Statistical Physics and Condensed Matter Theory 2

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Second take-home exercise set, April 9, 2008

1 Lattice models with 3-state variable

We consider a generalized Ising model, with variables taking three discrete values $S_i \in \{-1, 0, 1\}$. Derive and effective Quantum Field Theory and give a Mean Field analysis. Treat the following two cases

$$H_1 = J_1 \sum_{<ij>} S_i S_j, \qquad H_2 = J_2 \sum_{<ij>} \delta_{S_i, S_j}$$

where $< ij >$ indicates that sites i and j are nearest neighbors on the lattice. Include in the analysis external fields coupling as

$$\delta H = H_1 \sum_i S_i + H_2 \sum_i S_i^2 \ .$$

2 XY model in $2 + \epsilon$ dimensions

Repeating the RG analysis for an XY-model in $d = 2+\epsilon$ dimensions leads to the following flow equations

$$\begin{aligned} \frac{dJ^{-1}}{dl} &= -\epsilon J^{-1} + 4\pi^3 y^2 \\ \frac{dy}{dl} &= (2 - \pi J) y \end{aligned} \tag{1}$$

Questions

(a). Determine the non-trivial fixed point in the (J, y) plane and sketch the flow-diagram.

(b). Determine the eigenvalues of the linearized flow near the fixed point, to lowest order in ϵ, identify y_t and extract values for the exponents α and ν.

(c). Extrapolate to $d = 3$. [An accurate approximation for $d = 3$ gives $\nu \sim .6685$, $\alpha = 0$.]

3 Counting states for $d = 1$ fermions

(a). Derive the following expression for the partition sum of all particle-hole excitations near one of the two Fermi points of $d = 1$ spin-less fermions [as discussed by Schulz]

$$Z_F = \mathrm{Tr} q^{L_0} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{l=1}^{n} (1-q^l)^2}$$

(b). Show that the series expansions of Z_F and of the bosonic expression

$$Z_B = \frac{1}{\prod_{l=1}^{\infty} (1-q^l)}$$

agree to the first 10 powers of q.

(c). optional! Can you find a proof that Z_B and Z_F agree to all orders? (Feel free to consult the literature on this.)