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Distributional Wiener–Ikehara theorem and twin primes

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ABSTRACT

The Wiener–Ikehara theorem was devised to obtain a simple proof of the prime number theorem. It uses no other information about the zeta function $\zeta(z)$ than that it is zero-free and analytic for $\mathrm{Re}\, z \geqslant 1$, apart from a simple pole at $z = 1$ with residue 1. In the Wiener–Ikehara theorem, the boundary behavior of a Laplace transform in the complex plane plays a crucial role. Subtracting the principal singularity, a first order pole, the classical theorem requires uniform convergence to a boundary function on every finite interval. Here it is shown that local pseudofunction boundary behavior, which allows mild singularities, is necessary and sufficient for the desired asymptotic relation. It follows that the twin-prime conjecture is equivalent to pseudofunction boundary behavior of a certain analytic function.

1. INTRODUCTION

Relaxing the boundary behavior of the Laplace transform in the classical Wiener–Ikehara theorem [10,27]; cf. [15,16], we obtain an extension which includes a result in the opposite direction.

Theorem 1.1. *Let $S(t)$ vanish for $t < 0$, be nondecreasing, continuous from the right and such that the Laplace transform*

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$$f(z) = \mathcal{L}S(z) = \int_0^\infty S(t)e^{-zt}\,dt = \frac{1}{z}\int_{0-}^\infty e^{-zt}\,dS(t) \tag{1.1}$$

$$= \frac{1}{z}\mathcal{L}\,dS(z), \quad z = x + iy,$$

exists for $\operatorname{Re} z = x > 1$. *For some constant* A, *let* $g(z)$ *denote the analytic function given by*

$$g(x+iy) = f(x+iy) - \frac{A}{x+iy-1}, \quad x > 1. \tag{1.2}$$

(i) *Suppose that* $g(x+iy)$ *has a distributional limit* $g(1+iy)$ *as* $x \searrow 1$, *which on every finite interval* $\{-\mu < y < \mu\}$ *coincides with a pseudofunction* $g_\mu(1+iy)$. *Then*

$$e^{-t}S(t) \to A \quad \text{as } t \to \infty. \tag{1.3}$$

(ii) *Conversely, the limit relation* (1.3) *implies that* $g(x+iy)$ *converges distributionally to a pseudofunction* $g(1+iy)$ *as* $x \searrow 1$.

Locally uniform and local L^1 convergence imply distributional convergence. A pseudofunction on $\mathbb{R}$ is the distributional Fourier transform of a bounded function which tends to zero at $\pm\infty$. A continuous or integrable function $g(1+iy)$ on a compact interval $\{-\lambda \leqslant y \leqslant \lambda\}$ can be extended to an integrable function on $\mathbb{R}$, which is a special case of a pseudofunction.

The classical Wiener–Ikehara theorem dealt with $f^*(z) = \mathcal{L}\,dS(z)$ and $g^*(z) = f^*(z) - A/(z-1)$, rather than f and g. The distinction is unimportant until one considers a converse. Wiener and Ikehara postulated that $g^*(x+iy)$ extend analytically, or at least continuously, to the closed half-plane $\{x \geqslant 1\}$; cf. [28, Theorem 16]. Their theorem represented an important breakthrough: in related results, Landau, and Hardy and Littlewood, had to impose growth conditions on $g^*(x+iy)$ as $y \to \pm\infty$. For details, see [18, Section 66], [15,16].

The standard proofs for the Wiener–Ikehara theorem make use of the approximate identity given by the Fejér kernel for $\mathbb{R}$. The distributional case requires higher-order kernels.

2. FROM WIENER–IKEHARA TO PRIME NUMBER THEORY

Background material in number theory can be found in many books; classics are [18] and [7]. To obtain the PRIME NUMBER THEOREM (PNT) from Wiener–Ikehara, one takes $S(t)$ equal to

$$S_1(t) = \psi\big(e^t\big) = \sum_{1 \leqslant n \leqslant e^t} \Lambda(n), \tag{2.1}$$

where $\psi(v) = \sum_{n \leqslant v} \Lambda(n)$ is Chebyshev's function. The symbol $\Lambda(\cdot)$ stands for von Mangoldt's function,

$$\Lambda(n) = \begin{cases} 0 & \text{if } n = 1, \\ \log p & \text{if } n = p^\alpha \text{ with } p \text{ prime and } \alpha \geqslant 1, \\ 0 & \text{if } n \text{ has at least two different prime factors.} \end{cases} \tag{2.2}$$

Always taking $\operatorname{Re} z > 1$, one may use the Euler product for the zeta function,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}},$$

to obtain the generating function

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = \sum_{p \text{ prime}} \frac{p^{-z} \log p}{1 - p^{-z}} = -\frac{d}{dz} \log \zeta(z).$$

Writing $f_1(z)$ for $f(z)$ in the present case, one has

$$f_1(z) = \mathcal{L}S_1(z) = \frac{1}{z} \int_{0-}^{\infty} e^{-zt}\, d\psi\big(e^t\big) = \frac{1}{z} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{z\zeta(z)}. \tag{2.3}$$

The function $f_1(z)$ is analytic for $\{\operatorname{Re} z \geqslant 1\}$, except for a simple pole at the point $z = 1$ with residue 1. Indeed, $\zeta(z)$ is analytic and free of zeros for $\{\operatorname{Re} z \geqslant 1\}$, and has a simple pole at the point $z = 1$ with residue 1. Thus by the (classical) Wiener–Ikehara theorem,

$$S_1(t) \sim e^t \quad \text{as } t \to \infty, \qquad \psi(v) = \sum_{p^\alpha \leqslant v} \log p \sim v \quad \text{as } v \to \infty.$$

In the final summation, the powers p^α with $\alpha \geqslant 2$ may be omitted without affecting the asymptotic relation. Furthermore, $\log p \approx \log v$ for 'most' primes $p \leqslant v$. One thus obtains the PNT:

$$\pi(v) = \sum_{p \leqslant v} 1 \sim \frac{v}{\log v} \quad \text{as } v \to \infty. \tag{2.4}$$

Turning to *twin primes*: there is still no proof that there are infinitely many. However, there is ample numerical support for an even stronger statement: the TWIN-PRIME CONJECTURE (TPC) of Hardy and Littlewood [6, p. 42]; cf. [23–25]. Denoting the number of prime pairs $(p,\ p+2)$ with $p \leqslant v$ by $\pi_2(v)$, the TPC asserts that

$$\pi_2(v) \sim C_2 \frac{v}{\log^2 v} \quad \text{as } v \to \infty, \tag{2.5}$$

or equivalently,

$$\sum_{p,p+2 \text{ prime},\ p\leqslant v} \log p \cdot \log(p+2) \sim C_2 v. \tag{2.6}$$

Here $C_2 > 0$ is the so-called twin-prime constant,

$$C_2 = 2\prod_{p>2}\left\{1 - \frac{1}{(p-1)^2}\right\}. \tag{2.7}$$

A natural generating function associated with the twin-prime problem is the (adjusted) Dirichlet series

$$f_2(z) = \frac{1}{z}\sum_{n=1}^{\infty}\frac{\Lambda(n)\Lambda(n+2)}{n^z} \quad (\operatorname{Re} z > 1). \tag{2.8}$$

The TPC—see (2.6)—is equivalent to the relation

$$\psi_2(v) = \sum_{n\leqslant v}\Lambda(n)\Lambda(n+2) \sim C_2 v \quad \text{as } v \to \infty, \tag{2.9}$$

because the contribution of the prime powers p^α with $\alpha \geqslant 2$ can again be neglected.

In a recent preprint, Arenstorf [2] made a serious effort to derive the TPC from a Wiener–Ikehara theorem. As of this writing, it is not clear whether the classical Wiener–Ikehara theorem, involving continuous boundary values, will be adequate. The distributional form might provide a more promising approach. Indeed, set

$$S_2(t) = \psi_2(e^t), \quad \text{so that} \quad f_2(z) = \mathcal{L}S_2(z), \tag{2.10}$$

and define

$$g_2(z) = f_2(z) - \frac{C_2}{z-1} \quad (\operatorname{Re} z > 1). \tag{2.11}$$

Then Theorem 1.1 implies

Corollary 2.1. *The Twin-Prime Conjecture is equivalent to (local) pseudofunction boundary behavior of the function* $\{y \to g_2(x+iy)\}$ *as* $x \searrow 1$.

Cf. also the final remarks in Section 7.

3. AUXILIARY RESULTS ON DISTRIBUTIONS

We consider locally integrable functions and more general distributions on $\mathbb{R}$, continuous linear functionals on the testing space $C_0^\infty(\mathbb{R})$. The latter consists of the C^∞ functions ϕ of compact support, the testing functions, supplied with an appropriate notion of convergence. Applying distributions G to testing functions ϕ one obtains a bilinear functional, denoted by $\langle G, \phi\rangle$.

When we deal with holomorphic functions $G(z) = G(x + iy)$ on the half-plane $\{x > 0\}$, it is sometimes convenient to denote a boundary distribution by $G(iy)$. By definition, functions or distributions $G(x + iy) = G_x(iy)$ converge to a distribution $G(iy)$ as $x \searrow 0$ if

$$\langle G_x(iy), \phi(y)\rangle \to \langle G(iy), \phi(y)\rangle \tag{3.1}$$

for all testing functions ϕ. We need a standard result on the restriction of distributions to the class $C_0^\infty[-\lambda, \lambda]$ of the testing functions with support in $[-\lambda, \lambda]$. For $m \in \mathbb{N}_0$, let $C_0^m[-\lambda, \lambda]$ denote the Banach space of the C^m functions ϕ with support in $[-\lambda, \lambda]$, provided with the norm

$$\|\phi\| = \sup_y \left|D^m \phi(y)\right|.$$

Proposition 3.1. *Let $x \searrow 0$ through a sequence $\{x_j\}$ and suppose that corresponding distributions $G_x(iy)$ converge to $G(iy)$ in the sense of* (3.1)*. Then for every compact interval $[-\lambda, \lambda]$, there exists an integer $m = m(\lambda) \geqslant 0$ such that the functionals $G_x(iy)$, $G(iy)$ on $C_0^\infty[-\lambda, \lambda]$ can be extended to continuous linear functionals on the space $C_0^m[-\lambda, \lambda]$, and*

$$\langle G_x(iy), \phi(y)\rangle \to \langle G(iy), \phi(y)\rangle \quad \text{as } x = x_j \searrow 0 \tag{3.2}$$

for every function ϕ in $C_0^m[-\lambda, \lambda]$.

Cf. the books [22, Section III.6], [21, Theorem 6.8] and [9, Section 2.1].

Fourier transforms

Our aim is to prove Theorem 1.1, which involves pseudofunction boundary behavior of Laplace transforms $G_x(y) = G(x + iy)$. It is then convenient to use the theory of distributional Fourier transforms. Its natural setting is Schwartz's space of tempered distributions, the continuous linear functionals G on the Schwartz space $\mathcal{S}$. The space $\mathcal{S}$ consists of the functions ϕ in $C_0^\infty(\mathbb{R})$ and their limits under the family of seminorms

$$N_{rs}(\phi) = \sup_x \left|x^r D^s \phi(x)\right|, \qquad r, s = 0, 1, 2, \ldots;$$

the seminorms are used to define convergence in $\mathcal{S}$. The tempered distributions on $\mathbb{R}$ form the dual space $\mathcal{S}'$ of $\mathcal{S}$. It consists of the locally integrable functions of at most polynomial growth and their distributional derivatives of any order. Such functions or distributions $G_x(y)$ converge to a tempered distribution $G(y)$ as $x \searrow 0$ if

$$\int_{\mathbb{R}} G_x(y)\phi(y)\, dy \to \langle G(y), \phi(y)\rangle$$

for every function $\phi \in \mathcal{S}$.

The Fourier transform $\widehat{G}$ of $G \in \mathcal{S}'$ is defined by the relation

$$\langle \widehat{G}, \phi \rangle = \langle G, \hat{\phi} \rangle$$

for all testing functions ϕ. Fourier transformation on $\mathcal{S}'$ is continuous, one to one and onto; cf. [22,21,9]. If H is the Fourier transform of G, then G is equal to $1/(2\pi)$ times the reflected Fourier transform of H.

A tempered distribution G on $\mathbb{R}$ is called a *pseudomeasure* if it is the Fourier transform of a bounded (measurable) function $b(\cdot)$; it is called a *pseudofunction* if it is the Fourier transform of a bounded function which tends to zero at $\pm\infty$. Cf. Katznelson [12, Section 6.4]. By Fourier inversion and the Riemann–Lebesgue theorem, every function in $L^1(\mathbb{R})$ is a pseudofunction. A nontrivial example of a pseudomeasure on $\mathbb{R}$ is the distribution

$$\frac{-i}{y - i0} = \lim_{x \searrow 0} \frac{1}{x + iy} = \lim_{x \searrow 0} \int_0^\infty e^{-xt} e^{-iyt}\, dt. \tag{3.3}$$

It is the Fourier transform of the Heaviside function $1_+(t)$, which equals 1 for $t \geqslant 0$ and 0 for $t < 0$. Other examples are the delta distribution or Dirac measure, and the principal-value distribution, p.v. $(1/y)$. In the case of boundary singularities, and roughly speaking, first order poles correspond to pseudomeasures, slightly milder singularities to pseudofunctions.

Lemma 3.2. *Every pseudomeasure or pseudofunction $G = \hat{b}$ on $\mathbb{R}$ can be represented in the form*

$$G(y) = \lim_{x \searrow 0} G_x(y), \quad G_x(y) = \int_{\mathbb{R}} e^{-x|t|} b(t) e^{-iyt}\, dt \quad (x > 0). \tag{3.4}$$

One actually has

$$\langle G_x(y), \phi(y) \rangle \to \langle \hat{b}(y), \phi(y) \rangle, \quad \forall \phi \in C_0^2(\mathbb{R}). \tag{3.5}$$

Proof. Relation (3.4) follows immediately from the continuity of Fourier transformation: for bounded $b(\cdot)$ and $x \searrow 0$, the product $e^{-x|t|} b(t)$ tends to $b(t)$ in $\mathcal{S}'$.

The more precise relation (3.5) illustrates Proposition 3.1 and will be used later on. It follows from the observation that $\hat{\phi}(t) = \mathcal{O}\{1/(t^2+1)\}$ and an inversion of the order of integration:

$$\begin{aligned} \langle G_x(y), \phi(y) \rangle &= \int_{\mathbb{R}} e^{-x|t|} b(t) \hat{\phi}(t)\, dt \to \int_{\mathbb{R}} b(t) \hat{\phi}(t)\, dt \\ &= \langle \hat{b}(y), \phi(y) \rangle \quad \text{as } x \searrow 0. \qquad \square \end{aligned} \tag{3.6}$$

Formula (3.4) can be used to justify formal inversion of the order of integration in some situations. As an important consequence we have a *Riemann–Lebesgue type lemma* for pseudofunctions G:

Lemma 3.3. *For any pseudofunction G on $\mathbb{R}$ and any testing function ϕ,*

$$\langle G(y), \phi(y)e^{iuy}\rangle \to 0 \quad \textit{as } u \to \pm\infty. \tag{3.7}$$

For this it is sufficient if ϕ is in $C_0^2(\mathbb{R})$.

Proof. By representation (3.4),

$$\begin{aligned}\langle G(y), \phi(y)e^{iuy}\rangle &= \lim_{x\searrow 0}\left\langle \int_{\mathbb{R}} e^{-x|t|}b(t)e^{-iyt}\,dt, \phi(y)e^{iuy}\right\rangle \\ &= \lim_{x\searrow 0}\int_{\mathbb{R}} e^{-x|t|}b(t)\,dt \int_{\mathbb{R}} e^{-i(t-u)y}\phi(y)\,dy \\ &= \int_{\mathbb{R}} b(t)\hat{\phi}(t-u)\,dt.\end{aligned}$$

The final integral tends to zero as $u \to \pm\infty$; by the proof of Lemma 3.2, the result holds for every function ϕ in $C_0^2(\mathbb{R})$. $\square$

There is also a result in the other direction which can be used to recognize local pseudofunction behavior.

Lemma 3.4. *Let G be a tempered distribution such that* (3.7) *holds for any testing function ϕ. Then G is locally equal to a pseudofunction.*

Proof. Let τ be a testing function which is equal to 1 on $[-R, R]$ and define $G_1 = G\tau$. We now use the fact that for a distribution with compact support, the Fourier transform can be represented by a formula, similar to the one for an L^1 function; cf. [9, Theorem 7.1.14]. In particular, the inverse Fourier transform of our G_1 is equal to the function

$$H(t) = \frac{1}{2\pi}\langle G_1(y), \tau(y)e^{ity}\rangle.$$

Then $G_1(y) = \widehat{H}(y)$, and by (3.7),

$$2\pi H(t) = \langle G(y)\tau(y), \tau(y)e^{ity}\rangle = \langle G(y), \tau^2(y)e^{ity}\rangle \to 0 \quad \text{as } t \to \pm\infty.$$

Thus G_1 is a pseudofunction, and we have $G = G_1$ on $(-R, R)$. $\square$

4. A BASIC AUXILIARY RESULT

Wiener [27], and subsequently, Landau [19] and Bochner [3], based the proof of the classical Wiener–Ikehara theorem on a nonnegative *approximate identity* $\{K_\lambda\}$, $0 < \lambda \to \infty$, for which the Fourier transforms $\widehat{K_\lambda}$ have support in $[-\lambda, \lambda]$. They used the 'Fejér kernel for $\mathbb{R}$':

$$K_\lambda(t) = \lambda K(\lambda t) = \frac{1-\cos\lambda t}{\pi\lambda t^2} = \frac{\lambda}{2\pi}\left(\frac{\sin\lambda t/2}{\lambda t/2}\right)^2, \tag{4.1}$$

$$\widehat{K}_\lambda(y) = \int_{\mathbb{R}} K_\lambda(t)e^{-iyt}\,dt = \begin{cases} 1-|y|/\lambda & \text{for } |y|\leqslant\lambda, \\ 0 & \text{for } |y|>\lambda. \end{cases}$$

The functions K_λ, $\lambda\to\infty$, form an approximate identity in $L^1(\mathbb{R})$: for any L^1 function f, one has $K_\lambda * f \to f$, both in L^1 and in the sense of almost everywhere convergence. The functions in an approximate identity converge to the delta distribution.

We need nonnegative approximate identities whose Fourier transforms have a suitable degree of smoothness:

Lemma 4.1. *For* EVEN $q\in\mathbb{N}$ *and* $\lambda>0$*, let*

$$K_q(t) = c_q\left(\frac{\sin t/q}{t/q}\right)^q, \quad K_{q,\lambda}(t) = \lambda K_q(\lambda t), \tag{4.2}$$

where c_q is chosen such that $\int_{\mathbb{R}} K_q = 1$. Then for $0<\lambda\to\infty$, the functions $K_{q,\lambda}(t)$ form a nonnegative approximate identity. The Fourier transform $\widehat{K}_{q,\lambda}$ has support in $[-\lambda,\lambda]$ and is of class C^{q-2} on $\mathbb{R}$.

Proof. That the functions $K_{q,\lambda}$ form an approximate identity follows from the fact that they have integral 1, and for $\lambda\to\infty$ tend to zero uniformly outside any neighborhood of the point 0. In the case $q=4$ one obtains the 'Jackson kernel for $\mathbb{R}$'. For large q one has $c_q\approx c/\sqrt{q}$ with $c>0$. An exact formula for c_q goes back to Laplace; cf. [8], [26, p. 123] and [13].

For the second part one may change the scale and consider the qth power $M^q(t)$ of the simple kernel $M(t)=(\sin t)/(\pi t)$. The latter has Fourier transform $\widehat{M}(y)$ equal to 1 for $|y|<1$ and equal to 0 for $|y|>1$. Thus $\widehat{M^q}$ is the qth convolution power of $\widehat{M}$; by induction, it has support $[-q,q]$ and is of class C^{q-2}. □

Proposition 4.2. *Let $\sigma(t)$ vanish for $t<0$, be nonnegative for $t\geqslant 0$ and such that the Laplace transform*

$$F(z) = \mathcal{L}\sigma(z) = \int_0^\infty \sigma(t)e^{-zt}\,dt, \quad z=x+iy, \tag{4.3}$$

exists for $x>0$. Suppose that for $x\searrow 0$, the analytic function

$$G(x+iy) = F(x+iy) - A/(x+iy), \quad x>0, \tag{4.4}$$

has a distributional limit $G(iy)$, which on the finite interval $\{-\mu<y<\mu\}$ coincides with a pseudofunction $G_\mu(iy)$. Then for $0<\lambda<\mu$ and sufficiently large even $q=q(\lambda)$, the integral

$$(4.5)\qquad \int_{\mathbb{R}} K_{q,\lambda}(u-t)\sigma(t)\,dt = \int_{-\infty}^{\lambda u} \sigma(u-v/\lambda)K_q(v)\,dv$$

exists and tends to $A\int_{\mathbb{R}} K_q(v)\,dv = A$ *as* $u \to \infty$.

Proof. For $x > 0$, the convolution of the L^1 function $\sigma(t)e^{-xt}$ and $K_{q,\lambda}(t)$ has Fourier transform $F(x+iy)\widehat{K}_{q,\lambda}(y)$, so that by Fourier inversion

$$(4.6)\qquad \int_{\mathbb{R}} \sigma(t)e^{-xt}K_{q,\lambda}(u-t)\,dt = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} F(x+iy)\widehat{K}_{q,\lambda}(y)e^{iuy}\,dy.$$

The special function $F_1(x+iy) = 1/(x+iy)$ is the Laplace transform of the Heaviside function 1_+. Hence by (4.6) applied to $\sigma = 1_+$ and $F = F_1$,

$$\int_0^{\infty} e^{-xt}K_{q,\lambda}(u-t)\,dt = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} F_1(x+iy)\widehat{K}_{q,\lambda}(y)e^{iuy}\,dy.$$

One now subtracts A times this identity from (4.6) and then replaces $F - AF_1$ by G. The result may be written as

$$(4.7)\qquad \int_0^{\infty} \sigma(t)e^{-xt}K_{q,\lambda}(u-t)\,dt = A\int_0^{\infty} e^{-xt}K_{q,\lambda}(u-t)\,dt + \frac{1}{2\pi}\int_{-\lambda}^{\lambda} G(x+iy)\widehat{K}_{q,\lambda}(y)e^{iuy}\,dy.$$

Observe that $K_{q,\lambda}$ is in L^1 and that $\widehat{K}_{q,\lambda}$ is of class $C_0^{q-2}[-\lambda,\lambda]$. By the hypothesis $G(x+iy)$ has a distributional limit $G(iy)$ as $x \searrow 0$. We now fix an even $q \geqslant m+2$, where m is a number provided by Proposition 3.1 for a sequence $x = x_j \searrow 0$. We then use $K_{q,\lambda}(y)e^{iuy}$ as testing function $\phi(y)$ in (3.2). Then for $x = x_j \searrow 0$, the right-hand side of (4.7) has the finite limit

$$(4.8)\qquad A\int_0^{\infty} K_{q,\lambda}(u-t)\,dt + \frac{1}{2\pi}\big\langle G(iy), \widehat{K}_{q,\lambda}(y)e^{iuy}\big\rangle.$$

The left-hand side of (4.7) has the same limit. Since the product $\sigma(t)K_{q,\lambda}(u-t)$ is nonnegative, application of the monotone convergence theorem will show that this product is integrable over $(0,\infty)$. Letting $x = x_j \searrow 0$ in (4.7), one obtains the basic relation

$$(4.9)\qquad \int_0^{\infty} \sigma(t)K_{q,\lambda}(u-t)\,dt = A\int_0^{\infty} K_{q,\lambda}(u-t)\,dt + \frac{1}{2\pi}\big\langle G(iy), K_{q,\lambda}(y)e^{iuy}\big\rangle.$$

We finally use the fact that on $(-\mu, \mu)$, the distribution $G(iy)$ is equal to a pseudofunction $G_\mu(iy)$. Thus by the 'Riemann–Lebesgue' Lemma 3.3, with $G_\mu(iy)$ instead of $G(y)$, the last term in (4.9) tends to zero as $u \to \infty$. Substituting $u - t = v/\lambda$ and replacing $K_{q,\lambda}(v/\lambda)$ by $\lambda K_q(v)$, one concludes that

$$\lim_{u\to\infty} \int_{-\infty}^{\lambda u} \sigma(u - v/\lambda) K_q(v)\, dv = A \lim_{u\to\infty} \int_{-\infty}^{\lambda u} K_q(v)\, dv = A. \qquad \square$$

5. PROOF OF THEOREM 1.1, FIRST PART

We set $e^{-t} S(t) = \sigma(t)$. Then for $\operatorname{Re} z = x > 0$,

$$(5.1) \qquad F(z) \stackrel{\text{def}}{=} \mathcal{L}\sigma(z) = \int_0^\infty S(t) e^{-(z+1)t}\, dt = f(z+1),$$

$$G(z) \stackrel{\text{def}}{=} F(z) - \frac{A}{z} = f(z+1) - \frac{A}{z} = g(z+1);$$

cf. (1.1), (1.2). By the hypotheses of Theorem 1.1, part (i), the functions σ, F and G will satisfy the conditions of Proposition 4.2 for any $\mu > 0$ and $0 < \lambda < \mu$. Thus conclusion (4.5) can be applied to the present situation; for given μ and λ, we have to use a sufficiently large (even) number $q = q(\lambda)$. Because $\{K_{q,\lambda}\},\ \lambda \to \infty$, is an approximate identity, one expects that the first member of (4.5) will be close to $\sigma(u)$ when λ is large, provided $q(\lambda)$ can be kept under control! Let us investigate.

By the monotonicity of S, one has $\sigma(w') \geqslant \sigma(w) e^{w-w'}$ if $w' \geqslant w$. For any number $r > 0$, (4.5) thus shows that

$$A = \lim_{u\to\infty} \int_{-\infty}^{\lambda u} \sigma(u - v/\lambda) K_q(v)\, dv \geqslant \limsup_{u\to\infty} \int_{-r}^{r} \sigma(u - v/\lambda) K_q(v)\, dv$$

$$\geqslant \limsup_{u\to\infty} \sigma(u - r/\lambda) e^{-2r/\lambda} \int_{-r}^{r} K_q(v)\, dv.$$

As a result,

$$(5.2) \qquad \limsup_{t\to\infty} \sigma(t) \leqslant \frac{e^{2r/\lambda}}{\int_{-r}^{r} K_q(v)\, dv} A.$$

Conclusion for our fixed μ, λ, q and r: the function σ *is bounded*.

Referring to (5.1), it follows that the boundary distributions $F(iy) = \hat{\sigma}(y)$ and $G(iy)$ are pseudomeasures; A/z is the Laplace transform of the constant function A. Lemma 3.2 now shows that the limit relation

$$(5.3) \qquad \langle G(x+iy), \phi(y) \rangle \to \langle G(iy), \phi(y) \rangle \quad \text{as } x \searrow 0$$

holds for any function ϕ of class $C_0^2(\mathbb{R})$. We combine this information with the fact that $G(iy)$ is equal to a pseudofunction $G_\mu(iy)$ on any finite interval $(-\mu, \mu)$. As a result, the proof of Proposition 4.2 can be applied with $q = 4$ for any compact interval $[-\lambda, \lambda]$; the function $\widehat{K}_{4,\lambda}$ is of class $C_0^2[-\lambda, \lambda]$.

The conclusion is that we can use (5.2) with $q = 4$ for every pair of positive numbers λ and r. Taking $r = \sqrt{\lambda}$ one obtains an upper bound for $\limsup \sigma(t)$ of the form $C(\lambda)A$, where $C(\lambda)$ is close to 1 for large λ. Under the hypotheses of Theorem 1.1 we may indeed let λ go to ∞ to conclude that

$$\limsup_{t\to\infty} \sigma(t) \leqslant A. \tag{5.4}$$

Denote $\sup \sigma(t)$ by M. Taking $R > 0$ and observing that $K_4(v) \leqslant C/v^4$, one obtains an estimate from below:

$$\begin{aligned}
\liminf_{u\to\infty} \sigma(u + R/\lambda)e^{2R/\lambda} \int_{-R}^{R} K_4(v)\,dv &\geqslant \liminf_{u\to\infty} \int_{-R}^{R} \sigma(u - v/\lambda) K_4(v)\,dv \\
&\geqslant \lim_{u\to\infty} \int_{\mathbb{R}} \cdots - \limsup_{u\to\infty} \int_{-\infty}^{-R} \cdots - \limsup_{u\to\infty} \int_{R}^{\infty} \cdots \\
&\geqslant A - 2MC \int_R^\infty \left(1/v^4\right) dv = A - (2/3)MC/R^3.
\end{aligned}$$

This gives a lower bound for $\liminf_{t\to\infty} \sigma(t)$ which for large λ and related large R is as close to A as one wishes. Conclusion:

$$\liminf_{t\to\infty} \sigma(t) \geqslant A.$$

Since $\sigma(t) = S(t)e^{-t}$, this completes the proof of Theorem 1.1, part (i).

6. THE SECOND PART OF THEOREM 1.1

It is convenient to continue with the notation introduced in Section 5. In addition, we set

$$\sigma^*(t) \stackrel{\text{def}}{=} \sigma(t) - A \cdot 1_+(t) = e^{-t} S(t) - A \cdot 1_+(t). \tag{6.1}$$

Thus by the hypotheses of part (ii) in Theorem 1.1,

$$\sigma^*(t) = 0 \quad \text{for } t < 0, \qquad \sigma^*(t) \to 0 \quad \text{as } t \to \infty;$$

in particular $\sigma^*(t)$ is bounded. In view of these properties, the functions

$$e^{-xt}\sigma^*(t) \text{ converge to } \sigma^*(t)$$

boundedly on $\mathbb{R}$ as $x \searrow 0$, and hence, distributionally. It follows that the Fourier transform of $e^{-xt}\sigma^*(t)$, namely,

$$\text{(6.2)} \qquad \int_{\mathbb{R}} e^{-xt}\sigma^*(t)e^{-iyt}\,dt = \int_0^\infty \{\sigma(t) - A\}e^{-(x+iy)t}\,dt = F(x+iy) - \frac{A}{x+iy}$$
$$= G(x+iy) = g(1+x+iy),$$

converges distributionally to the Fourier transform $\hat{\sigma}^*(y)$ as $x \searrow 0$. We may of course denote the distributional limit $\hat{\sigma}^*(y)$ of $g(1+x+iy)$ by $g(1+iy)$. Since

$$\sigma^*(t) \to 0 \quad \text{as } t \to \pm\infty,$$

the distribution $g(1+iy)$ is a pseudofunction.

7. FINAL REMARKS

The prime number theorem. Newman [20] has given a nice proof for the PNT by complex analysis. He used a Tauberian theorem of Ingham [11] involving Dirichlet series, for which he found a clever proof by contour integration. The method is easily adapted to the case of Laplace transforms; cf. [1,14–16,29]. Newman's contour integration method can also be modified to obtain a proof of the classical Wiener–Ikehara theorem by complex analysis; see [17].

Twin primes. Developing his sieve method, Brun [4] showed that $\pi_2(v) = \mathcal{O}(v/\log^2 v)$, so that $\psi_2(v)$ in (2.9) is $\mathcal{O}(v)$; cf. [5, Theorem 3.11] . Thus the function

$$\sigma_2(t) = e^{-t}\psi_2(e^t) = e^{-t}S_2(t)$$

is bounded. It follows that the functions

$$e^{-xt}\sigma_2(t) \text{ converge to } \sigma_2(t)$$

boundedly on $\mathbb{R}$ as $x \searrow 0$, which implies distributional convergence of the Fourier transforms. By the definition of $f_2(\cdot)$ this means that $f_2(x+iy)$ converges distributionally to the Fourier transform $\hat{\sigma}_2(y)$ as $x \searrow 1$; cf. (2.8), (2.10). By the boundedness of $\sigma_2(t)$, the latter transform is a pseudomeasure. Hence $f_2(x+iy)$ converges to a pseudomeasure $f_2(1+iy)$ as $x \searrow 1$. To prove the twin-prime conjecture, one has to show that

$$g_2(1+iy) = f_2(1+iy) - \lim_{x\searrow 1} \frac{C_2}{x-1+iy} = \hat{\sigma}_2(y) + \frac{iC_2}{y-i0}$$

is (locally) equal to a pseudofunction; see Corollary 2.1 and cf. (3.3).

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