

## QUANTUM GROUPS AND KNOT THEORY: WEEK 48

This week we treat sections 3 and 4 of Chapter 6 of [2]. We give additional proofs and definitions.

### 1. MODULAR TENSOR CATEGORIES

One of the most important applications of the RT-functor is the construction of invariants of (closed, connected, oriented) 3-manifolds, the so-called Reshetikhin-Turaev-Witten invariants of 3-manifolds. However, the RT-invariants of ribbon links associated with colorings of links with objects of a ribbon category  $\mathcal{C}$  in week 46 and week 47 are far too general for this purpose. For this more ambitious goal we are forced to impose a very restrictive additional property of the ribbon category  $\mathcal{C}$  turning it into a so-called modular tensor category.

**1. Abelian categories.** Let  $k$  be a commutative ring. A category  $\mathcal{C}$  is called  $k$ -linear if all hom-sets are  $k$ -modules such that the composition is bilinear. In particular if  $X$  is an object of  $\mathcal{C}$  then the monoid  $\text{Hom}_{\mathcal{C}}(X)$  has the structure of an (associative, unital)  $k$ -algebra. We call a  $\mathbb{Z}$ -linear category  $\mathcal{C}$  *additive* if there exists a zero object  $0$  such that  $\text{Hom}(0, X) = \text{Hom}(X, 0) = 0$ , and moreover if every pair of objects  $(X, Y) \in \mathcal{C} \times \mathcal{C}$  has a *product*, i.e. there exists an object  $X \times Y \in \mathcal{C}$  which represents the functor  $\text{Hom}_{\mathcal{C}}(?, X) \times \text{Hom}_{\mathcal{C}}(?, Y)$  from  $\mathcal{C} \times \mathcal{C} \rightarrow \text{Mod}(k)$ . It is not difficult to show that in an additive category also every pair of objects  $(X, Y)$  defines a coproduct (or direct sum)  $X \oplus Y$  which by definition corepresents the functor  $\text{Hom}_{\mathcal{C}}(X, ?) \times \text{Hom}_{\mathcal{C}}(Y, ?)$ . In fact the product  $X \times Y$  and the direct sum  $X \oplus Y$  are canonically isomorphic.

An additive category  $\mathcal{C}$  is called *abelian* if all morphisms in  $\mathcal{C}$  admit kernels and cokernels, and if for every morphism  $f : A \rightarrow B$  the canonical morphism  $\bar{f} : \text{Coim}(f) \rightarrow \text{Im}(f)$  (where  $\text{Coim}(f)$  is defined by  $\text{Coim}(f) = \text{Coker}(\text{Ker}(f) \rightarrow A)$ , and  $\text{Im}(f) = \text{Ker}(B \rightarrow \text{Coker}(f))$ ) is an isomorphism. Equivalently, a morphism  $f$  is an isomorphism iff  $f$  is both an epimorphism and a monomorphism.

For example, if  $A$  is a  $k$ -algebra then the category of left modules over  $A$  is  $k$ -linear and abelian. In fact, it is known that any  $k$ -linear abelian category whose objects form a set (a *small* category) is equivalent to a full subcategory of the category of left  $A$ -modules of some  $k$ -algebra  $A$  (this is Mitchell's embedding theorem, which is based on Yoneda's embedding). Abelian categories are of fundamental importance since they provide the natural abstract framework in which homological algebra techniques can be considered.

Let  $\mathcal{C}$  be a  $k$ -linear abelian category. An object  $M$  of  $\mathcal{C}$  is called *simple* if  $M$  is nonzero, and for any monomorphism  $f : N \rightarrow M$  we have that either  $f = 0$  or else  $f$  is an isomorphism.

We have the easy but fundamental lemma of Schur:

**Proposition 1.1.** *Let  $\mathcal{C}$  be a  $k$ -linear abelian category. An object  $M \in \mathcal{C}$  is simple iff  $\text{End}_{\mathcal{C}}(M)$  is a division algebra over  $k$ . If  $M, N$  are both simple and  $f : N \rightarrow M$  is any morphism then either  $f = 0$  or  $f$  is an isomorphism.*

*Proof.* If  $M$  is simple and  $f \in \text{End}_{\mathcal{C}}(M)$  is nonzero then the canonical monomorphism  $\ker(f) \rightarrow M$  must be zero, hence  $f$  must be an isomorphism. The second assertion is proved similarly.  $\square$

In particular, if  $M$  is simple then  $m := \{x \in k \mid k.\text{Id}_M = 0\}$  is a maximal ideal in  $k$ . Moreover, if  $k$  is an algebraically closed field then we have an isomorphism  $\text{End}_{\mathcal{C}}(M) \simeq k.\text{Id}_M$ . A simple object  $M$  in a  $k$ -linear abelian category  $\mathcal{C}$  is called split if  $\text{End}_{\mathcal{C}}(M) \simeq k$ . If  $k$  is algebraically closed then simple objects are automatically split since the only division algebra over  $k$  is  $k$  itself.

An object of a  $k$ -linear abelian category is called semisimple if it is isomorphic to a direct sum of finitely many simple objects. We call a  $k$ -linear abelian category  $\mathcal{C}$  semisimple if every object of  $\mathcal{C}$  is semisimple.

**2. Modular tensor categories.** We will introduce modular tensor categories (sometimes also called *fusion categories*) under some rather restrictive conditions, but this is more than enough for our purposes.

**Definition 2.2.** *Let  $k$  be a field of characteristic zero. A strict modular tensor category over  $k$  is a  $k$ -linear semisimple abelian strict ribbon category  $\mathcal{C}$  such that*

- (a) *The set of isomorphism classes of simple objects is finite and all simple objects are split simple over  $k$ .*
- (b) *Let us parameterize by  $\{0, 1, \dots, N\}$  the set of isomorphy classes of simple objects, and for each  $i$  let  $V_i \in i$ . We choose the numbering such that 0 represents the class of the tensor unit. Consider the  $(N + 1) \times (N + 1)$  matrix  $\tilde{s}$  with entries  $\tilde{s}_{i,j} \in k$  given by figure 1. Then the  $S$ -matrix  $\tilde{s}$  is invertible.*

The name “modular tensor category” is justified by the fact that there exists a projective action of the modular group  $\text{SL}_2(\mathbb{Z})$  on the Grothendieck ring of the category, as we will see below. This appearance of the modular group is a reflection of the fact that a modular tensor category defines a (2+1)-dimensional topological quantum field theory. The modular group is the mapping class group of a two dimensional compact torus (see below), and therefore this group acts on the vector spaces attached to a two dimensional torus by the TQFT.

**3. The Verlinde algebra.** In a  $k$ -linear semisimple (strict) ribbon algebra we consider the  $k$ -algebra  $K := k \otimes_{\mathbb{Z}} K(\mathcal{C})$ , where  $K(\mathcal{C})$  is the Grothendieck ring of  $\mathcal{C}$ . The algebra structure is defined by  $[V][W] := [V \otimes W]$ . (We note that in any abelian ribbon category the tensor product bifunctor is exact in both factors, hence this definition makes sense even in the Grothendieck ring of any abelian ribbon category; in the semisimple situation this is of course trivial). From the axioms of braided monoidal categories it follows easily that  $K$  is unital (with unit  $[V_0]$ ), associative and commutative.

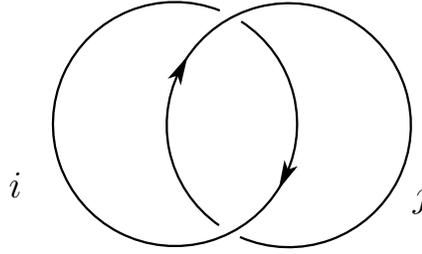


FIGURE 1.

The classes  $[V]$ , where  $V$  runs through a complete set of representatives of the equivalence classes of simple objects in  $\mathcal{C}$ , form a  $k$ -linear basis for  $K$ . The multiplication in  $K$  is completely determined by the so-called fusion coefficients  $N_{[U],[V]}^{[W]} \in \mathbb{Z}_{\geq 0}$  defined by

$$(3.1) \quad [U].[V] := \sum_W N_{[U],[V]}^{[W]} [W]$$

where the sum is over a complete set of representatives of equivalence classes of the simple objects. In the above situation where we have finitely many equivalence classes labelled by  $\{0, 1, \dots, N\}$  we write  $N_{ij}^k$  when  $U \simeq V_i$ ,  $V \simeq V_j$  and  $W \simeq V_k$ .

**Lemma 3.3.** *For all  $U, V, W$  in  $\mathcal{C}$  we have natural isomorphisms  $\text{Hom}_{\mathcal{C}}(U \otimes V, W) \rightarrow \text{Hom}_{\mathcal{C}}(U, W \otimes V^*)$  and  $\text{Hom}_{\mathcal{C}}(V^* \otimes U, W) \rightarrow \text{Hom}_{\mathcal{C}}(U, V \otimes W)$ . In particular, we have  $\text{Hom}_{\mathcal{C}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(I, U \otimes V^*)$ . Finally we have  $\dim \text{Hom}_{\mathcal{C}}(U, V) = \dim \text{Hom}_{\mathcal{C}}(U, V)$ .*

*Proof.* The statements about the natural isomorphism follow by graphical calculus and are true in any ribbon category. In particular The final assertion however uses the semisimplicity of  $\mathcal{C}$  and the self-duality of  $I$ . It follows from the fact that for any object  $W$  of  $\mathcal{C}$  one has  $\dim \text{Hom}_{\mathcal{C}}(I, W) = \dim \text{Hom}_{\mathcal{C}}(I, W^*)$ . Indeed, we have  $\dim \text{Hom}_{\mathcal{C}}(I, W) = d$  iff  $W \approx I^d \oplus W'$  with  $W'$  disjoint from  $I$  (i.e. all its simple summands are inequivalent to  $I$ ). But then also  $W^* \approx I^d \oplus W'^*$  with  $W'^*$  disjoint from  $I$ , by the self duality of  $I$ .  $\square$

Hence we have the following rules for the fusion coefficients (using the fact that  $N_{ij}^k = \dim \text{Hom}_{\mathcal{C}}(V_k, V_i \otimes V_j) = \dim \text{Hom}_{\mathcal{C}}(I, V_i \otimes V_j \otimes V_k^*)$ ):

$$(3.2) \quad N_{0j}^k = N_{j0}^k = \delta_j^k;$$

$$(3.3) \quad N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}$$

$$(3.4) \quad N_{ij}^0 = \delta_{i^*}^j$$

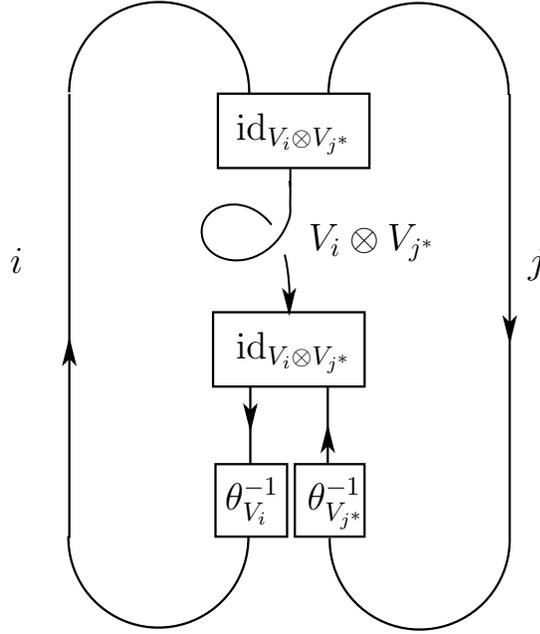


FIGURE 2.

Since the twist  $\theta$  is an endomorphism of the identity functor  $\theta$  must be (by Schur's lemma) a multiple  $\theta_V$  of the identity on each simple module  $V$ . We write  $\theta_i$  if  $V = V_i$ . The quantum dimension of  $V_i$  is denoted by  $d_i \in k = \text{End}_{\mathcal{C}}(I)$  (this last isomorphism is canonical and follows from Schur's lemma and (a) of definition 2.2). We have some easy consequences of the properties of traces and twists in ribbon categories:

$$(3.5) \quad \theta_0 = 1, \theta_{i^*} = \theta_i$$

$$(3.6) \quad d_0 = 1, d_{i^*} = d_i, d_i d_j = \sum_k N_{ij}^k d_k$$

Finally we have the following symmetries in the  $\tilde{s}_{ij}$  which are easily proved by the graphical calculus:

$$(3.7) \quad \tilde{s}_{ij} = \tilde{s}_{ji} = \tilde{s}_{i^*j^*} = \tilde{s}_{j^*i^*}; \tilde{s}_{i0} = \tilde{s}_{0i} = d_i$$

**Lemma 3.4.** *In a modular tensor category we have*

$$(3.8) \quad \tilde{s}_{ij} = \theta_i^{-1} \theta_j^{-1} \sum_k N_{i,j^*}^k \theta_k d_k$$

*Proof.* See figure 2 □

**Lemma 3.5.** *In a semisimple ribbon category  $\mathcal{C}$  we have  $d_i \neq 0$  for all simple objects  $V_i$ .*

*Proof.* Recall from week 46 the formula for the dimension  $d_i$ :

$$d_i := \text{tr}_{\mathcal{C}}(\text{id}_{V_i}) = d_{V_i} \circ (\text{id}_{V_i^*} \otimes \theta_{V_i}^{-1}) \circ c_{V_i^*, V_i}^{-1} \circ b_{V_i} = \theta_i^{-1} d_{V_i} \circ c_{V_i^*, V_i}^{-1} \circ b_{V_i}$$

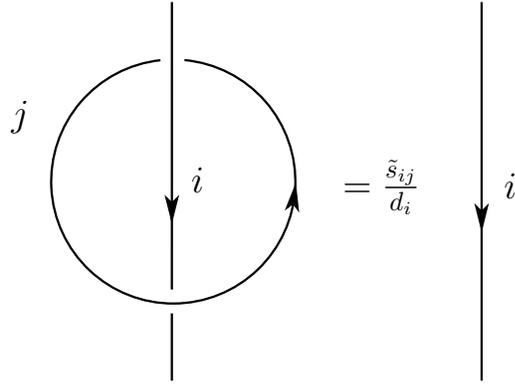


FIGURE 3.

We know that  $N_{i,i^*}^0 = 1$  so  $V_i \otimes V_{i^*} = X \oplus X'$  with  $X$  isomorphic to  $V_0 = I$  and  $X'$  disjoint from  $I$ . Similarly  $V_{i^*} \otimes V_i = Y \oplus Y'$  with  $Y$  isomorphic to  $I$  and  $Y'$  disjoint with  $I$ . The isomorphism  $c_{V_{i^*}, V_i}^{-1}$  therefore restricts to a direct sum of an isomorphism from  $Y \rightarrow X$  and an isomorphism from  $Y' \rightarrow X'$ . The evaluation  $d_{V_i} : V_{i^*} \otimes V_i \rightarrow I$  factors through a map  $Y \rightarrow I$  and similarly the co-evaluation  $b_{V_i} : I \rightarrow V_i \otimes V_{i^*}$  factors through a map  $I \rightarrow X$ . These maps are both nonzero, a fact that follows easily from the duality axioms. Since  $I$ ,  $X$  and  $Y$  are isomorphic and simple, Schur's lemma implies that these maps are isomorphisms. In particular, the above composition of maps is a composition of three isomorphisms and hence nonzero.  $\square$

**Lemma 3.6.** *See figure 3.*

*Proof.* Since  $V_i$  is irreducible we know that the left hand side is equal to a constant,  $a_{ij} \in k$  say, times  $\text{id}_{V_i}$ . To compute  $a_{ij}$  we close both sides of figure 3 to get figure 4. The left hand side is  $\tilde{s}_{ij}$ , proving the result (by using the previous Lemma).  $\square$

**Definition 3.7.** *We extend the graphical calculus for a  $k$ -linear modular tensor category by admitting coloring of the strands in a ribbon graphs by elements of the Verlinde algebra  $K$ . This simply represents the morphism in  $\mathcal{C}$  obtained by taking the associated  $k$ -linear combination of actual  $\mathcal{C}$ -colored ribbon graphs.*

*We will moreover use the convention in this graphical calculus that when we leave a strand uncolored this means that it is colored by the element in  $R \in K$  defined by  $R = \sum_i d_i [V_i]$ .*

**Lemma 3.8.** *See figure 5. Here  $\Delta^\pm := \sum_i \theta_i^\pm d_i^2$ .*

*Proof.* We prove the positive identity; the other sign being similar. Again it is clear that the left hand side equals  $\Delta^+ \theta_i^{-1} \text{id}_{V_i}$  for some constant  $\Delta^+ \in k$ . To compute this constant  $\Delta^+$  we close up the diagram, after multiplication by  $\theta_i$ . As in figure 2 we obtain figure 6 representing  $d_i \Delta^+$  (here we have used the self duality of  $R$  to reverse the arrow of the left

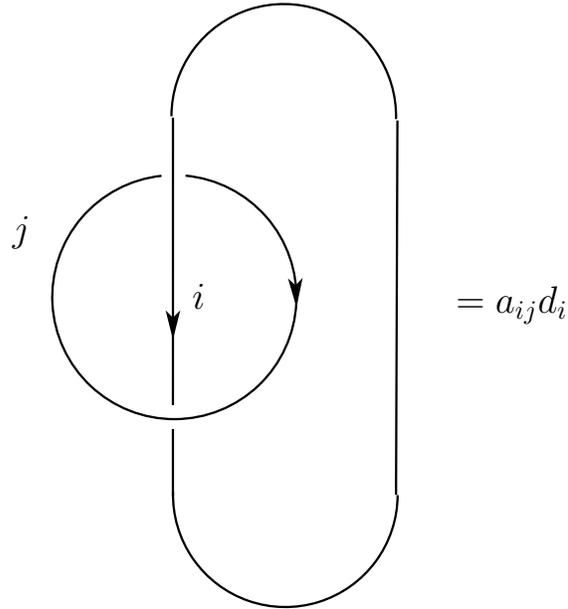


FIGURE 4.

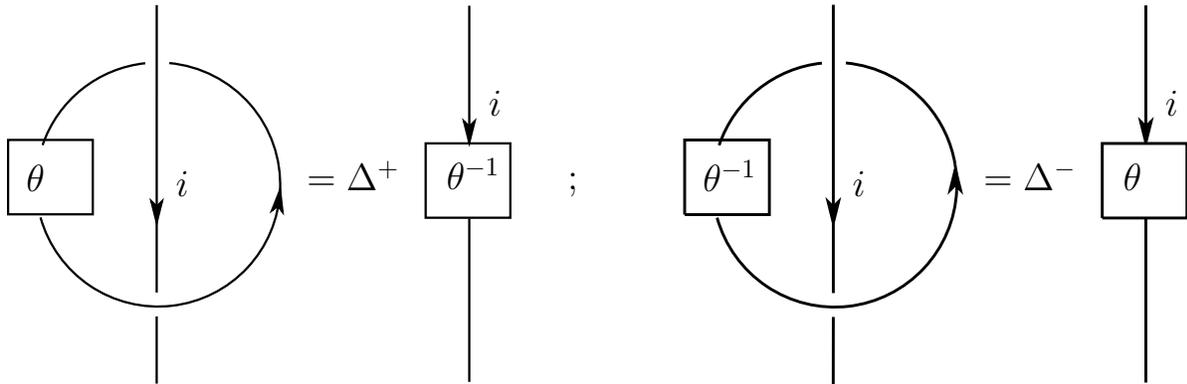


FIGURE 5.

hand side without changing the color  $R$ ). Decomposing  $R \otimes V_i$  in irreducibles and using (3.3), (3.6) we obtain:

$$(3.9) \quad d_i \Delta^+ = \sum_{j,k} d_j N_{ij}^k \theta_k d_k = \sum_{j,k} N_{ik}^{j*} d_j \theta_k d_k = \sum_k \theta_k d_i d_k^2$$

from which the assertion follows. □

**Corollary 3.9.** See figure 7.

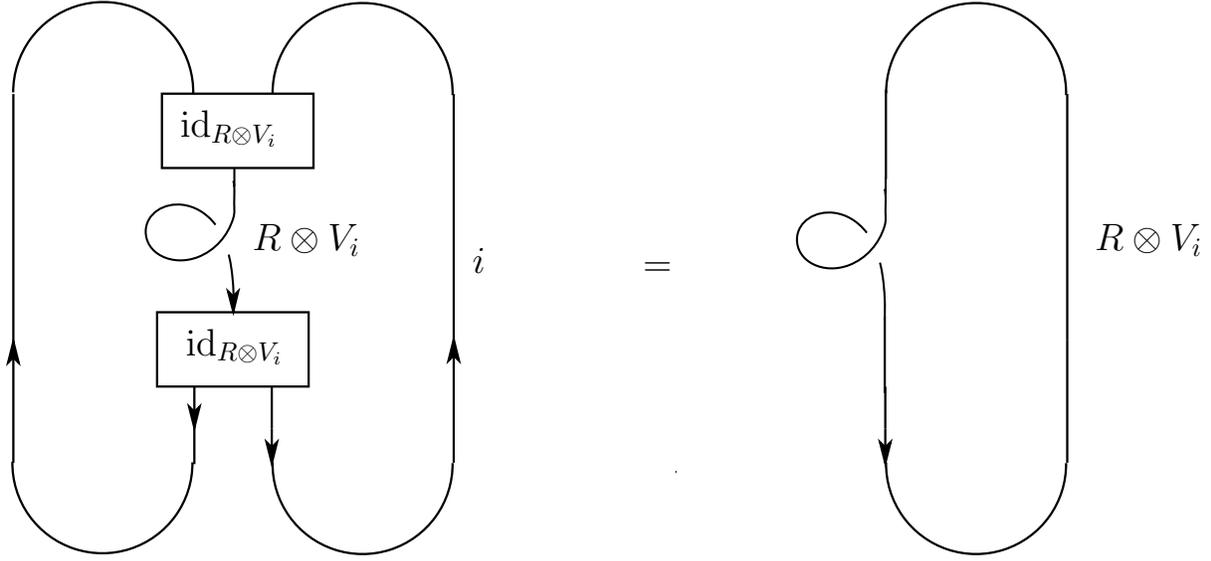


FIGURE 6.

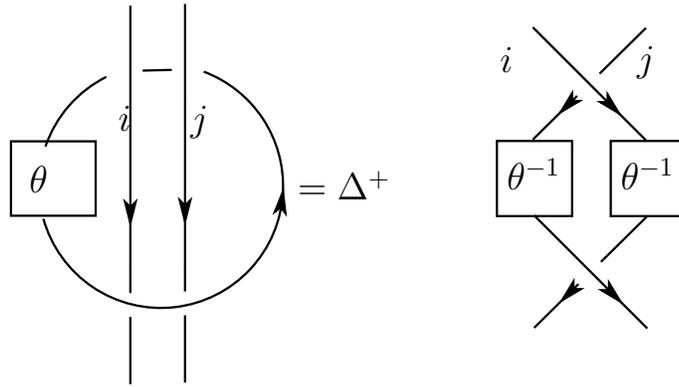


FIGURE 7.

**Theorem 3.10.** *Let  $\mathcal{C}$  be a  $k$ -linear modular tensor category. Consider the linear operators on  $K$  given in the standard basis  $\{[V_i]\}$  by the matrices  $\tilde{s} = (\tilde{s}_{ij})$ ,  $t = (t_{ij})$  and  $c = (c_{ij})$  (the “charge conjugation matrix”) where  $t_{ij} = \delta_{ij}\theta_i$  and  $c_{ij} = \delta_{ij^*}$ . These operators satisfy the following relations:*

$$(3.10) \quad (\tilde{s}t)^3 = \Delta^+ \tilde{s}^2$$

$$(3.11) \quad (\tilde{s}t^{-1})^3 = \Delta^- \tilde{s}^2 c$$

$$(3.12) \quad ct = tc, c\tilde{s} = \tilde{s}c, c^2 = 1$$

$$(3.13) \quad \tilde{s}^2 = \Delta^+ \Delta^- c$$

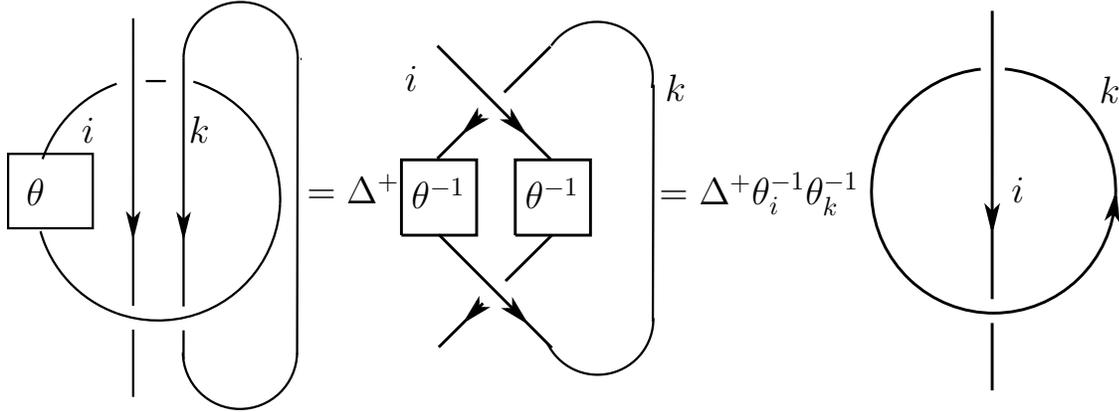


FIGURE 8.

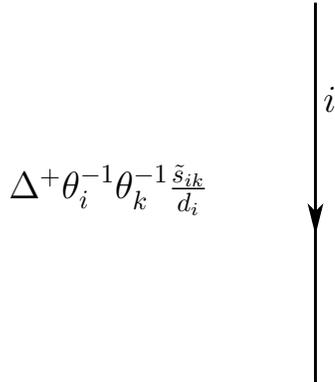


FIGURE 9.

*Proof.* Observe that (3.12) follows from the listed symmetries of  $t$  and  $\tilde{s}$ . We first show that (3.13) follows from (3.10), (3.11), (3.12) since  $\tilde{s}$  is invertible. Indeed, by (3.10), (3.11), (3.12) we see

$$(3.14) \quad t\tilde{s}t = \Delta^+ \tilde{s}^{-1} t^{-1} \tilde{s}$$

$$(3.15) \quad t^{-1} \tilde{s} t^{-1} = \Delta^- c \tilde{s}^{-1} t \tilde{s}$$

Now multiply these equations to obtain  $t\tilde{s}^2 t^{-1} = \Delta^+ \Delta^- c$ . Conjugation by  $t^{-1}$  finally gives the desired result. Therefore it remains to prove (3.10) and (3.11). We show the case of (3.10) (the other case being similar) using graphical calculus. Consider figure 8. The first identity follows from Corollary 3.9, and the second is obvious. Using Lemma 3.6 the right hand side is easily seen to be equal to the expression of figure 9. On the other hand we can rewrite the left hand side as in figure 10. Applying twice Lemma 3.6 one can

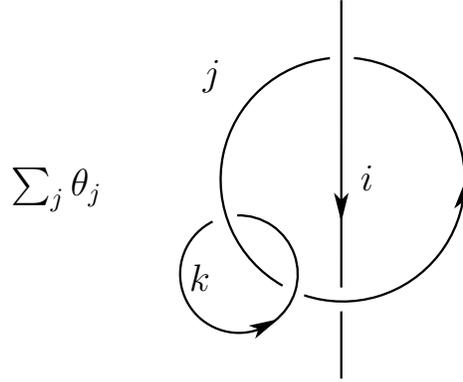


FIGURE 10.

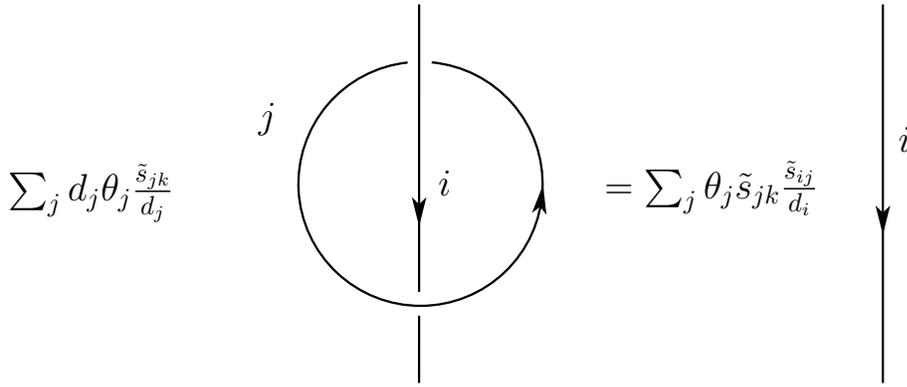


FIGURE 11.

rewrite this as in figure 11. We conclude that we have the following identity

$$(3.16) \quad \sum_j \tilde{s}_{ij} \theta_j \tilde{s}_{jk} = \Delta^+ \theta_i^{-1} \tilde{s}_{ik} \theta_k^{-1}$$

or

$$(3.17) \quad \tilde{s}t\tilde{s} = \Delta^+ t^{-1} \tilde{s}t^{-1}$$

proving (3.10). The proof of (3.11) is similar. □

**Corollary 3.11.** *In a  $k$ -linear modular category,  $\Delta^\pm$  are nonzero.*

**Definition 3.12.** *Let  $\mathcal{C}$  be a  $k$ -linear modular tensor category. We write  $\Delta := \sqrt{\Delta^+ \Delta^-}$  and  $\zeta := (\Delta^+ / \Delta^-)^{1/6}$ . Define  $s := \tilde{s} / \Delta$ .*

The relations of Theorem 3.10 translate as follows:

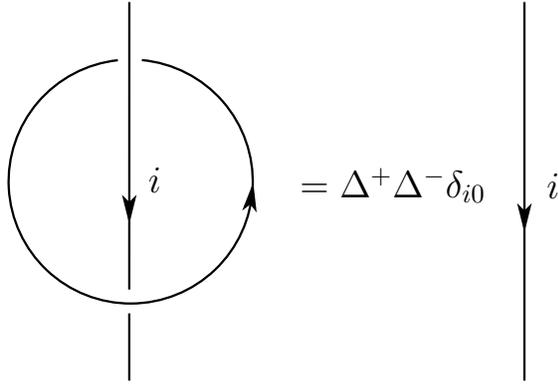


FIGURE 12.

**Theorem 3.13.** *We have the relations:*

$$(3.18) \quad (st)^3 = \zeta^3 s^2, \quad s^2 = c, \quad ct = tc, \quad c^2 = 1.$$

These relations are closely related to the modular group  $\mathrm{SL}_2(\mathbb{Z})$ . Indeed,  $\mathrm{SL}_2(\mathbb{Z})$  has generators  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . These satisfy the relations

$$(3.19) \quad (ST)^3 = S^2, \quad S^4 = 1$$

and we see easily that  $C = S^2$  is central (either from the explicit matrix for  $S$  or

$$(3.20) \quad CT = S^2T = (ST)^3T = (TS)^3T = T(ST)^3 = TC$$

using the above relations) and has order 2. It follows that the matrices  $s$  and  $t$  form a projective representation of  $\mathrm{SL}_2(\mathbb{Z})$  (in fact  $s$  and  $t/\zeta$  define a genuine representation of  $\mathrm{SL}_2(\mathbb{Z})$ ).

**Corollary 3.14.** *See figure 12.*

*Proof.* As usual it is clear that the identity is true up to a scalar multiple. To compute this constant we take traces on both sides of figure 12. Using Lemma 3.6 we see that this constant is

$$(3.21) \quad d_i^{-1} \sum_j d_j \tilde{s}_{ij} = d_i^{-1} \sum_j \tilde{s}_{ij} \tilde{s}_{j0} = d_i^{-1} \tilde{s}_{i0}^2 = d_i^{-1} \Delta^+ \Delta^- c_{i,0} = \Delta^+ \Delta^- \delta_{i0}$$

proving the result. □

**Corollary 3.15.** *We have  $\Delta^+ \Delta^- = \sum_i d_i^2$ .*

*Proof.* This is simply the special case  $i = 0$  in the previous corollary. □

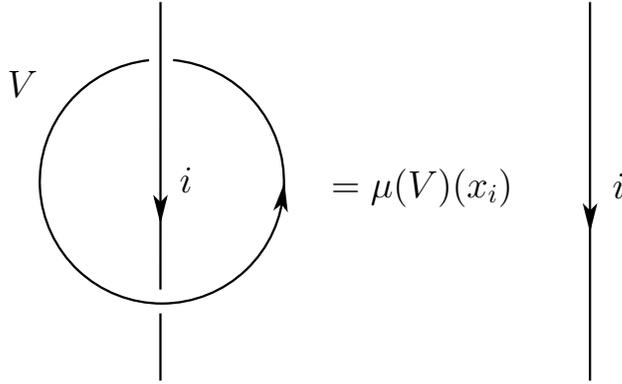


FIGURE 13.

**Theorem 3.16.** *Let  $\mathcal{C}$  be a  $k$ -linear modular tensor category and let  $F$  be the  $k$ -linear dual of  $K$ . We equip  $F$  with the unique unital associative commutative  $k$ -algebra structure in which the basis  $(v^i)$  of  $F$  which is dual to the basis  $(x_i)$  of  $K$  (where we have written  $x_i = [V_i]$  for brevity) consist of mutually orthogonal idempotents of  $F$ . We define a  $k$ -linear map  $\mu : K \rightarrow F$  defined by figure 13. Then  $\mu$  is an algebra isomorphism.*

*Proof.* It is clear from graphical calculus that  $\mu$  is an algebra homomorphism. When we renormalize the basis of  $F$  to  $\epsilon^i := v^i/s_{i0}$ , then by Lemma 3.6 the homomorphism  $\mu$  is given by  $\mu(x_j) = \sum_i \tilde{s}_{ij}/\tilde{s}_{i0}v^i = \sum_i s_{ij}\epsilon^i$ . The invertibility of  $\mu$  follows, completing the proof.  $\square$

The elements  $\epsilon_i := \mu^{-1}(\epsilon^i)$  too form a basis of  $K$ . In the proof of the previous theorem we saw that

$$(3.22) \quad x_j = \sum_i s_{ij}\epsilon_i$$

Combining this with the multiplication rule

$$(3.23) \quad \epsilon_i\epsilon_j = \delta_{ij}\epsilon_j/s_{j0}$$

we obtain that the multiplication by  $x_i$  in  $K$  in the new basis takes the diagonal form

$$(3.24) \quad x_i\epsilon_j = (s_{ij}/s_{0j})\epsilon_j$$

If we denote by  $M(x_i)$  the matrix of the left multiplication by  $x_i$  on  $K$  with respect to the basis  $(x_j)$ , and if  $D_i$  denotes the diagonal matrix  $(D_i)_{ab} = \delta_{ab}(s_{ia}/s_{0a})$  then we see that

$$(3.25) \quad sM(x_i)s^{-1} = (D_i)_{ab}$$

This fact is described by saying that “the s-matrix diagonalizes the fusion rules.” This gives a way to express the fusion coefficients in terms of the s-matrix:

**Theorem 3.17.** (*Verlinde formula*)

$$(3.26) \quad N_{ij}^k = \sum_r \frac{s_{ir}s_{jr}s_{k^*r}}{s_{0r}}$$

*Proof.* We have  $sM(x_i) = D_i s$  or, using the symmetry of  $s_{ij}$ ,

$$(3.27) \quad \sum_a N_{ij}^a s_{ar} = \frac{s_{ir}s_{jr}}{s_{0r}}$$

Now multiply both sides with  $s_{rk^*}$  and sum over  $r$ ; using that  $s^2 = c$  we get the desired result.  $\square$

## 2. RESHETIKHIN-TURAEV-WITTEN INVARIANTS OF 3-MANIFOLDS

Now we will use the acquired knowledge on the properties of the Verlinde algebra to construct the 3-manifold invariants of the TQFT attached to a  $k$ -linear modular tensor category  $\mathcal{C}$ .

1. **Surgery along framed links in  $S^3$ .** The geometric basis of the way of constructing 3-manifold invariants is the following deep theorem of Lickorish and (independently) Wallace:

**Theorem 1.1.** *Let  $M$  be a closed connected oriented topological 3-manifold. There exists a framed link  $L \subset S^3$  such that  $M$  is homeomorphic to the manifold obtained by applying “surgery” along  $L$  on  $S^3$ .*

To explain the surgery operation, let  $L_1, \dots, L_m$  be the components of the link. Recall that every link component  $L_i$  has a closed tubular neighborhood  $U_i$  which is homeomorphic to  $S^1 \times B^2$ , and we can choose these neighborhoods sufficiently small such that they are mutually disjoint. Choose a direction for each component (the result of the surgery will not depend on this choice, up to homeomorphisms). Obviously  $T_i = \partial(U_i)$  is a 2-dimensional torus. On  $T_i$  we can define a meridian  $\alpha_i$  (it is the boundary of a disk  $B^2$  in  $U_i$ , oriented according to the corkscrew rule relative to the direction of  $L_i$ ; this is a canonically defined homology class once the direction has been chosen) and the longitude  $\beta_i$  defined by the framing  $f_i$  of  $L_i$ . Observe that the basis  $([\alpha_i], [\beta_i])$  of  $H_1(T_i, \mathbb{Z})$  is positive with respect to the orientation of  $T_i$  by the outward normal vector with respect to  $U_i$ . Let  $U = S^1 \times B^2$  an abstract oriented solid torus. Choose orientations of the meridian  $\alpha = \{1\} \times S^1$  and longitude  $\beta = S^1 \times \{1\}$  such that  $([\alpha], [\beta])$  is a positive basis of  $H_1(T, \mathbb{Z})$  (i.e.  $[\alpha \wedge \beta]$  is positive).

Now we choose (orientation preserving) homeomorphisms  $g_i : \partial(U_i) \rightarrow \partial(U) = S^1 \times S^1$ , such that  $g_i(\alpha_i) = -\beta = -(S^1 \times \{1\})$  and  $g_i(\beta_i) = \alpha := \{1\} \times S^1$  (the minus sign refers to the orientation of the curves). This is possible and unique up to isotopy by the following classical result:

**Lemma 1.2.** *For any orientation preserving isomorphism  $G_i : H_1(T_i, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})$  there exists an orientation preserving homeomorphism  $g_i : T_i \rightarrow T$  such that  $G_i = (g_i)_*$ , the homomorphism induced by  $g_i$  on the first singular homology group. The isotopy class of the homeomorphism  $g_i$  is uniquely determined by its induced action  $G_i = (g_i)_*$  on*

$H_1(T_i, \mathbb{Z})$ . In particular, the mapping class group of isotopy classes of orientation preserving self-homeomorphisms of a compact two dimensional torus  $T$  (also called the Teichmüller modular group, or the homeotopy group of  $T$ ) is canonically isomorphic to  $\text{Aut}_{\mathbb{Z}}(H_1(T, \mathbb{Z})) \simeq \text{SL}_2(\mathbb{Z})$  via the isomorphism  $g \rightarrow g_*$ .

*Proof.* The universal covering space of a two dimensional compact torus  $T \approx S^1 \times S^1$  is  $\mathbb{R}^2$ , with as group of deck transformations the group of translations of a lattice  $X$  in  $\mathbb{R}^2$ . There is a natural isomorphism  $H_1(T, \mathbb{Z}) \approx X$ , and if we choose a base point  $e \in T$  then there is also a natural isomorphism  $\pi_1(T, e) \approx X$ . Choose base points  $e_i \in T_i$  and  $e \in T$ . We may in fact compose any homeomorphism  $T_i \rightarrow T$  with a translation in  $T$  without changing its isotopy type, hence we may always choose representatives of the isotopy classes of homeomorphisms which map  $e_i$  to  $e$ .

By elementary lifting results the homeomorphism  $h : \partial(U_i) \rightarrow \partial(U)$  gives rise to an isomorphism  $H_* : X_i \approx H_1(T_i, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z}) \approx X$  and a homeomorphism  $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  restricting to the isomorphism  $H_* : X_i \rightarrow X$ . On the other hand, any isomorphism  $H_* : X_i \rightarrow X$  can be extended linearly to a linear isomorphism  $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which descends to a homeomorphism  $h : T_i \rightarrow T$  such that  $H_*(h) = H$ . Hence given  $g_i$  we can construct the linear homeomorphism  $h : T_i \rightarrow T$  such that  $h(e_i) = e$  and such that  $H|_{X_i} = G_i|_{X_i}$ . In this situation we can use the classical result that two orientation preserving homeomorphisms between closed connected oriented surfaces are isotopic iff they are homotopic. But in the situation above it is clear that  $h$  and  $g_i$  are homotopic, by the homotopy of linear convex combinations of  $h$  and  $g_i$ . This finishes the proof.  $\square$

Let  $f_i : \partial(U_i) \rightarrow \partial(U)$  be the same homeomorphism as  $g_i$  but with the domain  $\partial(U_i)$  equipped with the opposite orientation (i.e. now we consider  $\partial(U_i)$  as a boundary component of the oriented 3-manifold  $S^3 \setminus \cup_i (\text{int}(U_i))$ ). Let  $f$  be the disjoint union of the maps  $f_i$ . The map  $f$  is an (orientation reversing!) homeomorphism of the boundary components of the 3-manifold  $S^3 \setminus \cup_i (\text{int}(U_i))$  to the boundary components of the  $m$ -fold disjoint union of the standard solid torus  $(S^1 \times B^2)^{\times m}$ , and is determined up to isotopy by the above recipe in terms of the framing of  $L$ .

**Definition 1.3.** Given a fixed homeomorphism  $f$  as described, we define

$$(1.1) \quad M_{L,f} := (S^3 \setminus \cup_i (\text{int}(U_i))) \cup_f (S^1 \times B^2)^{\times m}$$

This defines  $M_{L,f}$  up to homeomorphisms since it is well known that the identification space  $M_{L,f}$  only depends, up to homeomorphisms, on the isotopy class of  $f$  (see [1, Lemma 4.1.1](iii)). In fact if two gluing homeomorphisms  $f, f'$  are connected by an orientation preserving homeomorphism  $\phi$  of  $(S^1 \times S^1)^{\times m}$  with itself (i.e. we have  $f' = \phi \circ f$ ) such that  $\phi$  extends to a homeomorphism  $\Phi$  of  $(S^1 \times B^2)^{\times m}$  then the corresponding spaces  $M_{L,f}$  and  $M_{L,f'}$  are also homeomorphic. For one solid torus component  $U = S^1 \times B^2$  the set of isotopy classes of self homeomorphisms of the boundary  $T^2 = S^1 \times S^1$  which extend to self homeomorphisms of  $U$  is generated by the so-called Dehn-twists along the meridian curve  $\alpha$  (these correspond to the matrices  $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ ) and by the central element  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . Here

we have chosen the positive basis  $([\alpha], [\beta])$  for  $H_1(T, \mathbb{Z})$  with  $\alpha = \{1\} \times S^1$  the meridian, and  $\beta = S^1 \times \{1\}$  a longitude as usual. These remarks yield the following result:

**Theorem 1.4.** *Up to homeomorphisms  $M_{L,f}$  only depends on the class of  $L$  as a framed link. In particular,  $M_{L,f}$  is independent of the chosen directions of the components  $L_i$ , and also of the choices of the homeomorphisms  $f_i$  satisfying the description in terms of the framing of  $L$ . We denote the resulting isomorphism class of closed connected oriented 3-manifolds by  $M_L$ .*

1.1. *Examples.* The first important example is  $M_{L_0}$ , where  $L_0$  denotes the unframed unknot in  $S^3$ . We claim that  $M_{L_0} \approx S^1 \times S^2$ . Indeed, we have the Hopf decomposition  $S^3 \approx \partial(B^2 \times B^2) = (B^2 \times S^1) \cup_{\text{id}} (S^1 \times B^2)$  where  $\text{id}$  is the orientation reversing homeomorphism that interchanges the roles of meridian and longitude when we change the view from  $S^1 \times S^1$  as boundary of  $U_{L_0} = S^1 \times B^2$  to the boundary of the complement  $B^2 \times S^1$  of its interior in  $S^3$ . Now the homeomorphism  $g$  of Definition 1.3 corresponding to the 0-framing changes the role of meridian and longitude once more, sending  $\alpha_i \rightarrow -\beta$  and  $\beta_i \rightarrow \alpha$ . As we have argued above, we may in addition compose the gluing map with  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  without changing  $M_{L_0}$ , after which the gluing map  $f$  is prescribed by  $f_*([\alpha_0]) = -[\alpha]$  and  $f_*([\beta_0]) = [\beta]$ . Thus  $M_{L_0}$  consists of the two solid tori  $U_0 := S^3 \setminus \text{int}(U(L_0)) = B^2 \times S^1$  and  $U = S^1 \times B^2$  with the boundaries identified by the gluing map  $f$  such that meridian  $\alpha_0 = S^1 \times \{1\}$  maps to the opposite meridian  $-\alpha = -(\{1\} \times S^1)$  of  $U = S^1 \times B^2$ , and the longitude  $\beta_i = \{1\} \times S^1$  to  $\beta = S^1 \times \{1\}$ . In particular both solid tori have a common longitude onto which they project and the fiber of this projection consists of two disks  $B^2$  glued at the boundary by the identity map. Hence we obtain

$$(1.2) \quad M_{L_0} = (B^2 \cup_{\text{id}} B^2) \times S^1 = S^2 \times S^1$$

The second example is the case where we consider  $L_1$ , the unknot with framing 1. In that case one can show that one recovers  $M_{L_1} = S^3$ . Indeed, the orientation reversing map  $f : \partial(U_0) \rightarrow \partial(U)$  is now given by the matrix (with respect to the bases  $([\alpha_0], [\beta_0])$  and  $([\alpha], [\beta])$ )

$$(1.3) \quad f := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Indeed,  $[\alpha_0] = [\beta_1] - [\alpha_1]$  and  $[\beta_0] = [\alpha_1]$ . Hence  $[\alpha_0]$  maps to  $[\alpha] + [\beta]$  and  $[\beta_0]$  to  $-[\beta]$ . Again, since we are gluing two solid tori by means of the homeomorphism  $f$  of the boundaries we can compose  $f$  on both sides with orientation preserving homeomorphisms which extend to the full solid torus, without changing the space  $M_{L,f}$  up to homeomorphisms. In the bases we are using these are upper triangular matrices (since the meridian is the first basis element). In addition we may compose with  $-\text{id} \in \text{SL}_2(\mathbb{Z})$  without changing the space. Therefore the computation

$$(1.4) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

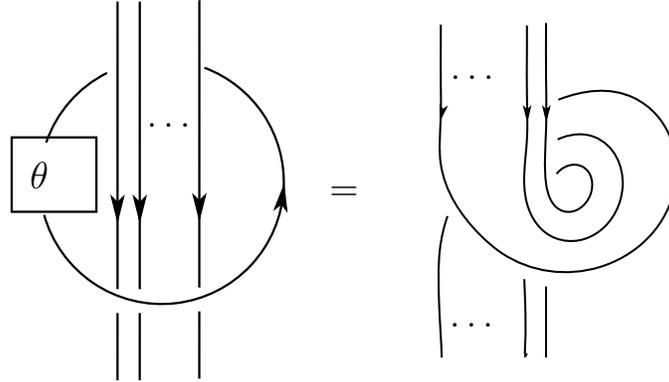


FIGURE 14.

shows it is equivalent to take  $f$  equal to the homeomorphism  $\text{id} : \partial(U_0) \rightarrow \partial(U)$  described in the beginning of the previous example, proving the assertion.

1.2. *Kirby calculus.* It is quite possible that we are given two inequivalent ribbon links  $L$  and  $L'$  while still  $M_L \approx M_{L'}$ . In fact we have seen an example of this phenomenon in the second example above, where we saw that surgery of  $S^3$  along a single unknot with framing 1 does not change  $S^3$  up to homeomorphisms. This is a special case of a so-called “Kirby move” which we will describe below. The existence of such homeomorphisms obstructs the definition of 3-manifold invariants using general invariants of ribbon links, since one needs additional requirements ensuring that whenever  $M_L \approx M_{L'}$  the ribbon link invariant in question will assign the same value to  $L$  and  $L'$ .

The quantum invariants of ribbon links (RT-invariants) associated with ribbon categories are in general much too strong for this and can not be used. However, if the RT-invariants are associated with the coloring of all strands by the regular representation  $R$  of a modular tensor category then essentially the RT-invariant descends to a 3-manifold invariant, as was shown by Reshetikhin and Turaev. The central technical result in the proof is the precise description of the ribbon links defining the same 3 manifold, due to Kirby (and proved by Fenn and Rourke).

**Theorem 1.5.** *Let  $L$  and  $L'$  be framed links in  $S^3$ . Then  $M_L \approx M_{L'}$  iff  $L$  and  $L'$  are connected by a sequence of “Kirby moves” (or their inverses). The Kirby moves are described by figure 14, where the number of strands inside the single ribbon band with framing 1 can be arbitrary (including 0).*

1.3. *Definition of the Reshetikhin-Turaev invariants.* We define the wreath number  $\sigma(L)$  of a framed link  $L \subset S^3$  as the signature of the canonical intersection pairing  $H_2(W_L, \mathbb{Z}) \times H_2(W_L, \mathbb{Z}) \rightarrow \mathbb{Z}$  on the second homology of the 4-manifold  $W_L$  which is obtained by gluing  $m$  handles  $B^2 \times B^2$  with their boundary component  $S^1 \times B^2$  to the tubular neighbourhood  $U_i$  of  $L_i$  in such a way that  $M_L = \partial(W_L)$  (see [2, Section 4]). It is known that this number  $\sigma(L)$  can be easily computed as the signature (i.e. the number of positive eigenvalues minus

the number negative eigenvalues) of the linking matrix of  $L$ , i.e. the  $m \times m$  symmetric integral matrix  $A$  with entries  $A_{i,j} = \text{Lk}(L_i, L_j)$  if  $i \neq j$  and  $A_{i,i} = \text{Fr}(L_i)$  where we chose (arbitrarily) directions and a numbering  $L_1, L_2, \dots, L_m$  of the components of  $L$ . Then

**Theorem 1.6.** *Let  $L \subset S^3$  be an  $m$ -component framed link. Let  $\mathcal{C}$  denote a  $k$ -linear modular tensor category, and let  $F_{RT}^{\mathcal{C}}$  be the associated RT-functor on  $\mathcal{T}_{\mathcal{C}}^G$ . The  $k$ -valued function*

$$(1.5) \quad \tau^{\mathcal{C}}(M_L) := \Delta^{-m-1} \zeta^{-3\sigma(L)} F_{RT}^{\mathcal{C}}(L_R)$$

*only depends on the 3-manifold  $M_L$  up to homeomorphisms. This defines an invariant of closed, connected, oriented 3-manifolds, the Reshetikhin-Turaev invariant  $\tau^{\mathcal{C}}$ .*

*Proof.* We need to show that  $\tau^{\mathcal{C}}$  is invariant for Kirby moves. If  $\Delta^+ = \Delta^- = \Delta$  this follows immediately from Lemma 3.6. The general case is not much harder, but one needs to study the behaviour of  $\sigma(L)$  under the Kirby moves.  $\square$

**Example 1.7.** *We have  $S^3 = M_{\emptyset}$ . Clearly  $\tau^{\mathcal{C}}(S^3) = \Delta^{-1}$  since  $\sigma(\emptyset) = 0$ . Indeed,  $S^3 = \partial(B^4)$  and  $H_2(B^4) = 0$ , so  $\sigma(\emptyset) = 0$ ,  $m = 0$  and  $F_{RT}^{\mathcal{C}}(\emptyset_R) = 1$ .*

**Example 1.8.** *As we have seen, we can also consider  $S^3$  as the surgery of  $S^3$  along the unknot with framing 1. Indeed, in that case we have  $\sigma(L) = 1$ ,  $m = 1$ , and  $F_{RT}^{\mathcal{C}}(L_R) = \Delta^+$ .*

**Example 1.9.** *Similarly we can also view  $S^3$  as obtained from the surgery of  $S^3$  along the unknot with framing  $-1$ . We leave it to the reader to verify this as in the above examples, and compute the invariant again in this way.*

**Example 1.10.** *For  $k \geq 2$  the Lens space  $L(k, 1)$  is by definition the 3-manifold obtained by surgery along the unknot with framing  $k$ . As in the above examples we see that*

$$(1.6) \quad \tau^{\mathcal{C}}(L(k, 1)) = \Delta^{-2} \zeta^{-3} \sum_i \theta_i^k d_i^2$$

**Exercise (a).** *Compute  $\tau^{\mathcal{C}}(S^2 \times S^1)$ .*

It is not difficult to see that  $M_{-L} = -M_L$  where  $-L$  denoted the mirror image of the framed link  $L$  and where, for a given oriented 3-manifold  $M$ , the notation  $-M$  is used to denote the same manifold with the opposite orientation. For example  $L(-k, 1) = -L(k, 1)$ .

**Exercise (b).** *Using the above, compute  $\tau^{\mathcal{C}}(-L(k, 1))$ .*

If  $L_1, L_2 \subset S^3$  are two disjoint, mutually unlinked links (i.e.  $L_1$  and  $L_2$  can be separated by a hyperplane in  $S^3$  after a suitable ambient isotopy) then it is clear that  $M_{L_1 \cup L_2} = M_{L_1} \# M_{L_2}$  (the connected sum of  $M_{L_1}$  and  $M_{L_2}$ ). Therefore it is easy to see that

**Theorem 1.11.** *For all closed connected oriented 3-manifolds  $M_1$  and  $M_2$  we have:*

$$(1.7) \quad \tau^{\mathcal{C}}(M_1 \# M_2) = \Delta \tau^{\mathcal{C}}(M_1) \cdot \tau^{\mathcal{C}}(M_2)$$

**Exercise (c).** *Prove Theorem 1.11.*

3. TILTING MODULES AND MODULAR TENSOR CATEGORIES FROM QUANTUM GROUPS AT ROOT OF 1

We have constructed a ribbon category from the ribbon algebra  $U_q(sl_2)$ . In this construction we worked over the field  $\mathbb{C}(q)$  of rational functions in a formal parameter  $q$ . To construct modular tensor categories we need to specialize the parameter  $q$  at a root of 1. In order to be able to do so, we first of all need to replace the field  $\mathbb{C}(q)$  by the subring  $A = \mathbb{C}[q, q^{-1}]$  in which specializations of  $q$  at nonzero complex numbers are well defined. For  $U_q(sl_2)$  this is elementary, we just take the  $A$ -subalgebra  $U_A$  of  $U_q(sl_2)$  generated by  $E, F, K^{\pm 1}$  and  $[K; 0]$  (see [2, Chapter 4, Section 4]).

**Proposition 0.1.** *Let  $\epsilon$  be a complex primitive  $l$ -th root of 1, where  $l$  is odd. Define  $u_\epsilon = U_A \otimes_A \mathbb{C}_\epsilon$ , where  $\mathbb{C}_\epsilon$  denotes the one dimensional complex  $A$ -module in which  $q$  acts as multiplication by  $\epsilon$ . The ribbon algebra structure of  $U_q(sl_2)$  gives rise to the structure of an abelian ribbon category on the category of finite dimensional modules of  $u_\epsilon$ .*

*Proof.* See [2, Chapter 7, Proposition 3.1]. □

There are finitely many finite dimensional irreducible representations  $L_\epsilon(\lambda)$  of  $u_\epsilon$ . These are parameterized by their highest weights  $\lambda \in \{0, \rho, 2\rho, \dots, (l-1)\rho\}$ . For our purpose the most important irreducible  $u_\epsilon$ -modules are the irreducible “tilting modules”, which are the  $L_\epsilon(\lambda)$  with  $\lambda \in \{0, \rho, 2\rho, \dots, (l-2)\rho\}$ . These irreducible tilting modules are simply the specializations at  $q = \epsilon$  of the finite dimensional modules of the generic algebra  $U_q(sl_2)$  with the same highest weight, in the following sense. There exist modules  $L_A(\lambda)$  (with  $\lambda$  in the above range) of  $U_A$  which are free as  $A$ -modules and such that  $L_A(\lambda) \otimes_A \mathbb{C}_q = L_\epsilon(\lambda)$  on the one hand, while on the other hand  $L_A(\lambda) \otimes_A \mathbb{C}(q) = L(\lambda)$ , the corresponding irreducible over  $U_q(sl_2)$ .

The  $L_\epsilon(\lambda)$  are the irreducible objects of the category of tilting modules of  $u_\epsilon$ , which are by definition modules of  $u_\epsilon$  which admit certain types of filtrations. The importance of the category of tilting modules in the theory of modular tensor categories was stressed by the work of the Danish mathematician H.H. Andersen, who worked out their fundamental properties in great generality (working with general simple Lie algebras and more general fields of definition). The tilting modules have the important property that tensor products of tilting modules are tilting, and summands of tilting modules are tilting. Moreover, the indecomposable tilting modules are parameterized by the set of dominant weights (in our situation the set  $\mathbb{Z}_{\geq 0}\rho$ ). Finally it was shown by Andersen that indecomposable tilting modules whose highest weight is above a certain level (in our case  $(l-2)\rho$ ) have the property that the quantum trace of all their endomorphisms is zero (while the irreducible tilting modules with highest weight below this level all have nonzero quantum dimensions). These results show that one can form a semisimple “reduced” tensor category with finitely many simple objects, namely the irreducible tilting modules described above, by defining the reduced tensor product of two tilting modules to be the quotient of the ordinary tensor product by the submodule which is the direct sum of all indecomposable tilting submodules with quantum dimension zero. This turns out to be a modular tensor category in the sense of Reshetikhin and Turaev.

There are many details to this story. We refer the interested reader to the original papers by Andersen for further reading.

#### REFERENCES

- [1] B. Bakalov, A. Kirillov, Jr., *Lectures on tensor categories and modular functors* (version May 2000).
- [2] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et synthèses **5**, Soc. Math. de France (1997)
- [3] V.G. Turaev, *Quantum invariants of Knots and 3-manifold invariants*, Walter de Gruyter (1994).