotor representation of boundaries, we get:

$$\begin{aligned}
T_{\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{n}]} &= 1 - e^0(\mathbf{p}_\mathcal{A}[\mathbf{n}] \oplus \mathbf{p}_\mathcal{B}[\mathbf{n}])/2 \\
&= \left(1 - e^0\mathbf{p}_\mathcal{A}[\mathbf{n}]/2\right)\left(1 - e^0\mathbf{p}_\mathcal{B}[\mathbf{n}]/2\right) \\
&= T_{\mathbf{p}_\mathcal{A}[\mathbf{n}]}\, T_{\mathbf{p}_\mathcal{B}[\mathbf{n}]}.
\end{aligned}$$

Therefore *wave propagation is represented as the product of boundary rotors* (at least if we overload the geometric product to work on sets of rotors, a straightforward extension).

1.4.3 A systems theory of wave propagation and collision

The above is analogous to what the Fourier transformation does for convolution: the convolution of two signals becomes multiplication of their frequency spectra (a complex number $A(\omega)\exp(i\phi(\omega))$ for each frequency ω); propagation of two boundaries has become multiplication of their 'direction spectra' $T_{\mathbf{p}[\mathbf{n}]}$ (a rotor $\exp(-e^0\mathbf{p}[\mathbf{n}]/2)$ for each direction $\mathbf{n}$).

This algebraic analogy permits a transfer of ideas from linear systems theory to the treatment of wave propagation, collision detection, and the other related problems of Section 1.1.3. For instance, for linear systems the delta-function is the function with which convolution reproduces the convolution kernel; it is the input function which allows measurement of the system's response function (for instance, the image of a point source gives the optical transfer function of a camera). In wave propagation, the point at the origin plays the same role: propagation from it provides the shape of the propagator; in robotics, you could measure the shape of the robot by tracking the position of a reference point as the robot collides with an infinitely thin pole at the origin. This is because its versor representation equals 1, or equivalently $\mathcal{R}(\mathbf{n}) = \mathbf{n}$); so this is the 'delta-boundary' for propagation.

In linear systems theory, if a delta-function is not available as a probe one may hope to reconstruct the system's response function by determining the common multiplicative factor in the Fourier transformation of a sufficiently rich set of responses (through a Wiener filter); this can then be used to sharpen that data by deconvolution. Similarly, given sufficiently rich collision data, one could determine a common multiplicative versor (or common additive $\mathbf{n}$-dependent function in the $\mathcal{R}(\mathbf{n})$-representation) by a similar procedure. This would enable determination of the shape of the atomic probe in the STM example of Section 1.1.3, and then of the actual unknown atomic surface by a 'de-dilation' of the measured surface. (But to do this fully, one would need to treat the unobservable parts of the collision boundary around the swallowtail catastrophes of Figure 1.1b properly – which we have not done yet.)

Such direct transfer of techniques from linear systems theory is possible because we have found a characteristic 'spectral' representation, which enables us to replace the involved effects of the collision operation as local operations in the spectral domain.

1.4.4 Matching tangents

The rotor result shows the algebraic analogy with the Fourier approach to convolution; but the equivalent 'addition of e^0-components' is actually simpler to implement. However, either result is somewhat deceptively simple, since the demand that both $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$ be written in terms of the *same* $\mathbf{n}$ may require an inversion and a reparametrization of either or both to obtain $\mathbf{p}$ as a function of $\mathbf{n}$ valid over a finite domain (if the boundaries were originally given in terms of parametrized position). Yet this can be done, if necessary numerically; and then the result is useful to construct the resulting boundary (in its $\mathbf{n}$-based representation form), and to derive its properties. Collision detection and wave propagation (and the other, equivalent operations mentioned in the introduction) can be done fully in this representation. (In some applications such as radar observations, the representation $\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{p} \cdot \mathbf{n})$ is even measured *directly*: a radar yields the distance $(\mathbf{p} \cdot \mathbf{n})$ of a tangent plane perpendicular to the direction $\mathbf{n}$ of the outgoing beam.) It is only when one desires the result to be drawn as a positional surface again that the rather involved inversion formula Equation (1.11) needs be invoked.

1.4.5 Examples of propagation

For a sphere at $\mathbf{c}$ with radius ρ, we found in Section 1.3.3 the representation $\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{c} \cdot \mathbf{n} - \rho)$. Therefore the dilation of two spheres (index 1 and 2) yields:

$$\mathcal{R}(\mathbf{n}) = \mathbf{n} - e^0(\mathbf{c}_1 \cdot \mathbf{n} - \rho_1) - e^0(\mathbf{c}_2 \cdot \mathbf{n} - \rho_2) = \mathbf{n} - e^0\left((\mathbf{c}_1 + \mathbf{c}_2) \cdot \mathbf{n} - (\rho_1 + \rho_2)\right),$$

which is immediately recognizable as the sphere with center $(\mathbf{c}_1 + \mathbf{c}_2)$ and radius $(\rho_1 + \rho_2)$. This is what we would expect as the result; but note that it is also valid for spherical holes (ρ negative). The propagation of a hole with radius ρ by a blob with radius ρ is a point (with radius zero). Or, in terms of robot collision avoidance, a spherical robot of radius ρ in a hole with radius ρ cannot move, its only permissible position is $(\mathbf{c}_1 + \mathbf{c}_2)$.

Figure 1.5a depicts the wave propagation on a parabolic concave boundary, and it was generated by adding the representations of a parabola $(\mathbf{x} \cdot \mathbf{e}_2) = \frac{1}{2}(\mathbf{x} \cdot \mathbf{e}_1)^2$ (with 'inside' in the $(-\mathbf{e}_2)$-direction) and a circular blob, and then 'inverting' the result to a positional boundary. Note the occurrence of swallowtail catastrophes in some of the wave fronts. Classically, these have been considered hard to treat, and even non-differentiable; however they are fully differentiable in our *directed* representation of the tangent space. In fact, these *spatial cusps correspond to inflection points* in the $\mathbf{n}$-representation, and are thus nothing more unusual than a sign change in the curvature of $\mathcal{R}(\mathbf{n})$. This is illustrated in Figure 1.5b, which shows how the boundaries and their representations coexist in $\mathcal{C}\ell_{m+1,1}$. The depiction is similar to Figure 1.2, but we have rescaled $\mathcal{R}(\mathbf{n})$ and drawn its intersection with the plane $\mathbf{e}_2 = -1$, where it is in fact the Legendre transformation of the boundary, see [1]. (In this example, we obtain $\mathbf{n}' - \frac{1}{2}e^0(\mathbf{n}'^2 - 1)$ where $\mathbf{n}' = \mathbf{n}/(\mathbf{n} \cdot \mathbf{e}_2)$.) The shift in e^0 (by the radius ρ of the circle, since the centered circular blob has $\mathcal{R}(\mathbf{n}) = \mathbf{n} + e^0\rho$) results in the development of inflection points in the representation, which correspond to the cusps. The locations where the boundary surface self-intersects (important for the analysis of the 'millability' of surfaces) correspond to non-local properties of the representation; the intersection point at $\frac{1}{2}(1 + \rho^2)\mathbf{e}_2$ for $\rho > 1$ corresponds to a straight part of the convex hull of $\mathcal{R}(\mathbf{n})$ (for some more details see [1]).

1.4.6 Analysis of propagation

Now that we have a convenient representation of wave propagation, we can derive many properties of the geometry of the result, for instance:

> *The propagated boundary $\mathcal{C} = \mathcal{A}\check{\oplus}\mathcal{B}$ obeys the 'velocity law' which relates velocities on the propagated boundary to those on the propagators:*
>
> $$\underline{\mathbf{n}}^{-1}_{\mathcal{A}\check{\oplus}\mathcal{B}}(\mathbf{m})[\mathbf{p}_\mathcal{A} \oplus \mathbf{p}_\mathcal{B}] = \underline{\mathbf{n}}^{-1}_\mathcal{A}(\mathbf{m})[\mathbf{p}_\mathcal{A}] \oplus \underline{\mathbf{n}}^{-1}_\mathcal{B}(\mathbf{m})[\mathbf{p}_\mathcal{B}]. \qquad (1.17)$$
>
> *The result is $\emptyset$ for $\mathbf{m}$* not *in the common range of $\underline{\mathbf{n}}_\mathcal{A}(\cdot)[\mathbf{p}_\mathcal{A}]$ and $\underline{\mathbf{n}}_\mathcal{B}(\cdot)[\mathbf{p}_\mathcal{B}]$.*

Proof: Introduce three tangent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, to measure the derivative on each of the surfaces, and use the chain rule of [3] to rewrite them in terms of

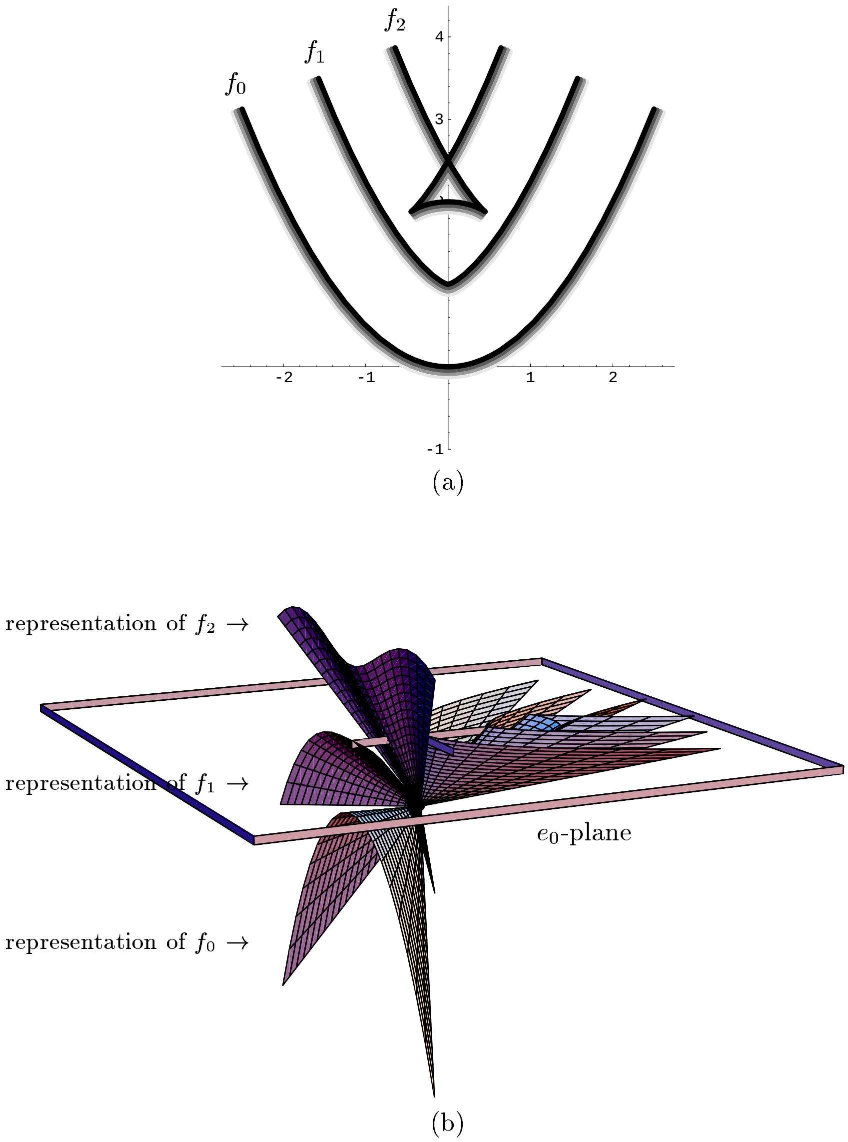


FIGURE 1.5. Circular wave propagation of the parabola-shaped boundary f_0 (see text for explanation).

derivatives of $\mathbf{p}[\mathbf{n}]$: $\mathbf{a} = \mathbf{P}_{\mathcal{A}}(\mathbf{a}) = (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{A}} = (\underline{\mathbf{n}}_{\mathcal{A}}(\mathbf{a}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\, \mathbf{p}_{\mathcal{A}}[\mathbf{n}]$ and similarly $\mathbf{b} = \mathbf{P}_{\mathcal{B}}(\mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{B}} = (\underline{\mathbf{n}}_{\mathcal{B}}(\mathbf{b}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\, \mathbf{p}_{\mathcal{B}}[\mathbf{n}]$ and $\mathbf{c} = \mathbf{P}_{\mathcal{C}}(\mathbf{c}) = (\mathbf{c} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{C}} = (\underline{\mathbf{n}}_{\mathcal{C}}(\mathbf{c}) \cdot \boldsymbol{\partial}_{\mathbf{n}})\, \mathbf{p}_{\mathcal{C}}[\mathbf{n}]$. Now select these such that $\underline{\mathbf{n}}_{\mathcal{A}}(\mathbf{a}) = \underline{\mathbf{n}}_{\mathcal{B}}(\mathbf{b}) = \underline{\mathbf{n}}_{\mathcal{C}}(\mathbf{c}) = \mathbf{m}$. We then find from the above that these tangents add as position vectors: $\mathbf{c} = (\mathbf{m} \cdot \boldsymbol{\partial}_{\mathbf{p}})\mathbf{p}_{\mathcal{C}}[\mathbf{n}] = (\mathbf{m} \cdot \boldsymbol{\partial}_{\mathbf{p}})\,(\mathbf{p}_{\mathcal{A}}[\mathbf{n}] + \mathbf{p}_{\mathcal{B}}[\mathbf{n}]) = \mathbf{a} + \mathbf{b}$. Our selection of $\mathbf{m}$ implies that $\mathbf{a} \in \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m})$, $\mathbf{b} \in \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m})$ and $\mathbf{c} \in \underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m})$; therefore, over all possibilities of choosing $\mathbf{a}$ and $\mathbf{b}$ given $\mathbf{c}$, we obtain $\underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m}) = \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m}) \oplus \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m})$. The right hand side produces $\emptyset$ for any element not common to both sets contributing to the Minkowski sum; hence only elements in both ranges contribute – which implies that $\mathbf{m}$ must be in the common range of $\underline{\mathbf{n}}_{\mathcal{A}}(\cdot)$ and $\underline{\mathbf{n}}_{\mathcal{B}}(\cdot)$ at $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$, respectively. □

This interaction of the local differential geometries can produce involved results, especially for surfaces with torsion. However, there is an interestingly simple property when we 'lump' over all tangent directions at $\mathbf{p}$:

In m-dimensional wave propagation, Gaussian curvatures add reciprocally:

$$\boldsymbol{\kappa}_{\mathcal{C}}^{-1} = \boldsymbol{\kappa}_{\mathcal{A}}^{-1} + \boldsymbol{\kappa}_{\mathcal{B}}^{-1} \tag{1.18}$$

(locally, at every triple of corresponding points).

Proof: Extending Equation (1.17) as a linear outermorphism to all of $\mathcal{G}(\mathbf{I})$ at the appropriate points we get: $\underline{\mathbf{n}}_{\mathcal{A}\oplus\mathcal{B}}^{-1}(\mathbf{I}) = \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{I}) + \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{I})$ for each triple of corresponding points. The Gaussian curvature is related to $\underline{\mathbf{n}}(\mathbf{I})$ by Equation (1.3): $\underline{\mathbf{n}}(\widehat{\mathbf{I}}) = \boldsymbol{\kappa}\mathbf{I}^{-1}$, so that $\underline{\mathbf{n}}^{-1}(\mathbf{I}) = \widehat{\mathbf{I}}/\boldsymbol{\kappa}$, at each point, and Equation (1.18) follows after division by $\widehat{\mathbf{I}}$. □

One of the consequences is that locally flat parts (where $\boldsymbol{\kappa} = 0$) stay locally flat after propagation.

1.5 Conclusions

This Chapter demonstrates that the rather involved operation of wave front propagation in m-dimensional space can be represented as a geometric product of direction-dependent rotors. These rotors represent boundaries in Euclidean m-space, within a Minkowski space of dimension $(m+1, 1)$, as *direction-dependent translations* of the point object at the origin. This representation combines well with Euclidean operations on the boundaries, as Figure 1.3 showed. We plan to use it to analyze differential properties of the propagation operation (some first results were shown).

Alternatively, and computationally somewhat more convenient, propagation can be represented as an addition of scalar support functions on the Gauss sphere of directions; in our representation this is a sphere in

Minkowski space, with the support function geometrically represented as the e^0-component. Such representation have been used before (e.g. [6]); but their relevance for the propagation-type interactions of boundaries appears to be new; and we now have them for arbitrary dimensionality.

We hope to apply this spectral representation of wave propagation to some of the practical problems of Section 1.1.3 which have, in essence, the same mathematical structure; notably to the prevention of robot collisions which was our original motivation.

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