# QUANTUM GROUPS AND KNOT THEORY: WEEK 46

This week we treat Chapter 5 of [2]. We give some additional proofs and definitions.

## 1. The Jones polynomial and skein categories

### 1. **Knots, Links and Link Diagrams.**

1.1. *Framed links.*

**Definition 1.1.** *A framed link $L_f$ in $\mathbb{R}^3$ is a smooth link $L$ equipped with a nonzero normal vector field $f$ on $L$. The framing of $L$ determined by $f$ is the homotopy class of $f$ (within the space of nonzero normal vector fields). An isotopy of framed links is a smooth ambient isotopy of the underlying unframed links such that the framings correspond under the isotopy.*

An alternative way to think about framed links is as *ribbon links*. A ribbon link in $\mathbb{R}^3$ is a finite collection of disjoint images of smooth embeddings of $S^1 \times I$ in $\mathbb{R}^3$, equipped with an orientation. The correspondence between these notions is such that the framed link of a ribbon link is the collection of core curves, each framed by the positive unit normal vector field of the ribbon.

**Definition 1.2.** *The framing (or self-linking) number* $\mathrm{Fr}(K_f) \in \mathbb{Z}$ *of a framed knot $K_f$ is the linking number $Lk(K_+, K'_+)$ where $K'$ denotes the knot obtained by moving $K$ over a distance $\epsilon$ in the direction of the normal vector field (with $\epsilon$ sufficiently small) and $+$ refers to an (arbitrarily chosen) orientation of $K$.*

**Exercise (a).** *Let $K_f$ be a framed, oriented knot, and $R$ be the associated ribbon knot. Prove $Fr(K_f) = Lk(R_l, R_r)$ if $R_l, R_r$ are the left and right boundary components of $R$ (both oriented in the same way as $K$ is oriented).*

**Proposition 1.3.** *The framing number is well defined. Let $K$ be a knot in $\mathbb{R}^3$. For every $k \in \mathbb{Z}$ there exists a unique framing $f = f_k$ of $K$ such that* $\mathrm{Fr}(K_f) = k$.

*Proof.* Let us show that the normal bundle $N$ of $K \subset \mathbb{R}^3$ is trivial: First choose a smooth parametrization $\phi : S^1 \to K$ of $K$, and let $t$ be the nonvanishing vector field on $K$ obtained by applying $d(\phi)$ to the positive unit tangent vector field on $S^1$. Furthermore choose a smooth nonzero normal vector field $e_1$ on $K$. Define $e_2 = t \times e_1$. Then $(e_1, e_2)$ is a global basis for $N$ and hence trivializes $N$.

By results from elementary differential topology (for the interested reader: see e.g. [1, Lemma 2.6, Theorem 12.11]) there exists $\epsilon > 0$ such that the map $S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ given by $(x, (\lambda, \mu)) \to \phi(x) + \lambda e_1(\phi(x)) + \mu e_2(\phi(x))$ restricts to an orientation preserving diffeomorphism $\Phi : S^1 \times \mathbb{D}_\epsilon \to U$ (with $\mathbb{D}_\epsilon$ the open $\epsilon$ disk centered at the origin in $\mathbb{R}^2$) where the image $U$ is an open neighborhood of $K$ (a "tubular neighborhood").

Fix an orientation preserving diffeomorphism $\Phi$ as above. We see that a nonzero normal vector field $f$ on $K$ yields (after a suitable scaling so that $f(x) \in U$ for all $x \in K$) a well defined homotopy class $[F(f)]$ of smooth maps $F(f) : S^1 \to \mathbb{D}_\epsilon^\times$. Conversely a map $F : S^1 \to \mathbb{D}_\epsilon^\times$ defines a nonzero normal vector field $f(F)$ on $K$ by application of $\Phi$. This gives rise to a bijection between framings $[f]$ of $K$ and homotopy classes $[F]$ of maps $F : S^1 \to \mathbb{D}_\epsilon^\times$. These are in turn in canonical bijection with $\mathbb{Z}$ through the winding number map $[F] \to \mathrm{Wind}(F)$. It is easy to see that $\mathrm{Fr}(K_{f(F)}) - \mathrm{Wind}(F)$ is independent of $[F]$. Hence $\mathrm{Fr}(K_f)$ can assume any integral value and this number uniquely determines the framing $[f]$. □

1.2. *Link diagrams.* This was treated in week 37. A link diagram $D$ is a 4-valent planar graph with over/under information at each vertex. Every link $L \in \mathbb{R}^3$ is isotopic to a link which is regular in the sense that the projection to $R^2 \times \{0\}$ is regular. The projection of a regular link determines a link diagram in the obvious way. Conversely each link diagram $D$ determines a unique isotopy class $[L(D)]$ of links.

**Definition 1.4.** *Let $D$ be a link diagram, and let $L(D)$ be a link which projects regularly onto $D$ with matching over/under crossing data. In particular the plane $N_x$ normal to $L(D)$ at $x \in L(D)$ is never horizontal. Given $x \in L(D)$ we define a (nonzero) vector $bf(x) \in N_x$ by projecting $(0, 0, 1)$ to $N_x$. Then $L_{bf}(D)$ is a framed link. One can easily show that the isotopy class $[L_{bf}(D)]$ is uniquely determined, and this framing is called the blackboard framing of $[L(D)]$.*

**Definition 1.5.** *Define the writhe $w(D) \in \mathbb{Z}$ of a knot diagram $D$ by $w(D) = Fr(L_{bf}(D))$.*

**Exercise (b).** *Let $D$ be a knot diagram. Choose an orientation of $D$. Show that $w(D) = \sum_{c \in V(D)} \epsilon(c)$ where $V(D)$ is the vertex set of $D$ (hint: use exercise (a)).*

We use the result of this exercise to extend writhe to oriented link diagrams:

**Definition 1.6.** *Let $D$ be an oriented link diagram. We define its writhe $w(D)$ to be $w(D) = \sum_{c \in V(D)} \epsilon(c)$.*

1.3. *Reidemeister moves.* Recall Reidemeister's Theorem (week 37):

**Theorem 1.7.** *Let $D, D'$ be link diagrams. We have $[L(D)] = [L(D')]$ iff $D'$ can be obtained from $D$ by a finite sequence of steps consisting of*

(1) *Planar ambient isotopy.*
(2) *The Reidemeister moves $\Omega_1^{\pm 1}, \Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$.*

1.4. *Reidemeister's theorem for ribbon links.* As we have seen above we can attach framed links to link diagrams by using the blackboard framing. By Proposition 1.3 a framing of a link $L$ is given by attaching a framing number ($\in \mathbb{Z}$) to each of the components of $L$. Since we can change the writhe of components of $L(D)$ at will by adding positive or negative curls to $D$ (without changing the (unframed) isotopy class of $L(D)$) it is clear that we can represent any framed link using the blackboard framing of an appropriate link diagram. The Reidemeister theorem [2, Theorem 1.4] for such framed link diagrams requires the weakening of the first Reidemeister move $\Omega_1$ to $\Omega_0$ (see [2, Figure 1.4]):

**Lemma 1.8.** *The equivalence relation on link diagrams defined by finite sequences of steps of the form*

(1) *Planar ambient isotopy.*
(2) *The Reidemeister moves* $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}$ *and* $\Omega_3^{\pm 1}$.

*is equal to the equivalence relation obtained by steps of the form*

(1) *Planar ambient isotopy.*
(2) *The Reidemeister moves* $\tilde{\Omega}_0^{\pm 1}, \Omega_2^{\pm 1}$ *and* $\Omega_3^{\pm 1}$.

*where* $\tilde{\Omega}_0$ *consists of replacing the positive right handed curl by the positive left handed curl.*

*Proof.* We leave this as a pleasant exercise for the reader. □

**Theorem 1.9.** *(see* [2, Theorem 1.4]*) Let* $D, D'$ *be link diagrams. We have* $[L_{bf}(D)] = [L_{bf}(D')]$ *iff* $D$ *and* $D'$ *are equivalent in the sense of Lemma 1.8.*

*Proof.* By Proposition 1.3 the framing of $L_{bf}(D)$ is completely determined by the writhes $w(D_i)$ where $D_i$ runs over the set of components of $D$. Since the steps used to define the equivalence described in Lemma 1.8 do not change the writhes of the components the "if" part follows.

The "only if" is more complicated. Assume that $L(D)_{bf}$ and $L(D')_{bf}$ are isotopic as framed links. In particular $L(D)$ and $L(D')$ are isotopic as unframed links, so $D'$ can be obtained from $D$ by a sequence $S$ consisting of Reidemeister moves and planar isotopies. Suppose that somewhere in the sequence $S$ we need to apply a $\Omega_1$ step. Then we correct this by adding a positive or negative curl in the same strand in such a way that in total we apply a $\Omega_0$ step. We can reduce the size of this additional curl arbitrarily by planar isotopy. We will refer to such a "very small curl" as a "cable kink". On the other hand if we need to apply a $\Omega_1^{-1}$ step in $S$ then instead of removing the curl we just reduce its size and start treating it as an additional cable kink. The crucial observation about these cable kinks is that we may pass them (if necessary) through a crossing of the diagram at the cost of additional $\Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$ steps (which are allowed within our equivalence relation). In this way we are free to move the "cable kinks" up and down the strands in order to keep them from interfering with other steps in $S$. Hence we can ignore their presence until we are finished with the sequence $S$. Working through the steps of $S$ in this adapted sense we are only using Reidemeister steps of type $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$, and when we are finished we are presented with a diagram $D''$ which is equal to $D'$ except for the presence of a number of these cable kinks. From the assertion it follows however that in each component of $D''$ the number of positive kinks is equal to the number of negative kinks. We group the kinks in each component of $D''$ together on a pieces of strand without crossings (again at the cost of $\Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$ steps). Using additional $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$ steps we can now safely remove all of them by cancellation. □

**Exercise (c).** *See* [2, Exercise 1.5(i)].

**Exercise (d).** *See* [2, Exercise 1.5(ii)].

## 2. **The Jones polynomial of oriented links.**

2.1. *Skein classes of modules.* We do not follow the notation of the book here, since this is somewhat confusing. We define:

**Definition 2.10.** *Let $D$ be a link diagram. We denote by $[D](a) \in E(a)$ its class in the skein module.*

We give a direct definition of the bracket polynomial of a link diagram:

**Definition 2.11.** *Let $D$ be a link diagram. We denote by $\langle D\rangle(a)$ the bracket polynomial given by*

$$\langle D\rangle(a) = \sum_{S}(-a^2 - a^{-2})^{c(S)-1}\langle D, S\rangle(a) \tag{2.1}$$

*where the sum runs over all smoothings $S$ of the diagram $D$, $c(S)$ denotes the number of components of $S$, and where for each $S$ we put*

$$\langle D, S\rangle(a) := \prod_{c\in V(D)} a^{n(S,c)} \tag{2.2}$$

*with $V(D)$ the vertex set of $D$, and $n(S,c) = 1$ if $S$ opens the $a$ channel at $c$ and $n(S,c) = -1$ if $S$ opens the $a^{-1}$ channel at $c$ (see* [2, Figure 2.1]*).*

2.2. *The dimension of $E(a)$.* Using the above definition it is an elementary matter to verify the following:

**Proposition 2.12.** *The bracket polynomial $\langle D\rangle(a)$ only depends on the skein class $[D](a)$ of $D$.*

We use this proposition to prove:

**Theorem 2.13.** *(see* [2, Theorem 2.2, Chapter 5]*) $\dim(E(a)) = 1$.*

*Proof.* The argument in [2] shows that the dimension is at most one. By the preceding proposition there exists a unique linear functional $\phi$ on $E(a)$ such that $\phi([D](a)) = \langle D\rangle(a)$ for all diagrams $D$. In particular $\phi([O](a)) = 1$, and we conclude that $E(a)$ must be at least one dimensional. □

2.3. *The skein class is a ribbon link invariant.* The result [2, Theorem 2.3] is very fundamental. It shows that the skein class $[D](a)$ of a link diagram $D$ is an invariant of the ribbon link $L(D)_{bf}$ represented by $D$.

2.4. *The bracket polynomial.* In our notation we have for any link diagram $D$ the following relation in $E(a)$:

$$[D](a) = \langle D\rangle(a)[O](a). \tag{2.3}$$

2.5. *The Jones polynomial.* We define for an oriented link diagram $D$ the polynomial:

$$f_D(a) := (-a^{-3})^{w(D)}\langle D\rangle(a) \tag{2.4}$$

**Theorem 2.14.** *$f_D$ is an isotopy invariant of* oriented *links, and $f_D \in \mathbb{Z}[a^{-2}, a^2]$.*

*Proof.* It is clear that $f_D$ is invariant for planar isotopy and for the Reidemeister moves $\Omega_0^{\pm1}, \Omega_2^{\pm1}, \Omega_3^{\pm1}$ since this is true for the bracket polynomial $\langle D\rangle(a)$ as well as for the writhe $w(D)$. The computation [2, Figure 2.2] proves that $f_D$ is invariant for $\Omega_1^{\pm1}$ as well.

The second assertion follows easily from the explicit expansion in the definition of the bracket polynomial $\langle D\rangle(a)$. □

**Remark 2.15.** *$f_D$ is independent of the orientation of $D$ if $D$ is a knot diagram.*

**Definition 2.16.** *The Jones polynomial $V_L \in \mathbb{Z}[q, q^{-1}]$ of an oriented link $L$ defined by a diagram $D$ is defined by*

$$V_L(q) = f_D(q^{-1/2}) \tag{2.5}$$

2.6. *Conway triples.* The Jones polynomial as defined above satisfies the usual Conway type skein relation and is normalized by $V_O(q) = 1$. This characterizes the Jones polynomial completely (see [2, Theorem 2.6]).

2.7. *Exercises.*

**Exercise (e).** *Prove* [2, Theorem 2.6(iii)].

**Exercise (f).** *See* [2, Exercise 2.7(ii)].

## 3. **Skein module of tangles.**

3.1. *Tangles, framed tangles (or ribbon tangles).*

3.2. *Tangle Diagrams.*

3.3. *Reidemeister's theorem for framed tangles.*

3.4. *The skein module $E_{k,l}(a)$.*

**Theorem 3.17.** *The skein module $E_{k,l}(a)$ has dimension* 0 *if $k + l$ is not even, and if $k+l = 2n$ is even then it has dimension $C_n :=$* binom$(2n, n)/(n+1)$ *(the Catalan number).*

*Proof.* It is clear that there exist $k, l$ tangle diagrams only if the total number $k + l$ of end points is even. Moreover $E_{k,l}(a)$ is the linear span of the skein classes of smooth (i.e. no crossings) $k, l$ tangle diagrams. By induction on $n = (k + l)/2$ we easily see that two smooth $k, l$ tangle diagrams which determine the same pair matching of the $2n$ end points are planar isotopic. Hence the planar isotopy classes of smooth diagrams yield a linear basis of $E_{k,l}(a)$. Thus $\dim(E_{k,l}(a))$ is equal to the number of pair matchings of the $2n$ end points which are admissible in the sense that they arise as the pair matching of a smooth diagram. Using a planar isotopy we can move the $2n$ end points of the $k, l$ tangle diagrams to $2n$ points on a circle, preserving their circular order. This gives a bijective correspondence between admissible pair matchings of $2n$ points on the circle and admissible pair matchings of the $2n$ end points of $k, l$ tangles. In particular the number of such admissible pair matchings depends only on $n$. Let us denote this number by $a_n$. Fix a circular numbering of $p_1, p_2, \dots, p_{2n}$ of the end points. If we match $p_1$ with $p_{2+i}$ then we divide the remaining $2n-2$ points in two groups of size $i$ and $2n-2-i$ which can no longer be matched by any smooth diagram which connects $p_1$ and $p_{2+i}$. Hence the number of distinct admissible pair matchings in which $p_1$ is matched with $p_{2+i}$ is equal to 0 if $i$ is odd, and equal to $a_k a_{n-k-1}$ if $i = 2k$. Whence the recurrence relation

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1} \tag{3.1}$$

Moreover we have $a_0 = 1$ (by Theorem 2.13). Consider the generating power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. From the above recurrence relation and initial condition we see that $zf(z)^2 = f(z) - 1$, or $(2zf)^2 - 2(2zf) + 4z = 0$. Using the binomial expansion theorem we get

$$2zf(z) = 1 - (1-4z)^{1/2} \tag{3.2}$$

$$= 1 - \sum_{k=0}^{\infty} \mathrm{binom}(1/2, k)(-4z)^k \tag{3.3}$$

$$= 2\sum_{k=1}^{\infty} C_{k-1} z^k \tag{3.4}$$

proving the theorem. □

3.5. *The skein class as isotopy invariant of framed tangles.*

## 4. **Categories of Tangles.**

4.1. *The category of framed tangles.* The category of framed tangles $\mathcal{T}$ is a strict monoidal category.

There exists a notion of a presentation of a strict monoidal category by means of generators and relations, analogous to presentations of groups (or better still, monoids). This plays a mayor role in the theory of the Reshetikhin-Turaev invariants. In the paragraph we discuss the definition of a presentation of a strict monoidal category, and we will study some simple examples (in particular, the ribbon tangle category $\mathcal{T}$).

**Definition 4.18.** *Let $\mathcal{M}$ be a strict monoidal category and let $G$ be a set of morphisms in $\mathcal{M}$. Let $\mathcal{A}_G$ be the set of morphisms of $\mathcal{M}$ consisting of all identities of $\mathcal{M}$ and all morphisms of the type $\mathrm{id}_V \otimes g \otimes \mathrm{id}_W$ with $g \in G$ and $V, W \in \mathcal{M}$. We call the elements of $\mathcal{A}_G$ elementary morphisms (with respect to the set $G$). We say that $\mathcal{M}$ is generated (as a strict monoidal category) by $G$ if any morphism of $\mathcal{M}$ can be written as a composition of elementary morphisms with respect to $G$.*

**Example 4.19.** *The braid category $\mathcal{B}^{alg}$ is generated as a strict monoidal category by the set $G = \{X^{\pm}\}$ where $X^+ = c_{1,1} \in \mathcal{B}_2$ is the (positive) generator of $\mathcal{B}_2$ and $X^-$ is its inverse.*

We have a notion of presentation of a strict monoidal category by means of generators and relations, analogous to presentations of groups (or better still, monoids). This plays a mayor role in the theory of the Reshetikhin-Turaev invariants, so we will study the "baby example" of the ribbon tangle category in this section.

Consider the set $\mathcal{W}_G$ of *admissible* words in the alphabet formed by the identities in $\mathcal{M}$ and the above elementary morphisms (i.e. concatenations of such letters such that consecutive letters in a word are composable morphisms in $\mathcal{M}$). If $G$ generates $\mathcal{M}$ then any morphism $m$ of $\mathcal{M}$ can be obtained from an admissible word $w \in \mathcal{W}_G$ by replacing the concatenations in $w$ by compositions in $\mathcal{M}$. We write $m = [w]$ to express that a morphism $m$ in $\mathcal{M}$ is represented in such a way by the word $w \in \mathcal{W}_G$. This yields an equivalence relation $\sim_{\mathcal{M}}$ on $\mathcal{W}_G$ by defining $w \sim_{\mathcal{M}} w'$ iff $[w] = [w']$. We denote the concatenation of words $w_1, w_2$ representing composable morphisms by $w_1 * w_2$.

Observe that the alphabet $\mathcal{A}_G$ is invariant for tensoring by an identity morphism $\mathrm{id}_V$ on the left or the right hand side. Obviously these operations respect the composability of elementary morphisms. Hence for each object $V \in \mathcal{M}$ and for each word $w \in \mathcal{W}_G$ we can define new words $\mathrm{id}_V \otimes w$ and $w \otimes \mathrm{id}_V$ in $\mathcal{W}_G$.

Let $R \subset \mathcal{W}_G \times \mathcal{W}_G$ be a set of pairs $(r, r')$ of (admissible) words with the property that the source and target objects of $r$ and of $r'$ are equal. We define an equivalence relation $\sim_R$ generated by $R$ as follows. First we form a bigger set $R_1 \subset \mathcal{W}_G \times \mathcal{W}_G$ by adding to $R$ all pairs $(r_1, r_2)$ of the following form: Let $g_1 : U \to U'$ and $g_2 : W \to W'$ be elements of $G$. We put

$$r_1 = (g_1 \otimes \mathrm{id}_{V\otimes W'}) * (\mathrm{id}_{U\otimes V} \otimes g_2) \tag{4.1}$$

and

$$r_2 = (\mathrm{id}_{U'\otimes V} \otimes g_2) * (g_1 \otimes \mathrm{id}_{V\otimes W}) \tag{4.2}$$

(where $V$ is an arbitrary object of $\mathcal{M}$).

Next we define the set $\tilde{R} \subset \mathcal{W}_G \times \mathcal{W}_G$ obtained from $R_1$ by tensoring the pairs in $R_1$ on both sides with arbitrary identities of $\mathcal{M}$. Finally we say that $w \sim_R w'$ if and only if $w'$ can be obtained from $w$ by a finite sequence of steps consisting of

(1) Deleting or inserting an identity morphism in the concatenation.
(2) Replacing a word of the form $w_1 * r * w_2$ by a word of the form $w_1 * r' * w_2$ where the pair $(r, r')$ or $(r', r)$ belongs to $\tilde{R}$.

It is clear that $\sim_R$ so defined is an equivalence relation on $\mathcal{W}_G$. It is easy to see that $\sim_R$ is stronger than $\sim_{\mathcal{M}}$ iff all pairs $(r_1, r_2) \in R$ satisfy $[r_1] = [r_2]$.

**Definition 4.20.** *Let $\mathcal{M}$ be a strict monoidal category and let $G$ be a set of morphisms of $\mathcal{M}$. Let $R \subset \mathcal{W}_G \times \mathcal{W}_G$ be a set of pairs $(r, r')$ of admissible words in the alphabet $\mathcal{A}_G$. We say that $(G, R)$ is a presentation of $\mathcal{M}$ if $G$ is a set of generators of $\mathcal{M}$ and if the equivalence relations $\sim_{\mathcal{M}}$ and $\sim_R$ agree on $\mathcal{W}_G$.*

We know from week 37 (in a different language):

**Example 4.21.** *The braid category $\mathcal{B}^{alg}$ has a presentation $(G, R)$ with $G = \{X^{\pm}\}$ (as above) and $R = \{\Omega_2, \Omega_3\}$. In other words, $R$ consists of the pairs $(X^+ * X^-, \mathrm{id}_2)$ and the Yang Baxter relation $((\mathrm{id}_1 \otimes X^+)*(X^+\otimes \mathrm{id}_1)*(\mathrm{id}_1 \otimes X^+), (X^+\otimes \mathrm{id}_1)*(\mathrm{id}_1 \otimes X^+)*(X^+\otimes \mathrm{id}_1))$.*

We usually represent a pair $(r_1, r_2)$ in the set $R$ of relations by the equality $[r_1] = [r_2]$ where both sides are expressed as the appropriate compositions of identities and elementary morphisms.

**Theorem 4.22.** *The ribbon tangle category $\mathcal{T}$ is generated as a strict monoidal category by the set $G = \{\cup, \cap, X^{\pm}\}$ with set of relations $R$ consisting of*

(1) $(\mathrm{id}_1 \otimes \cap) \circ (\cup \otimes \mathrm{id}_1) = (\cap \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \cup) = \mathrm{id}_1$,
(2) $(\cap \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes X^{\pm}) = (\mathrm{id}_1 \otimes \cap) \circ (X^{\mp} \otimes \mathrm{id}_1)$,
(3) $(\mathrm{id}_1 \otimes X^{\pm}) \circ (\cup \otimes \mathrm{id}_1) = (X^{\mp} \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \cup)$,
(4) $\tilde{\Omega}_0 : (\cap \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes X^{\pm}) \circ (\cup \otimes \mathrm{id}_1) = (\mathrm{id}_1 \otimes \cap) \circ (X^{\pm} \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \cup)$,
(5) $\Omega_2$,
(6) $\Omega_3$.

*Proof.* A tangle diagram $D$ in $\mathbb{R} \times I$ is called generic if its crossings are transversal double points, if the critical points of the height function on $D$ are all nondegenerate local extrema and if all the crossings and local extrema occur at different heights. We call the finite set of local extrema and crossings of $D$ the set of singular points.

Any ribbon tangle $T$ can be represented by a generic tangle diagram $D$. Cutting $D$ in horizontal strips containing at most one singular point each and applying planar ambient isotopies we see that $T$ is a composition of elementary morphisms from the set $\mathcal{A}_G$. The listed relations are true in $\mathcal{T}$, so we know in particular that $\sim_R$ is stronger than $\sim_{\mathcal{T}}$.

It remains to show that if $w_1, w_2 \in \mathcal{W}_G$ and $w_1 \sim_{\mathcal{T}} w_2$ then $w_1 \sim_R w_2$. Now recall [2, Theorem 3.3] (Reidemeister's theorem for ribbon tangles): $w_1 \sim_{\mathcal{T}} w_2$ implies that $w_1$ and $w_2$ can be obtained from each other by a finite sequence of steps of the form

(1) Planar diagram isotopies,
(2) $\tilde{\Omega}_0, \Omega_2, \Omega_3$.

The steps of the last kind are all explicitly included in $R$ so these steps can be performed by $R$-equivalences. For the steps of the first kind we need to analyze the situation a bit further. First notice that a generic tangle diagrams determines an element of $\mathcal{W}_G$ (up to insertions of identities) and vice versa. Suppose that we need to apply a planar isotopy to go from a generic tangle diagram $T$ to a generic tangle diagram $T'$. This can be achieved by a finite sequence of changes in the representing word in $\mathcal{W}_G$ of the following kind (see Week 6, Theorem 1.11):

(a) Interchange the order of two elementary morphisms as described by (4.1), (4.2),
(b) Birth or annihilation of a pair of local extrema,
(c) Move a crossing from one side of a local extremum to the other side.

Now (a) is included in $\sim_R$ (these type of equivalences were added to $R$ to get $R_1$), and (b),(c) are given by (1)-(3). This concludes the proof. $\square$

We can reformulate the above result by formally introducing a strict tensor category $\mathcal{T}^{alg}$ with objects $\{0, 1, 2, \dots\}$ and morphisms built from the $R$-equivalence classes in $\mathcal{W}_G$. The content of the above theorem is then an isomorphism of categories given by $\overline{w} \to [w]$ from $\mathcal{T}^{alg}$ to $\mathcal{T}$ compatible with the strict monoidal structures.

Let us look at this more carefully in the general case of a strict tensor category $\mathcal{M}$ with presentation $(G, R)$. First we form the category $\mathcal{F}(G)$ by applying the above constructions with $R = \emptyset$. Hence $\mathrm{Obj}(\mathcal{F}(G)) = \mathrm{Obj}(\mathcal{M})$ and the set $\overline{\mathcal{W}}_G$ of morphisms is obtained from the class $\mathcal{W}_G$ of admissible words by taking the equivalence classes for the relation $\sim_\emptyset$. The composition of morphisms in $\mathcal{F}(G)$ is given by concatenation (of words that represent composable morphisms). Now we define a strict tensor structure on $\mathcal{F}(G)$. Recall that the operation of tensoring a word $w \in \mathcal{W}_G$ by an identity of $\mathcal{M}$ on the left or right is well defined. We use this to define a strict monoidal structure on $\mathcal{F}(G)$: Its tensor unit and the tensor product on the level of objects are the same as in $\mathcal{M}$. If $w_1 : U \to U'$ and $w_2 : V \to V'$ are words of length $a$ and $b$ respectively then we define $\overline{w}_1 \otimes \overline{w}_2 \in \overline{\mathcal{W}}_G$ by $\overline{w}_1 \otimes \overline{w}_2 := \overline{(w_1 \otimes \mathrm{id}^a_{V'}) * (\mathrm{id}^b_U \otimes w_2)}$. Using that $\mathcal{M}$ is a strict tensor category, and using (4.1), (4.2) it is not hard to show that this defines a strict monoidal structure on $\mathcal{F}(G)$. The crucial point is that we have the identity

$$\overline{(w_1 \otimes \mathrm{id}^a_{V'}) * (\mathrm{id}^b_U \otimes w_2)} = \overline{(\mathrm{id}^b_{U'} \otimes w_2) * (w_1 \otimes \mathrm{id}^a_V)} \tag{4.3}$$

and this relation easily implies the functoriality of $\otimes$ on $\overline{\mathcal{W}}_G$.

**Exercise (g).** *Show (4.3) and that $\otimes$ defines a strict monoidal structure on $\mathcal{F}(G)$.*

The equivalence relation $\sim_R$ on the morphisms $\mathcal{W}_G$ of $\mathcal{F}(G)$ is compatible with the tensor product and composition of morphisms. Hence the class of $\sim_R$-equivalence classes of words in $\mathcal{W}_G$ can also be equipped with a tensor product and a composition law, defining a strict tensor category $\mathcal{F}_{(G,R)}$.

**Proposition 4.23.** *If all the relations of $R$ hold in $\mathcal{M}$ then there exists a canonical functor $\phi : \mathcal{F}_{(G,R)} \to \mathcal{M}$ which is the identity on objects and such that $\phi((w)_R) = [w]$ (where $(w)_R$ denotes the $\sim_R$-equivalence class of $w \in \mathcal{W}_G$). Moreover $\phi$ is a strict tensor functor (see below) and $\phi$ is an isomorphism of categories iff $(G, R)$ is a presentation of $\mathcal{M}$.*

**Exercise (h).** *Prove the above Proposition.*

4.2. *$\mathcal{T}$ is a braided strict monoidal tensor category.* As we have seen in Exercise 3.5(b) of [2, Chapter 2] there exists a unique braiding $c$ of the braid category $\mathcal{B}$ such that $c_{1,1} = \sigma_1 \in \mathcal{B}_2$. In a formula this braiding is given by

$$c_{m,n} = (\sigma_n \dots \sigma_{m+n-1})(\sigma_{n-1} \dots \sigma_{m+n-2}) \dots (\sigma_1 \dots \sigma_m) \tag{4.4}$$

Since $\mathcal{B}$ is a subcategory of $\mathcal{T}$ we only need to show the naturality of the braiding with respect to the additional generators $\cap, \cup$ in order to show that $c$ also defines a braiding for $\mathcal{T}$. This is obvious topologically, or it can be shown using the relations of Theorem 4.22.

**Exercise (i).** *Show using Theorem 4.22 and (4.4) that*

$$(\mathrm{id}_2 \otimes \cup) c_{1,1} = c_{3,1} (\mathrm{id}_1 \otimes \cup \otimes \mathrm{id}_1) \tag{4.5}$$

4.3. *Strict tensor functor.* A strict tensor functor is a functor $F : \mathcal{M} \to \mathcal{N}$ where $\mathcal{M}, \mathcal{N}$ are strict tensor categories, such that $F$ respects the tensor product. This means that $F$ maps the tensor unit of $\mathcal{M}$ to the tensor unit of $\mathcal{N}$, $F(V \otimes W) = F(V) \otimes F(W)$ for all objects of $\mathcal{M}$, and $F(\alpha \otimes \beta) = F(\alpha) \otimes F(\beta)$ for all morphisms $\alpha, \beta$ of $\mathcal{M}$.

Now we come to an important existence criterion for strict tensor functors on strict tensor categories with a presentation:

**Theorem 4.24.** *Let $\mathcal{M}, \mathcal{N}$ be strict tensor categories and suppose that $(G, R)$ is a presentation of $\mathcal{M}$. Suppose that we are given a map $f_0 : \mathrm{Obj}(\mathcal{M}) \to \mathrm{Obj}(\mathcal{N})$ such that $f_0(I_{\mathcal{M}}) = I_{\mathcal{N}}$ and such that $f_0(V \otimes W) = f_0(V) \otimes f_0(W)$ for all objects $V, W$ in $\mathcal{M}$. Suppose that we are moreover given a map $f_1 : G \to \mathrm{Hom}(\mathcal{N})$ with the property that if $g : U \to V$ belongs to $G$, then $f_1(g) \in \mathrm{Hom}_{\mathcal{N}}(f_0(U), f_0(V))$. There exists a strict tensor functor $F : \mathcal{M} \to \mathcal{N}$ such that $F$ coincides with $f_0$ on the level of objects and with $f_1$ on $G$ iff for all pairs $(r_1, r_2) \in R$ we get an identity of morphisms in $\mathcal{N}$ if we replace the elementary morphisms of the form $\mathrm{id}_V \otimes g \otimes \mathrm{id}_W$ which occur as letters of $r_1$ and $r_2$ by the morphism $\mathrm{id}_{f_0(V)} \otimes f_1(g) \otimes \mathrm{id}_{f_0(W)}$, and similarly replace the identities in $r_1$ and $r_2$ by the corresponding identities in $\mathcal{N}$ and finally replace the concatenations by compositions in $\mathcal{N}$. In this case the functor $F$ is uniquely determined.*

*Proof.* It is obvious that there exists a (unique) strict tensor functor $\tilde{F} : \mathcal{F}(G) \to \mathcal{N}$ which restricts to $f_0$ on objects, and to $f_1$ on $G$. In view of Proposition 4.23 we therefore need to verify that $\tilde{F}$ factors through $\phi$. This amounts to checking that the images under $\tilde{F}$ of $\overline{r}_1$ and $\overline{r}_2$ are equal in $\mathcal{N}$ for all pairs $(r_1, r_2) \in R$. But this was the assertion. $\square$

4.4. *Skein category $\mathcal{S}$ and skein functor.* We now take the vector space $E_{k,l}(a)$ of $\mathbb{C}$-linear combinations of the skein classes $[D](a)$ of the tangle diagrams of $k, l$-tangles as the space of morphisms from $k$ to $l$. This defines a category $\mathcal{S}$ with composition law and tensor product defined analogously to $\mathcal{T}$. Since the skein class of a diagram is invariant for the Reidemeister moves $\Omega_0, \Omega_2, \Omega_3$ we obtain a functor $P : \mathcal{T} \to \mathcal{S}$. It is obviously a strict tensor functor.

**Definition 4.25.** *Let $a \in \mathbb{C}^{\times}$. The Temperley-Lieb algebra $TL_n(a)$ is the complex algebra $\mathrm{End}_{\mathcal{S}}(n)$.*

## References

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