QUANTUM GROUPS AND KNOT THEORY LECTURE: WEEK 48

This week in class we treat quantum traces and dimensions in general ribbon categories (from the syllabus of week 47, see also [2, Chapter 6]) and we define the general notion of ribbon algebras and discuss how these give rise relation to ribbon categories. This material will be used in later to give some important explicit constructions of ribbon categories. The syllabus of this week 48 contains material that is not covered in [2] (in particular, this week the numbering of the syllabus does not refer to [2]). A good source for additional reading is [1].

1. Ribbon algebras and ribbon categories

1. **Recollection of some results for braided Hopf algebras.** Let k be a field and let $A = (A, \mu, \eta, \Delta, \epsilon, R)$ be a braided bi-algebra a over k with universal R-matrix $R \in A \otimes A$. Let $\mathbf{Mod}_A$ denote the category of A-modules, i.e. the category whose objects are A-modules and whose morphisms are A-linear maps between A-modules. Let $\mathbf{Mod}_A^f$ be the full subcategory whose objects are the A-modules which are finite dimensional as a k-vector space. Recall from Week 41 that we can equip $\mathbf{Mod}_A^f$ with the structure of a *braided tensor category* where

(1) The monoidal structure of the category $\mathbf{Mod}_A^f$ is given by the the monoidal structure of the underlying category of vector spaces (in particular, it is not strict).
(2) The commutativity constraint (or braiding) c is given by the natural family of A-module isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ (where V, W are f.d. A-modules) defined by:

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\to \tau(R(v \otimes w)) \end{aligned}$$

Here τ denotes the flip of the two tensor legs as usual, and the action of $A \otimes A$ on $V \otimes W$ is defined by $(a \otimes b)(v \otimes w) = av \otimes bw$.

Hence, if $R = \sum_i s_i \otimes t_i \in A \otimes A$ then

$$c_{V,W}(v \otimes w) = \sum_i t_i w \otimes s_i v \tag{1.1}$$

Next assume that A is a braided Hopf algebra with *invertible* antipode S (as in week 41). We introduced in week 41 the element $u \in A$ by

$$u := \sum_i S(t_i) s_i \in A \tag{1.2}$$

and proved that u is invertible with inverse

$$u^{-1} = \sum_i S^{-1}(t_i)S(s_i) \tag{1.3}$$

and the property that $S^2(a) = uau^{-1}$ for all $a \in A$. As a consequence (see week 41, Corollary 2.6), the element $D = uS(u) = S(u)u$ is *central* in A. Recall week 41, Proposition 2.7:

Proposition 1.1. (a) $\epsilon(u) = 1$.
(b) $\Delta(u) = (R_{2,1}R)^{-1}(u \otimes u) = (u \otimes u)(R_{1,2}R)^{-1}$.
(c) $\Delta(S(u)) = (R_{2,1}R)^{-1}(S(u) \otimes S(u)) = (S(u) \otimes S(u))(R_{2,1}R)^{-1}$.
(d) $\Delta(D) = (R_{2,1}R)^{-2}(D \otimes D) = (D \otimes D)(R_{1,2}R)^{-2}$.

The elements u and D of A play an important role in the next paragraph on duality and twist in $\mathbf{Mod}_A^f$.

1.1. *Ribbon algebras and ribbon categories.* Assume that $A = (A, \mu, \eta, \Delta, \epsilon, R, S)$ is a braided Hopf algebra with universal R-matrix R and invertible antipode S as above. We call such algebra also a ribbon algebra, because the category $\mathbf{Mod}_A^f$ of *finite dimensional* A-module, which is a braided monoidal category as we saw above, can be equipped with the structure of a *ribbon category* as we discussed in week 47. This is done as follows:

Let $\mathcal{C}$ be the (braided) monoidal category $\mathbf{Mod}_A^f$ of finite dimensional modules over A.

Definition 1.2. *If $V, W \in \mathcal{C}$ we define an A-module structure on* $\mathrm{Hom}_k(V, W)$ *as follows:*

$$af(v) := \sum_{(a)} a'f(S(a'')v) \tag{1.4}$$

Exercise (a). *Prove that this defines an A-module structure on* $\mathrm{Hom}_k(V, W)$.

In the special case $V^* = \mathrm{Hom}_k(V, k)$ the A-action simplifies to $a\phi = \phi \circ m(S(a))$:

$$\begin{aligned} a\phi(v) &= \sum_{(a)} \epsilon(a')\phi(S(a'')v) \\ &= \phi(S(\sum_{(a)} \epsilon(a')a'')v) \\ &= \phi(S(a)v) \end{aligned}$$

In fact the A-action on $\mathrm{Hom}_k(V, W)$ of Definition 1.2 is just the A action on $W \otimes V^*$ viewed via the usual vector space isomorphism between $W \otimes V^*$ and $\mathrm{Hom}_k(V, W)$ (NB: here we use the assumption that V is finite dimensional!) according to the following:

Proposition 1.3. *The vector space isomorphism*

$$\begin{aligned} \lambda_W^V : W \otimes V^* &\to \mathrm{Hom}_k(V, W) \\ w \otimes \phi &\to \{v \to \phi(v)w\} \end{aligned}$$

is A-linear.

Exercise (b). *Prove Proposition 1.3*

Definition 1.4. *Let $V \in \mathcal{C}$ and let $(v_i)_{i=1}^n$ be a k-bases for V. Let $(v^i)_{i=1}^n$ denote the dual bases of V^*. Define a k-linear map $b_V : k \to V \otimes V^*$ by*

$$b_V(1) = \sum_{i=1}^{n} v_i \otimes v^i$$

(extended to k by linearity) and a k-linear map $d_V : V^* \otimes V \to k$ by

$$d_V(\phi \otimes v) = \phi(v)$$

Theorem 1.5. *The maps b_V and d_V are A-linear and the triple $(*, b_V, d_V)$ is a left duality on $\mathcal{C}$.*

Proof. We first remark that b_V and d_V well defined. The expression for $b_V(1)$ is independent of the basis (v_i) of V, which can for example be seen by the remark that

$$\lambda_V^V(\sum_{i=1}^{n} v_i \otimes v^i) = \mathrm{id}_V \tag{1.5}$$

The description of d_V uses the universal property of tensor products of vector spaces (see week 40): d_V is the unique k-linear map such that the bilinear map $V^* \times V \to k$ given by $(\phi, v) \to \phi(v)$ is equal to $(\phi, v) \to d_V(\phi \otimes v)$.

Next we show that b_V, d_V are A-linear. In the case of b_V this means that we need to show that $ab_V(1) = b_V(a.1) = \epsilon(a)b_V(1)$. In order to show this we use equation (1.5) and Proposition 1.3. This reduces our task to showing that $\mathrm{id}_V \in \mathrm{End}_V$ (with A-module structure given by Definition 1.2) satisfies $a\mathrm{id}_V = \epsilon(a)\mathrm{id}_V$. Indeed, for all $v \in V$ we have

$$\begin{aligned}(a\mathrm{id}_V)(v) &= \sum_{(a)} a'\mathrm{id}_V(S(a'')v) \\ &= \sum_{(a)} a'S(a'')v = \epsilon(a)v = (\epsilon(a)\mathrm{id}_V)(v)\end{aligned}$$

by definition of the antipode S.

The A-linearity of d_V follows from a direct computation:

$$\begin{aligned}d_V(a(\phi \otimes v)) &= d_V(\sum_{(a)} \phi \circ m(S(a')) \otimes a''v) \\ &= \sum_{(a)} \phi(S(a')a''v) \\ &= \phi(\sum_{(a)} S(a')a''v) \\ &= \epsilon(a)\phi(v)\end{aligned}$$

Finally we need to show that the triple $(*, b_V, d_V)$ forms a left duality. Recall from week 47 that *in a strict monoidal category* this means that the following identities hold for all $V \in \mathcal{C}$:

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V \tag{1.6}$$
$$(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

Since we have already shown that b_V and d_V are A-linear with the usual A-module structures on the tensor products and duals it suffices simply to show that $(*, b_V, d_V)$ satisfies (1.6) in the category of finite dimensional vector spaces. This follows from a simple direct computation, but before we go into this computation we need to make a remark at this point. Strictly speaking the identities (1.6) are not true in the category of finite dimensional vector spaces (or $\mathcal{C}$ for that matter) since this is a monoidal category which is *not* strict. This has the effect that we need to correct the strict versions (1.6) of the left duality axioms by inserting appropriate unit and associativity constraints of $\mathbf{Vect_k}$ in order for these identities to be admissible compositions of morphisms. Two remarks are in order here. First of all, a priori there could be many essentially different ways to insert unit and associativity constraints to get an admissible composition of morphisms. Fortunately this is not the case. All possible admissible compositions of identities, unit and associativity constraints from V to W (with V, W aribtrary objects in a monoidal category) give the same morphism, as a result of Mac Lane's coherence theorem (see week 40, and [1, Section XI.5]). For example, Mac Lane's theorem implies that there is only one morphism $V \to k \otimes V$ which can be built from identities and unit and associativity constraints (namely l_V^{-1}). The second remark is that we can do this in fact inside $\mathcal{C}$, as we know that the unit and associativity constraints of $\mathbf{Vect}_k$ are A-linear (this is part of Theorem 1.1 of Week 41). After all these "ifs and buts" let us now finally do the computation. We show only the first identity, leaving the second one to the reader:

$$\begin{aligned}
r_V(\mathrm{id}_V \otimes d_V)a_{V,V^*,V}(b_V \otimes \mathrm{id}_V)l_V^{-1}(v) &= r_V(\mathrm{id}_V \otimes d_V)a_{V,V^*,V}(b_V \otimes \mathrm{id}_V)(1 \otimes v)\\
&= r_V(\mathrm{id}_V \otimes d_V)a_{V,V^*,V}(\sum_{i=1}^{n}(v_i \otimes v^i) \otimes v)\\
&= r_V(\mathrm{id}_V \otimes d_V)(\sum_{i=1}^{n} v_i \otimes (v^i \otimes v))\\
&= r_V(\sum_{i=1}^{n} v_i \otimes v^i(v))\\
&= \sum_{i=1}^{n} v^i(v)v_i\\
&= v
\end{aligned}$$

□

The issue of the non-strictness of $\mathcal{C}$ will be mostly ignored in the rest of this syllabus.

1.2. *Twisting elements.* Finally we add the last bit of structure to A in order to make A a ribbon algebra:

Definition 1.6. *A* ribbon algebra *A over a field k is a braided Hopf algebra over k with invertible antipode and a* central, *invertible "twisting element" $\theta \in A$ such that the following relations are satisfied:*

$$\Delta(\theta) = (R_{2,1}R)^{-1}(\theta \otimes \theta), \ \epsilon(\theta) = 1, \quad \text{and } S(\theta) = \theta \tag{1.7}$$

where R denotes the universal R-matrix, ϵ the co-unit, and S the antipode of A.

The main result on ribbon algebras is that they produce ribbon categories:

Theorem 1.7. *(see* [1, Propositin XIV.6.2]*) Let A be a ribbon algebra over a field k. Then the category $\mathcal{C} = \mathbf{Mod}_A^f$ of finite dimensional A-modules is a ribbon category whose ribbon structure is defined as follows:*

(a) *As a monoidal category $\mathcal{C}$ is a monoidal subcategory of $\mathbf{Vect}_k$, where the A-module structure of $V \otimes W$ is given by $a(v \otimes w) = \Delta(a)(v \otimes w)$.*
(b) *The left duality $(*, b, d)$ is the restriction to $\mathcal{C}$ of the usual left duality in the category $\mathbf{Vect}_k^f$ of finite dimensional vector spaces. Here the A-module action on V^* is defined by $a\phi(v) := \phi(S(a))v$.*
(c) *The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is the natural family of A-linear isomorphisms defined by $c_{V,W}(v \otimes w) := \tau(R(v \otimes w))$.*
(d) *The twist $\theta_V : V \to V$ is the natural family of A-linear isomorphisms that is defined by $\theta_V(v) := \theta^{-1}v$ (notice the inverse!).*

Proof. We have already shown that the structures defined in (a)-(c) satisfy the required axioms to justify their naming. We now must show that θ_V satisfies the axioms of a twist, and that the duality and the twist are compatible.

It is clear that the family of isomorphisms θ_V is A-linear, because θ^{-1} is an element of the center of A. The family $\{\theta_V\}_V$ of isomorphisms is natural because the application of $\theta^{-1} \in A$ obviously commutes with the application of any A-linear map defined on V. The twist property follows from:

$$\begin{aligned}(\theta_V \otimes \theta_W)c_{W,V}c_{V,W}(v \otimes w) &= (\theta^{-1} \otimes \theta^{-1})(R_{2,1}R)(v \otimes w) \\ &= \Delta(\theta^{-1})(v \otimes w) = \theta_{V \otimes W}(v \otimes w)\end{aligned}$$

Finally we need to show that the twist is compatible with the duality, i.e. we need to show that $\theta_V^* = \theta_{V^*}$. Notice that because of (b), the transpose $f^* : W^* \to V^*$ of a morphism $f : V \to W$ in $\mathcal{C}$ is simply its transpose as a k-linear map. Let $V \in \mathcal{C}$ with $\phi \in V^*$ and $v \in V$.

$$\begin{aligned}\theta_V^*(\phi)(v) &= \phi(\theta_V(v)) = \phi(\theta^{-1}v) \\ &= \phi(S(\theta^{-1})v) = (\theta^{-1}\phi)(v) = (\theta_{V^*}\phi)(v)\end{aligned}$$

whence $\theta_V^* = \theta_{V^*}$. This finishes the proof. □

Corollary 1.8. *Let A be a ribbon algebra. Then the graphical calculus (see week 47) applies to $\mathcal{C}$ (or strictly speaking, to its strictification).*

1.3. *The square of the twist and the quantum trace in $\mathcal{C}$.* We retain the assumptions that $\mathcal{C}$ is the ribbon category of finite dimensional modules over a ribbon algebra A over k.

Theorem 1.9. *The element $\theta^2 \in A$ acts in objects of $\mathcal{C}$ as multiplication by the central element $D = uS(u) \in A$.*

Proof. By graphical calculus in $\mathcal{C}$ we have for any $V \in \mathcal{C}$ (we intentionally omit non-strict structural information like brackets, unit constraints, associators etc.; these can be added in a unique way):

$$\phi_V^{-2} = (\mathrm{id}_V \otimes d_V)(\mathrm{id}_V \otimes c_{V,V^*})(c_{V,V}^{-1} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes b_V) \tag{1.8}$$

Indeed, the proof is Figure 1. We compute the action of the right hand side of this identity

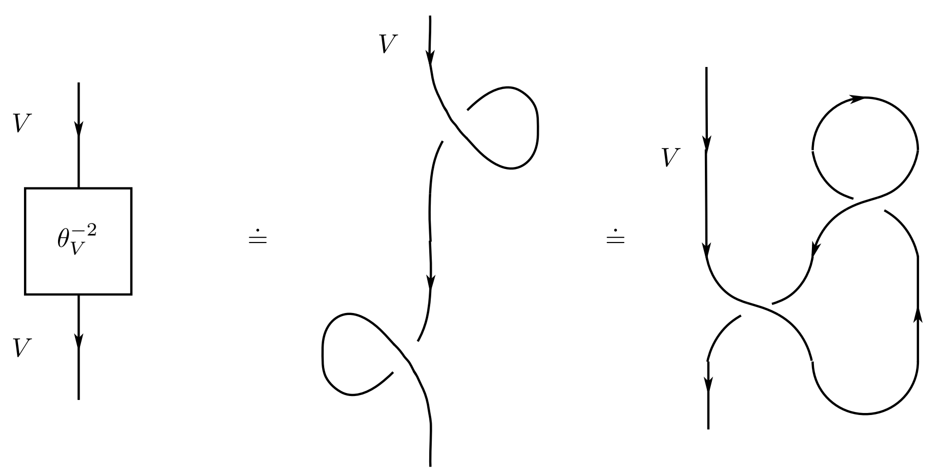

FIGURE 1. Graphical formula for θ_V^{-2}

on $v \in V$:

$$
\begin{aligned}
&(\mathrm{id}_V \otimes d_V)(\mathrm{id}_V \otimes c_{V,V^*})(c^{-1}_{V,V} \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes b_V)(v) \\
&\qquad = (\mathrm{id}_V \otimes d_V)(\mathrm{id}_V \otimes c_{V,V^*})(c^{-1}_{V,V} \otimes \mathrm{id}_{V^*})(\sum_i v \otimes v_i \otimes v^i) \\
&\qquad = (\mathrm{id}_V \otimes d_V)(\mathrm{id}_V \otimes c_{V,V^*})(\tau_{V,V}(m \otimes m)(\mathrm{id}_A \otimes S)(R_{2,1}) \otimes \mathrm{id}_{V^*})(\sum_i v \otimes v_i \otimes v^i) \\
&\qquad = (\mathrm{id}_V \otimes d_V)(\mathrm{id}_V \otimes c_{V,V^*})(\sum_{i,j} S(s_j)v_i \otimes t_j v \otimes v^i) \\
&\qquad = (\mathrm{id}_V \otimes d_V)(\sum_{i,j,k} S(s_j)v_i \otimes t_k v^i \otimes s_k t_j v) \\
&\qquad = (\mathrm{id}_V \otimes d_V)(\sum_{i,j,k} S(s_j)v_i \otimes (v^i \circ m(S(t_k))) \otimes s_k t_j v) = \sum_{i,j,k} v^i(S(t_k)s_k t_j v))S(s_j)v_i \\
&\qquad = \sum_{i,j,k} S(s_j)(v^i(S(t_k)s_k t_j v))v_i) = \sum_{j,k} S(s_j)S(t_k)s_k t_j v = \sum_j S(s_j)u t_j v \\
&\qquad = \sum_j S(s_j)S^2(t_j)uv = \sum_j S(S(t_j)s_j))uv = S(u)uv = uS(u)v
\end{aligned}
$$

□

Remark 1.10. *If A is not finite dimensional it does not follow that $\theta^2 = D$ in A.*

Proposition 1.11. *(see* [1, Proposition XIV.6.4]*) Let A be a ribbon algebra over k and let $\mathcal{C}$ be the ribbon category of finite dimensional modules over A. Let $V \in \mathcal{C}$, and let $f \in \mathrm{End}_{\mathcal{C}}(V)$. Then*

$$\mathrm{tr}_{\mathcal{C}}(f) := \mathrm{tr}_k(v \to \theta^{-1}uf(v)) \tag{1.9}$$

where tr_k refers to the usual trace in the category of finite dimensional vector spaces over the ground field k. In particular, the quantum dimension of $V \in \mathcal{C}$ is the vector space trace of the action of the element $\theta^{-1}u$ in V.

Proof. We have by definition (see Figure 10, week 47):

$$\mathrm{tr}_{\mathcal{C}}(f) = d^-_V(f \otimes \mathrm{id}_{V^*})b_V \tag{1.10}$$

Looking at Figure 19 of week 47 and using the functor F we see that

$$d^-_V = d_V c_{V,V^*}(\phi_V \otimes \mathrm{id}_{V^*}) \tag{1.11}$$

so that

$$\begin{aligned}\mathrm{tr}_{\mathcal{C}}(f) &= d_V c_{V,V^*}(\phi_V f \otimes \mathrm{id}_{V^*}) b_V = d_V(\sum_i c_{V,V^*}(\theta^{-1} f(v_i) \otimes v^i)) \\ &= d_V(\sum_{i,j}(t_j v^i \otimes s_j \theta^{-1} f(v_i))) = d_V(\sum_{i,j}(v^i \circ m(S(t_j)) \otimes s_j \theta^{-1} f(v_i))) \\ &= \sum_{i,j} v^i(S(t_j) s_j \theta^{-1} f(v_i)) = \sum_i v^i(u\theta^{-1} f(v_i)) = \mathrm{tr}_k(v \to \theta^{-1} u f(v))\end{aligned}$$

$\square$

Exercise (c). *Let A be a finite abelian group (with group operation $+$ and with unit element 0) and let k be a field. Let $\mathrm{Fun}_k(A)$ be the Hopf algebra of k-valued functions on A, i.e. the Hopf algebra which is dual to the group algebra $k[A]$. Explicitly the Hopf algebra structure of $\mathrm{Fun}_k(A)$ is given as follows. For $x \in A$ write $\delta_x \in \mathrm{Fun}_k(A)$ for the function on A with value one at x and zero elsewhere. The multiplication in $\mathrm{Fun}_k(A)$ is pointwise multiplication, hence given by $\delta_a \delta_b = \delta_{a,b}\delta_a$. The co-multiplication is given by $\Delta(f) = \sum_{x,y \in A} f(x+y)\delta_x \otimes \delta_y$. The unit element is $1 = \sum_{x \in A} \delta_x$ and the co-unit ϵ is given by $\epsilon(f) = f(0)$. Finally the antipode S is given by $S(f)(x) = f(-x)$.*

(1) *Show that $\mathrm{Fun}_k(A)$ is a braided Hopf algebra with universal R-matrix $\mathcal{R} \in \mathrm{Fun}_k(A) \otimes \mathrm{Fun}_k(A)$ if and only if*

$$\mathcal{R} = \sum_{a,b \in A} \gamma(a,b) \delta_a \otimes \delta_b \tag{1.12}$$

satisfies $\gamma(a,b) \in k^\times$ for all $a, b \in A$, and moreover $\gamma : A \times A \to k^\times$ is a bi-character of A, i.e.

$$\gamma(a+a', b) = \gamma(a,b)\gamma(a',b), \qquad \gamma(a, b+b') = \gamma(a,b)\gamma(a,b')$$

for all $a, a', b, b' \in A$.

(2) *Let $\chi : A \to k^\times$ be a character (i.e. a group homomorphism from A to the multiplicative group $k^\times$ of k). Assume that $\chi(x)^2 = 1$ for all $x \in A$. Prove that $\theta = \sum_{x \in A} \chi(x)\gamma(x,-x)\delta_x$ gives $\mathrm{Fun}_k(A)$ the structure of a ribbon algebra.*

(3) *Let $\mathcal{C}$ be the ribbon category of finite dimensional modules over $\mathrm{Fun}_k(A)$. Consider the one-dimensional module $V_a \in \mathcal{C}$ of $\mathrm{Fun}_k(A)$ defined by setting $V_a = k$ as a k-vector space, with $\mathrm{Fun}_k(A)$-action given by $f1 = f(a)$. Prove that $\dim_{\mathcal{C}}(V_a) = \chi(a)$.*

References

[1] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)

[2] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)