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The prime-pair conjectures of Hardy and Littlewood

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Abstract

By (extended) Wiener–Ikehara theory, the prime-pair conjectures are equivalent to simple pole-type boundary behavior of corresponding Dirichlet series. Under a weak Riemann-type hypothesis, the boundary behavior of weighted sums of the Dirichlet series can be expressed in terms of the behavior of certain double sums $\Sigma^*_{2k}(s)$. The latter involve the complex zeros of $\zeta(s)$ and depend in an essential way on their differences. Extended prime-pair conjectures are true if and only if the sums $\Sigma^*_{2k}(s)$ have good boundary behavior. Equivalently, a more general sum $\Sigma^*_{\omega}(s)$ (with real $\omega > 0$) should have a boundary function (or distribution) that is well-behaved, apart from a pole $R(\omega)/(s - 1/2)$ with residue $R(\omega)$ of period 2. [$R(\omega)$ could be determined for $\omega \leq 2$.]

Keywords: Prime-pair conjecture; Wiener–Ikehara theorem; Zeta zeros

1. Introduction

Most mathematicians believe that there are infinitely many *prime twins* $(p, p + 2)$, although this has not been proved. In fact, there is strong numerical support for the prime-pair conjectures ("PPC's") B and D of Hardy and Littlewood [12]. Conjecture B asserts that the number $\pi_{2r}(x)$ of prime pairs $(p, p + 2r)$ with $p \leq x$ satisfies the asymptotic relation

$$\pi_{2r}(x) \sim 2C_{2r}\mathrm{li}_2(x) = 2C_{2r}\int_2^x \frac{dt}{\log^2 t} \sim 2C_{2r}\frac{x}{\log^2 x} \tag{1.1}$$

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Table 1
Counting prime pairs $(p, jp \pm 2r)$ with $p \le x$.

prprs $\backslash x$	10^3	10^4	10^5	10^6	10^7	10^8	C_{2jr}/C_2
$(p, p+2)$	35	205	1224	8 169	58 980	440 312	1
$(p, p+4)$	41	203	1216	8 144	58 622	440 258	1
$(p, p+6)$	74	411	2447	16 386	117 207	879 908	2
$(p, p+8)$	38	208	1260	8 242	58 595	439 908	1
$(p, p+10)$	51	270	1624	10 934	78 211	586 811	4/3
$(p, p+12)$	70	404	2421	16 378	117 486	880 196	2
$(p, p+14)$	48	245	1488	9 878	70 463	528 095	6/5
$(p, p+16)$	39	200	1233	8 210	58 606	441 055	1
$(p, p+30)$	99	536	3329	21 990	156 517	1 173 934	8/3
$(p, p+210)$	107	641	3928	26 178	187 731	1 409 150	16/5
$(p, 3p+2)$	64	352		15 136		828 477	2
$(p, 3p-2)$	64	362		15 007		826 250	2
$(p, 9p+2)$	57	342		14 003		780 760	2
$(p, 9p-2)$	52	310		13 928		781 433	2
$L_2(x)$:	46	214	1249	8 248	58 754	440 368	

as $x \to \infty$. Here

$$C_2 = \prod_{p>2 \text{ prime}} \left\{1 - \frac{1}{(p-1)^2}\right\} \approx 0.6601618, \tag{1.2}$$

and

$$C_{2r} = C_2 \prod_{q>2 \text{ prime};\, q|r} \frac{q-1}{q-2}. \tag{1.3}$$

Thus, for example, $C_4 = C_8 = C_2$, $C_6 = 2C_2$, $C_{10} = (4/3)C_2$. We mention the curious fact that the prime-pair constants C_{2r} have *mean value* 1. Bombieri and Davenport [4], and later, Friedlander and Goldston [8], gave precise estimates; Tenenbaum [26] recently found a simple proof.

On the Internet one finds counts of twin primes for p up to 10^{16} by Nicely [22]. In Amsterdam, prime pairs $(p, p+2r)$ have been counted by Fokko van de Bult [29] and Herman te Riele [21]; the latter has also counted certain prime pairs $(p, jp \pm 2r)$ [23]. Table 1 shows a very small part of their work; the bottom line shows (rounded) values $L_2(x)$ of the comparison function $2C_2\mathrm{li}_2(x)$. Tables support the strong conjecture that for every r and $\varepsilon > 0$,

$$\pi_{2r}(x) - 2C_{2r}\mathrm{li}_2(x) = \mathcal{O}\left\{x^{(1/2)+\varepsilon}\right\}. \tag{1.4}$$

[The corresponding conjecture for $\pi(x)$, the number of primes $p \le x$, is equivalent to Riemann's Hypothesis (RH).]

Among other things, the Hardy–Littlewood Conjecture D deals with prime pairs $(p, jp \pm 2r)$, where j is *prime* to $2r$. The corresponding counting functions $\pi_{j,\pm 2r}(x)$ for pairs with $p \le x$ should be roughly comparable to $2C_{2jr}\, \mathrm{li}_2(x)$, but see (1.8). Conjectures by later authors involved still more general prime pairs; we mention Schinzel and Sierpinski [25], Bateman and Horn [2,3] and Schinzel [24]; cf. also the survey by Hindry and Rivoal [15].

It is a classical result of Brun [5], obtained by applying what is now called Brun's sieve, that $\pi_2(x) = \mathcal{O}(x/\log^2 x)$. Using more advanced sieves, Jie Wu [33] has shown that $\pi_2(x) < 6.8\, C_2\, x/\log^2 x$ for all sufficiently large x. There are related results for prime pairs $(p, jp \pm 2r)$.

In particular, for every $\varepsilon > 0$ there is a number $x_0 = x_0(\varepsilon)$ independent of j and r such that

$$\pi_{j,\pm 2r}(x) \leq (8+\varepsilon)C_{2jr}\, x/\log^2 x \tag{1.5}$$

for all $x \geq x_0$; see Halberstam and Richert [11].

The best result in the other direction is Chen's [6]: if $N(x)$ denotes the number of primes $p \leq x$ for which $p+2$ has at most two prime factors, then $N(x) \geq cx/\log^2 x$ for some $c > 0$. Recently Goldston, Pintz and Yildirim [9] proved that there are infinitely many pairs of primes (p, q) with $2 \leq q - p \leq 16$ by assuming a form of the Elliott–Halberstam conjecture [7]. The latter postulates a certain uniformity of the distribution of primes in arithmetic progressions.

In terms of sums

$$\psi_{j,\pm 2r}(x) \stackrel{\text{def}}{=} \sum_{n\leq x} \Lambda(n)\Lambda(jn \pm 2r) \approx \sum_{p\leq x;\ p,\ jp\pm 2r \text{ prime}} \log p \log(jp \pm 2r) \tag{1.6}$$

the PPC's take the simpler form

$$\psi_{j,\pm 2r}(x) \sim 2C_{2jr}\, x \quad \text{as } x \to \infty. \tag{1.7}$$

Here $\Lambda(k)$ denotes von Mangoldt's function: $\Lambda(k) = \log p$ if $k = p^\alpha$ with p prime, and $\Lambda(k) = 0$ if k is not a prime power. Hence the product $\Lambda(n)\Lambda(jn \pm 2r)$ is different from zero only when both n and $jn \pm 2r$ are powers of primes. Now the number of pairs $(p^\alpha, q^\beta = jp^\alpha \pm 2r)$ with p, q prime, $p^\alpha \leq x$ and $\alpha \geq 2$ or $\beta \geq 2$, is found to be $\mathcal{O}(x^{1/2})$, hence their contribution to $\psi_{j,\pm 2r}(x)$ is $\mathcal{O}(x^{1/2}\log^2 x)$.

For $\pi_{j,\pm 2r}(x)$, the number of prime pairs $(p, jp \pm 2r)$ with $p \leq x$, relation (1.7) leads to the comparison

$$\pi_{j,\pm 2r}(x) \approx \int_2^x \frac{d\psi_{j,\pm 2r}(t)}{\log t\, \log jt} \approx 2C_{2jr}\int_2^x \frac{dt}{\log t\, \log jt} \tag{1.8}$$

when x is large; cf. Table 1. [The final integral might be called $\mathrm{li}_2(x; j)$.]

A Tauberian approach to the twin-prime problem has been advocated by, among others, Golomb [10] and Arenstorf [1]. For prime pairs $(p, jp \pm 2r)$ the Wiener–Ikehara theorem below leads one to study *prime-pair functions* given by Dirichlet-type series:

$$D_{j,\pm 2r}(s) \stackrel{\text{def}}{=} \sum_{n>n_1} \frac{\Lambda(n)\Lambda(jn\pm 2r)\, j^s}{n^s(jn\pm 2r)^s} \quad (s = \sigma + i\tau,\ \sigma > 1/2). \tag{1.9}$$

[We will usually write $D_{1,2r}(s)$ as $D_{2r}(s)$.] For the PPC's one wishes to investigate the behavior of $D_{j,\pm 2r}(s)$ close to the line $\{\sigma = 1/2\}$. Setting

$$D_{j,\pm 2r}(s) - \frac{C_{2jr}}{s - 1/2} = G_{j,\pm 2r}(s), \tag{1.10}$$

(1.7) would follow from good boundary behavior of $G_{j,\pm 2r}(s)$ as $\sigma \searrow 1/2$. Indeed, modulo a 'good' function, $D_{j,\pm 2r}(s)$ has the same boundary behavior as $\sum \Lambda(n)\Lambda(jn \pm 2r)/n^{2s}$. Setting $2s = w$ one may now apply the Wiener–Ikehara theorem ([16,31,32], cf. [17,18]):

Theorem 1.1. *Let* $\sum_{n=1}^\infty a_n/n^w$ *with* $a_n \geq 0$ *converge to a sum function* $f(w)$ *for* $w = u + iv$ *with* $u > 1$. *Then*

$$\sum_{n\leq x} a_n \sim Ax \quad \text{as } x \to \infty \tag{1.11}$$

if for $u \searrow 1$, *the difference*

$$f(u+iv) - \frac{A}{(u+iv)-1} = g(u+iv) \tag{1.12}$$

tends to a continuous function $g(1+iv)$, *uniformly on every finite interval* $\{-B < v < B\}$.

More precisely, one has (1.11) if and only if for $u \searrow 1$, the difference $g(u+iv)$ has a distributional limit $g(1+iv)$, which on every finite interval $\{-B < v < B\}$ coincides with a pseudofunction (that may a priori depend on B). We will then say that $g(w)$ has "good boundary behavior" (for $u \searrow 1$), and that $f(w)$ "has residue A" (at $w = 1$); cf. Korevaar [19]. The condition $\sum_{n \le x} a_n = \mathcal{O}(x)$ would ensure that $f(u+iv)$ and $g(u+iv)$ have a distributional limit as $u \searrow 1$. A pseudofunction is the distributional Fourier transform of a bounded function which tends to zero at $\pm\infty$; locally, such a distribution is given by trigonometric series with coefficients that tend to zero. Continuous and locally integrable functions are simple examples.

CONVENTIONS. The letters p and q are reserved for primes; s, z and w denote complex variables with the standard decompositions

$$s = \sigma + i\tau, \qquad z = x + iy, \qquad w = u + iv;$$

and δ, ε and η always denote small positive numbers. We say that a function $F_1(X)$ is majorized by a positive function $F_2(X)$ for $X \in \Omega$, and write

$$F_1(X) \ll F_2(X) \quad (\text{on } \Omega),$$

if there is a constant C such that

$$|F_1(X)| \le C F_2(X), \quad \forall X \in \Omega.$$

Starred summation Σ_n^{*2r} refers to a sum over all positive integers n prime to $2r$. The symbol "$\cong$" denotes an *equivalence* relative to functions $H(s)$ that are holomorphic for $\sigma = \operatorname{Re} s > 1/2$ and have *good boundary behavior* as $\sigma \searrow 1/2$. (Local pseudofunction boundary behavior.)

2. Present results

As we saw, the prime-pair conjectures of Hardy and Littlewood have an equivalent formulation in terms of the boundary behavior of Dirichlet-type series $D_{j,\pm 2r}(s)$. In Section 4 we identify a natural comparison function $D_0(s, j, \pm 2r)$ for $D_{j,\pm 2r}(s)$ that has the "right" pole-type boundary behavior. It is analogous to a comparison function of Arenstorf [1] for the case of twin primes.

More important, we consider certain extensions of the Hardy–Littlewood conjectures. They involve generalized prime-pair functions as follows, cf. Sections 3, 5 and 11:

$D_0^2(s) = D_0(s, 2) \cong C_2/(s - 1/2)$: see (4.13)
$D_2^2(s) \cong D(s, 2) \cong D_2 = D_2(s)$: see (3.6)
$D_4^2(s) \cong D(s, 4) \cong D_4 = D_4(s)$
$D_6^2(s) \cong D_6 - D_2/3^{2s} - (D_{3,2} + D_{3,-2})/9^{2s} - (D_{9,2} + D_{9,-2})/27^{2s} - \cdots$: (11.3)
$D_8^2(s) \cong D_8$
$D_{10}^2(s) \cong D_{10} - D_2/5^{2s} - (D_{5,2} + D_{5,-2})/25^{2s} - (D_{25,2} + D_{25,-2})/125^{2s} - \cdots,$

etc. We *know* that $D_0^2(s) - C_2/(s - 1/2)$ has good boundary behavior as $\sigma \searrow 1/2$. Under the Hardy–Littlewood conjectures, the same will hold for all the other differences $D_{2k}^2(s) - C_2/(s - 1/2)$. We call the conjecture that $D_{2k}^2(s) - C_2/(s - 1/2)$ has good boundary behavior the *extended*

Hardy–Littlewood conjecture for D^2_{2k}. The extended H–L conjectures follow from the original ones; see Sections 11 and 14–16.

For the following results it is assumed that $\zeta(z)$ is zero-free in *some* strip $\{1-\delta < x < 1\}$ ["weak" RH]. Elaborate complex analysis then shows that weighted sums of differences $D^2_{2j}(s) - C_2/(s-1/2)$ are *equivalent*, for $\sigma \searrow 1/2$, $|\tau| < B$ and any number B, to certain analytic functions $\Sigma^*_{2k}(s, B)$. The latter are represented by infinite series that involve the zeros of the zeta function with real part $>(1/2)-\eta$, and with imaginary part of absolute value $>B$, see (2.3) and Sections 11 and 12.

The extended H–L conjectures are equivalent to good boundary behavior of the functions $\Sigma^*_{2k}(s, B)$ as $\sigma \searrow 1/2$.

The formula for $\Sigma^*_{2k}(s, B)$ requires some preliminary definitions:

$$Q_2(z, w) \stackrel{\text{def}}{=} \prod_{p>2\,\text{prime}} \left\{1 - \frac{p^{-z-w}}{(1-p^{-z})(1-p^{-w})}\right\}; \tag{2.1}$$

note that $Q_2(1, 1) = C_2$. Next,

$$M(z) \stackrel{\text{def}}{=} \Gamma(-z-1)\sin(\pi z/2); \tag{2.2}$$

one has $M(x+iy) \ll (|y|+1)^{-x-3/2}$ for $|y| \ge 1$, $|x| \le C$. For any $\omega > 0$ we now define

$$\Sigma^*_\omega(s, B) \stackrel{\text{def}}{=} \frac{1}{2\pi} \sum^*_{\rho,\rho'} \Gamma(\rho - s)\Gamma(\rho' - s) Q_2(\rho, \rho') \times \omega^{\rho+\rho'-2s+1} M(\rho+\rho'-2s)\cos\{\pi(\rho-\rho')/2\}, \tag{2.3}$$

where Σ^* stands for a (double) sum over the zeros ρ, ρ' of $\zeta(\cdot)$ with real part $>(1/2)-\eta$ and imaginary part of absolute value $>B$.

The extended H–L conjectures are also equivalent to pole-type boundary behavior of $\Sigma^*_\omega(s, B)$ *with period* 2 *in* ω. [We know the residue for $\omega \le 2$.]

It may be noted that quite different relations between certain prime-pair conjectures and complex zeros of L-functions have been studied by Turán [28] and Heath-Brown [14].

3. Prime pairs $(p, jp \pm 2r)$

In [10], Golomb used a precursor to Proposition 3.1 and a real Tauberian theorem to study the twin-prime conjecture, (1.1) for $r = 1$. Aiming to apply the classical Wiener–Ikehara theorem, Arenstorf [1] obtained a further proposition and corollaries for the twin-prime case. We extend these results to prime pairs $(p, jp \pm 2r)$, where j is *prime* to $2r$.

If $(n, 2jr) > 1$ then $\Lambda(n)\Lambda(jn \pm 2r) = 0$ for all $n >$ some n_1. We need

Proposition 3.1. *Let n be prime to $2jr$ and >1. Then*

$$2\Lambda(n)\Lambda(jn+2r) = \sum_{m|n(jn+2r)} \mu(m)\log^2 m, \tag{3.1}$$

and similarly with $jn - 2r$ instead of $jn + 2r$ provided $jn > 2r$.

Proof. The Möbius function $\mu(n)$ is equal to $(-1)^k$ if n is the product of k different primes and $\mu(n) = 0$ if n contains a multiple prime factor. From the Euler product for $\zeta(z)$ one obtains the

Dirichlet series

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_1^\infty \frac{\Lambda(n)}{n^z} \quad \text{and} \quad \frac{1}{\zeta(z)} = \sum_1^\infty \frac{\mu(n)}{n^z}.$$

Substituting these series in the identity

$$\left\{\frac{\zeta'(z)}{\zeta(z)}\right\}^2 = \zeta(z)\left\{\frac{1}{\zeta(z)}\right\}'' + \left\{\frac{\zeta'(z)}{\zeta(z)}\right\}',$$

one finds that

$$\sum_{m|k} \Lambda(m)\Lambda(k/m) = \sum_{m|k} \mu(m)\log^2 m + \Lambda(k)\log k. \tag{3.2}$$

Now set $k = n(jn + 2r)$. Then k cannot be a prime power p^α because n is prime to $2r$. Thus $\Lambda(k) = 0$. Also, $\Lambda(m)\Lambda(k/m) = 0$ for $m|k$ unless m and k/m are both prime powers, $m = p^\alpha$ and $k/m = q^\beta$, say, with $\alpha,\ \beta \geq 1$ and $q \neq p$. Since n and $jn+2r$ are relatively prime, the latter occurs only if either $n = p^\alpha = m$ and $jn + 2r = q^\beta$, or $jn + 2r = p^\alpha = m$ and $n = p^\beta$. □

By (1.9) and Proposition 3.1, taking $\sigma = \mathrm{Re}\, s > 1/2$,

$$\begin{aligned} D_{j,2r}(s) &\cong \sum_{n>n_1;\,(n,2jr)=1} \frac{\Lambda(n)\Lambda(jn+2r)j^s}{n^s(jn+2r)^s} \\ &= \frac{1}{2}\sum_{n>n_1;\,(n,2jr)=1} n^{-s}j^s(jn+2r)^{-s} \sum_{m|n(jn+2r)} \mu(m)\log^2 m \\ &\cong \frac{1}{2}\sum_m^{*2jr} \mu(m)(\log^2 m) \sum_{n;\,n(jn+2r)\equiv 0\,(\mathrm{mod}\,m)}^{*2jr} n^{-s}j^s(jn+2r)^{-s}. \end{aligned} \tag{3.3}$$

The next proposition describes solutions of a certain congruence.

Proposition 3.2. *Let $m \in \mathbb{N}$ be square-free and prime to $2jr$. Then there is a one-to-one correspondence between the (positive) solutions n prime to $2jr$ of the congruence*

$$n(jn + 2r) \equiv 0\,(\mathrm{mod}\, m),$$

and the integers n of the form ak, where k varies over the divisors of m, while for fixed k, setting $m/k = l$, a runs over the first member of the (positive) solution pairs (a, b) of the equations

$$ajk - bl = -2r, \quad (a, 2jr) = 1.$$

Interchanging k and l, one obtains a corresponding result involving the equations $ak - bjl = 2r$, $(a, 2jr) = 1$.

The congruence $n(jn - 2r) \equiv 0\,(\mathrm{mod}\ m)$ similarly leads to the equations $ajk - bl = 2r$, $(a, 2jr) = 1$ and $ak - bjl = -2r$, $(a, 2jr) = 1$.

Proof. (i) Let n be a solution of the congruence that is prime to $2jr$. Define

$$(n, m) = k \quad \text{and} \quad l = m/k, \quad \text{so that } (k, l) = 1.$$

Since kl divides $n(jn + 2r)$ and $(n, kl) = k, l$ must divide $jn + 2r$. Define a and b by $n = ak$, $jn + 2r = bl$, so that a and b are prime to $2jr$ [recall that $(j, 2r) = 1$]. Then

$$ajk - bl = -2r.$$

To the given solution n of the congruence we have assigned unique $k,\ l,\ a,\ b$ prime to $2jr$ with $kl = m$ and $ajk - bl = -2r$.

(ii) Conversely, let k be a divisor of m and $l = m/k$. Let a and b be arbitrary (positive) solutions of the equation $ajk - bl = -2r$ that are prime to $2jr$. [Using congruences, it is not difficult to prove that there are such numbers a and b, but that is not essential to the argument.] Now form the integer $\tilde{n} = ak$. Then $\tilde{n}$ is prime to $2jr$, and

$$\tilde{n}(j\tilde{n} + 2r) = ak(ajk + 2r) = akbl \equiv 0 \pmod{m}.$$

To the given $a,\ b,\ k,\ l$ as described we have thus assigned a unique solution $\tilde{n}$ of the congruence that is prime to $2jr$. □

Thinking of $n = ak$ and $jn + 2r = bl$, or the other way around, one obtains

Corollary 3.3. *Let m be square-free and prime to $2jr$. Then*

$$\sum_{n>n_1;\, n(jn+2r)\equiv 0 \,(\mathrm{mod}\, m)}^{*2jr} n^{-s}(jn+2r)^{-s} \cong \sum_{k,l;\, kl=m}^{*2jr} \ \sum_{a,b>0;\, ajk-bl=-2r}^{*2jr} (akbl)^{-s}$$
$$\text{and also } \cong \sum_{k,l;\, kl=m}^{*2jr} \ \sum_{a,b>0;\, ak-bjl=2r}^{*2jr} (akbl)^{-s}. \tag{3.4}$$

Similarly with $jn - 2r$ instead of $jn + 2r$.

In the case of classical prime pairs $(p, p + 2r)$, or $j = 1$, it is convenient to take the average of the second and third expression in (3.4):

$$R(s, 2r, m) \overset{\text{def}}{=} \frac{1}{2} \sum_{k,l;\, kl=m}^{*2r} \ \sum_{a,b>0;\, ak-bl=\pm 2r}^{*2r} (akbl)^{-s}.$$

We now introduce a sieving factor $E_{2r}(\nu)$ to replace the awkward restricted summation over $a,\ b$ by unlimited summation over variables $a,\ b > 0$ prime to $2r$. Setting $E_{2r}(\nu) = 1/2$ for $\nu = \pm 2r$ and $E_{2r}(\nu) = 0$ for all other even integers ν, one has

$$R(s, 2r, m) = \sum_{k,l;\, kl=m}^{*2r} \ \sum_{a,b>0}^{*2r} (akbl)^{-s} E_{2r}(ak - bl). \tag{3.5}$$

In view of (3.3) and (3.4), our original prime-pair function $D_{2r}(s)$ in (1.9) for the case $j = 1$ has the same pole-type boundary behavior as the following adjusted prime-pair function:

$$D(s, 2r) \overset{\text{def}}{=} \frac{1}{2} \sum_{m}^{*2r} \mu(m)(\log^2 m) R(s, 2r, m) \quad (\sigma = \operatorname{Re} s > 1/2). \tag{3.6}$$

Corollary 3.4. *The H–L conjecture (1.1) for prime-pairs $(p, p + 2r)$ is true if and only if the difference*

$$D(s, 2r) - \frac{C_{2r}}{s - 1/2} \tag{3.7}$$

has 'good' (that is, local pseudofunction) boundary behavior as $\sigma \searrow 1/2$.

We will do something similar in the case $j > 1$, but then the equations $ajk - bl = \pm 2r$ will correspond to two different functions, namely, $D_{j,\pm 2r}(s)$. However, in the subsequent theory we always encounter the sum of those two functions, so that it makes sense to introduce their average. For m prime to $2jr$, we generalize (3.5) to

$$R(s, j, 2r, m) \stackrel{\text{def}}{=} j^s \sum_{k,l;\, kl=m}^{*2jr} \sum_{a,b>0}^{*2jr} (akbl)^{-s} E_{2r}(ajk - bl). \tag{3.8}$$

Always taking $\sigma = \operatorname{Re} s > 1/2$, we next set

$$D(s, j, 2r) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m}^{*2jr} \mu(m)(\log^2 m) R(s, j, 2r, m). \tag{3.9}$$

This function will have the same pole-type boundary behavior as the average of the functions $D_{j,\pm 2r}(s)$ of (1.9).

Corollary 3.5. *The "average" H–L conjecture for prime pairs $(p.jp + 2r)$ and $(p.jp - 2r)$ (with $j > 1$ prime to $2r$) is true if and only if*

$$D(s, j, 2r) - \frac{C_{2jr}}{s - 1/2} \tag{3.10}$$

has good boundary behavior.

4. A comparison function $D_0(s, 2r)$ for $D(s, 2r)$

Studying the case of twin primes, prime pairs $(p, p+2)$, Arenstorf [1] proposed a comparison function $D_0(s, 2)$ for $D(s, 2)$ which we generalize. The comparison function $D_0(s, 2r)$ for the case of prime pairs $(p, p + 2r)$ will depend only on the different odd prime factors of r. In order to keep the notation simple, it is assumed (only) in this section that $2r$ does not contain multiple prime factors.

The positive solutions a and b of the equation

$$ak - bl = 2r \tag{4.1}$$

that are prime to $2r$ have the form $a = a_0 + hl, b = b_0 + hk$, where h runs over the positive integers prime to $2r$. Here $a_0 = a_0(k, l, r)$ and $b_0 = b_0(k, l, r)$ are the solutions of (4.1) that are divisible by $2r$ and such that $-rl < a_0 < rl$ and $-rk < b_0 < rk$. Observe that the qualifying positive solutions of the equation $ak - bl = -2r$ are given by $a = -a_0 + hl, b = -b_0 + hk$, with h prime to $2r$. Hence by (3.5), with m square-free and prime to $2r$,

$$R(s, 2r, m) = \frac{1}{2} \sum_{k,l;\, kl=m}^{*2r} (kl)^{-2s} \sum_{h}^{*2r} h^{-2s} \left\{ \left(1 + \frac{a_0}{hl}\right)^{-s} \left(1 + \frac{b_0}{hk}\right)^{-s} + \left(1 - \frac{a_0}{hl}\right)^{-s} \left(1 - \frac{b_0}{hk}\right)^{-s} \right\}. \tag{4.2}$$

It is convenient to introduce functions

$$\zeta_{2r}(z) \stackrel{\text{def}}{=} \sum_{n}^{*2r} n^{-z} = \zeta(z)(1 - 2^{-z}) \prod_{q>2 \text{ prime};\, q|r} (1 - q^{-z}). \tag{4.3}$$

[The formula will also be used for general r.] Then by (4.2), writing $d(m)$ for the number of divisors of m,

$$R(s,2r,m) = m^{-2s} d(m) \sum_{h}^{*2r} h^{-2s}\bigl\{1+\mathcal{O}(h^{-2})\bigr\}$$
$$= m^{-2s} d(m)\zeta_{2r}(2s) + \mathcal{O}\bigl\{m^{-2\sigma} d(m)\bigr\}, \tag{4.4}$$

uniformly for $\sigma > 1/2$ and $|s| \le C$. Introducing a sieving function $E_0(\nu)$ that is equal to 1 for $\nu = 0$ and equal to 0 for all other even integers ν, we now define

$$R_0(s,2r,m) \stackrel{\text{def}}{=} \sum_{k,l;\, kl=m}^{*2r} \sum_{a,b}^{*2r} (akbl)^{-s} E_0(ak-bl) = \sum_{k,l;\, kl=m}^{*2r} \sum_{h;\, a=hl,\, b=hk}^{*2r} (akbl)^{-s}. \tag{4.5}$$

Hence, cf. (4.3) and (4.4),

$$R_0(s,2r,m) = m^{-2s} d(m)\zeta_{2r}(2s). \tag{4.6}$$

Thus the function $R_0(s,2r,m)$ is analytic for $\sigma \ge 1/2$, except for a first-order pole given by

$$\frac{d(m)}{m}\,\frac{1}{2} \prod_{q>2\,\text{prime};\, q|r} \frac{q-1}{q}\,\frac{1}{2s-1}.$$

It is clear that $R(s,2r,m)$ shows the same pole-type boundary behavior as $R_0(s,2r,m)$. It thus appears reasonable to *expect* that $D(s,2r)$ in (3.6) has the same pole-type boundary behavior as

$$D_0(s,2r) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m}^{*2r} \mu(m)(\log^2 m) R_0(s,2r,m)$$
$$= \frac{1}{2} \sum_{m}^{*2r} \mu(m) d(m)(\log^2 m) m^{-2s}\zeta_{2r}(2s). \tag{4.7}$$

In order to evaluate the Dirichlet series in the final member we will compute the auxiliary function

$$K_{2r}(z) \stackrel{\text{def}}{=} \sum_{m}^{*2r} \mu(m) d(m) m^{-z} = \prod_{p\,\text{prime};\, (p,2r)=1} (1-2p^{-z}). \tag{4.8}$$

In terms of $K_{2r}(z)$, the formula for $D_0(s,2r)$ becomes

$$D_0(s,2r) = \frac{1}{2} K''_{2r}(2s)\zeta_{2r}(2s). \tag{4.9}$$

To verify the product representation in (4.8), observe that for square-free numbers m one has $d(m) = 2^{\nu(m)}$, where $\nu(m)$ is the number of prime factors of m. Now the arithmetic function $a(m) = \mu(m)2^{\nu(m)}$ is multiplicative, and for primes p one has $a(p) = -2$, while $a(p^\alpha) = 0$ for $\alpha \ge 2$. Hence by standard factorization, cf. [13],

$$\sum_{m}^{*2r} a(m) m^{-z} = \prod_{(p,2r)=1} \{1 + a(p)p^{-z} + a(p^2)p^{-2z} + \cdots\}$$
$$= \prod_{(p,2r)=1} (1-2p^{-z}).$$

Formula (4.8) defines the function $K_{2r}(z)$ only for $x = \mathrm{Re}\, z > 1$, but we need its behavior close to the line $\{x = 1\}$. The function can be continued analytically through multiplication by $\zeta_{2r}^2(z)$ or $\zeta_2^2(z)$, cf. (4.3):

$$K_{2r}(z)\zeta_2^2(z) = \prod_q (1 - 2q^{-z})^{-1} \prod_{p>2} (1 - 2p^{-z}) \prod_{p>2} (1 - p^{-z})^{-2}$$
$$= \prod_q (1 - 2q^{-z})^{-1} \prod_{p>2} \left\{1 - \frac{1}{(p^z - 1)^2}\right\}. \tag{4.10}$$

[Here and in the following, q runs over the odd prime divisors of r.] Since the final member of (4.10) is analytic for $x = \mathrm{Re}\, z > 1/2$ and $\zeta_2^{-1}(z)$ is analytic for $x \geq 1$, it follows that $K_{2r}(z)$ is analytic for $x \geq 1$ [and for $x > 1/2$ under RH]. Expansion about the point $z = 1$ gives

$$\zeta_2^{-1}(z) = 2(z-1) + \cdots,$$
$$K_{2r}(z) = \prod_q (1 - 2q^{-z})^{-1} \prod_{p>2} \left\{1 - \frac{1}{(p^z - 1)^2}\right\} \zeta_2^{-2}(z)$$
$$= \prod_q \frac{q}{q-2} \prod_{p>2} \left\{1 - \frac{1}{(p-1)^2}\right\} 4(z-1)^2 + \cdots.$$

Hence by the definition of C_2 in (1.2),

$$K_{2r}''(z) = 8C_2 \prod_q \frac{q}{q-2} + c(z-1) + \cdots. \tag{4.11}$$

Finally expanding about the point $s = 1/2$, the conclusion from (4.11), (4.9) and (4.3) is that

$$D_0(s, 2r) = \frac{1}{2} K_{2r}''(2s)\zeta_{2r}(2s)$$
$$= \frac{1}{2} 8C_2 \prod_q \frac{q}{q-2} \cdot \frac{1}{2} \prod_q \frac{q-1}{q} \frac{1}{2s-1} + \cdots = \frac{C_{2r}}{s - 1/2} + \cdots; \tag{4.12}$$

cf. (1.3). Summarizing, we have proved.

Theorem 4.1. *The difference*

$$D_0(s, 2r) - \frac{C_{2r}}{s - 1/2} = G_{2r}^*(s) \tag{4.13}$$

is holomorphic for $\sigma \geq 1/2$, and for $\sigma > 1/4$ under RH.

Hence the natural comparison function $D_0(s, 2r)$ for $D(s, 2r)$ indeed has the same pole-type behavior for $\sigma \searrow 1/2$ as that, expected for $D(s, 2r)$; see Corollary 3.4. The theorem thus supports the Hardy–Littlewood conjecture for prime pairs $(p, p + 2r)$.

Note that $D_0(s, 2r)$ depends only on the different prime factors of $2r$, so that, for example, $D_0(s, 4) = D_0(s, 2)$. However, it is not at all clear that $D(s, 4)$ and $D(s, 2)$ have the same pole-type boundary behavior; cf. (3.6), (4.1) and (4.2). The series for $K_2''(1)$, and more generally $K_{2r}''(1)$:

$$K_{2r}''(1) = \sum_m^{*2r} \frac{\mu(m)d(m)\log^2 m}{m} = 8C_{2r} \prod_{q>2 \text{ prime};\, q|2r} \frac{q}{q-1}, \tag{4.14}$$

fails to be absolutely convergent.

A COMPARISON FUNCTION FOR $D(s, j, 2r)$. The comparison function for $D(s, 2r)$ can be generalized to a comparison function for $D(s, j, 2r)$. It will depend only on the different odd prime factors of $2r$; as before we assume that $2r$ has no multiple prime factors. Recall also that j and $2r$ must be relatively prime. Analysis as above shows that the qualifying solutions of the equation $ajk - bl = 2r$ have the form

$$a = a_0 + hl, \qquad b = b_0 + hjk,$$

where a_0 and b_0 are solutions "around" zero that are multiples of $2jr$ and h runs over the positive integers prime to $2jr$. Thus we are in the same situation as before, except that $2r$ has now been replaced by $2jr$.

The logical comparison function is given by

$$D_0(s, j, 2r) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m}^{*2jr} \mu(m)(\log^2 m) R_0(s, j, 2r, m), \tag{4.15}$$

where for square-free m prime to $2jr$,

$$R_0(s, j, 2r, m) \stackrel{\text{def}}{=} j^s \sum_{k,l;\, kl=m} \sum_{h}^{*2jr} (akbl)^{-s} \Big|_{a=hl,\, b=hjk}. \tag{4.16}$$

One thus finds

$$R_0(s, j, 2r, m) = m^{-2s} d(m) \zeta_{2jr}(2s),$$

$$D_0(s, j, 2r) = \frac{1}{2} \sum_{m}^{*2jr} \mu(m) d(m)(\log^2 m) m^{-2s} \zeta_{2jr}(2s). \tag{4.17}$$

The result is equal to $D_0(s, 2jr)$. It is analytic for $\sigma \geq 1/2$, except for a first-order pole at $s = 1/2$ with residue C_{2jr}. Hence by Corollary 3.5 we have

Theorem 4.2. *The natural comparison function $D_0(s, 2jr)$ for $D(s, j, 2r)$ has the same pole-type behavior for $\sigma \searrow 1/2$ as that, expected for $D(s, j, 2r)$.*

The theorem thus supports the "average" Hardy–Littlewood conjecture for prime pairs $(p, jp \pm 2r)$.

5. Generalized sieving and prime-pair functions

In [20] the author reduced the pole-type boundary behavior of certain combinations of prime-pair functions to that of double series of functions which involve the complex zeros of the zeta function. Here we will use representations such as (3.6), (3.9) and (4.7) to obtain more refined results for a general class of functions including $D(s, j, 2r)$ and $D_0(s, j, 2r)$.

For simplicity we will use continuous piecewise linear sieving functions; they can be represented by integrals as follows. Taking $\lambda > 0$, set

$$E^\lambda(x) \stackrel{\text{def}}{=} \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \lambda t}{\lambda t^2} \cos xt \, dt = \begin{cases} 1 - |x|/\lambda & \text{for } |x| \leq \lambda, \\ 0 & \text{for } |x| \geq \lambda. \end{cases} \tag{5.1}$$

[The formula may be verified by computing the inverse Fourier transform of the right-hand side.] Observe that $E^\lambda(\nu)$ can serve as sieving function $E_0(\nu)$ in (4.5) provided $\lambda \in (0, 2]$.

For any real numbers $\kappa \geq 0$ and $\lambda > 0$ we define a *generalized sieving* function $E_\kappa^\lambda(\nu)$ by substituting $\kappa \pm \nu$ for x in (5.1) and averaging the results:

$$E_\kappa^\lambda(\nu) \stackrel{\text{def}}{=} (1/2)\big[E^\lambda\{|\kappa-\nu|\} + E^\lambda\{\kappa+\nu\}\big] = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda t}{\lambda t^2}\cos\kappa t\,\cos\nu t\,dt. \tag{5.2}$$

Note that for any $\lambda \in (0, 2]$, $E_{2r}^\lambda(\nu)$ is equal to $1/2$ for $\nu = \pm 2r$ and equal to 0 for all other even integers ν, as required of the sieving function $E_{2r}(\nu)$ in (3.5). For $\lambda > 2$ the situation is more complicated. The only values that matter for a sieving function are the values on the set of the even integers. For example, looking at the graphs, one finds that

$$E_2^4 = E_2^2 + (1/2)E_0^2 + (1/2)E_4^2, \qquad E_0^4 = E_0^2 + E_2^2. \tag{5.3}$$

We need the Mellin transform of the kernel of the cosine transform in (5.2) that is formed by the factor $\cos\nu t$:

Proposition 5.1. *For $\kappa \geq 0$, $\lambda > 0$ and $-1 < x = \operatorname{Re} z < 1$,*

$$M_\kappa^\lambda(z) \stackrel{\text{def}}{=} \frac{2}{\pi}\int_0^\infty \frac{(1-\cos\lambda t)\cos\kappa t}{\lambda t^2}\,t^{-z}dt = \frac{1}{\pi\lambda}\big\{|\kappa-\lambda|^{z+1} - 2\kappa^{z+1} + (\kappa+\lambda)^{z+1}\big\}\Gamma(-z-1)\sin(\pi z/2). \tag{5.4}$$

The function $M_\kappa^\lambda(z)$ has a meromorphic extension to the complex plane, with poles (of the first order) at $z = -1$ (if $\kappa = 0$), $1, 3, \ldots$. The residue at $z = 1$ equals $-\lambda/\pi$. One has $M_\kappa^\lambda(0) = 0$ if $\kappa \geq \lambda$; otherwise $M_\kappa^\lambda(0) = 1 - \kappa/\lambda$. For fixed κ and λ,

$$M_\kappa^\lambda(z) \ll (|y|+1)^{-x-3/2} \quad \text{when } |x| \leq C,\ |y| \geq 1. \tag{5.5}$$

Proof. For $0 < \alpha < 1$ and $\beta > 0$, the improper integral for $\Gamma(\alpha)$ implies that

$$\int_0^{\infty-} t^{\alpha-1}\sin\beta t\,dt = \Gamma(\alpha)\beta^{-\alpha}\sin(\pi\alpha/2).$$

Integrating with respect to β, one finds

$$\int_0^\infty t^{\alpha-1}\frac{1-\cos\beta t}{t}\,dt = -\Gamma(\alpha-1)\beta^{1-\alpha}\sin(\pi\alpha/2).$$

From this one obtains (5.4) by forming suitable combinations.

Since $\Gamma(-z-1)$ is holomorphic except for first-order poles at the points $z = -1, 0, 1, 2, \ldots$, it is clear that $M_\kappa^\lambda(z)$ has a meromorphic extension to the whole complex plane. The poles at $z = 0, 2, \ldots$ are canceled by zeros of $\sin(\pi z/2)$; if $\kappa > 0$ the pole at $z = -1$ is also canceled. To calculate the value of $M_\kappa^\lambda(z)$ at $z = 0$ and the residue at $z = 1$ one may use the recurrence relation $\Gamma(z+1) = z\Gamma(z)$. The order estimate (5.5) follows from the standard inequalities

$$\Gamma(z) \ll |y|^{x-1/2}e^{-\pi|y|/2}, \qquad \sin(\pi z/2) \ll e^{\pi|y|/2}, \tag{5.6}$$

which are valid for $|x| \leq C$ and $|y| \geq 1$; cf. [30]. □

In terms of the sieving function $E_\kappa^\lambda(\nu)$ of (5.2) we define *generalized prime-pair* functions [all analytic for $\sigma > 1/2$] by

$$D_\kappa^\lambda(s, j, 2r) \stackrel{\text{def}}{=} \frac{1}{2} j^s \sum_{k,l}^{*2jr} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2jr} (ab)^{-s} E_\kappa^\lambda(ajk - bl). \tag{5.7}$$

For $\kappa = 2r > 0$ and $\lambda \le 2$, the new function reduces to $D(s, j, 2r)$ of (3.6), (3.9). If $j = 1$ we write $D_\kappa^\lambda(s, 2r)$ for $D_\kappa^\lambda(s, j, 2r)$. For $\kappa = 0$ and $\lambda \le 2$, the right-hand side of (5.7) then reduces to $D_0(s, 2r)$ of (4.7).

6. Repeated complex integrals for sieving functions

Extending and refining Arenstorf's work on twin primes [1], we will introduce a repeated complex integral for $E_\kappa^\lambda(\alpha - \beta)$ in which α and β occur separately. It will involve the Mellin transform $M_\kappa^\lambda(z)$ of (5.4).

The factor $1/(2\pi i)$ in complex integrals is omitted. Denoting the 'vertical' line $\{x = c\}$ by $L(c)$ we set

$$\int_{L(c)} f(z)dz \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z)dz. \tag{6.1}$$

Here the integral would usually be a principal-value integral, $\lim_{R\to\infty} \int_{c-iR}^{c+iR}$. It is, however, essential for us to have absolutely convergent integrals, and therefore we introduce special paths of integration. They replace the line $L(c)$ and have the form $L(c, B) = L(c_1, c_2, B)$ where $c_1 < c_2$ and $B > 0$ (cf. Fig. 1):

$$L(c, B) = \begin{cases} \text{the half-line} & \{x = c_1,\ -\infty < y \le -B\} \\ +\ \text{the segment} & \{c_1 \le x \le c_2,\ y = -B\} \\ +\ \text{the segment} & \{x = c_2,\ -B \le y \le B\} \\ +\ \text{the segment} & \{c_2 \ge x \ge c_1,\ y = B\} \\ +\ \text{the half-line} & \{x = c_1,\ B \le y < \infty\}. \end{cases} \tag{6.2}$$

Proposition 6.1. *Let $\alpha, \beta > 0, -1/2 < c_1 < 0 < c_2 < 1/2$ and $B > 0$. Then for $\kappa \ge 0$ and $\lambda > 0$,*

$$E_\kappa^\lambda(\alpha - \beta) = \int_{L(c,B)} \Gamma(z)\alpha^{-z}dz \int_{L(c,B)} \Gamma(w)\beta^{-w} M_\kappa^\lambda(z + w)\cos\{\pi(z - w)/2\}\, dw. \tag{6.3}$$

The absolute convergence of the repeated integral may be derived from the inequalities (5.5) and (5.6). They show that the integrand is majorized by

$$(|y| + 1)^{c_1 - 1/2}(|v| + 1)^{c_1 - 1/2}(|y + v| + 1)^{-2c_1 - 3/2}, \tag{6.4}$$

provided z, w and $z + w$ stay away from singular points. One finally uses a simple lemma:

Lemma 6.2. *For real constants a, b, c, the function*

$$\phi(y, v) = (|y| + 1)^{-a}(|v| + 1)^{-b}(|y + v| + 1)^{-c}$$

is integrable over $\mathbb{R}^2$ if and only if $a + b > 1, a + c > 1, b + c > 1$ and $a + b + c > 2$. For integrability over $\mathbb{R}^2_+$ the condition $a + b > 1$ may be dropped.

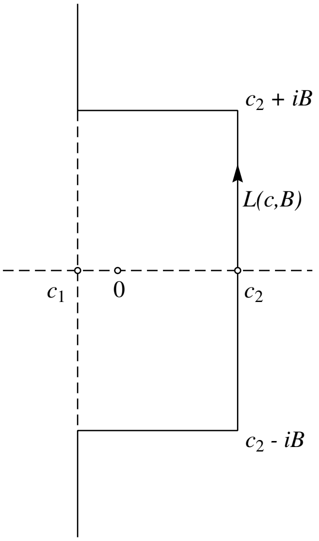

Fig. 1. The path $L(c_1, c_2, B)$.

We outline the proof of the proposition; for a detailed discussion of a related result see [20]. Mellin inversion of a cosine integral related to the Gamma function gives

$$\cos \alpha t = \int_{L(c,B)} \Gamma(z)(\alpha t)^{-z} \cos(\pi z/2) dz,$$

and similarly for $\sin \alpha t$. For absolute convergence one would need $c_1 < -1/2$; the formulas may be verified by moving the path $L(c, B)$ across the poles of $\Gamma(z)$. Because of the possible pole of $M_\kappa^\lambda(z+w)$ when $z + w = -1$, we take $-1/2 < c_1 < 0$ and use a principal value integral. Omitting the part of $L(c, B)$ with $|y| > R$ $(> B)$ we write $L_R(c, B)$. Also using the corresponding integrals for $\cos \beta t$ and $\sin \beta t$, but with variable of integration w, one obtains

$$\begin{aligned} \cos\{(\alpha - \beta)t\} &= \cos \alpha t \cos \beta t + \sin \alpha t \sin \beta t \\ &= \lim_{R\to\infty} \int_{L_R(c,B)} \Gamma(z)\alpha^{-z} t^{-z} dz \\ &\quad \times \int_{L_R(c,B)} \Gamma(w)\beta^{-w} t^{-w} \cos\{\pi(z - w)/2\} dw. \end{aligned}$$

Substituting this result in formula (5.2) for $E_\kappa^\lambda(\nu)$ with $\nu = \alpha - \beta$ and using the definition of $M_\kappa^\lambda(z)$ in (5.4), one obtains (6.3).

7. Repeated complex integral for $D_\kappa^\lambda(s, 2r)$

We will use Proposition 6.1 to obtain a repeated complex integral for the function $D_\kappa^\lambda(s, 2r) = D_\kappa^\lambda(s, 1, 2r)$ of (5.7). The representation will require a function $K_{2r}(z, w)$ of *two complex variables* that is related to our earlier function $K_{2r}(z)$ of (4.8). [The latter will be equal to $K_{2r}(z, z)$.] The new function is

$$\begin{aligned} K_{2r}(z, w) &\stackrel{\text{def}}{=} \sum_{k,l}^{*2r} \mu(kl) k^{-z} l^{-w} = \prod_{(p,2r)=1} (1 - p^{-z} - p^{-w}) \\ &= Q_{2r}(z, w)\zeta_{2r}^{-1}(z)\zeta_{2r}^{-1}(w), \end{aligned} \tag{7.1}$$

where

$$Q_{2r}(z,w) \stackrel{\text{def}}{=} \prod_{p \text{ prime}, (p,2r)=1} \left\{1 - \frac{p^{-z-w}}{(1-p^{-z})(1-p^{-w})}\right\}. \tag{7.2}$$

Taking $x, u > 1$, the infinite product in (7.1) may be obtained as follows. Because of the factor $\mu(kl)$ one may assume that k and l are square-free and relatively prime. Fixing N for a moment, set $k = p_1 \cdots p_\kappa$ and $l = \tilde{p}_1 \cdots \tilde{p}_\lambda$, where $p_1, \ldots, p_\kappa$ and $\tilde{p}_1, \ldots, \tilde{p}_\lambda$ are non-overlapping increasing sequences of primes $\le N$ that do not divide $2r$. Summing over all such finite sequences of primes, one finds that the corresponding partial sum S_N of the series in (7.1) can be written as a product:

$$\begin{aligned} S_N &= \sum_{k,l \text{ special}}^{*2r} \mu(kl) k^{-z} l^{-w} = \sum (-1)^{\kappa+\lambda} (p_1 \cdots p_\kappa)^{-z} (\tilde{p}_1 \cdots \tilde{p}_\lambda)^{-w} \\ &= \prod_{(p,2r)=1;\, p \le N} (1 - p^{-z} - p^{-w}). \end{aligned}$$

Letting $N \to \infty$ absolute convergence gives the desired result.

From here on we assume a weak form of Riemann's Hypothesis (*"weak" RH*), namely, that $\zeta(z)$ is zero-free in a strip $\{1-\delta < x < 1\}$. Since $Q_{2r}(z,w)$ is analytic for $x > 0,\ u > 0,\ x+u > 1$ it follows from (7.1) that $K_{2r}(z,w)$ can be considered as analytic for $x,\ u > 1/2$ except at the zeros of $\zeta(z)$ and $\zeta(w)$. Under weak RH, it will be of small growth for $|y|, |v| \to \infty$ when $x,\ u$ are close to 1. Indeed, for any $\varepsilon > 0$ (cf. [27] for the case of RH):

$$\zeta(x+iy),\ \zeta^{-1}(x+iy) \ll |y|^\varepsilon \text{ when } x \ge 1-\delta+\eta,\ |y| \ge 1.$$

Using products one finds that $Q_{2r}(z,w)\zeta_{2r}(z+w)$ is analytic and bounded for $x,\ u \ge \eta$ if also $2x+u,\ x+2u \ge 1+\eta$. Under weak RH it follows that $Q_{2r}(z,w)$ and its derivatives will be analytic and $\mathcal{O}(|y|^\varepsilon + |v|^\varepsilon)$ for $x,\ u \ge \delta/3$ if we require in addition that $x+u \ge 1-\delta/4$, say.

It will be convenient to write $(\partial/\partial z + \partial/\partial w) K_{2r}(z,w) = K'_{2r}(z,w)$, etc. Then by (7.1)

$$K''_{2r}(z,w) = \sum_{k,l}^{*2r} \mu(kl) k^{-z} l^{-w} (\log^2 kl).$$

We now extend a representation used by Arenstorf [1]. Starting with (5.7), the integral for $E_\kappa^\lambda(\alpha-\beta)$ in (6.3) shows that for values of $s = \sigma + i\tau$ with $\sigma > 1/2$, and for $-1/2 < c_1 < 0 < c_2 < 1/2$ and any $B > 0$,

$$\begin{aligned} D_\kappa^\lambda(s,2r) &= \frac{1}{2} \sum_{k,l}^{*2r} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2r} (ab)^{-s} E_\kappa^\lambda(ak-bl) \\ &= \frac{1}{2} \sum_{k,l}^{*2r} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2r} (ab)^{-s} \int_{L(c,B)} \Gamma(z)(ak)^{-z}\, dz \\ &\quad \times \int_{L(c,B)} \Gamma(w)(bl)^{-w} M_\kappa^\lambda(z+w) \cos\{\pi(z-w)/2\} dw. \end{aligned}$$

Fixing s with $\sigma > 1 - c_1$ (> 1) and appealing to absolute convergence, we invert the order of summation and integration to get

$$\begin{aligned} D_\kappa^\lambda(s, 2r) &= \frac{1}{2}\int_{L(c,B)}\int_{L(c,B)} \Gamma(z)\Gamma(w)M_\kappa^\lambda(z+w)\cos\{\pi(z-w)/2\} \\ &\quad \times \sum_{k,l}^{*2r} \mu(kl)k^{-z-s}l^{-w-s}(\log^2 kl)\sum_{a,b}^{*2r} a^{-z-s}b^{-w-s}\,dzdw \\ &= \frac{1}{2}\int_{L(c,B)}\int_{L(c,B)} \Gamma(z)\Gamma(w)M_\kappa^\lambda(z+w)\cos\{\pi(z-w)/2\} \\ &\quad \times K_{2r}''(z+s, w+s)\zeta_{2r}(z+s)\zeta_{2r}(w+s)\,dzdw. \end{aligned} \tag{7.3}$$

Observe that on the paths we have $x + \sigma \geq c_1 + \sigma > 1$ and similarly for $u + \sigma$, so that in the final double integral, the product

$$K_{2r}''(z+s, w+s)\zeta_{2r}(z+s)\zeta_{2r}(w+s)$$

remains bounded. Absolute convergence then follows as in the case of (6.3).

Using majorization for the integrand and a uniqueness theorem for analytic functions, one obtains

Theorem 7.1. *Let* $-1/2 < c_1 < 0 < c_2 < 1/2$ *and* $B > 0$. *Then under weak RH, formula* (7.3) *provides a holomorphic representation of* $D_\kappa^\lambda(s, 2r)$ *for* $s = \sigma + i\tau$ *with* $\sigma > \max\{1 - \delta - c_1, 1 - c_2\}$. *A similar representation for* $D_\kappa^\lambda(s, j, 2r)$ *is obtained through replacement of* $2r$ *by* $2jr$.

The condition $\sigma > 1 - \delta - c_1$ is required for absolute convergence, cf. (6.4); the condition $\sigma > 1 - c_2$ ensures that for z and w on the paths, $z + s$ and $w + s$ stay away from the pole of $\zeta_{2r}(\cdot)$. Analyticity of the integral follows from locally uniform convergence in s.

8. First reduction of $D_\kappa^\lambda(s, 2r)$

By Cauchy's theorem the paths of integration in (7.3) may be shifted. Passing a singular point of the integrand one then picks up a residue. This process was initiated by Arenstorf [1], and carried further by the author, to split off parts of the integral with known pole-type boundary behavior. With the kernel $K_{2r}(z, w)$, which involves $Q_{2r}(z, w)$, the situation is more complicated than in Korevaar [20].

For just a moment it is convenient to introduce the notation

$$K^*(z, w) = K_{2r}''(z, w)\zeta_{2r}(z)\zeta_{2r}(w). \tag{8.1}$$

The kernel $K^*(z, w)$ can be written as a sum of 'good' terms $X(z, w)$, which involve at most one of the expressions $\zeta_{2r}'(z)/\zeta_{2r}(z)$ and $\zeta_{2r}'(w)/\zeta_{2r}(w)$, and one 'bad' term $Y(z, w)$, which involves both. The good terms $X(z, w)$ are

$$\left[\left\{\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\right\}^2 - \left\{\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\right\}'\right] Q_{2r}(z, w), \qquad -2\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\, Q_{2r}'(z, w),$$

$$\left[\left\{\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}\right\}^2 - \left\{\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}\right\}'\right] Q_{2r}(z, w), \qquad -2\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}\, Q_{2r}'(z, w), \quad Q_{2r}''(z, w).$$

The more difficult mixed term is

$$Y(z,w) = 2\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}\,Q_{2r}(z,w). \tag{8.2}$$

For a proof one may write (7.1) in the form

$$K_{2r}(z,w) = Q_{2r}(z,w)\zeta_{2r}^{-1}(z)\zeta_{2r}^{-1}(w) = Q/Z,$$

say. Then

$$K_{2r}' = (Q/Z)' = Q'/Z - (Q/Z)(Z'/Z),$$
$$K_{2r}'' = Q''/Z - 2(Q'/Z)(Z'/Z) + (Q/Z)(Z'/Z)^2 - (Q/Z)(Z'/Z)'.$$

Multiplying by Z one obtains the desired decomposition of $K^*(z,w)$.

Using the small growth of $Q_{2r}(z,w)$ and its derivatives under the conditions indicated in Section 7 one can show the following. Let $X(z,w)$ stand for any of the good functions above. Then under weak RH, taking $-1/2 < c_1 < 0 < c_2 < 1/2$ and $B > 0$, the corresponding function

$$H_X(s) \stackrel{\text{def}}{=} \int_{L(c,B)} \Gamma(z)dz \int_{L(c,B)} \Gamma(w)X(z+s,w+s)M_\kappa^\lambda(z+w)\cos\{\pi(z-w)/2\}dw,$$

which is holomorphic for $\sigma > \max\{1-\delta-c_1, 1-c_2\}$, has a holomorphic extension to the closed half-plane $\{\sigma \geq 1/2\}$.

For the proof one changes the paths of integration and appeals to Lemma 6.2, recalling that under weak RH there are inequalities such as, cf. [27],

$$\zeta'(x+iy)/\zeta(x+iy) \ll \log|y| \quad \text{when } x \geq 1-\delta+\eta,\ |y| \geq 2.$$

Take for example the case where $X(z,w)$ does not involve $\zeta_{2r}'(w)/\zeta_{2r}(w)$. Moving the remote part of the z-path to the line $\{x = (1-\delta)/2\}$ and the remote part of the w-path to a line $\{u = -(3-2\delta)/6+\eta\}$ with very small η, one obtains an integrable majorant for the integrand when $\sigma \geq (1/2)-\eta$. Indeed, on the remote parts of the paths one will have $x+u = -(\delta/6)+\eta < 0$, $x+\sigma \geq 1-(\delta/2)-\eta$ and $u+\sigma \geq \delta/3$, so that $x+u+2\sigma \geq 1-(\delta/6)-\eta \geq 1-\delta/4$. These inequalities will imply suitable bounds on $(\zeta_{2r}'/\zeta_{2r})(z+s)$ and $Q_{2r}(z+s,w+s)$.

Thus for the study of the pole-type behavior of $D_\kappa^\lambda(s,2r)$ as $\sigma \searrow 1/2$, one may in (7.3) replace

$$K_{2r}''(z,w)\zeta_{2r}(z)\zeta_{2r}(w) \quad \text{by} \quad 2\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}\,Q_{2r}(z,w).$$

Corollary 8.1. *Under weak RH, the function $D_\kappa^\lambda(s,2r)$ of (5.7) or (7.3) has the same pole-type behavior for $\sigma \searrow 1/2$ as the function which for $-1/2 < c_1 < 0 < c_2 < 1/2$ and $\sigma > \max\{1-\delta-c_1, 1-c_2\}$ is given by*

$$D_\kappa^{\lambda,1}(s,2r) = \int_{L(c,B)} \Gamma(z)\frac{\zeta_{2r}'(z+s)}{\zeta_{2r}(z+s)}dz \int_{L(c,B)} \Gamma(w)\frac{\zeta_{2r}'(w+s)}{\zeta_{2r}(w+s)}$$
$$\times\, Q_{2r}(z+s,w+s)M_\kappa^\lambda(z+w)\cos\{\pi(z-w)/2\}dw. \tag{8.3}$$

The difference $D_\kappa^\lambda - D_\kappa^{\lambda,1}$ is holomorphic for $\sigma \geq 1/2$.

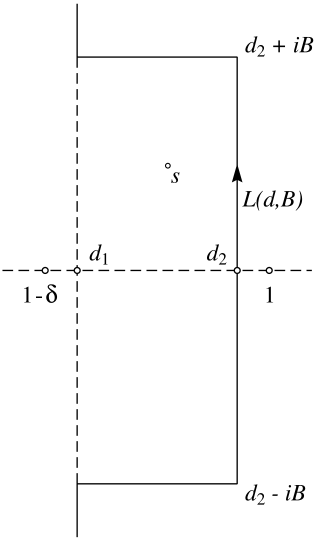

Fig. 2. The path $L(d_1, d_2, B)$.

9. Subtraction of a crucial pole at $s = 1/2$

In the present section, the paths of integration in (8.3) will be moved across the pole of ζ'_{2r}/ζ_{2r} at the point 1. Here it is convenient to change variables; we set $z + s = z'$, $w + s = w'$, and subsequently drop the primes on the variables. This results in new paths of integration $L(c', B')$, where initially $c'_1 = c_1 + \sigma, c'_2 = c_2 + \sigma$ and $B' = B + \tau$; we will require $|\tau| < B$. By the usual estimates and Cauchy's theorem, one may make c' and B' independent of s; we will take $c'_1 = 1 - \delta + \eta, c'_2 = 1 + \eta$ and B' equal to a new constant B. Then for $1 - \delta + \eta < \sigma < 1 + \eta$ and $|\tau| < B$,

$$D_\kappa^{\lambda,1}(s, 2r) = \int_{L(c',B)} \Gamma(z-s)\frac{\zeta'_{2r}(z)}{\zeta_{2r}(z)}\,dz \int_{L(c',B)} \Gamma(w-s)\frac{\zeta'_{2r}(w)}{\zeta_{2r}(w)} \times Q_{2r}(z,w)M_\kappa^\lambda(z+w-2s)\cos\{\pi(z-w)/2\}dw. \tag{9.1}$$

Observe that henceforth, the point s will be to the left of the paths.

We are now ready to move the paths $L(c', B)$ across the poles $z = 1$ and $w = 1$ to $L(d, B)$, where $d_1 = c'_1 = 1 - \delta + \eta, d_2 = 1 - \eta$ with $\eta < \delta/2$; cf. Fig. 2. Starting with the w-path, the residue theorem gives

$$D_\kappa^{\lambda,1}(s, 2r) = \int_{L(c',B)} \cdots\, dz \int_{L(d,B)} \cdots\, dw + U_\kappa^\lambda(s, 2r), \tag{9.2}$$

where

$$U_\kappa^\lambda(s, 2r) = -\int_{L(c',B)} \Gamma(z-s)\frac{\zeta'_{2r}(z)}{\zeta_{2r}(z)}\,\Gamma(1-s)Q_{2r}(z,1)M_\kappa^\lambda(z+1-2s)\sin(\pi z/2)dz. \tag{9.3}$$

In the final integral we move the path $L(c', B)$ across the pole $z = 1$ to the line $L(d_2) = \{x = d_2\}$. Varying $d_2 > d_1$ and $d_1 > 1 - \delta$ without letting $L(d_2)$ cross any singularities, the integral along

$L(d_2)$ defines a function $H_3(s)$ which is holomorphic for $0 < \sigma < 1$. There is also a residue $V_\kappa^\lambda(s, 2r)$ due to the singular point $z = 1$:

$$V_\kappa^\lambda(s, 2r) = \Gamma^2(1-s)Q_{2r}(1, 1)M_\kappa^\lambda(2-2s). \tag{9.4}$$

By Proposition 5.1 the residue function is holomorphic for $0 < \sigma < 1$ except for a first order pole at $s = 1/2$ with residue $(\lambda/2)Q_{2r}(1, 1)$.

Returning to the repeated integral in (9.2), we move its z-path (after inverting order of integration) to $L(d, B)$. Besides a residue, which defines a holomorphic function $H_4(s)$ for $0 < \sigma < 1$, this gives a new repeated integral which (after inversion) takes the form

$$D_\kappa^{\lambda,2}(s, 2r) \overset{\text{def}}{=} \int_{L(d,B)} \Gamma(z-s)\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\,dz \int_{L(d,B)} \Gamma(w-s)\frac{\zeta_{2r}'(w)}{\zeta_{2r}(w)}$$
$$\times\, Q_{2r}(z, w)M_\kappa^\lambda(z+w-2s)\cos\{\pi(z-w)/2\}dw. \tag{9.5}$$

On the remote parts of the paths, the integrand is majorized by

$$|y|^{d_1-\sigma+\varepsilon-1/2}|v|^{d_1-\sigma+\varepsilon-1/2}(|y+v|+1)^{-2d_1+2\sigma-3/2},$$

and this holds for any $\varepsilon > 0$. Thus for absolute convergence (which is locally uniform in s) one will take $1-\delta+\eta < \sigma < 1-\eta$ and $|\tau| < B$, but η can be taken small and B large.

Corollary 9.1. *Under weak RH, the function* $D_\kappa^{\lambda,2}(s, 2r)$ *of* (9.5) *has the same pole-type behavior for* $\sigma \searrow 1/2$ *(when* $|\tau| < B$*) as the difference*

$$\tilde{D}_\kappa^{\lambda,2}(s, 2r) = D_\kappa^{\lambda,1}(s, 2r) - \frac{(\lambda/2)Q_{2r}(1, 1)}{s-1/2}, \tag{9.6}$$

where $D_\kappa^{\lambda,1}(s, 2r)$ *is given by* (8.3).

The difference $D_\kappa^{\lambda,2} - \tilde{D}_\kappa^{\lambda,2}$ is holomorphic for $1/2 \le \sigma < 1$.

10. Reduction of $D_\kappa^{\lambda,2}(s, 2r)$ to a sum involving zeta's zeros

Assuming "strong RH" ($\delta = 1/2$), we start with $D_\kappa^{\lambda,2}(s, 2r)$ in (9.5), and one by one move the paths $L(d, B)$ across the complex zeros ρ of $\zeta(\cdot)$ with $|\mathrm{Im}\,\rho| > B$. Taking multiplicities into account, the zeros are enumerated as

$$\rho = \rho_n = (1/2) + i\gamma_n, \quad 0 < \gamma_1 \approx 14 < \gamma_2 \approx 21 \le \cdots, \ \gamma_{-n} = -\gamma_n. \tag{10.1}$$

We allow any B different from all γ_n and use new paths $L(d', B)$ with $1/4 < d_1' < 1/2$ and $1/2 < d_2' < 1$. By the residue theorem we will then obtain holomorphic decompositions

$$D_\kappa^{\lambda,2}(s, 2r) = H_5(s, 2r) + V_\kappa^{\lambda,2}(s, 2r, B)$$
$$= H_5(s, 2r) + H_6(s, 2r) + \Sigma_\kappa^\lambda(s, 2r, B). \tag{10.2}$$

Here $H_5(s, 2r)$ stands for an integral similar to the one for $D_\kappa^{\lambda,2}(s, 2r)$ in (9.5), but with z-path $L(d, B)$ and w-path $L(d', B)$. Varying d and d', and using estimates of the same type as before, one finds that $H_5(s, 2r)$ defines a holomorphic function for $3/8 < \sigma < 1$ and $|\tau| < B$. The 'residue integral' $V_\kappa^{\lambda,2}(s, 2r, B)$ has the form

$$V_\kappa^{\lambda,2}(s, 2r, B) = \int_{L(d,B)} \Gamma(z-s)\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\,\Sigma_\kappa^\lambda(z, s, 2r)dz, \tag{10.3}$$

where

$$\Sigma_\kappa^\lambda(z,s,2r) = \sum_{|\mathrm{Im}\,\rho|>B} \Gamma(\rho-s)Q_{2r}(z,\rho)M_\kappa^\lambda(z+\rho-2s)\cos\{\pi(z-\rho)/2\}. \tag{10.4}$$

To justify the application of the residue theorem one would start with w-integrals over a sequence of closed contours W_R, $B < R = R_k \to \infty$, which are obtained from $L(d,B) - L(d',B)$ as follows. The parts where $|v| > R$ are deleted and replaced by the horizontal segments from $d_1 + iR$ to $d_1' + iR$ and $d_1' - iR$ to $d_1 - iR$; see Fig. 3. Here the numbers R are chosen 'away from the numbers γ_n', so that $\zeta'(w)/\zeta(w)$ remains $\mathcal{O}(\log^2|v|)$ on the family of remote horizontal segments; cf. [27]. Although in (10.3) we now have a combination of an integral and a sum, the necessary estimates are of the same type as before. One can use the fact that

$$|\rho_n| \sim \gamma_n \sim 2\pi n/\log n \quad \text{as } n \to \infty$$

and may then appeal to an appropriate analog of Lemma 6.2.

Next moving the path $L(d,B)$ in the integral (10.3) for $V_\kappa^{\lambda,2}(s,2r,B)$ to $L(d',B)$, one obtains a decomposition

$$V_\kappa^{\lambda,2}(s,2r,B) = H_6(s,2r) + \Sigma_\kappa^\lambda(s,2r). \tag{10.5}$$

Here

$$H_6(s,2r) = \int_{L(d',B)} \Gamma(z-s)\,\frac{\zeta_{2r}'(z)}{\zeta_{2r}(z)}\,\Sigma_\kappa^\lambda(z,s,2r)\,dz \tag{10.6}$$

defines a holomorphic function for $3/8 < \sigma < 1$ and $|\tau| < B$, and

$$\begin{aligned}\Sigma_\kappa^\lambda(s,2r) &= \Sigma_\kappa^\lambda(s,2r,B) = \sum_{|\mathrm{Im}\,\rho'|>B} \Gamma(\rho'-s)\Sigma_\kappa^\lambda(\rho',s,2r)\\ &= \sum_{|\mathrm{Im}\,\rho|>B,\ |\mathrm{Im}\,\rho'|>B} \Gamma(\rho-s)\Gamma(\rho'-s)\\ &\quad\times Q_{2r}(\rho,\rho')M_\kappa^\lambda(\rho+\rho'-2s)\cos\{\pi(\rho-\rho')/2\}.\end{aligned} \tag{10.7}$$

By the usual estimates, the double series will converge absolutely (and locally uniformly in s) for $1/2 < \sigma < 1$.

Corollary 10.1. *Assume RH. Then for any $B > 2$, and for $s = \sigma + i\tau$ with $1/2 < \sigma < 1$ and $|\tau| < B$, there is a holomorphic decomposition*

$$D_\kappa^{\lambda,2}(s,2r) = \Sigma_\kappa^\lambda(s,2r,B) + H_7(s,2r), \tag{10.8}$$

where $H_7(s,2r)$ is holomorphic for $3/8 < \sigma < 1$ and $|\tau| < B$.

Combining Corollary 10.1 with Corollaries 3.4, 8.1 and 9.1 and Theorem 4.1, and referring to Theorem 7.1 for $j > 1$, we obtain the following result.

Theorem 10.2. *Assume RH. Then the* generalized *prime-pair function $D_\kappa^\lambda(s,j,2r)$ of (5.7) has a pole at $s = 1/2$ with residue $R(\kappa,\lambda)$ if and only if, for every B, the sum $\Sigma_\kappa^\lambda(s,2jr,B)$ has a pole at $s = 1/2$ with residue*

$$R(\kappa,\lambda) - (\lambda/2)Q_{2jr}(1,1).$$

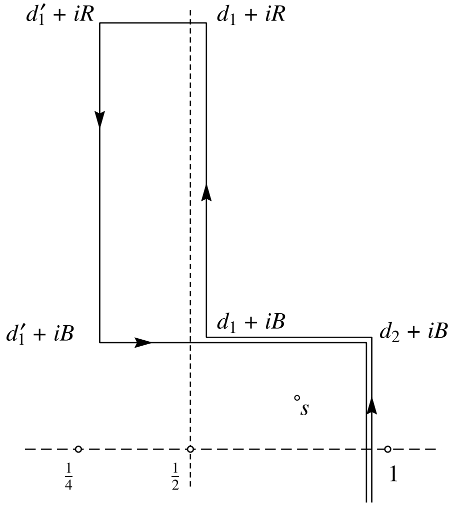

Fig. 3. Upper half of W_R.

In particular, the Hardy–Littlewood conjecture for prime pairs $(p, p + 2r)$ *will be true if and only if for* $\kappa = 2r$ *and some (or every) number* $\lambda \in (0, 2]$ *(so that* $D^{\lambda}_{2r} \cong D_{2r}$*), and every number* $B > 0$*, the difference*

$$\Sigma^{\lambda}_{\kappa}(s, 2r, B) - \frac{C_{2r} - (\lambda/2) Q_{2r}(1, 1)}{s - 1/2} \tag{10.9}$$

has good (local pseudofunction) boundary behavior for $|\tau| < B$ *as* $\sigma \searrow 1/2$*.*

In the special case $\kappa = 0$ *and* $\lambda \leq 2$*, the above difference* DOES *have good boundary behavior, independently of H–L.*

Remarks 10.3. A typical case of prime pairs $(p, jp \pm 2r)$ with $j > 1$ will be treated in Section 15.

Hypothesis "weak" RH, namely, that $\zeta(z)$ is zero-free in *some* strip $1 - \delta < x < 1$, will suffice for an adjusted form of Corollary 10.1. For the summation of an adjusted [not necessarily absolutely convergent] double series (10.7) one may use rectangular partial sums.

Taking $B = 2$, say, one *could* move the paths of integration in the integral (9.5) across the point s as well as the points ρ. The additional residue would be

$$\left\{\frac{\zeta'_{2r}(s)}{\zeta_{2r}(s)}\right\}^2 Q_{2r}(s, s) M^{\lambda}_{\kappa}(0) + 2\frac{\zeta'_{2r}(s)}{\zeta_{2r}(s)} \sum_{\rho} \Gamma(\rho - s) Q_{2r}(s, \rho)$$
$$\times M^{\lambda}_{\kappa}(\rho - s) \cos\{\pi(\rho - s)/2\}.$$

This residue defines a function that cancels the poles ρ, ρ' in the sum (10.7). We have not carried out this move because it would obscure the fact that the distant points ρ, ρ' in (10.7) may generate spurious poles, such as a pole at $s = 1/2$ if the H–L conjecture for $D^{\lambda}_{\kappa}(s)$ would be false.

11. Results for special functions $D^{\lambda}_{\kappa}(s, 2)$

In this section we restrict ourselves to the case where $j = 1$ and $2r = 2$ in (5.7), while $\kappa \geq 0$ and $\lambda > 0$ are even integers. Thus $Q_{2r}(z, w) = Q_2(z, w)$ and we may simplify the notation $D_\kappa^\lambda(s,2)$ to $D_\kappa^\lambda(s)$. Fixing B, we also simplify the notation $\Sigma_\kappa^\lambda(s,2,B)$ to $\Sigma_\kappa^\lambda(s)$. The only differences in boundary behavior will come from the entries $M_\kappa^\lambda(z)$; see (10.7). Omitting a normalizing initial factor $1/(2\pi)$ and the common factor $\Gamma(-z-1)\sin(\pi z/2)$, the remaining *critical part* of $M_\kappa^\lambda(z)$ in (5.4) is

$$\frac{2}{\lambda}\left\{|\kappa-\lambda|^{z+1}-2\cdot\kappa^{z+1}+(\kappa+\lambda)^{z+1}\right\}. \tag{11.1}$$

(i) For $\kappa=0,\ \lambda=2$, the critical factor is

$$2\cdot 2^{z+1},$$

and by Theorem 10.2, the residue of $\Sigma_0^2(s)$ at $s=1/2$ is equal to

$$\text{res}\, D_0^2(s)-Q_2(1,1)=\text{res}\, D_0(s,2)-C_2=0;$$

cf. (7.2), (1.2) and Theorem 4.1.

(ii) For $\kappa=0,\ \lambda=4$, the critical factor is

$$(1/2)\cdot 2\cdot 4^{z+1}=4^{z+1}.$$

Since $E_0^4=E_0^2+E_2^2$ one has $D_0^4(s)=D_0^2(s)+D_2^2(s)$. By Theorem 10.2 the residue of $\Sigma_0^4(s)$ equals res $D_0^4(s)-(4/2)Q_2(1,1)$, or

$$\text{res}\, D_0^2(s)-C_2+\text{res}\, D_2^2(s)-C_2=\text{res}\, D_2(s)-C_2.$$

(iii) For $\kappa=0,\ \lambda=6$, the critical factor is

$$(1/3)2\cdot 6^{z+1}=(2/3)6^{z+1}.$$

Since $E_0^6=E_0^2+(4/3)E_2^2+(2/3)E_4^2$, one has

$$D_0^6=D_0^2+(4/3)D_2^2+(2/3)D_4^2.$$

Subtracting $(6/2)Q_2(1,1)$, it follows that the residue of Σ_0^6 equals

$$\begin{aligned}&C_2-C_2+(4/3)\{\text{res}\, D_2^2-C_2\}+(2/3)\{\text{res}\, D_4^2-C_2\}\\&\quad=(2/3)\{2(\text{res}\, D_2-C_2)+\text{res}\, D_4-C_2\}.\end{aligned}$$

(iv) For $\kappa=0,\ \lambda=8$, the critical factor is

$$(1/4)(2\cdot 8^{z+1})=(1/2)8^{z+1}.$$

Analyzing E_0^8 one finds that $D_0^8=D_0^2+(3/2)D_2^2+D_4^2+(1/2)D_6^2$. Subtracting $(8/2)Q_2(1,1)$, it follows that the residue of Σ_0^8 equals

$$C_2-C_2+(3/2)\{\text{res}\, D_2-C_2\}+\{\text{res}\, D_4-C_2\}+(1/2)\{\text{res}\, D_6^2-C_2\}.$$

If one assumes the PPC for D_2 and D_4, the result equals $(1/2)\{\text{res}\, D_6^2-C_2\}$. Observe that $D_6^2=D_6^2(s,2)$ does *not* correspond to the original prime-pair function D_6; cf. (3.6). Indeed, we have not required that m, a and b be prime to 6, as needed in (3.5). We will often write $D_6^*(s)$ for $D_6^2(s,2)$.

THE FUNCTION $D_6^2(s,2)$. It seems that the expected value of res $D_6^2-C_2$ cannot be obtained by using other combinations of λ and κ. Choices such as $\kappa=2,4,6$ and $\lambda=2$ give nothing new.

One can get some new information from the function $D_0^2(s, 6) = D_0(s, 6)$; the corresponding sum $\Sigma_0^2(s, 6)$ will have residue $(2/3)C_2$. However, it involves the factor $Q_6(\rho, \rho')$ instead of $Q_2(\rho, \rho')$, and hence this new information is not immediately useful.

By the general definition (5.7) one has

$$D_6^2(s, 2) = \frac{1}{2} \sum_{k,l}^{*2} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2}(ab)^{-s} E_6^2(ak - bl). \tag{11.2}$$

Here k, l, a and b must be prime to 2, while $E_6^2(\nu) = E_6(\nu)$ is equal to $1/2$ for $\nu = \pm 6$, and equal to zero for all other even integers ν. Arguing as in Section 4, the logical comparison function for $D_6^2(s, 2)$ would be $D_0^2(s, 2)$. Thus one would expect res $D_6^2(s, 2) - C_2$ to be zero.

The function $D_6^2(s, 2)$ can, modulo "good functions", be expressed in terms of standard prime-pair functions. Indeed, the analysis in Sections 14–16 will establish the important decomposition

$$D_6^2 \cong D_6 - D_2/3^{2s} - (D_{3,2} + D_{3,-2})/9^{2s} - (D_{9,2} + D_{9,-2})/27^{2s} - \cdots. \tag{11.3}$$

The sum $D_{3,2}(s) + D_{3,-2}(s)$ is associated with prime pairs $(p, 3p \pm 2)$ and each term has conjectured residue $2C_2$. Similarly for $D_{9,2} + D_{9,-2}$, etc. On the basis of the H–L conjectures, the infinite sum in (11.3) will indeed have residue C_2. For termwise evaluation one may appeal to dominated convergence of a corresponding series of terms $x^{-1}\psi_{j,\pm 2r}(x) - 2C_{2jr}$; cf. (1.5).

We will call the conjecture that $D_2^6(s, 2) - C_2/(s - 1/2)$ has good boundary behavior the *extended* Hardy–Littlewood *conjecture* for $D_6^* = D_6^2$, and similarly for the generalized prime-pair functions D_{10}^*, D_{12}^*, etc. The extended H–L conjectures *follow* from the original conjectures; see Sections 14–16.

12. The principal results

We continue with $2r = 2$ in (5.7) and even integers λ and κ. For $\lambda = 2$ and positive $\kappa = 2k$ we are dealing with the functions $D_{2k}^2(s, 2)$. These are like $D_{2k}(s)$ if k is a power of 2, and are denoted by $D_{2k}^*(s)$ otherwise.

Looking back at the "critical factors" in parts (i)–(iv) of Section 11 and the corresponding "Σ residues", one notices that the results become nicer if one multiplies the factor in (iii) by $3/2$ and the factor in (iv) by $4/2$. For a factor $(2k)^{z+1}$ we will denote the corresponding function Σ by $\Sigma_{2k}^*(s, B)$ or $\Sigma^*(s, 2k, B)$. It is obtained from $\Sigma_\kappa^\lambda(s, 2, B)$ in (10.7) through replacement of the critical factor in $M_\kappa^\lambda(z)$ by $(2k)^{z+1}$; cf. (11.1) and (12.1). Continuing the work begun in Section 11 one obtains the following list of factors and corresponding Σ^* residues:

TABULATION.

2^{z+1}: 0.
4^{z+1}: res $D_2 - C_2$.
6^{z+1}: $2(\text{res } D_2 - C_2) + \text{res } D_4 - C_2$.
8^{z+1}: $3(\text{res } D_2 - C_2) + 2(\text{res } D_4 - C_2) + \text{res } D_6^* - C_2$.
10^{z+1}: $4(\text{res } D_2 - C_2) + 3(\text{res } D_4 - C_2) + 2(\text{res } D_6^* - C_2) + \text{res } D_8 - C_2$.
12^{z+1}: $5(\text{res } D_2 - C_2) + 4(\text{res } D_4 - C_2) + \cdots + \text{res } D_{10}^* - C_2$.
$(2k)^{z+1}$: $(k - 1)(\text{res } D_2 - C_2) + (k - 2)(\text{res } D_4 - C_2) + \cdots + \text{res } D_{2k-2}^* - C_2$.

It is useful to introduce more general functions Σ^*_ω with $\omega \in \mathbb{R}_+$:

$$\Sigma^*_\omega(s, B) = \Sigma^*(s, \omega, B) \stackrel{\text{def}}{=} \frac{1}{2\pi} \sum_{|\operatorname{Im}\rho|>B,\, |\operatorname{Im}\rho'|>B} \Gamma(\rho - s)\Gamma(\rho' - s)Q_2(\rho, \rho')$$
$$\times\, \omega^{\rho+\rho'-2s+1}\Gamma(2s - 1 - \rho - \rho')\sin\{\pi(\rho + \rho' - 2s)/2\}$$
$$\times\, \cos\{\pi(\rho - \rho')/2\}. \tag{12.1}$$

Under RH one may use asymptotic analysis to obtain a more transparent equivalent function; see Section 13.

Theorem 12.1. *Under a weak form of RH, the (extended) H–L prime-pair conjecture is true for each of the functions $D^*_{2k}(s) = D^2_{2k}(s, 2)$, $k = 1, 2, \ldots$, if and only if for every k and B, the sum $\Sigma^*(s, 2k, B)$ has good (local pseudofunction) boundary behavior for $|\tau| < B$ as $\sigma \searrow 1/2$.*

In view of (11.3) we have the following special result:

Corollary 12.2. *The prime-pair conjectures for D_2, D_4 and the combination $D_6 - (D_{3,2} + D_{3,-2})/9 - (D_{9,2} + D_{9,-2})/27 - \cdots$ are true if and only if the sums $\Sigma^*(s, 2k, B)$ have good boundary behavior for $2k = 4, 6, 8$.*

We now *compute* the *residue* $R(\omega)$ of the function $\Sigma^*(s, \omega, B)$ for $\omega = 2k + \alpha$ (with $k \in \mathbb{N}_0$ and $0 < \alpha \le 2$) under the extended H–L conjectures. Analysis of $E_0^{2k+\alpha}$ shows that $D_0^\alpha = D_0^2$ and for $k \ge 1$,

$$D_0^{2k+\alpha} = D_0^2 + \frac{2k - 2 + \alpha}{2k + \alpha} 2D_2^2 + \frac{2k - 4 + \alpha}{2k + \alpha} 2D_4^2 + \cdots + \frac{\alpha}{2k + \alpha} 2D_{2k}^2.$$

This function will have "H–L residue"

$$\left\{1 + \frac{2k(k - 1 + \alpha)}{2k + \alpha}\right\} C_2.$$

According to Theorem 10.2, the residue of the sum $\Sigma_0^{2k+\alpha}(s, 2, B)$ is obtained by subtracting $(2k + \alpha)C_2/2$; the result is $(\alpha - \alpha^2/2)C_2/(2k + \alpha)$. The "critical factor" in $\Sigma_0^{2k+\alpha}(s, 2, B)$ is $2(2k + \alpha)^{-1} \cdot 2(2k + \alpha)^{z+1}$. For critical factor $(2k + \alpha)^{z+1}$, combination gives the residue

$$R(2k + \alpha) = (1/8)\alpha(2 - \alpha)C_2 : \tag{12.2}$$

$R(\omega)$ will be periodic with period 2 ! [(12.2) holds unconditionally for $k = 0$.]

Theorem 12.3. *Under a weak form of RH, the extended Hardy–Littlewood conjectures for the functions $D^2_{2k}(s.2)$ are true if and only if the pole-type boundary behavior of the functions $\Sigma^*(s, \omega, B)$ is periodic with period 2.*

Indeed, the periodicity would imply that the difference $\Sigma^*(s, 2k+2, B) - \Sigma^*(s, 2k, B)$ has good boundary behavior for every k and B.

Corollary 12.4. *Suppose that there are few prime pairs, in the precise sense that all the functions $D^2_{2k}(s, 2)$ have residue zero. Then the residue of $\Sigma^*(s, \omega, B)$ would behave like $-\omega^2 C_2/2$ as $\omega \to \infty$.*

Looking at $\Sigma^*(s, \omega, B)$, such a large negative residue would seem unlikely.

13. Transformation of $\Sigma^*(s, \omega, B)$ under RH

It follows from Lemma 6.2 that the part of the series for $\Sigma^*(s, \omega, B)$, in which $\operatorname{Im} \rho = \gamma$ and $\operatorname{Im} \rho' = \gamma'$ have the *same* sign, defines a meromorphic function for $0 < \sigma \le 1$, with poles at the points ρ. Hence for the boundary behavior of $\Sigma^*(s) = \Sigma^*(s, \omega, B)$ as $\sigma \searrow 1/2$, we need only the part $\Sigma_1^*(s)$ where $\operatorname{Im} \rho = \gamma$ and $\operatorname{Im} \rho'$ have *opposite* sign. By symmetry we may take $\gamma > 0$ and $\operatorname{Im} \rho' = -\gamma' < 0$, provided we multiply the resulting sum by 2.

Taking possible multiplicities into account, let $N(t)$ denote the number of zeta's zeros $\rho = (1/2) + i\gamma$ with $0 < \gamma \le t$. Then

$$\begin{aligned} &N(t) = L(t) + S(t), \quad \text{where} \\ &2\pi L(t) = 2 \operatorname{Im} \log \Gamma\{(1/4) + (1/2)it\} - t \log \pi + 2\pi \\ &\qquad\quad = t \log t - (1 + \log 2\pi)t + (7/4)\pi + \mathcal{O}\{1/(t+1)\}, \\ &S(t) = (1/\pi) \arg \zeta\{(1/2) + it\} = \mathcal{O}\{\log(t+1)\}; \end{aligned} \tag{13.1}$$

cf. Titchmarsh's book [27].

It is convenient to write

$$M(z) \quad \text{for } \Gamma(-1-z)\sin(\pi z/2). \tag{13.2}$$

Then by the preceding, assuming RH, and fixing ω and B,

$$\Sigma^*(s, \omega, B) \cong \Sigma_1^*(s) = 2 \iint_{y,\, v > B} F(y, v, s)\, dN(y)\, dN(v), \tag{13.3}$$

where by (12.1),

$$\begin{aligned} F(y, v, s) = {}& (1/2\pi)\Gamma(iy - s + 1/2)\Gamma(-iv - s + 1/2) Q_2(iy + 1/2, -iv + 1/2) \\ & \times \omega^{2-2s+i(y-v)} M\{1 - 2s + i(y - v)\} \cosh\{\pi(y + v)/2\}. \end{aligned} \tag{13.4}$$

We now use Stirling's uniform asymptotic formula for $|\arg z| \le \pi/2$ and $|z| > 2$:

$$\log \Gamma(z) = (z - 1/2)\log z - z + (1/2)\log(2\pi) + \mathcal{O}(1/|z|); \tag{13.5}$$

cf. Whittaker and Watson [30]. Setting $s = (1/2) + \eta + iA$ with $\eta < 1/8$, say, (13.5) will imply that for $y,\ v > B > 2A$,

$$\begin{aligned} F(y, v, s) = {}& (yv)^{-s} \exp\left\{ i \int_v^y \log t \, dt \right\} Q_2(iy + 1/2, -iv + 1/2) \\ & \times \omega^{2-2s+i(y-v)} M\{1 - 2s + i(y - v)\} \big\{1 + \mathcal{O}(1/y) + \mathcal{O}(1/v)\big\}. \end{aligned} \tag{13.6}$$

Note also that one has

$$Q_2(z, w) = H(z, w)/\zeta(z + w), \tag{13.7}$$

where $H(z, w) = Q_2(z, w)\zeta(z + w)$ is holomorphic and bounded (with bounded derivatives) for $|\tau| < A$ and $x, u > 3/8$; cf. Section 7.

Lemma 13.1. *The pole-type boundary behavior of $\Sigma^*(s, \omega, B)$ for $\sigma \searrow 1/2$ (or $\eta \searrow 0$) and $|\tau| < A < B/2$ is the same as that of the reduced function*

$$\Sigma_2^*(s) = \iint_{y,\, v>B;\, |y-v|<y^{3/4}} (yv)^{-s+i(y-v)/2} Q_2(iy+1/2, -iv+1/2) \times \omega^{2-2s+i(y-v)} M\{1-2s+i(y-v)\} dN(y)dN(v). \tag{13.8}$$

Proof. In the discussion of the integral of F in (13.6) one may ignore the quantities $\mathcal{O}(1/y)$ and $\mathcal{O}(1/v)$; by Lemma 6.2 they lead to bounded functions of s. Simple majorization will next show that the integral I_1 of $|F(y, v, \eta)|dN(y)dN(v)$ over the set Ω_1, where $y,\ v > B$ and $|y-v| \geq y^{3/4}$, is bounded for $0 < \sigma - 1/2 = \eta < 1/8$. Indeed, by (5.5), fixing an $\varepsilon < 1/8$,

$$\begin{aligned} I_1 &\ll \iint_{\Omega_1} |F(y, v, s)|\, dN(y)dN(v) \\ &\ll \int_B^\infty y^{-\eta-1/2}(\log y)dy \int_{v>B, |y-v|\geq y^{3/4}} v^{-\eta-1/2}(|y-v|+1)^{2\eta+\varepsilon-3/2}(\log v)dv \\ &\ll \int_B^\infty y^{-\eta-1/2}(\log y)\cdot y^{(\eta/2)+\varepsilon-5/8}(\log y)dy. \end{aligned}$$

It follows that we may restrict ourselves to the part I_2 of the integral in (13.3) over the set Ω_2, where $y,\ v > B$ and $|y-v| < y^{3/4}$. On this set the function

$$v^{-\eta-1/2} = y^{-\eta-1/2}\{1+(v-y)/y\}^{-\eta-1/2} = y^{-\eta-1/2} + \mathcal{O}(y^{-\eta-3/4})$$

might be replaced by $y^{-\eta-1/2}$; the error term gives rise to a bounded function of $\eta = \sigma - 1/2$. We finally observe that on Ω_2,

$$\int_v^y \log t\, dt = (y-v)\log\sqrt{yv} + \mathcal{O}\{|y-v|^2/y^2\},$$

hence

$$\exp\left\{i\int_v^y \log t\, dt\right\} = (yv)^{i(y-v)/2}\big[1+\mathcal{O}\{|y-v|^2/y^2\}\big].$$

The contribution to I_2 due to the final $\mathcal{O}$-term is uniformly bounded for our values $s = (1/2)+\eta+i\tau$. Thus as regards its pole-type boundary behavior, the function $\Sigma_1^*(s)$ can be reduced to $\Sigma_2^*(s)$.

With the new integrand, the integration may also be extended to the whole set $\{y, v > B\}$; the additional contribution due to Ω_1 will remain bounded. □

From here it is only a small step to

Theorem 13.2. *Under RH, the pole-type boundary behavior of $\Sigma^*(s, \omega, B)$ as $\sigma \searrow 1/2$ and $|\tau| < A < B/2$ is the same as that of the function*

$$\Sigma_3^*(s) = \sum_{\gamma,\gamma'>B} \omega^{2-2s+i(\gamma-\gamma')}(\gamma\gamma')^{-s+i(\gamma-\gamma')/2} \times Q_2(i\gamma+1/2, -i\gamma'+1/2)M\{1-2s+i(\gamma-\gamma')\}. \tag{13.9}$$

Here one may in addition require that $|\gamma-\gamma'| < \gamma^{3/4}$.

Question 13.3. Going back to formula (13.8), observe that $dN(t)$ is the sum of an absolutely continuous part $dL(t)$ and a singular part $dS(t)$. Using integration by parts one can show that the combinations $dL(y)dL(v)$ and $dS(y)dL(v)$ do not give rise to poles as $\sigma \searrow 1/2$. Possible singularities must be due to the combination $dS(y)dS(v)$. What does the periodic pole-type boundary behavior of $\Sigma^*(s, \omega, B)$ say about the function $S(t)$?

14. Special functions of mixed type

A proof that the original Hardy–Littlewood conjectures for prime pairs $(p, jp \pm 2r)$ imply the extended conjectures for the functions $D^*_{2k} = D^2_{2k}$ requires careful analysis. Here we start on the case of

$$D^2_6(s, 2) = \frac{1}{2} \sum_{k,l}^{*2} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2} (ab)^{-s} E^2_6(ak - bl). \tag{14.1}$$

Modulo 'good' functions, D^2_6 will be expressed in terms of D_6, D_2 and functions related to prime pairs $(p, 3p \pm 2)$, $(p, 9p \pm 2)$, $(p, 27p \pm 2), \ldots$; cf. formula (11.3).

For the analysis we begin with the equations

$$ak - bl = \pm 6, \tag{14.2}$$

where $kl = m$ is square-free, and m, a, b must be prime to 2.

One has to consider several cases.

(1) kl prime to 6.

(1.1) a and b also prime to 6 (if one is, so is the other). The corresponding part of $D^2_6(s, 2)$ is

$$D^2_6(s, 6) = \frac{1}{2} \sum_{m}^{*6} \mu(m)(\log^2 m)R(s, 6, m), \quad \text{where}$$

$$R(s, 6, m) = \sum_{k,l;\, kl=m}^{*6} \sum_{a,b}^{*6} (akbl)^{-s} E^2_6(ak - bl). \tag{14.3}$$

$D^2_6(s, 6)$ is equivalent to the old function $D_6(s)$; cf. Section 3. The H–L conjecture gives residue $2C_2$.

(1.2) a and b divisible by 3: $a = 3a_1$, $b = 3b_1$, with odd a_1, b_1. Our equation becomes

$$a_1k - b_1l = \pm 2.$$

The corresponding sum is

$$D^2_2(s, 6, 2) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{k,l}^{*6} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a_1,b_1}^{*2} (9a_1b_1)^{-s} E^2_2(a_1k - b_1l). \tag{14.4}$$

The method of Section 4 gives likely residue C_2.

(2) kl divisible by (one factor) 3.

(2.1) Say $k = 3k_1$, with k_1, l prime to 6. The equation $3ak_1 - bl = 6$ requires $b = 3b_1$, hence (14.2) becomes

$$ak_1 - b_1l = \pm 2.$$

The corresponding sum is

$$\frac{1}{2} \sum_{k_1,l}^{*6} \mu(3k_1l)(3k_1l)^{-s}(\log^2 3k_1l) \sum_{a,b_1}^{*2} (3ab_1)^{-s} E^2_2(ak_1 - b_1l).$$

It is equivalent to $-D_2^2(s,6,2)$. Indeed, $\mu(3k_1l) = -\mu(k_1l)$ and the two factors 3^{-s} give a factor 9^{-s}. Finally, of the factor $\log^2 3k_1l = (\log k_1l + \log 3)^2$, only $\log^2 k_1l$ gives a function with a singularity at $s = 1/2$. To verify this, one may go back to the method of Section 3. Starting with the identity $\sum_{m|k} \mu(m)\log m = \Lambda(k)$ and taking $k = n(n \pm 2)$, one finds that

$$\sum_{k,l}^{*2} \mu(kl)(kl)^{-s}(\log kl) \sum_{a,b}^{*2} (ab)^{-s} E_2^2(ak-bl) \cong \sum_{n>2}^{*2} \frac{\Lambda\{n(n\pm 2)\}}{n^s(n\pm 2)^s} \cong 0. \tag{14.5}$$

The corresponding sum over k, l prime to 6 will also be equivalent to 0.

(2.2) The case $l = 3l_1$, k, l_1 prime to 6 goes exactly like (2.1).

(3) Adding the preceding results, we find that D_6^2 is equivalent to $D_6 - D_2^2(s,6,2)$, hence we have to look more closely at $D_2^2(s,6,2)$. For the residue at $1/2$ we may replace 9^{-s} by $1/3$. The numbers kl prime to 6 can be obtained by taking the numbers prime to 2, and taking away the odd multiples of 3. Thus $D_2^2(s,6,2)$ is equivalent to $9^{-s}(D^* - D^{**})$, where

$$D^*(s) = \frac{1}{2}\sum_{k,l}^{*2} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2}(ab)^{-s} E_2^2(ak-bl),$$

$$D^{**}(s) = \frac{1}{2}\sum_{k,l;\, kl\equiv 0 \pmod 3}^{*2} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2}(ab)^{-s} E_2^2(ak-bl). \tag{14.6}$$

(3.1) Here $D^* = D_2^2$ is like D_2, with H–L residue C_2.

(3.2) For D^{**} one has to consider two cases. Either $k = 3k_1$, with k_1 and l prime to 6, or $l = 3l_1$, with k and l_1 prime to 6. This leads to the equations

$$3ak_1 - bl = \pm 2 \quad \text{and} \quad ak - 3bl_1 = \pm 2, \tag{14.7}$$

with a, b odd. In the first case the corresponding homogeneous equation requires $b = 3b_1$, and since $\mu(3k_1l) = -\mu(k_1l)$, the by now standard approach will give likely residue $-C_2$. The same holds for the second case, $a = 3a_1$. Thus the likely residue of $D^{**}(s)$ is $-2C_2$. Our aim is to derive this from the H–L conjectures.

Replacing k by $3k_1$ and then leaving off the subscripts, the function D^{**} leads one to consider twice the function

$$D(s,6,2;3) \stackrel{\text{def}}{=} \frac{1}{2} 3^{-s} \sum_{k,l}^{*6} \mu(kl)(kl)^{-s}(\log^2 kl) \sum_{a,b}^{*2} (ab)^{-s} E_2^2(3ak-bl). \tag{14.8}$$

Here the final 3 in the argument of $D(s,6,2;3)$ refers both to the factor 3^{-s} and the 3 in the equation $3ak - bl = \pm 2$. We will see below that $D(s,6,2;3)$ can be associated with certain prime pairs $(p, jp \pm 2)$.

So far we have found that

$$\begin{aligned} D_6^2 &\cong D_6 - D_6^2(s,6,2) \cong D_6 - 9^{-s}D_2 + 9^{-s}D^{**} \\ &\cong D_6 - 9^{-s}D_2 - 2\cdot 9^{-s} D(s,6,2;3). \end{aligned} \tag{14.9}$$

15. Prime pairs $(p, 3p \pm 2)$, $(p, 9p \pm 2)$, etc.

For the study of prime pairs $(p, 3p + 2)$ it is natural to start with the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(3n+2)}{n^{2s}}.$$

Here we may take $n > 1$ and prime to 6 without affecting pole-type behavior as $\sigma \searrow 1/2$. Proceeding as in Section 3, we obtain the equivalent function

$$D_{3,2}(s) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{n}^{*6} \frac{3^s}{n^s(3n+2)^s} \sum_{m|n(3n+2)}^{*6} \mu(m) \log^2 m$$
$$= \frac{1}{2} \sum_{m}^{*6} \mu(m)(\log^2 m) R_{3.2}(s, m), \tag{15.1}$$

where

$$R_{3.2}(s, m) = 3^s \sum_{k,l;\, kl=m}^{*6} \sum_{a,b;\, 3ak-bl=-2}^{*6} (akbl)^{-s}. \tag{15.2}$$

We also need $D_{3,-2}(s)$. For the expected boundary behavior of the average $D(s, 3, 2)$ [cf. (3.9)] we proceed as in Section 4. Thus we replace the final sum by a sum over $a = hl$ and $b = 3hk$ with h prime to 6. The comparison function will be equal to

$$D_0(s, 3, 2) = \frac{1}{2} \sum_{m}^{*6} \mu(m) d(m) m^{-2s} (\log^2 m) \zeta_6(2s). \tag{15.3}$$

This is just the function $D_0(s, 6)$ of (4.12). It follows that the expected residue of $D_0(s, 3, 2)$ is $2C_2$. The counting function for the prime pairs $(p, 3p \pm 2)$ with $p \le x$ would then be asymptotic to $4C_2 x/\log^2 x$. This agrees with Conjecture D in Hardy and Littlewood [12, p. 45].

Remarks 15.1. For any $\nu \ge 1$, the study of prime pairs $(p, 3^\nu p \pm 2)$ leads to the function

$$D(s, 3^\nu, 2) = \frac{1}{2} \sum_{m}^{*6} \mu(m)(\log^2 m) R(s, 3^\nu, 2, m), \quad \text{where}$$
$$R(s, 3^\nu, 2, m) = 3^{\nu s} \sum_{k,l;\, kl=m}^{*6} \sum_{a,b;\, 3^\nu ak-bl=\mp 2}^{*6} (akbl)^{-s}. \tag{15.4}$$

The expected residue will always be $2C_2$.

For the completion of our program we have to analyze the function $D(s, 6, 2; 3)$ of formula (14.8).

16. The decomposition (11.3) of D_6^2, etc.

We return to $D(s, 6, 2; 3)$. Proceeding in a now standard manner, we find that for $kl = m$ with m prime to 6 and $3ak - bl = \pm 2$, either a, b are prime to 6, or $a = 3a_1$ with a_1 odd. Thus

$$\sum_{a,b}^{*2} (ab)^{-s} E_2^2(3ak - bl) = \sum_{a,b}^{*6} (ab)^{-s} E_2^2(3ak - bl) + \sum_{a_1,b}^{*2} (3a_1 b)^{-s} E_2^2(9a_1 k - bl).$$

Putting the first part

$$\frac{1}{2}\,3^{-s}\sum_{k,l}^{*6}\mu(kl)(kl)^{-s}(\log^2 kl)$$

of $D(s,6,2;3)$ in front of the above sums, one finds that

$$D(s,6,2;3)\cong 3^{-2s}D_{3,2}(s)+D(s,6,2;9). \tag{16.1}$$

Here we have first used (15.2) and its analog for $R_{3,-2}(s,m)$, and next the analog $D(s,6,2;9)$ to $D(s,6,2;3)$ in (14.8). The final 9 in the argument of $D(s,6,2;9)$ refers both to a factor 9^{-s} and to the 9 in the equation $9a_1k-bl=\pm 2$.

Continuing in this manner, one arrives at the identity

$$D(s,6,2;3)\cong D_{3,2}(s)/3^{2s}+D_{9,2}(s)/9^{2s}+D_{27,2}(s)/27^{2s}+\cdots. \tag{16.2}$$

On the basis of the H–L conjectures one expects the residue to be

$$\big\{(1/3)+(1/9)+\cdots\big\}2C_2=C_2.$$

This result shows that all the heuristic residues in Section 14 are in accordance with the H–L conjectures. It also proves that $D_6^2(s,2)$ is equivalent to the sum given in formula (11.3), and it completes the proof of Corollary 12.2.

For functions of the form D_{2q}^2 ($q>3$ prime), one readily obtains a decomposition analogous to the one for D_6^2:

$$D_{2q}^2\cong D_{2q}-D_2/q^{2s}-(D_{q,2}+D_{q,-2})/q^{4s}-(D_{q^2,2}+D_{q^2,-2})/q^{6s}-\cdots. \tag{16.3}$$

The situation is more complicated for functions D_{2k}^2 with composite k. Here a form of induction shows that decomposition is always possible, and one may use the method of Section 4 to keep track of the likely residues at each step.

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