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DISTRIBUTION PROOF OF WIENER'S TAUBERIAN THEOREM¹

JACOB KOREVAAR

1. **Introduction.** Let f be an L^1 function whose Fourier transform \hat{f} is free of (real) zeros. We will refer to such a function f as a Wiener kernel, and write $f \in \mathcal{W}$. Let s be an L^∞ function which is slowly oscillating:

$$s(x) - s(y) \rightarrow 0 \quad \text{as } x \rightarrow \infty \text{ and } x - y \rightarrow 0.$$

Finally suppose that

$$(f * s)(x) = \int_{-\infty}^{\infty} f(x-t)s(t) dt \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Then

$$s(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

This is Pitt's form [4], [5] of Wiener's Tauberian theorem [7], [8].

The above Tauberian theorem is easily derived from a closure theorem, also due to Wiener (loc. cit.), which asserts that for any $f \in \mathcal{W}$ the finite linear combinations of translates $f(x+\lambda)$, λ real, are dense in L^1 . Thus by the continuous linear functionals test, Wiener's Tauberian theorem is a consequence of the following

THEOREM A. *For any Wiener kernel f , the equation*

$$(1) \quad f * g = 0, \quad g \in L^\infty,$$

implies that $g=0$.

It is also possible, as indicated by Beurling [1], to prove directly that Wiener's theorem is a consequence of Theorem A.

A heuristic proof of Theorem A goes as follows. By Fourier transformation, equation (1) becomes $\hat{f}\hat{g}=0$. Thus since \hat{f} is free of zeros one must have $\hat{g}=0$, and hence $g=0$. The only difficulty with this approach is that for arbitrary $g \in L^\infty$, the Fourier transform \hat{g} is a (tempered) distribution, and the product $\hat{f}\hat{g}$ is not defined in the usual theory (cf. [3], however). In the present note we indicate how one can get around this problem by replacing f with a suitable testing function of rapid descent, that is, a function belonging to Schwartz's

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space S [6]. (It should be mentioned that Beurling has given several proofs of Theorem A [1], [2], and that the second one cited also employs a generalized Fourier transform of g .)

2. **Two simple lemmas.** Let ϕ be any testing function of rapid descent, and define functions ϕ_n by setting

$$(2) \quad \phi_n(x) = \frac{1}{n} \phi\left(\frac{x}{n}\right), \quad n = 1, 2, \dots$$

LEMMA 1. For $f \in L^1$ and ϕ_n as above,

$$\|f * \phi_n - \hat{f}(0)\phi_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

PROOF. The norm in question is given by

$$\int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} f(t) \{ \phi_n(x-t) - \phi_n(x) \} dt \right| \leq \int_{-\infty}^{\infty} |f(t)| \rho_n(t) dt,$$

where

$$\rho_n(t) = \| \phi_n(x-t) - \phi_n(x) \| = \| \phi(y-t/n) - \phi(y) \|.$$

It is clear that $\rho_n(t) \rightarrow 0$ for every fixed t , while $\rho_n(t) \leq 2\|\phi\|$. The lemma thus follows from Lebesgue's dominated convergence theorem.

LEMMA 2. Suppose that u and v belong to L^1 and that $\|v\| < 1$. Then $u/(1+v)$ is the Fourier transform of an L^1 function w .

PROOF. Consider the series

$$(3) \quad u - u * v + u * v * v - \dots$$

Since

$$\|u * v^{*n}\| \leq \|u\| \|v\|^n,$$

the sum of the norms of the terms in (3) is finite. It follows that the series converges in L^1 to a function w which has the desired Fourier transform.

3. **Proof of Theorem A.** Suppose that $f \in \mathcal{W}$, and that $g \in L^\infty$ satisfies equation (1). We introduce a testing function ϕ of rapid descent whose Fourier transform $\hat{\phi}$ is equal to 1 on $[-1, 1]$ and equal to 0 outside $(-2, 2)$.

Defining ϕ_n by equation (2), Lemma 1 shows that we can choose an index p so large that

$$\|f * \phi_p - \hat{f}(0)\phi_p\| < |\hat{f}(0)|.$$

It is clear that $\hat{\phi}_p(x) = \hat{\phi}(px) = 1$ for $|x| \leq 1/p$; we also note that $\hat{\phi}_{2p}(x) = 0$ for $|x| \geq 1/p$.

We now set

$$u = \frac{1}{\hat{f}(0)} \phi_{2p}, \quad v = \frac{1}{\hat{f}(0)} \{f * \phi_p - \hat{f}(0)\phi_p\}.$$

By Lemma 2 the quotient

$$\frac{\hat{u}}{1 + \hat{v}} = \frac{\hat{\phi}_{2p}}{\hat{f}(0) + \hat{f}\hat{\phi}_p - \hat{f}(0)\hat{\phi}_p} = \frac{\hat{\phi}_{2p}}{\hat{f}}$$

is the Fourier transform of an L^1 function w . For this w we will have $w * f = \phi_{2p}$, hence, by equation (1),

$$(4) \quad \phi_{2p} * g = w * f * g = 0.$$

Since ϕ_{2p} is a testing function of rapid descent we can take Fourier transforms in (4) to obtain

$$(5) \quad \hat{\phi}_{2p}\hat{g} = 0.$$

Observing that the testing function $\hat{\phi}_{2p}$ is equal to 1 for $|x| \leq 1/2p$, one derives from (5) that the distribution \hat{g} is equal to 0 at least on the open interval $|x| < 1/2p$.

So far we have only used the nonvanishing of $\hat{f}(0)$. However, equation (1) shows that the convolution of $f(x)e^{-icx}$ and $g(x)e^{-icx}$ is equal to 0 for every real number c , hence, by the preceding argument, the nonvanishing of $\hat{f}(c)$ implies that \hat{g} vanishes in a neighborhood of the arbitrary point c . We conclude that $\hat{g} = 0$ on $(-\infty, \infty)$ and, hence, $g = 0$.

REFERENCES

1. A. Beurling, *Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel*, Acta Math. **77** (1945), 127-136.
2. ———, *Sur une classe de fonctions presque-périodiques*, C. R. Acad. Sci. Paris **225** (1947), 326-328.
3. J. Korevaar, *Fourier transforms of generalized functions*, Symposium on Harmonic Analysis and Related Integral Transforms, Cornell University, Ithaca, N. Y., 1956.
4. H. R. Pitt, *General Tauberian theorems*, Proc. London Math. Soc. (2) **44** (1938), 243-288.
5. ———, *Tauberian theorems*, Oxford Univ. Press, London, 1958.
6. L. Schwartz, *Théorie des distributions*, Vols. I, II, Hermann, Paris, 1950-51.
7. N. Wiener, *Tauberian theorems*, Ann. of Math. (2) **33** (1932), 1-100.
8. ———, *The Fourier integral and certain of its applications*, Cambridge Univ. Press, Cambridge, 1933.