

Below chapters 1 through 5 of the lectures notes ‘Kondo Problem’ (K. Schoutens - UvA/WINS - 1996). Hard copy of the complete version (eight chapters), including figures, references and problems, available on request.

1 Introduction and summary

The course is centered around the study of the so-called Kondo model (or $s-d$ exchange model) for dilute magnetic alloys. The formulation of this Kondo model is particularly simple: we have a free electron gas which is coupled to a single magnetic impurity via a spin-spin interaction term. The hamiltonian reads (see chapter 2 for notations)

$$H_K = \sum_{\vec{k}, \alpha} \epsilon_{\vec{k}} n_{\vec{k}, \alpha} + J \sum_{\vec{k}, \vec{k}', \alpha, \alpha'} \vec{S}_{\text{imp}} \cdot (c_{\vec{k}, \alpha}^\dagger \vec{\sigma}_{\alpha \alpha'} c_{\vec{k}', \alpha'}) . \quad (1.1)$$

The physical system that is modeled by this hamiltonian is a metal containing a low concentration of magnetic impurities, *i.e.* atoms or ions with a permanent magnetic moment. One may think of gold (*Au*) with a low concentration (< 1 at. %) magnetic impurities in the form of iron (*Fe*). In 1964 Kondo showed that the presence of magnetic impurities explains a remarkable property, namely the occurrence of a resistance minimum at low temperature, of good metals such as gold.

Already in 1934, de Haas, de Boer and van den Berg (who worked in Leiden) found that the resistance of metals such as gold passes through a minimum when the temperature is lowered (see the figure on the cover, which is taken from their original paper). The minimum occurs at a characteristic temperature which, in the original measurements by de Haas et al. on gold, was in the order of a few Kelvin.

In the usual picture, electrical resistance in metals is caused by the scattering of conduction electrons on phonons, which are vibrations of the ionic lattice. When temperature decreases, such phonons will be less and less excited and we expect that the electrical resistance will decrease monotonically. A theoretical result for this is (see, *e.g.* , Ch. 26 of [AM] or Ch. 9 of [Ki])

$$R(T) = a T^5 \quad (\text{phonon} - \text{mechanism at low } T) . \quad (1.2)$$

So we are facing the challenge of understanding why it is that the presence of magnetic impurities, which we describe with the hamiltonian (1.1), leads to a resistance which, for low enough temperatures, grows with decreasing temperatures!

We shall quickly discover that the analysis of the Kondo model is not so easy. The reason for this is, in brief, that we have to deal with a many-body problem which can

not be reduced to a one-body problem. (We will clarify this somewhat cryptical remark in later chapters!) Before we start our analysis we give a brief historical overview of the steps that have been taken to master this intriguing problem.

The obvious first step in the analysis of the Kondo model is to try to use perturbation theory in the coupling J . In such an approach, we start from a free electron gas and we add the interaction term as a small perturbation. In 1964 Kondo found that the corrections to the free electron gas that are obtained in this manner are logarithmically divergent for temperatures $T \rightarrow 0$. Kondo's computations (which we shall discuss in chapter 4) explain the occurrence of a resistance minimum and are thus of great importance. On the other hand, divergences at low temperatures can not be physical and we have to conclude that perturbation theory will break down at low enough temperatures. We shall thus need an entirely different approach to learn about the behavior of the Kondo model at very low temperatures.

The analysis of the low-temperature regime of the Kondo model, where perturbation theory fails, is known as 'the Kondo problem'. It is a highly non-trivial problem, and many of the approaches that have been proposed have utterly failed. In the words of Tsvelick en Wiegmann ([TW], pag. 464)

‘Readers familiar with the history of studies in the Kondo and Anderson models could not miss noticing what a frightening debris one is trapped in, while attacking by inadequate methods the phenomena which were eventually described by such simple mathematics.’

This should put you on your guard! In this course, we shall avoid the historical pitfalls and give a survey of those methods that are appropriate for the analysis of the Kondo problem! Indeed we shall find that the essential physics can be understood in relatively simple terms.

Among the first to understand the essence of the Kondo problem were Anderson and Yuval (1969), who successfully applied scale transformations and placed the problem in the context of the Renormalization Group (RG).

In 1974 Wilson performed numerical RG calculations which allowed him to study the behavior of the Kondo model in the entire range from high temperatures down to very low temperatures. Shortly thereafter, Nozières explained how the low temperature behavior can be derived from a simple effective theory.

After these developments the Kondo problem was *solved exactly* in 1980-1981. This was done by using the so-called Bethe Ansatz, which had previously been employed for the solution of the one-dimensional spin- $\frac{1}{2}$ Heisenberg anti-ferromagnet and other one-dimensional quantum systems such as the Hubbard model. We shall turn to the exact solution in chapter 8.

In the early nineties, the analysis of the Kondo problem was further streamlined by applying methods from Conformal Field Theory to the low temperature regime. This has been particularly illuminating in the case of the ‘multi-channel Kondo problem’ (see below for some comments on the multi-channel case). Conformal Field Theory was used to confirm the physical picture developed by Nozières and Blandin and to provide more detailed information on, for example, correlation functions. We shall come to this approach in our chapter 7.

Before closing this chapter we would like to spend a few words on the significance of the work on the Kondo problem for Modern Physics.

In the course of these lectures, we shall see that the theoretical predictions on the Kondo model are in excellent agreement with experimental observations on dilute magnetic alloys. However, for many actual materials the hamiltonian (1.1) is too simple and we have to refine the model in order to be able to make meaningful predictions. In this direction some progress has been made: there are a number of models that are somewhat more realistic than the original single-impurity Kondo model, and whose behavior is quite well understood. We mention here the *Anderson model* and the *two-impurity* version of the Kondo model. The behavior of conduction electrons in the presence of a *high density* of magnetic impurities is much less understood and is certainly not under analytic control. It is known, for example, that in certain cases the impurity spins form a spin-glass phase which is notoriously hard to study. The situation where impurities are located on a regular lattice can be studied using a *periodic Anderson model* or a *Kondo-lattice* model. For these models, exact solutions are not (yet) available, and one largely relies on numerical methods such as Monte Carlo and variational methods.

Of special interest is the work on the so-called multi-channel Kondo model, where the electrons couple to the impurity via a number (k) of independent channels. This situation arises when one tries to give a realistic description of an impurity atom, taking into account the orbital angular momentum of the electrons. In the simplest case this leads to $k = 2l + 1$ channels for an electron of angular momentum $L^2 = \hbar^2 l(l + 1)$. The multi-channel Kondo problem is also of great interest because of applications to Heavy Fermion physics, which enjoys a lot of interest here in the van der Waals-Zeeman laboratory. In particular, there is a concrete proposal for a description of Uranium-based Heavy Fermion materials using the quadrupolar ($k = 2$) Kondo effect.

Last but not least a remark from a theorist’s point of view. The techniques which have been applied to, and developed in the context of, the Kondo problem are of great use in a variety of situations in theoretical physics. Apart from other applications in Condensed Matter Theory, where the Renormalization Group, Conformal Field Theory and the Bethe Ansatz are quite prominent these days, there are interesting

parallels with entirely different areas of theoretical physics. For example, the Renormalization Group analysis leading to a low energy strong-coupling fixed point in the anti-ferromagnetic Kondo model (see chapter 5) is strikingly similar to an analogous analysis for QCD (Quantum Chromo-Dynamics, the theory of quarks, gluons and the strong interaction), where it leads to the phenomenon of quark confinement! The context of the Kondo problem is relatively simple and offers a testing ground where one can practice working with these advanced techniques and learn how to extract the physics from the formulas that one writes. This last aspect is certainly one of the motivations behind this course!

2 Electrons and spins, a brief review

In this chapter we briefly review the behavior of free electrons and free quantum mechanical spins at low temperatures. For the electron gas we introduce the so-called second quantized notation which we shall frequently use in later chapters. In section 2.3, we give an expression for the conductivity of an electron gas in the presence of scatterers. References for the material in this chapter are, for example, [AG], [Ki], [AM], [He].

2.1 Electron gas

At low temperatures, electrons form a strongly degenerate quantum gas with properties that are to a large extent dictated by the Pauli principle. Starting point for the description are

1. The dispersion relation $\epsilon_{\vec{k}}$, which expresses the energy in terms of the wave vector \vec{k} . For free electrons

$$\epsilon_{\vec{k}} = \frac{\hbar^2}{2m} |\vec{k}|^2 . \quad (2.1)$$

For the Kondo problem one typically thinks of electrons in a conduction band, which have a rather different dispersion relation. Our main conclusions will not depend on the details of the band structure, and when doing computations we will work with a simplified ‘band’ (see chapters 4,5).

2. The density of states $\mathcal{D}(\epsilon)$, which is defined such that the number of available states for electrons of energy between ϵ and $\epsilon + d\epsilon$ equals $\mathcal{D}(\epsilon)d\epsilon$. For free electrons in a volume V in three dimensions the result is

$$\mathcal{D}(\epsilon) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{\epsilon} . \quad (2.2)$$

3. The Fermi-Dirac distribution function

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1} . \quad (2.3)$$

Here k_B is the Boltzmann constant, T the temperature and μ the chemical potential, which at $T = 0$ equals the Fermi energy ϵ_F . This distribution function gives the fraction of the states at energy ϵ which are actually occupied at temperature T . For $T = 0$ this is a step-function which jumps from 1 to 0 at energy ϵ_F ; for $T > 0$ we have a smeared-out version of this.

Two important quantities for the electron gas are the specific heat C_b and the magnetic susceptibility χ_b (the letter b stands for bulk). Without derivation we give the results for these two quantities in the limit of low temperatures, ($k_B T \ll \epsilon_F$)

$$\begin{aligned} C_b &= \frac{\pi^2}{3} \mathcal{D}(\epsilon_F) k_B^2 T , \\ \chi_b &= \mathcal{D}(\epsilon_F) \left(\frac{1}{2} g \mu_B\right)^2 . \end{aligned} \quad (2.4)$$

Here g is the gyromagnetic ratio for electrons ($g = 2$ for free electrons) and $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton. Note that the ratio

$$\left(\frac{\chi_b T}{C_b}\right) = \frac{3}{\pi^2} \frac{\left(\frac{1}{2} g \mu_B\right)^2}{k_B^2} \quad (2.5)$$

is independent of the details of the density of states and is entirely expressed in fundamental constants.

About notations the following. We will be using so-called *second-quantized notation* to describe the electron gas. [The term ‘second quantization’ is traditionally a source of confusion. For our purposes it suffices to realize that ‘second quantization’ is a way of writing the theory which makes it possible to describe states with different total numbers of electrons in a single formalism. We are *not* introducing a second Planck’s constant!] The central ingredient are the so-called creation and annihilation operators $c_{\vec{k},\alpha}^\dagger$ and $c_{\vec{k},\alpha}$, which satisfy the following *anti-commutation relations*

$$\{c_{\vec{k},\alpha}^\dagger, c_{\vec{k}',\alpha'}^\dagger\} = 0 , \quad \{c_{\vec{k},\alpha}, c_{\vec{k}',\alpha'}\} = 0 , \quad \{c_{\vec{k},\alpha}^\dagger, c_{\vec{k}',\alpha'}\} = \delta_{\vec{k},\vec{k}'} \delta_{\alpha,\alpha'} . \quad (2.6)$$

The labels \vec{k} and α give the wave-vector and spin ($\alpha = \uparrow, \downarrow$) of the electrons. In a cubic box of length L , the allowed values of the components k_x, k_y en k_z are $0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \dots$. The number operator $n_{\vec{k},\alpha} = c_{\vec{k},\alpha}^\dagger c_{\vec{k},\alpha}$ counts the number of electrons with wave-vector \vec{k} and spin α and has 0 and 1 as its possible eigenvalues.

Using the Fourier transform

$$c_\alpha(\vec{x}) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} c_{\vec{k},\alpha} \quad (2.7)$$

we can write the electron creation and annihilation operators in the position representation. The spin of the electron gas at position \vec{x} can be written as

$$\vec{S}(\vec{x}) = \frac{1}{2} c_\alpha^\dagger(\vec{x}) \vec{\sigma}_{\alpha\alpha'} c_{\alpha'}(\vec{x}) , \quad (2.8)$$

where the 2×2 matrices $\vec{\sigma}$ are the well-known Pauli spin matrices. Note that the interaction term in the Kondo hamiltonian can be written as als

$$H_{K,int} = 2J \vec{S}_{\text{imp}} \cdot \vec{S}(0) . \quad (2.9)$$

2.2 Quantum spin

In addition to the electron gas, the Kondo situation contains a magnetic impurity, which couples to the electron gas through its spin-operator \vec{S}_{imp} . In quantum mechanics, \vec{S}_{imp} is an operator that acts on a Hilbert space of dimension $2s + 1$ ($s = \frac{1}{2}$ for the Kondo hamiltonian). We shall be interested in the magnetic susceptibility, which we shall denote by χ_i (i stands for impurity). The so-called *Curie law* expresses χ_i for a free spin in terms of the temperature

$$\chi_i = \frac{s(s+1)}{3} \frac{(g\mu_B)^2}{k_B T} . \quad (2.10)$$

Clearly, the susceptibility diverges as $T \rightarrow 0$ because in that limit nothing stops the spin from aligning itself with an external magnetic field. In the presence of the Kondo interaction this will change: we will see that in that situation χ_i assumes a finite value at zero temperature.

2.3 Electrical resistance

The electrical resistance of a metal is entirely caused by the presence of scatterers that form an obstruction for the conduction electrons that carry the current. [In the absence of scatterers an electron gas is a perfect conductor; this has nothing to do with superconductivity!]

A simple picture for the conductivity of an electron gas is provided by the so-called Drude model. Here we start from the observation that in the presence of an external electric field an electron is accelerated according to

$$m \frac{d}{dt} \vec{v} = e \vec{E} . \quad (2.11)$$

In this way a so-called drift velocity is built up. Working against this are collisions with scatterers (phonons, impurities, etc.). In the absence of an electric field these will lead to a relaxation of the drift velocity with a characteristic relaxation time τ :

$\vec{v}(t) \sim \exp(-t/\tau)$, *i.e.*, $\frac{d}{dt}\vec{v} = -\frac{1}{\tau}\vec{v}$. In the presence of \vec{E} there is a balance between the two processes, so that $\vec{v}_{av} = \frac{e\tau}{m}\vec{E}$. This gives a current density

$$\vec{J} = \frac{n_e e^2 \tau}{m} \vec{E} \quad (2.12)$$

($n_e = N_e/V$ is the density of the electrons), which leads to a conductivity

$$\sigma = \frac{n_e e^2 \tau}{m} . \quad (2.13)$$

We shall later need a generalization of this formula for the case where the relaxation time τ depends on energy and spin. Using a Boltzmann equation, we can derive the following result (this is an approximation, but it will be good enough for our purposes in this course!)

$$\sigma = - \sum_{\alpha} \frac{e^2}{6} N(\epsilon_F) v_F^2 \int d\epsilon \frac{df}{d\epsilon} \tau_{\alpha}(\epsilon) . \quad (2.14)$$

Here $N(\epsilon) = \mathcal{D}(\epsilon)/V$ is the density of states per volume and v_F is the Fermi velocity. It is an instructive exercise to check that for a free electron gas with constant τ this expression reduces to the Drude formula (2.13).

The quantity τ can be related to the differential cross-section $\sigma(\theta)$ via

$$\frac{1}{v_F \tau} = 2\pi N_i \int_0^{\pi} (1 - \cos \theta) \sigma(\theta) \sin \theta d\theta . \quad (2.15)$$

We shall use these formulas in the chapters 4 and 6, where we will compute the impurity resistance for the Kondo model.

3 From Anderson to Kondo

In this chapter we discuss the origin of the Kondo hamiltonian as an effective description of magnetic impurities in a metal. We shall start from a somewhat more detailed description, the so-called Anderson model, and show that there is a parameter regime where this model reduces to the Kondo model.

The magnetic impurities in a Kondo system (atoms such as Mn , Fe) are atoms with an incomplete internal shell (typically d or f) with total spin different from zero. Such an atom has an orbital which is singly occupied (one electron) for an atom in isolation. When the atom is placed inside a metal and is thus brought into contact with a band of conduction electrons, there is the possibility of *hybridization*: one of the conduction electrons can jump to the impurity orbital (which has room for two electrons!) and spend some time there, or, the electron that occupied the impurity orbital can temporarily jump to the conduction band. These processes can easily be described in second quantized language: we introduce creation and annihilation operators $c_{i,\alpha}^\dagger$ and $c_{i,\alpha}$ for the impurity orbital and add to the hamiltonian of the electron gas the term

$$\sum_{\vec{k},\alpha} \left(V_{\vec{k}} c_{\vec{k},\alpha}^\dagger c_{i,\alpha} + V_{\vec{k}}^* c_{i,\alpha}^\dagger c_{\vec{k},\alpha} \right) . \quad (3.1)$$

Due to the Coulomb interaction, a second electron in the impurity orbital will pick up some electrostatic potential energy, which we model by adding a so-called Hubbard term $U n_{i,\uparrow} n_{i,\downarrow}$ to the hamiltonian. A ‘complete’ description is then given by the Anderson hamiltonian

$$H_A = \sum_{\vec{k},\alpha} \epsilon_{\vec{k}} n_{\vec{k},\alpha} + \sum_{\alpha} \epsilon_{\text{imp}} c_{i,\alpha}^\dagger c_{i,\alpha} + U c_{i,\uparrow}^\dagger c_{i,\uparrow} c_{i,\downarrow}^\dagger c_{i,\downarrow} + \sum_{\vec{k},\alpha} \left(V_{\vec{k}} c_{\vec{k},\alpha}^\dagger c_{i,\alpha} + V_{\vec{k}}^* c_{i,\alpha}^\dagger c_{\vec{k},\alpha} \right) . \quad (3.2)$$

To make contact with the Kondo model, we shall assume that the energy of the first electron in the impurity orbital, ϵ_{imp} , is well below the Fermi energy ϵ_F of the conduction electrons while the energy of a second electron, $\epsilon_{\text{imp}} + U$, is well above ϵ_F . In that case, the average occupation number of the impurity orbital will be one and the states with 0 or 2 impurity electrons will give rise to an effective interaction with the conduction electrons.

We can get an impression of the effect of this interaction by using second order perturbation theory (in first order the interaction term does not contribute to the energy!). Before actually computing the effect we can quite easily tell what the result will be by using the following reasoning. As a general result, the second-order correction to the energy E_n of the n^{th} eigenstate $|n\rangle$ of a hamiltonian $H = H_0 + V$ is given

by

$$E_n^{(2)} = - \sum_{k \neq n} \frac{|V_{nk}|^2}{E_k^{(0)} - E_n^{(0)}} . \quad (3.3)$$

Note that this correction will be negative if the intermediate states $|k\rangle$ have an energy which is higher than the energy of $|n\rangle$. In other words, an interaction term that couples a given state to states that lie higher in the spectrum tends to *lower* the energy of that state! For the application to the Anderson hamiltonian, the state $|n\rangle$ will be a state with one single electron in the impurity orbital, while the states k will have 0 or 2 impurity electrons. In this way, the hybridization interaction (3.1) lowers the energy of those states for which a transition of an electron into or out of the impurity orbital is possible. Whether or not this is the case depends on the occupation numbers of the impurity orbital and of the electron gas at position $\vec{x} = 0$ (we assume for now that $V_{\vec{k}}$ does not depend strongly on \vec{k} , so that the hybridization interaction (3.1) is mostly in terms the conduction electrons at $\vec{x} = 0$). If, for example, both locations have an electron of spin up sitting there, the Pauli principle forbids a transition. If two electrons of opposite spins are present, a transition is possible! This then shows that a state with anti-parallelly aligned spins is energetically favored over the analogous state with parallel spins. This in turn means that the charge fluctuations between the impurity and the conduction band will lead to an effective exchange interaction between \vec{S}_{imp} and $\vec{S}(0)$ which is *anti-ferromagnetic*. We therefore expect an effective Kondo-type hamiltonian with $J > 0$.

We can be a bit more precise by doing a little computation. [Note that, in general, it pays to do a ‘hand-waving’ investigation, such as the one sketched in the above, before diving into an actual computation. It is usually extremely hard to get a computation right when one does not have a good feeling for the kind of physics that is going on. Really!] We can write the Schrödinger equation for our problem in the following form

$$\begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{10} & H_{11} & H_{12} \\ 0 & H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} . \quad (3.4)$$

Here ψ_j denotes a state with occupation number j for the impurity orbital and we have

$$\begin{aligned} H_{00} &= H_0 , & H_{11} &= H_0 + \epsilon_{\text{imp}} , & H_{22} &= H_0 + 2 \epsilon_{\text{imp}} + U \\ H_{01} &= H_{12} = \sum_{\vec{k}, \alpha} V_{\vec{k}} c_{\vec{k}, \alpha}^\dagger , & H_{10} &= H_{21} = \sum_{\vec{k}, \alpha} V_{\vec{k}}^* c_{\vec{k}, \alpha} \end{aligned} \quad (3.5)$$

with $H_0 = \sum_{\vec{k}, \alpha} \epsilon_{\vec{k}} n_{\vec{k}, \alpha}$. We can now eliminate ψ_0 and ψ_2 from these equations to find

$$\left(H_{11} + H_{12}(E - H_{22})^{-1}H_{21} + H_{10}(E - H_{22})^{-1}H_{01} \right) \psi_1 = E \psi_1 . \quad (3.6)$$

The second and third term of the effective hamiltonian can be written as

$$\begin{aligned} H_{12}(E - H_{22})^{-1}H_{21} &= \sum_{\vec{k}, \vec{k}', \alpha, \alpha'} V_{\vec{k}}^* V_{\vec{k}'} \frac{1}{U + \epsilon_{\text{imp}} - \epsilon_{\vec{k}'}} c_{\vec{k}, \alpha}^\dagger c_{i, \alpha} c_{\vec{k}', \alpha'} c_{i, \alpha'}^\dagger , \\ H_{10}(E - H_{00})^{-1}H_{01} &= \sum_{\vec{k}, \vec{k}', \alpha, \alpha'} V_{\vec{k}}^* V_{\vec{k}'} \frac{1}{\epsilon_{\vec{k}} - \epsilon_{\text{imp}}} c_{\vec{k}', \alpha'} c_{i, \alpha'}^\dagger c_{\vec{k}, \alpha}^\dagger c_{i, \alpha} . \end{aligned} \quad (3.7)$$

Note that we have assumed $|E - \epsilon_{\text{imp}} - H_0| \ll \Delta$ where $\Delta = U + \epsilon_{\text{imp}} - \epsilon_{\vec{k}'}$ or $\epsilon_{\vec{k}} - \epsilon_{\text{imp}}$. This means that the difference ΔE between the true energy E and the energy $H_0 + \epsilon_{\text{imp}}$ of the non-interacting model should be much smaller than the two gaps that separate the Fermi level from the two impurity levels. We expect ΔE to be of the order of magnitude of the line width $\Gamma \sim \mathcal{D}(\epsilon_F) V^2$. This condition then defines the parameter regime for which the Kondo model is a good approximation to the Anderson model.

What remains is some ‘gymnastics’ to re-order the spin indices into a more recognizable form. A useful identity for this is

$$\sum_{a=1}^3 \sigma_{\alpha\beta}^a \sigma_{\gamma\delta}^a = 2 \delta_{\alpha\delta} \delta_{\beta\gamma} + \delta_{\alpha\beta} \delta_{\gamma\delta} . \quad (3.8)$$

This leads to the following effective form of the interaction term in the effective hamiltonian for the states ψ_1

$$\begin{aligned} &\sum_{\vec{k}, \vec{k}', \alpha, \alpha'} V_{\vec{k}}^* V_{\vec{k}'} \left(\frac{1}{U + \epsilon_{\text{imp}} - \epsilon_{\vec{k}'}} + \frac{1}{\epsilon_{\vec{k}} - \epsilon_{\text{imp}}} \right) c_{\vec{k}, \alpha}^\dagger \vec{\sigma}_{\alpha\alpha'} c_{\vec{k}', \alpha'} \cdot \vec{S}_{\text{imp}} \\ &+ \frac{1}{2} \sum_{\vec{k}, \vec{k}', \alpha} V_{\vec{k}}^* V_{\vec{k}'} \left(\frac{-1}{U + \epsilon_{\text{imp}} - \epsilon_{\vec{k}'}} + \frac{1}{\epsilon_{\vec{k}} - \epsilon_{\text{imp}}} \right) c_{\vec{k}, \alpha}^\dagger c_{\vec{k}', \alpha} . \end{aligned} \quad (3.9)$$

In the first term we recognize a modified Kondo interaction, with positive (antiferromagnetic) coupling J . The second term is independent of spin and describes so-called potential scattering. This term is not important for the Kondo effect and we shall largely ignore it in what follows.

The above result, which was first obtained by Schrieffer and Wolff (1966), gives a beautiful example of an effective exchange (spin-spin) interaction which has its origin in

charge fluctuations of electrons. Another example of this is the spin-spin interaction term in an effective model (t-J model or, for the half filled band, Heisenberg anti-ferromagnet) for a Hubbard model with $U \gg t$.

Referring to chapter 8, we remark that the Anderson model has been solved exactly by Bethe Ansatz. Using this solution, the full phase diagram, including those regimes where the Kondo limit is not appropriate, can be understood. Other techniques that we will apply to the Kondo model (such as scaling) have also been applied to the various regimes of the Anderson model.

Of great interest for Heavy Fermion physics is the so-called periodic Anderson model, in which a large number of impurities, placed on a regular lattice, are taken into account. The understanding of this model, and of the analogous Kondo-lattice, is far less complete than that of the one-impurity Anderson and Kondo models.

4 Perturbation Theory

After three introductory chapters, we are now ready to start our analysis of the Kondo hamiltonian (1.1). In this chapter we make the logical first step, which is to study Kondo's perturbative analysis, which explains the resistance minimum and gives a good approximation for the physics at temperatures well above the Kondo temperature. A few remarks before we dive into this computation.

In section 2.1 we already mentioned that the details of the band structure of the electrons will not be very important for us. However, one aspect is quite crucial, namely that there is a *finite bandwidth*, which we will denote by D in this course. This bandwidth sets an energy scale for, among other things, the Kondo temperature. The simplest model description will be to assume a density of states which is constant and equal to $\mathcal{D}(\epsilon_F)$ for energies in the range $\epsilon_F - D < \epsilon < \epsilon_F + D$ and zero elsewhere.

A second remark concerns the scattering cross-section that serves as input for the computation of the resistivity in (2.14), (2.15). As a general result, the differential cross-section for the scattering of an electron of wave vector \vec{k} is of the form (this is for free electrons in unit volume)

$$d\sigma(k) = \frac{m^2}{4\pi^2\hbar^4} \frac{1}{4} \sum_{s,s'} \sum_{\alpha,\alpha'} |T_{(\vec{k}',\alpha',s'),(\vec{k},\alpha,s)}|^2 d\Omega . \quad (4.1)$$

This formula gives the cross-section as the square of a matrix element of the so-called T -matrix, which is summed over the final spin states and averaged over the initial spin states. The T -matrix has the expansion

$$T_{fi} = V_{fi} + \sum_m V_{fm} \frac{1}{(\epsilon_i - \epsilon_m + is)} V_{mi} + \dots \quad (4.2)$$

where the labels f, i denote the final and initial states and the m are intermediate states. The notation V_{fi} is shorthand for the matrix element of the interaction term in the hamiltonian between the states f and i . The expansion (4.2) for the T -matrix is found by an iterative solution of the so-called Lippmann-Schwinger equation of scattering theory. Using the relation

$$\sum_{\vec{k}'} |T_{\vec{k}'\vec{k}}|^2 \delta(\epsilon_{\vec{k}'} - \epsilon_{\vec{k}}) = \frac{1}{(2\pi)^3} \frac{mk}{\hbar^2} \int |T_{\vec{k}'\vec{k}}|^2 d\Omega , \quad (4.3)$$

we can write

$$\sigma(k) = \frac{m}{\hbar k} \frac{2\pi}{\hbar} \sum_{\vec{k}'} |T_{\vec{k}'\vec{k}}|^2 \delta(\epsilon_{\vec{k}'} - \epsilon_{\vec{k}}) . \quad (4.4)$$

Figuur 4.1: Second order contributions to the transition amplitude.

In lowest order, replacing T by V , we then recognize the ‘Golden Rule’ of time-dependent perturbation theory.

For the Kondo hamiltonian, the relevant matrix element of T (which we call a transition amplitude) is in lowest order given by

$$\langle \vec{k}', \sigma', s' | T | \vec{k}, \sigma, s \rangle = J \vec{S}_{s's} \cdot \vec{\sigma}_{\alpha'\alpha} \quad (4.5)$$

The corresponding cross-section is found to be proportional to

$$\frac{1}{4} J^2 \sum_{s,s'} \sum_{\alpha,\alpha'} |\vec{S}_{s's} \cdot \vec{\sigma}_{\alpha'\alpha}|^2 = \frac{3}{4} J^2 . \quad (4.6)$$

Here we used

$$\text{tr}(\sigma^a \sigma^b) = 2 \delta^{ab} , \quad \text{tr}(S^a S^a) = \frac{3}{2} . \quad (4.7)$$

This cross-section is independent of energy and temperature and therefore the corresponding impurity resistance is independent of T . Similar, T -independent contributions to the resistivity will in general arise from spin-independent interaction terms, such as the second term in (3.9), which lead to so-called potential scattering. It will be clear that these contributions do not lead to a resistance minimum. For that, we need to go to the next order in perturbation theory.

In formula, the second order in perturbation theory is represented by the second term in the expansion (4.2). It is instructive to draw diagrams that show the physical processes described by this term. In an obvious convention we find the two diagrams given in fig. 4.1.

In these diagrams, the time flow is from left to right. In diagram (i), the incoming electron, of momentum \vec{k} , scatters off the impurity into an intermediate state of

momentum \vec{k}_1 which then scatters into a final state of momentum \vec{k}' . Depending on the choice of the interaction terms, the spins of both the conduction electron and the impurity may flip at each of the two scattering moments. The formula that goes with the diagram (i) is (we suppress the indices s, s' in our notation)

$$-J^2 \sum_{\alpha_1} \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{(\vec{S} \cdot \vec{\sigma}_{\alpha'\alpha_1})(\vec{S} \cdot \vec{\sigma}_{\alpha_1\alpha}) c_{\vec{k}',\alpha'}^\dagger c_{\vec{k}_1,\alpha_1} c_{\vec{k}_1,\alpha_1}^\dagger c_{\vec{k},\alpha}}{\epsilon_{\vec{k}_1} - \epsilon_{\vec{k}}}. \quad (4.8)$$

In the sequence of events for diagram (ii), the outgoing electron of momentum \vec{k} is first created and only after that is the incoming electron annihilated. That this diagram is possible at all is clearly a many body effect. What happens at the first scattering is the creation of an electron-hole pair from the Fermi sea which is the ‘background’ for these scattering diagrams. At the second scattering, the hole absorbs the incoming electron of momentum \vec{k} . In formula

$$-J^2 \sum_{\alpha_1} \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{(\vec{S} \cdot \vec{\sigma}_{\alpha_1\alpha})(\vec{S} \cdot \vec{\sigma}_{\alpha'\alpha_1}) c_{\vec{k}_1,\alpha_1}^\dagger c_{\vec{k},\alpha} c_{\vec{k}',\alpha'}^\dagger c_{\vec{k}_1\alpha_1}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}_1}}. \quad (4.9)$$

From this point on it is an exercise in computation to extract the contribution of these two diagrams to the resistivity of the material. As a first step, we may re-order the creation and annihilation operators in the above formulas and use the following expectation values

$$\begin{aligned} \langle c_{\vec{k}_1\alpha_1} c_{\vec{k}_1\alpha_1}^\dagger \rangle &= 1 - f(\epsilon_{\vec{k}_1}), \\ \langle c_{\vec{k}_1\alpha_1}^\dagger c_{\vec{k}_1\alpha_1} \rangle &= f(\epsilon_{\vec{k}_1}). \end{aligned} \quad (4.10)$$

The fact that $1 - f(\epsilon_{\vec{k}_1})$ enters in the formula for diagram (i) expresses the fact that the process depicted there is only possible if the state (\vec{k}_1, α_1) was empty in the background Fermi sea, which happens with probability $1 - f(\epsilon_{\vec{k}_1})$. Notice that the $f(\epsilon_{\vec{k}_1})$ factors lead to an explicit temperature dependence of the scattering amplitudes.

The spin indices in our formulas can be worked out using

$$\begin{aligned} \sum_{ab} \sum_{\alpha_1} S^a \sigma_{\alpha'\alpha_1}^a \sigma_{\alpha_1\alpha}^b S^b &= \frac{3}{4} \delta_{\alpha\alpha'} - \vec{S} \cdot \vec{\sigma}_{\alpha'\alpha}, \\ \sum_{ab} \sum_{\alpha_1} S^a \sigma_{\alpha_1\alpha}^a \sigma_{\alpha'\alpha_1}^b S^b &= \frac{3}{4} \delta_{\alpha\alpha'} + \vec{S} \cdot \vec{\sigma}_{\alpha'\alpha}, \end{aligned} \quad (4.11)$$

which follow by using

$$\sum_{\alpha_1} \sigma_{\alpha\alpha_1}^a \sigma_{\alpha_1\alpha'}^b = i \sum_c \epsilon^{abc} \sigma_{\alpha\alpha'}^c + \delta^{ab} \delta_{\alpha\alpha'}. \quad (4.12)$$

The total contribution of the second order terms has now been brought into the form

$$J^2 c_{\vec{k}', \alpha'}^\dagger c_{\vec{k}, \alpha} \int \frac{d\vec{k}_1}{(2\pi)^3} \left(\frac{3}{4} \frac{\delta_{s's} \delta_{\alpha'\alpha}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}_1}} + \frac{2f(\epsilon_{\vec{k}_1}) - 1}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}_1}} \vec{S}_{s's} \cdot \vec{\sigma}_{\alpha'\alpha} \right). \quad (4.13)$$

The first term in this expression is temperature independent and will give a contribution that is of the same form as the first order scattering term but smaller by an order of magnitude. We therefore focus our attention on the second term.

Trading the $d\vec{k}_1$ integration for an integral over energy, and writing $\xi = \epsilon_{\vec{k}'} - \mu$ and $\xi_1 = \epsilon_{\vec{k}_1} - \mu$, the second term becomes

$$\frac{1}{2} J^2 c_{\vec{k}', \alpha'}^\dagger c_{\vec{k}, \alpha} \int d\xi_1 \mathcal{D}(\xi_1) \frac{2f(\xi_1) - 1}{\xi_1 - \xi} \vec{S}_{s's} \cdot \vec{\sigma}_{\alpha'\alpha}. \quad (4.14)$$

From here the canonical procedure would be to evaluate this amplitude as a function of both the energy ξ and temperature T and to process the result using (2.14), which involves an integration over the energy ξ . We will here take the (justifiable) shortcut of approximating the integration in (2.14) by the evaluation at $\xi = 0$ (though somewhat crude, this approximation does not affect the leading $\ln(k_B T/D)$ behavior which we will find below).

If we now assume the rectangular density of states that we introduced in the beginning of this chapter, we can simply take the density of states $\mathcal{D}(\xi_1)$ out of the integration in (4.14) so that we are faced with the integral

$$I(\xi, T) = \int_{-D}^D d\xi_1 \frac{2f(\xi_1) - 1}{\xi_1 - \xi}. \quad (4.15)$$

This integration looks singular around the point $\xi_1 = \xi$ but with a principal value prescription the result is finite *as long as* $T > 0$. The integral is sensitive to the cut-off value D and would in fact diverge if we were to work with an infinite band. By using a partial integration, one finds the following T -dependence of the integral at $\xi = 0$ (assuming $D \gg k_B T$)

$$I(\xi = 0, T) = -2 \ln(k_B T/D) + \text{const.} \quad (4.16)$$

Adding the lowest order contribution (4.5), we end up with a spin-dependent scattering amplitude at $\xi = 0$, *i.e.* at $\epsilon_{\vec{k}} = \epsilon_{\vec{k}'} = \epsilon_F$

$$t(T) = J \vec{S} \cdot \vec{\sigma} [1 - J \mathcal{D}(\epsilon_F) \ln(k_B T/D)]. \quad (4.17)$$

Figuur 4.2: The comparison of experimental results for the resistivity of dilute Fe in Au at very low temperatures with the logarithmic form (4.18), reproduced from Kondo's 1964 paper.

Via (2.14) and (2.15), a similar logarithmic temperature dependence occurs in the formula for the resistance. Adding the phonon contribution, see (1.2), we have a total resistance of the form

$$R(T) = aT^5 + c_{\text{imp}}R_0 - c_{\text{imp}}R_1 \ln(k_B T/D) . \quad (4.18)$$

Using this formula, Kondo was able to fit some of the experimental curves for the resistance of dilute magnetic alloys; see fig. 4.2 where the drawn lines are based on the formula (4.18).

From (4.18) we easily find that a resistance minimum occurs at temperature T_{\min} ,

$$T_{\min} = \left(\frac{R_1}{5a} \right)^{1/5} c_{\text{imp}}^{1/5}, \quad (4.19)$$

which is in general agreement with experimental observations.

Having come this far, we should have a critical look at the result obtained. The expression (4.18) for the total resistance diverges for temperature $T \rightarrow 0$, which is clearly an unphysical result (both with regard to physical common sense and with regard to the experimental data, which show a behavior $R(T) \sim C_0 - C_2 T^2 + \dots$ near zero temperature, see chapter 6). Clearly, perturbation theory breaks down for very low temperatures.

Inspecting our formula (4.17) we can see the problem loud and clear: at low enough temperature the second term, which is meant to be a small perturbative correction to the first term, becomes of equal magnitude as the first, leading term. The two are equal for a temperature such that

$$k_B T_K = D e^{-1/(J\mathcal{D}(\epsilon_F))}. \quad (4.20)$$

We shall call this characteristic temperature the *Kondo temperature*. It signals the regime where, coming from high temperatures, perturbation theory breaks down and non-perturbative effects set in.

There is of course the logical possibility that the problem signaled here goes away if we continue to higher orders in perturbation theory. This has been worked out and the answer turns out to be negative. For example, after summing the most divergent contributions from higher orders, Abrikosov derived the following expression for the spin-dependent scattering rate

$$t(T) = J \vec{S} \cdot \vec{\sigma} \frac{1}{[1 + J\mathcal{D}(\epsilon_F) \ln(k_B T/D)]}. \quad (4.21)$$

While at high temperatures this certainly gives a better result for the resistivity, there is still an unphysical divergence, which now happens for $T = T_K$. This again confirms that perturbation theory breaks down near $T = T_K$.

Note that, since the resistance minimum arises from the superposition of two contributions to the resistance (phonons and magnetic impurities), the temperature T_{\min} where the minimum occurs is not directly related to the Kondo temperature T_K (except, of course, for the trend that materials with a low T_K will tend to have a low value for T_{\min}). The Kondo temperature can be ‘read off’ from graphs displaying the

behavior of properties that are intrinsic to the Kondo system, such as the impurity susceptibility or the impurity specific heat.

The fact that the dependence of T_K on the coupling constant J and the density of states $D(\epsilon_F)$ is exponential suggests that a wide range of values can be realized in nature. Indeed, choosing gold (Au) as the host metal, we find Kondo temperatures in the order of 1 mK for Cr impurities, 0.2 K for Fe impurities and 1000 K for Ni impurities [MN].

5 Scaling and Wilson's analysis

In defining the Kondo model we have introduced two independent energy scales, which are the coupling strength J and the band width D . [By using the inverse density of states at the Fermi level as a reference energy, these correspond to two dimensionless quantities $\mathcal{D}(\epsilon_F)J$ and $\mathcal{D}(\epsilon_F)D$.] One thus expects that all physical quantities will be functions of J and D , and of course of temperature. However, it turns out that for $k_B T \ll D$ (the so-called *scaling regime*) all quantities in fact depend on J and D through a single variable, which is precisely the Kondo temperature T_K given in (4.20).

Some first evidence for this comes from the perturbative results of the previous section. It is easily checked that the scattering amplitude (4.21) can be written as

$$t(T) = \vec{S} \cdot \vec{\sigma} \frac{1}{\mathcal{D}(\epsilon_F) \ln(T/T_K)} + \dots \quad (5.1)$$

Other physical quantities can be written in a similar manner. For example, in perturbation theory the spin susceptibility has the following expansion for $T \gg T_K$

$$\chi_i(T) = \frac{(g\mu_B)^2}{4k_B T} \left(1 - \frac{1}{\ln(T/T_K)} + \dots \right). \quad (5.2)$$

We shall later see that in the regime where T is of the order of or smaller than T_K the dependence on J and D is of the same universal form.

Obviously, there is a reason for this particular dependence on J and D , and this brings us to the notion of scaling and the Renormalization Group, and to the work of Anderson and Wilson on the Kondo problem.

Let us start by recalling how the band width D made its appearance in our analysis. This happened when we summed over the momentum \vec{k}_1 of the intermediate state in the diagrams in fig. 4.1, which correspond to second order perturbation theory. The fact that these diagrams are sensitive to the bandwidth D shows that the entire band contributes to the scattering amplitudes. Clearly, electrons (holes) of energies of the order $\epsilon_F + D$ ($\epsilon_F - D$) will not be thermally excited (remember $D \gg k_B T$), so that their only contribution to the physics is precisely as virtual intermediate states. We may now wonder what would happen if we slightly change the band width D by reducing it to $D + \delta D$ with $\delta D < 0$. (see fig. 5.1). This will affect the outcome of perturbative computations. However, since those are always proportional to powers of the coupling constant J (J^2 in second order) we can try to compensate for the change in the amplitudes by adjusting the value of J to, say, $J + \delta J$.

Figure 5.1: The particle and hole states which are removed from the conduction band on reducing the band width by $|2\delta D|$.

We will here not go through the detailed analysis of the procedure outlined above, which can be found *e.g.* in Hewson's book. The result is that it can be shown that the results of second order perturbative calculations are invariant if we accompany the change δD in the bandwidth by a corresponding change

$$\delta J = -\mathcal{D}(\epsilon_F) J^2 \frac{\delta D}{D} \quad (5.3)$$

in the coupling strength. The global structure of this formula is clear enough: the change in energy that is absorbed in δJ is of the form J^2/D , which is the energy gained by temporarily occupying a state of excess energy D , times a factor $\mathcal{D}(\epsilon_F)|\delta D|$, which represents the number of available virtual states that are eliminated.

The idea that a change of energy scale (represented here by D) can be compensated for by a change of the parameters of a physical theory (in this case J) is a very general one, which has found applications in a variety of physical theories, and which goes under the name Renormalization Group. Outside of Condensed Matter Theory it has applications in Statistical Mechanics (in the context of phase transitions) and in High Energy Physics where it leads to so-called running coupling constants which vary with the energy scale at which a theory is probed.

Returning to the Kondo problem, we can now understand why it is that physical quantities only depend on J , D via T_K . Namely if we vary the expression (4.20) for T_K according to (5.3) we find

$$\delta T_K = \delta \left(D e^{-1/J\mathcal{D}(\epsilon_F)} \right) = \delta D e^{-1/J\mathcal{D}(\epsilon_F)} + D e^{-1/J\mathcal{D}(\epsilon_F)} \frac{1}{J^2 \mathcal{D}(\epsilon_F)} \delta J = 0 . \quad (5.4)$$

In other words, the Kondo temperature is an *invariant* of the scale transformations and as such it represents the single combination of J and D which has physical meaning. Stated differently: two different materials that have different couplings J and different band widths D , but which have the same value for T_K , will have identical physical properties! [This is for temperatures such that $k_B T \ll D$.] The above ‘derivation’ of this is based on perturbation theory and it thereby has a limited regime of validity. However, the same conclusion has been reached in numerical studies by Wilson (see below) and in the exact solution based on Bethe Ansatz (the resulting expression for T_K is of course a little different from (4.20)).

The scaling relation (5.3) has very important physical consequences to which we now turn. Let us introduce dimensionless quantities $d = \mathcal{D}(\epsilon_F)D$ and $j = \mathcal{D}(\epsilon_F)J$ and study the relation (5.3) between the two. We easily find the following differential equation

$$\frac{dj}{dd} = -\frac{j^2}{d} . \quad (5.5)$$

Let us start from initial parameters d_0 and $j_0 = j(d_0)$ and compute the trajectory that solves this equation. The result is

$$j(d) = \frac{j_0}{1 - j_0 \ln(d_0/d)} . \quad (5.6)$$

If we now assume that $j_0 < 0$, we see that upon decreasing d down from d_0 , the coupling j approaches 0! [Note that in this (ferromagnetic) case, perturbation theory is self-consistent in the sense that a small initial value of $|j|$ remains small during the flow to lower and lower energies.] The physical meaning of this result is that, if we probe a ferro-magnetic Kondo system at low energies, *i.e.* at low temperatures, we will observe a very small coupling J ! This means that the impurity simply decouples, so that, for example, the impurity susceptibility follows the Curie law (2.10). We have thus completely understood the ferro-magnetic Kondo model in the low temperature regime!

The analysis in the anti-ferromagnetic case ($j_0 > 0$), which is the case of physical interest, is quite a bit harder. In this case, the coupling j increases upon decreasing the scale d . This means that the system approaches a strong coupling regime when temperature is lowered, and this then implies that the flow equation (5.3), which was derived in perturbation theory, becomes unreliable. Actually, the solution (5.6) for $j(d)$ has a singularity for

$$d = d_0 e^{-1/j_0} , \quad (5.7)$$

which corresponds to a band width equal to k_B times the Kondo temperature set by d_0 and j_0 . It has been found that this singularity is an artefact of the perturbative

approach. The true flow of J , which was found by Wilson, is smooth at the Kondo temperature, but diverges when D is reduced to 0!

This last observation is absolutely crucial since it holds the clue for understanding the behavior of the anti-ferro magnetic Kondo model at very low temperatures. In brief, these will follow by studying the system in the vicinity of the $J = \infty$ fixed point that is reached at zero temperature. We shall come to this in the chapters 6 and 7.

The above scaling approach to the Kondo problem, which was pioneered by Anderson and Anderson and Yuval (1969) is known under the name ‘a poor man’s scaling’. This is because, though conceptually very appealing, this scaling approach is still rooted in perturbation theory.

A full treatment of the Kondo problem was first given by Wilson, who in 1975 combined the Renormalization Group with numerical computations and was able to compute physical quantities over the full temperature range. Wilson’s analysis is a beautiful piece of work, which combines fundamental physical insight with extensive computations. We shall here not go into the technicalities of Wilson’s work, but only cite the main conclusions.

Most importantly, Wilson found that under the Renormalization Group flow there are two fixed points for the coupling J , namely $J = 0$ and $J = \infty$. When we start the flow at high energies (temperatures), we are in the vicinity of the $J = 0$ fixed point. In this regime, Wilson’s results confirm the picture from the ‘poor man’s scaling’. [In this regime, the flow is dominated by a marginal interaction term, which leads to a logarithmic dependence on the energy scale.] At energies of the order of $k_B T_K$, Wilson’s results show a *cross-over*, and for energies well below $k_B T_K$ the results are dominated by the $J = \infty$ fixed point, which is finally reached for energy (temperature) 0.

As an example of the results obtained by Wilson, we show in fig. 5.2 the explicit curve for the impurity susceptibility $\chi_i(T)$ as a function of temperature, in the range $10^{-1} < T/T_K < 10^5$. The asymptotic behavior is in agreement with (5.2). Note the cross-over near the Kondo temperature.

At temperatures $T \ll T_K$, Wilson obtained accurate results for χ_i and the impurity specific heat $C_i(T)$. To within the numerical accuracy, he found the value

$$R_W = \frac{\chi_i/\chi_b}{C_i/C_b} = \frac{4\pi^2 k_B^2}{3(g\mu_B)^2} \frac{\chi_i T}{C_i} = 2 \quad (5.8)$$

for what is now called the *Wilson ratio* R_W . This means that the ratio $(\chi_i T)/C_i$ is precisely twice the value for free electrons! In our next chapter, we shall present a simple derivation, first given by Nozières, of this remarkable result. Wilson also

Figuur 5.2: The universal susceptibility curve for the $s = \frac{1}{2}$ Kondo hamiltonian (Wilson, 1975).

obtained a numerical value for the value of $\chi_i(T = 0)$

$$\chi_i(T = 0) = w \frac{(g\mu_B)^2}{4k_B T_K}, \quad w = 0.4128 \pm 0.002 . \quad (5.9)$$

In chapter 8, we shall use the Bethe Ansatz to derive an exact result for the number w !