On Newman's Quick Way to the Prime Number Theorem

J. Korevaar

1. Introduction and Overview

There are several interesting functions in number theory whose tables look quite irregular, but which exhibit surprising asymptotic regularity as $x \to \infty$. A notable example is the function $\pi(x)$ which counts the number of primes p not exceeding x.

1.1. The Famous Prime Number Theorem

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty, \tag{1.1}$$

was surmised already by Legendre and Gauss. However, it took a hundred years before the first proofs appeared, one by Hadamard and one by de la Vallée Poussin (1896). Their and all but one of the subsequent proofs make heavy use of the Riemann zeta function. (The one exception is the long so-called elementary proof by Selberg [11] and Erdös [4].)

For Re $s > 1$ the zeta function is given by the Dirichlet series

D. J. Newman

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}. \tag{1.2a}$$

By the unique representation of positive integers n as products of prime powers, the series may be converted to the Euler product (cf. [5])

$$\zeta(s) = \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \ldots\right)\left(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \ldots\right)\ldots$$
$$= \prod_p \frac{1}{1 - p^{-s}}. \tag{1.2b}$$

The above function element is analytic for Re $s > 1$ and can be continued across the line Re $s = 1$ (Fig. 1). More precisely, the difference

$$\zeta(s) - \frac{1}{s-1}$$

can be continued analytically to the half-plane Re $s > 0$ (cf. § B.1 in the box on p. 111) and in fact to all of $\mathbb{C}$. The essential property of $\zeta(s)$ in the proofs of the prime number theorem is its non-vanishing on the line Re $s = 1$

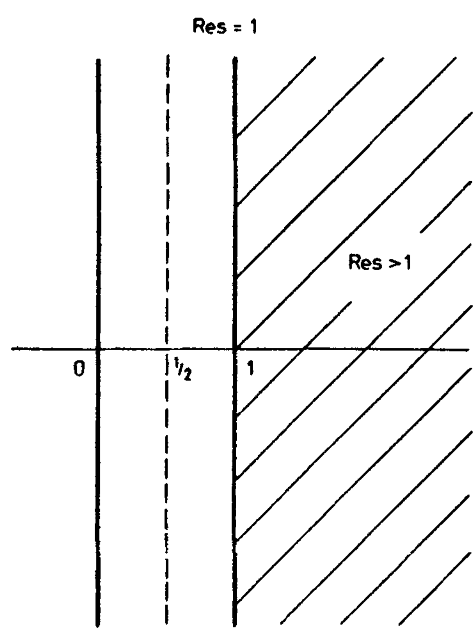

Figure 1

(cf. § B.2). [The zeta function has many zeros in the strip $0 < \text{Re } s < 1$. Riemann's conjecture (1859) that they all lie on the central line $\text{Re } s = \frac{1}{2}$ remains unproven to this day.]

For about fifty years now, the standard proofs of the prime number theorem have involved some form of Wiener's Tauberian theory for Fourier integrals, usually the Ikehara-Wiener theorem of § 1.2 (see Wiener [14] and cf. various books, for example Doetsch [2], Chandrasekharan [1], Heins [6]). Thus the proof of the prime number theorem has remained quite difficult until the recent breakthrough by D. J. Newman [10].

In 1980, he succeeded in replacing the Wiener theory in the proof by an ingenious application of complex integration theory, involving nothing more difficult than Cauchy's integral formula, together with suitable estimates. We present Newman's method in § 2 (applying it to Laplace integrals instead of Dirichlet series). In this method, the Ikehara-Wiener Tauberian theorem is replaced by a poor man's version which also readily leads to the prime number theorem.

Excellent accounts of the history of the prime number theorem and the zeta function may be found in the books of Landau [9], Ingham [7], Titchmarsh [13] and Edwards [3].

1.2. A Gem from Ingham's Work

Newman's method leads directly to the following pretty theorem which is already contained in work of Ingham [8]. However, Ingham used Wiener's method to prove his (more general) results.

Auxiliary Tauberian theorem. *Let $F(t)$ be bounded on $(0, \infty)$ and integrable over every finite subinterval, so that the Laplace transform*

$$G(z) = \int_0^\infty F(t)e^{-zt}dt \tag{1.3}$$

is well-defined and analytic throughout the open half-plane $\text{Re } z > 0$. *Suppose that $G(z)$ can be continued analytically to a neighborhood of every point on the imaginary axis. Then*

$$\int_0^\infty F(t)dt \text{ exists} \tag{1.4}$$

as an improper integral [and is equal to $G(0)$].

Under the given hypothesis, the Laplace integral (1.3) will converge everywhere on the imaginary axis. For the conclusion (1.4), it is actually sufficient that $G(z)$ have a continuous extension to the closed half-plane $\text{Re } z \geqslant 0$ which is smooth at $z = 0$: see § 2.

At first glance, the above theorem looks quite different from the

Ikehara-Wiener theorem [14]: Let $f(x)$ be nonnegative and nondecreasing on $[1, \infty)$ and such that the Mellin transform

$$g_0(s) = \int_1^\infty x^{-s}df(x) = -f(1) + s\int_1^\infty f(x)x^{-s-1}dx$$

exists for $\text{Re } s > 1$. Suppose that for some constant c, the function

$$g_0(s) - \frac{c}{s-1}$$

has a continuous extension to the closed half-plane $\text{Re } s \geqslant 1$. Then

$$f(x)/x \to c \quad \text{as} \quad x \to \infty.$$

This is an extremely useful theorem, but what could we do with the auxiliary theorem in the same direction? We will show that the latter has a corollary which is just as good for the application that we want to make.

1.3. A Poor Man's Ikehara-Wiener Theorem

We will establish the following

Corollary to the auxiliary theorem. *Let $f(x)$ be nonnegative, nondecreasing and $O(x)$ on $[1, \infty)$, so that its Mellin transform*

$$g(s) = s\int_1^\infty f(x)x^{-s-1}dx \tag{1.5}$$

is well-defined and analytic throughout the half-plane $\text{Re } s > 1$. *Suppose that for some constant c, the function*

$$g(s) - \frac{c}{s-1} \tag{1.6}$$

can be continued analytically to a neighborhood of every point on the line $\text{Re } s = 1$. *Then*

$$f(x)/x \to c \quad \textit{as} \quad x \to \infty. \tag{1.7}$$

Derivation from the auxiliary theorem. Let $f(x)$ and $g(s)$ satisfy the hypotheses of the corollary. We set $x = e^t$ and define

$$e^{-t}f(e^t) - c = F(t),$$

so that $F(t)$ is bounded on $(0, \infty)$. Its Laplace transform will be

$$G(z) = \int_0^\infty \{e^{-t}f(e^t) - c\}\, e^{-zt}dt$$

$$= \int_1^\infty f(x)x^{-z-2}dx - \frac{c}{z} = \frac{1}{z+1}\left\{g(z+1) - \frac{c}{z} - c\right\}.$$

Thus by the hypothesis of the corollary, $G(z)$ can be continued analytically to a neighborhood of every point on the imaginary axis. We may now apply the auxiliary theorem from § 1.2.

What does its conclusion tell us? Setting $t = \log x$ we find that the improper integrals

$$\int_0^\infty \{e^{-t}f(e^t) - c\}\, dt = \int_1^\infty \frac{f(x) - cx}{x^2}\, dx \tag{1.8}$$

exist. Using the fact that $f(x)$ is an increasing function, one readily derives that $f(x) \sim cx$ in the sense of (1.7).

Indeed, suppose for a moment that $\limsup f(x)/x > c$ $(\geqslant 0)$. Then there would be a positive constant δ such that for certain arbitrarily large numbers y

$$f(y) > (c + 2\delta)y.$$

It would follow that

$$f(x) > (c + 2\delta)y > (c + \delta)x \quad \text{for} \quad y < x < \rho y$$

where $\rho = (c + 2\delta)/(c + \delta)$. But then

$$\int_y^{\rho y} \frac{f(x) - cx}{x^2}\, dx > \int_y^{\rho y} \frac{\delta}{x}\, dx = \delta \log \rho$$

for those same numbers y, contradicting the existence of (1.8).

One similarly disposes of the contingency $\liminf f(x)/x < c$ (in this case c would have to be positive and one would consider intervals $\theta y < x < y$ with $\theta < 1$ where $f(x) < (c - \delta)x$). Thus $f(x)/x \to c$.

1.4. Corollary ⇒ Prime Number Theorem

This step is routine to number theorists. One takes $f(x) = \psi(x)$, where $\psi(x)$ is that well-known function from prime number theory,

$$\psi(x) = \sum_{p^m \leqslant x} \log p \tag{1.9}$$

(the summation is over all prime powers not exceeding x). It is a simple fact (first noticed by Chebyshev) that $\pi(x) = 0(x/\log x)$ or equivalently, $\psi(x) = 0(x)$ (cf. § B.4 in the box for more details). Thus $f(x)$ is as the corollary wants it.

What about its Mellin transform $g(s)$? A standard calculation based on the Euler product in (1.2) shows that

$$g(s) = -\frac{\zeta'(s)}{\zeta(s)}, \quad \operatorname{Re} s > 1$$

(cf. § B.3). Since $\zeta(s)$ behaves like $1/(s-1)$ around $s = 1$, the same is true for $g(s)$. The analyticity of $\zeta(s)$ at the points of the line $\operatorname{Re} s = 1$ (different from $s = 1$) and its non-vanishing there imply that $g(s)$ can be continued analytically to a neighborhood of every one of those points (cf. §§ B.1, B.2). Thus

$$g(s) - \frac{1}{s-1}$$

has an analytic continuation to a neighborhood of the closed half-plane $\operatorname{Re} s \geqslant 1$.

The conclusion of the corollary now tells us that

$$\psi(x)/x \to 1 \quad \text{as} \quad x \to \infty,$$

and this is equivalent to the prime number theorem (1.1) (cf. § B.4).

2. Newman's Beautiful Method

2.1. Proof of the Auxiliary Tauberian Theorem

Let $F(t)$ be bounded on $(0, \infty)$ and such that its Laplace transform $G(z)$ can be continued to a function (still called $G(z)$) which is analytic in a neighborhood of the closed half-plane $\operatorname{Re} z \geqslant 0$. We may and will assume that

$$|F(t)| \leqslant 1, \quad t > 0.$$

For $0 < \lambda < \infty$ we write

$$G_\lambda(z) = \int_0^\lambda F(t)e^{-zt}dt. \tag{2.1}$$

Observe that $G_\lambda(z)$ is analytic for all z. We will show that

$$G_\lambda(0) = \int_0^\lambda F(t)dt \to G(0) \quad \text{as} \quad \lambda \to \infty.$$

Some details left out in 1.4

We begin with the necessary facts about the zeta function.

B.1. Analytic continuation of $\zeta(s)$. Simple transformations show that for Re $s > 2$

$$\zeta(s) = \sum_1^\infty \frac{n}{n^s} - \sum_1^\infty \frac{n-1}{n^s} = \sum_1^\infty \frac{n}{n^s} - \sum_1^\infty \frac{n}{(n+1)^s} = \sum_1^\infty n\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} = \sum_1^\infty ns \int_n^{n+1} x^{-s-1}dx = s\sum_1^\infty \int_n^{n+1} [x]x^{-s-1}dx =$$

$$= s\int_1^\infty [x]x^{-s-1}dx, \tag{B.1}$$

where $[x]$ denotes the largest integer $\leqslant x$. Since first and final member are analytic for Re $s > 1$, the integral formula holds throughout that half-plane.

It is reasonable to compare the integral with

$$s\int_1^\infty x \cdot x^{-s-1}dx = \frac{s}{s-1} = 1 + \frac{1}{s-1}. \tag{B.2}$$

Combination of (B.1) and (B.2) gives

$$\zeta(s) - \frac{1}{s-1} = 1 + s\int_1^\infty ([x] - x)x^{-s-1}dx. \tag{B.3}$$

The new integral converges and represents an analytic function throughout the half-plane Re $s > 0$. Thus (B.3) provides an analytic continuation of the left-hand side to that half-plane.

B.2. Non-vanishing of $\zeta(s)$ *for* Re $s \geqslant 1$. The Euler product in (1.2) shows that $\zeta(s) \neq 0$ for Re $s > 1$. For Re $s = 1$ we will use Mertens's clever proof of 1898. The key fact is the inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geqslant 0, \quad \theta \text{ real}. \tag{B.4}$$

Suppose that $\zeta(1 + ib)$ would be equal to 0, where b is real and $\neq 0$. Then the auxiliary analytic function

$$\varphi(s) = \zeta^3(s)\zeta^4(s + ib)\zeta(s + 2ib)$$

would have a zero for $s = 1$: the pole of $\zeta^3(s)$ could not cancel the zero of $\zeta^4(s + ib)$. It would follow that

$$\log|\varphi(s)| \to -\infty \quad \text{as} \quad s \to 1. \tag{B.5}$$

We now take *s real* and > 1. By the Euler product,

$$\log|\zeta(s + it)| = -\text{Re}\sum_p \log(1 - p^{-s-it}) = \text{Re}\sum_p \{p^{-s-it} + \tfrac{1}{2}(p^2)^{-s-it} + \tfrac{1}{3}(p^3)^{-s-it} + \ldots\} = \text{Re}\sum_1^\infty a_n n^{-s-it} \quad \text{with} \quad a_n \geqslant 0.$$

Thus

$$\log|\varphi(s)| = \text{Re}\sum_1^\infty a_n n^{-s}(3 + 4n^{-ib} + n^{-2ib}) = \sum_1^\infty a_n n^{-s}\{3 + 4\cos(b\log n) + \cos(2b\log n)\} \geqslant 0$$

because of (B.4), contradicting (B.5).

B.3. Representations for $\zeta'(s)/\zeta(s)$. Logarithmic differentiation of the Euler product in (1.2) gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{p^{-s}}{1 - p^{-s}}\log p = \sum_p (p^{-s} + p^{-2s} + \ldots)\log p = \sum_1^\infty \Lambda(n)n^{-s}, \tag{B.6}$$

where $\Lambda(n)$ is the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m, \\ 0 \quad \text{ if } n \text{ is not a prime power.} \end{cases}$$

The corresponding partial sum function is equal to $\psi(x)$:

$$\psi(x) = \sum_{p^m \leqslant x} \log p = \sum_{n \leqslant x} \Lambda(n). \tag{B.7}$$

Proceeding as in (B.1), the series (B.6) leads to the integral representation

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi(x) x^{-s-1} dx, \qquad \text{Re } s > 1. \tag{B.8}$$

The integral converges and is analytic for Re $s > 1$ since by (B.7), $\psi(x) \leqslant x \log x$.

B.4. Relation between $\psi(x)$ *and* $\pi(x)$. By (B.7), $\psi(x)$ counts $\log p$ (for fixed p) as many times as there are powers $p^m \leqslant x$, hence

$$\psi(x) = \sum_{p \leqslant x} \left[\frac{\log x}{\log p}\right] \log p \leqslant \log x \sum_{p \leqslant x} 1 = \pi(x) \log x. \tag{B.9}$$

On the other hand, when $1 < y < x$,

$$\pi(x) = \pi(y) + \sum_{y<p \leqslant x} 1 \leqslant \pi(y) + \sum_{y<p\leqslant x} \frac{\log p}{\log y} < y + \frac{\psi(x)}{\log y}.$$

Taking $y = x/\log^2 x$ one thus finds that

$$\pi(x) \frac{\log x}{x} < \frac{1}{\log x} + \frac{\psi(x)}{x} \frac{\log x}{\log x - 2 \log \log x}. \tag{B.10}$$

Combination of (B.9) and (B.10) shows that

$$\lim \pi(x) \frac{\log x}{x} = 1 \quad \text{ if and only if } \quad \lim \frac{\psi(x)}{x} = 1. \tag{B.11}$$

We finally indicate a standard proof of the estimate

$$\psi(x) = 0(x). \tag{B.12}$$

For positive integral n, the binomial coefficient $\binom{2n}{n}$ must be divisible by all primes p on $(n, 2n]$. Hence

$$\prod_{n<p\leqslant 2n} p \leqslant \binom{2n}{n} < 2^{2n},$$

so that

$$\sum_{2^{k-1}<p\leqslant 2^k} \log p \leqslant 2^k \log 2.$$

It follows that

$$\sum_{p \leqslant 2^k} \log p \leqslant (2^k + 2^{k-1} + \ldots + 1) \log 2 < 2^{k+1} \log 2$$

and hence there is a constant C such that

$$\sum_{p \leqslant x} \log p \leqslant Cx.$$

Since the prime powers higher than the first contribute at most a term $0(x^{1/2+\epsilon})$ to $\psi(x)$, inequality (B.12) follows.

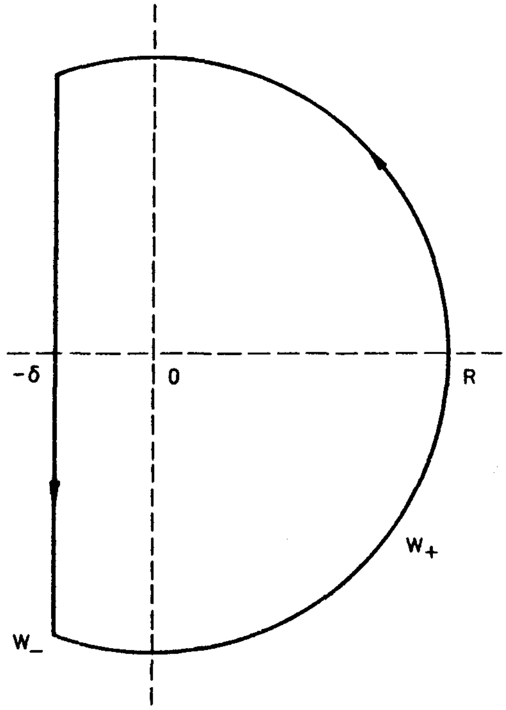

Figure 2

First idea. We try to estimate $G(0) - G_\lambda(0)$ with the aid of Cauchy's formula. Thus we look for a suitable path of integration W around 0. The simplest choice would be a circle, but we can not go too far into the left half-plane because we know nothing about $G(z)$ there. So for given $R > 0$, the positively oriented path W will consist of an arc of the circle $|z| = R$ and a segment of the vertical line $\mathrm{Re}\, z = -\delta$ (Fig. 2). Here the number $\delta = \delta(R) > 0$ is chosen so small that $G(z)$ is analytic on and inside W. We denote the part of W in $\mathrm{Re}\, z > 0$ by W_+, the part in $\mathrm{Re}\, z < 0$ by W_-. By Cauchy's formula,

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_W \{G(z) - G_\lambda(z)\} \frac{1}{z} dz. \tag{2.2}$$

We have the following simple estimates:

for $x = \mathrm{Re}\, z > 0$,

$$|G(z) - G_\lambda(z)| = |\int_\lambda^\infty F(t) e^{-zt} dt| \leqslant \int_\lambda^\infty e^{-xt} dt = \frac{1}{x} e^{-\lambda x}; \tag{2.3}$$

for $x = \mathrm{Re}\, z < 0$,

$$|G_\lambda(z)| = |\int_0^\lambda F(t) e^{-zt} dt| \leqslant \int_0^\lambda e^{-xt} dt < \frac{1}{|x|} e^{-\lambda x}. \tag{2.4}$$

Second idea. Observe the similarity between the bounds obtained in (2.3) and (2.4)! It will be advantageous to multiply $G(z)$ and $G_\lambda(z)$ in (2.2) by $e^{\lambda z}$. This will not affect the left-hand side, but in estimating on W, the exponential $e^{-\lambda x}$ (large on W_-) will disappear from (2.3) and (2.4).

Third idea. Could we also get rid of the troublesome factor $1/|x|$ in the estimates which is bad near the imaginary axis? Yes, this can be done by adding the term z/R^2 to $1/z$ in (2.2), again without affecting the left-hand side. (For the experts: this trick is used also in Carleman's formula for the zeros of an analytic function in a half-plane, cf. [12].) The resulting modified formula is

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_W \{G(z) - G_\lambda(z)\}\, e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2}\right) dz. \tag{2.5}$$

Let us start harvesting. On the circle $|z| = R$,

$$\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2}. \tag{2.6}$$

Thus on W_+ the integrand $I(z)$ in (2.5) may be estimated as follows (see (2.3)):

$$|I(z)| \leqslant \frac{1}{x} e^{-\lambda x} e^{\lambda x} \frac{2x}{R^2} = \frac{2}{R^2}.$$

The corresponding integral is harmless:

$$|\frac{1}{2\pi i} \int_{W_+} I(z) dz| \leqslant \frac{1}{2\pi} \frac{2}{R^2} \pi R = \frac{1}{\mathrm{R}}! \tag{2.7}$$

Fourth idea. We now turn to the part of (2.5) due to W_-. Since $G_\lambda(z)$ is analytic for all z, we may replace the integral over W_- involving $G_\lambda(z)$ by the corresponding integral over the semi-circle

$$W_-^* : \{|z| = R\} \cap \{\mathrm{Re}\, z < 0\}$$

(Fig. 3). Cauchy's theorem and inequality (2.4) readily give

$$|\frac{1}{2\pi i} \int_{W_-} G_\lambda(z) e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2}\right) dz| = |\frac{1}{2\pi i} \int_{W_-^*} \ldots dz| < \frac{1}{\mathrm{R}}. \tag{2.8}$$

We finally tackle the remaining integral

$$\frac{1}{2\pi i} \int_{W_-} G(z) e^{\lambda z} \left(\frac{1}{z} + \frac{z}{R^2}\right) dz. \tag{2.9}$$

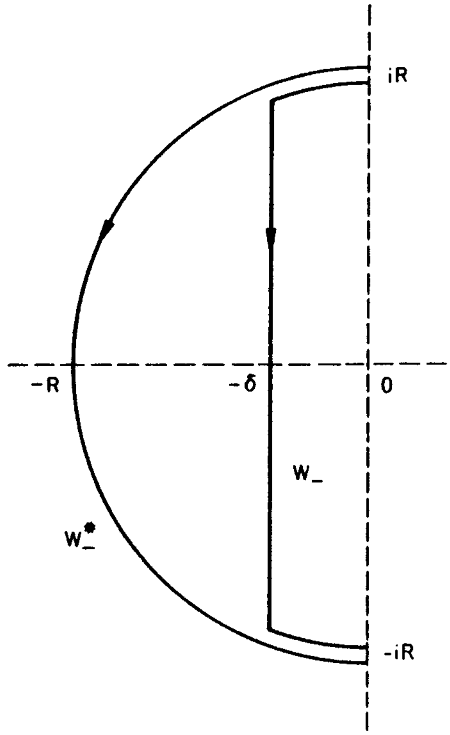

Figure 3

By the analyticity of $G(z)$ on W_- there will be a constant $B = B(R, \delta)$ such that

$$|G(z)\left(\frac{1}{z} + \frac{z}{R^2}\right)| \leqslant B \quad \text{on} \quad W_-.$$

It follows that

$$|G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)| \leqslant Be^{\lambda x}.$$

Hence on the part of W_- where $x \leqslant -\delta_1 < 0$, the integrand in (2.9) tends to zero uniformly as $\lambda \to \infty$. On the remaining small part of W_- (we take $\delta_1 < \delta$ small), the integrand is bounded by B. Thus for fixed W, the integral in (2.9) tends to zero as $\lambda \to \infty$.

Conclusion. For given $\epsilon > 0$ one may choose $R = 1/\epsilon$. One next chooses δ so small that $G(z)$ is analytic on and inside W. One finally determines λ_0 so large that (2.9) is bounded by ϵ for all $\lambda > \lambda_0$. Then by (2.5) and (2.7)–(2.9),

$$|G(0) - G_\lambda(0)| < 3\epsilon \quad \text{for} \quad \lambda > \lambda_0.$$

In other words, $G_\lambda(0) \to G(0)$ as $\lambda \to \infty$.

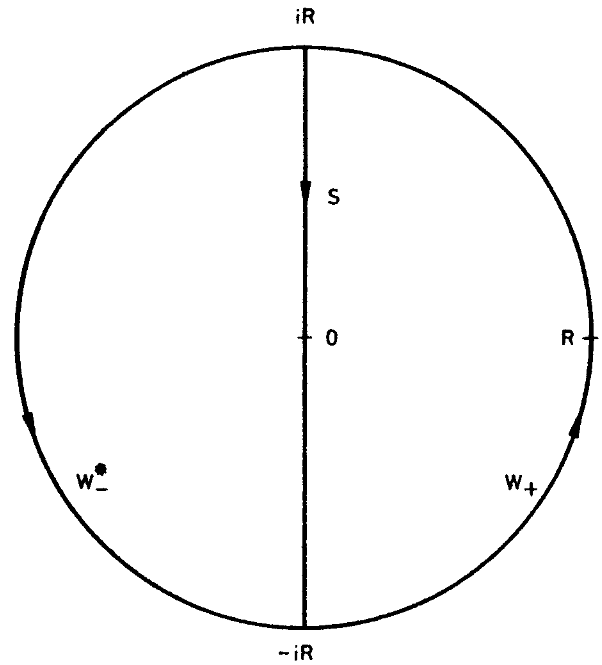

Figure 4

2.2. Relaxing the Conditions on G(z)

In the above proof, it is not really necessary to take $G(z)$ into the left half-plane. By modifying $F(t)$ on some finite interval one may assume that $G(0) = 0$. Then $G(z)/z$ will be analytic for $\operatorname{Re} z \geqslant 0$ and thus

$$G(0) = 0 = \frac{1}{2\pi i} \int_{W_+ \cup S} G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz,$$

where S is the segment $[iR, -iR]$ of the imaginary axis (Fig. 4). For $G_\lambda(0)$ we integrate over the circle $|z| = R$. Subtracting, one obtains

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_{W_+} \{G(z) - G_\lambda(z)\}e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz$$
$$+ \frac{1}{2\pi i} \int_{iR}^{-iR} G(z)e^{\lambda z} \ldots dz - \frac{1}{2\pi i} \int_{W_-^*} G_\lambda(z) \ldots dz. \qquad (2.10)$$

The first and third integral are just as before. To deal with the second integral one may apply integration by parts or the Riemann-Lebesgue lemma.

In order to arrive at (2.10), we have not used any analyticity of $G(z)$ at points of the imaginary axis. It would be more than enough to know that $G(z)/z$ can be extended continuously to $\operatorname{Re} z \geqslant 0$. The Riemann-Lebesgue lemma will then handle the second integral.

Conclusion. In the auxiliary Tauberian theorem, it is sufficient to require that $\{G(z) - G(0)\}/z$ can be extended continuously to the closed half-plane Re $z \geqslant 0$.

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J. Korevaar
Mathematisch Instituut
Universiteit van Amsterdam
Roeterstraat 15
Amsterdam