QUANTUM DOUBLE AS A BRAIDED HOPF ALGEBRA AND QUANTUM $\mathfrak{sl}_2$

In this text we complete the treatment on the generalized quantum double construction. We introduce the quantized universal enveloping algebra U of the Lie algebra $\mathfrak{sl}_2$ and we realize it as a quotient Hopf algebra of an explicit generalized quantum double. Next week we use this observation to show that U is braided. This text complements parts of Chapter 3 and 4 of [3].

Convention: We fix an algebraically closed field k of characteristic zero.

We give several exercises in the text.
The **homework exercise** is 1.9.

1. The universal R-matrix of the quantum double

For Hopf algebras A and B with invertible antipodes, we constructed in the last lecture the generalized double $\mathcal{D}_\varphi(A,B)$ associated to a Hopf pairing $\varphi : A \times B \to k$. It is $A \otimes B$ as a vector space with Hopf algebra structure characterized by

1. The canonical linear embeddings $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ of A and B in $\mathcal{D}_\varphi(A,B)$ are Hopf algebra morphisms.

2. For $a \in A$ and $b \in B$ we have $(a \otimes 1)(1 \otimes b) = a \otimes b$ and we have the "straightening rule"

$$(1 \otimes b)(a \otimes 1) = \sum_{(a),(b)} \varphi\big(S^{-1}(a_{(1)}), b_{(1)}\big)\varphi\big(a_{(3)}, b_{(3)}\big)(a_{(2)} \otimes 1)(1 \otimes b_{(2)}).$$

For the remainder of the section we assume that A and B are finite dimensional Hopf algebras with invertible antipodes. We furthermore assume that $\varphi : A \times B \to k$ is a nondegenerate Hopf pairing (if you prefer, you may without loss of generality take $B = A^{*,cop}$ and φ the canonical Hopf pairing, see Exercise 2.4 of the lecture notes of last week).

Let $\{a_i\}_i$ be a fixed linear basis of A and denote $\{b_i\}_i$ for the associated dual linear basis of B with respect to φ, so that $\varphi(a_i, b_j) = \delta_{i,j}$ (the Kronecker delta function $\delta_{i,j}$ is one if $i = j$ and zero otherwise). Some elementary facts are listed in the following

Lemma 1.1. (i) *For $a \in A$ and $b \in B$,*

$$\sum_i \varphi(a, b_i)a_i = a, \qquad \sum_i \varphi(a_i, b)b_i = b.$$

In particular, $\sum_i \epsilon(b_i)a_i = 1$ and $\sum_i \epsilon(a_i)b_i = 1$.
(ii) *$\sum_i a_i \otimes b_i \in A \otimes B$ is independent of the choice of basis $\{a_i\}_i$ of A.*

Proof. **(i)** Set $a' = \sum_i \varphi(a, b_i)a_i$. For a dual basis element b_j we have

$$\varphi(a', b_j) = \sum_i \varphi(a, b_i)\delta_{i,j} = \varphi(a, b_j),$$

hence $a' = a$ because φ is nondegenerate. Since $\varphi(1, b_i) = \epsilon(b_i)$ by the counit axiom for φ, we obtain $\sum_i \epsilon(b_i)a_i = 1$ by considering the latter equation for $a = 1$. In the same way one verifies the second identity.
(ii) The assigment $a \otimes b \mapsto \varphi(a, \cdot)b$ defines a linear isomorphism $A \otimes B \to \mathrm{End}_k(B)$. By the first part of the lemma the element $\sum_i a_i \otimes b_i$ is mapped to Id_B, which is independent of the choice of basis $\{a_i\}_i$.

□

Denote

$$R = \sum_i (a_i \otimes 1) \otimes (1 \otimes b_i) \in \mathcal{D}_\varphi(A, B)^{\otimes 2}.$$

By the previous lemma, it is independent of the choice of basis $\{a_i\}_i$ of A.

Lemma 1.2. *The element $R \in \mathcal{D}_\varphi(A, B)^{\otimes 2}$ is invertible with inverse*

$$R^{-1} = \sum_i (S(a_i) \otimes 1) \otimes (1 \otimes b_i). \tag{1.1}$$

Proof. Write $T \in \mathcal{D}_\varphi(A, B)^{\otimes 2}$ for the right hand side of (1.1). We compute in $\mathcal{D}_\varphi(A, B)^{\otimes 2}$

$$\begin{aligned}
RT &= \sum_{i,j} (a_i S(a_j) \otimes 1) \otimes (1 \otimes b_i b_j) \\
&= \sum_{i,j,l} \varphi\big(a_i S(a_j), b_l\big)(a_l \otimes 1) \otimes (1 \otimes b_i b_j) \\
&= \sum_{i,j,l} \sum_{(b_l)} \varphi\big(a_i, b_{l(2)}\big)\varphi\big(S(a_j), b_{l(1)}\big)(a_l \otimes 1) \otimes (1 \otimes b_i b_j) \\
&= \sum_{i,j,l} \sum_{(b_l)} \varphi\big(a_i, b_{l(2)}\big)\varphi\big(a_j, S^{-1}(b_{l(1)})\big)(a_l \otimes 1) \otimes (1 \otimes b_i b_j) \\
&= \sum_{l} \sum_{(b_l)} (a_l \otimes 1) \otimes \big(1 \otimes b_{l(2)} S^{-1}(b_{l(1)})\big) \\
&= \sum_{l} (\epsilon(b_l)a_l \otimes 1) \otimes (1 \otimes 1) \\
&= (1 \otimes 1)^{\otimes 2}.
\end{aligned}$$

Here the previous lemma gives the second, fifth and seventh equality, the (co)multiplication axiom of the pairing φ implies the third equality, the antipode axiom of the pairing implies the fourth equality, and the antipode axiom in B gives the sixth equality.

The identity $TR = (1 \otimes 1)^{\otimes 2}$ is proved in a similar fashion. □

The invertible element $R \in \mathcal{D}_\varphi(A,B)^{\otimes 2}$ is our candidate for the universal R-matrix for $\mathcal{D}_\varphi(A,B)$. As a first step we show the hexagon relations.

Lemma 1.3. *In $\mathcal{D}_\varphi(A,B)^{\otimes 3}$ we have*

$$(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23},$$
$$(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12}.$$

Proof. We compute in $\mathcal{D}_\varphi(A,B)$,

$$\begin{aligned}
R_{13}R_{23} &= \sum_{i,j} (a_i \otimes 1) \otimes (a_j \otimes 1) \otimes (1 \otimes b_i b_j) \\
&= \sum_{i,j,l} \varphi\big(a_l, b_i b_j\big)(a_i \otimes 1) \otimes (a_j \otimes 1) \otimes (1 \otimes b_l) \\
&= \sum_{i,j,l} \sum_{(a_l)} \varphi\big(a_{l(1)}, b_i\big)\varphi\big(a_{l(2)}, b_j\big)(a_i \otimes 1) \otimes (a_j \otimes 1) \otimes (1 \otimes b_l) \\
&= \sum_{l} \sum_{(a_l)} (a_{l(1)} \otimes 1) \otimes (a_{l(2)} \otimes 1) \otimes (1 \otimes b_l) \\
&= (\Delta \otimes \mathrm{Id})(R).
\end{aligned}$$

The second identity is proved analogously. □

Exercise 1.4. *Prove that* $(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12}$ *in* $\mathcal{D}_\varphi(A,B)^{\otimes 3}$.

In order for R to be the universal R-matrix of $\mathcal{D}_\varphi(A,B)$ it remains to prove that

$$R\Delta(a \otimes b) = \Delta^{op}(a \otimes b)R \tag{1.2}$$

in $\mathcal{D}_\varphi(A,B)^{\otimes 2}$ for all $a \in A$ and $b \in B$. The following lemma gives a convenient alternative criterion for (1.2).

Lemma 1.5 (Radford)**.** *Let H be a Hopf algebra with invertible antipode and fix an element $R = \sum_r \alpha_r \otimes \beta_r \in H \otimes H$. Let $H' \subseteq H$ be a Hopf subalgebra. Then the condition*

$$R\Delta(h) = \Delta^{op}(h)R \qquad \forall h \in H' \tag{1.3}$$

is equivalent to

$$R(1 \otimes h) = \sum_{r,(h)} h_{(3)}\alpha_r S^{-1}(h_{(1)}) \otimes h_{(2)}\beta_r \qquad \forall h \in H'. \tag{1.4}$$

Proof. Suppose that (1.3) holds. It implies

$$\sum_{r,(h)} h_{(1)} \otimes \alpha_r h_{(2)} \otimes \beta_r h_{(3)} = \sum_{r,(h)} h_{(1)} \otimes h_{(3)}\alpha_r \otimes h_{(2)}\beta_r$$

for all $h \in H'$. Applying $(\mu^{op} \otimes \mathrm{Id})(S^{-1} \otimes \mathrm{Id} \otimes \mathrm{Id})$ we get

$$\sum_{r,(h)} \alpha_r h_{(2)} S^{-1}(h_{(1)}) \otimes \beta_r h_{(3)} = \sum_{r,(h)} h_{(3)}\alpha_r S^{-1}(h_{(1)}) \otimes h_{(2)}\beta_r$$

for all $h \in H'$. Applying the antipode axiom and the counit axiom, the left hand side reduces to $R(1 \otimes h)$.

Conversely, suppose that (1.4) holds. Then we have for $h \in H'$,

$$\begin{aligned} R\Delta(h) &= \sum_{(h)} R(1 \otimes h_{(2)})(h_{(1)} \otimes 1) \\ &= \sum_{(h),r} h_{(4)}\alpha_r S^{-1}(h_{(2)})h_{(1)} \otimes h_{(3)}\beta_r \\ &= \sum_{(h)} h_{(2)}\alpha_r \otimes h_{(1)}\beta_r \\ &= \Delta^{op}(h)R, \end{aligned}$$

where the second equality is due to (1.4) and the third equality follows from the antipode axiom and the counit axiom in H'. □

Exercise 1.6. *In the set-up of Radford's lemma, show that* (1.5) *is also equivalent to*

$$(h \otimes 1)R = \sum_{r,(h)} \alpha_r h_{(2)} \otimes S(h_{(1)})\beta_r h_{(3)} \qquad \forall h \in H'. \tag{1.5}$$

Theorem 1.7 (Drinfeld)**.** *Let A and B be finite dimensional Hopf algebras with invertible antipodes. Let $\varphi : A \times B \to k$ be a nondegenerate Hopf pairing.*

Then the Hopf algebra $\mathcal{D}_\varphi(A, B)$ is braided with universal R-matrix R.

Proof. It remains to show (1.2). Since Δ is an algebra homomorphism it suffices to prove that

$$R\Delta(a \otimes 1) = \Delta^{op}(a \otimes 1)R, \qquad \forall a \in A \tag{1.6}$$

and

$$R\Delta(1 \otimes b) = \Delta^{op}(1 \otimes b)R, \qquad \forall b \in B. \tag{1.7}$$

We consider (1.6) first. Since $A \otimes 1 \subset \mathcal{D}_\varphi(A, B)$ is a Hopf subalgebra, Radford's lemma applied to $H' := A \otimes 1$ as Hopf subalgebra in $H := \mathcal{D}_\varphi(A, B)$ implies that (1.6) is equivalent to

$$R((1 \otimes 1) \otimes (a \otimes 1)) = \sum_{i,(a)} (a_{(3)} a_i S^{-1}(a_{(1)}) \otimes 1) \otimes (a_{(2)} \otimes b_i), \qquad \forall a \in A. \tag{1.8}$$

Straightening the left hand side we get

$$\begin{aligned} R((1 \otimes 1) \otimes (a \otimes 1)) &= \sum_i (a_i \otimes 1) \otimes ((1 \otimes b_i)(a \otimes 1)) \\ &= \sum_{i,(a)} \varphi\big(S^{-1}(a_{(1)}), b_{i(1)}\big)\varphi\big(a_{(3)}, b_{i(3)}\big)(a_i \otimes 1) \otimes (a_{(2)} \otimes b_{i(2)}). \end{aligned} \tag{1.9}$$

The formula (1.9) is seen to be equal to the right hand side of (1.8) by expanding the first tensor component in the right hand side of (1.8) in terms of the basis $\{a_j\}_j$. The relevant formula is a special case of the following identity in the vector space $A \otimes B$:

$$\sum_i a a_i a' \otimes b_i = \sum_j \sum_{(b_j)} \varphi\big(a', b_{j(1)}\big)\varphi\big(a, b_{j(3)}\big) a_j \otimes b_{j(2)} \tag{1.10}$$

for $a, a' \in A$. This follows from the computation

$$\begin{aligned}
\sum_i a a_i a' \otimes b_i &= \sum_{i,j} \varphi\big(a a_i a', b_j\big) a_j \otimes b_i \\
&= \sum_{i,j} \sum_{(b_j)} \varphi\big(a, b_{j(3)}\big)\varphi\big(a_i, b_{j(2)}\big)\varphi\big(a', b_{j(1)}\big) a_j \otimes b_i \\
&= \sum_j \sum_{(b_j)} \varphi\big(a, b_{j(3)}\big)\varphi\big(a', b_{j(1)}\big) a_j \otimes b_{j(2)},
\end{aligned}$$

where the first and third equality follow from Lemma 1.1 and the second equality from the (co)multiplication axiom of φ.

The verification of (1.7) follows by a similar argument, now using Exercise **(b)** instead of Radford's lemma. □

Example. The quantum double $\mathcal{D}(G)$ of a finite group is a braided Hopf algebra, with universal R-matrix given by

$$R = \sum_{g \in G} (g \otimes 1) \otimes (1 \otimes e_g).$$

Exercise 1.8. *Complete the proof of Theorem 1.7 by deriving formula* (1.7)*.*

Exercise 1.9. *Let H be a finite dimensional braided bialgebra with universal R-matrix $R \in H \otimes H$. Show that the bilinear form $\varphi : H^* \times H^* \to k$ defined by*

$$\varphi(f, g) := (g \otimes f)(R), \qquad f, g \in H^*$$

is a bialgebra pairing. **Hint:** *You may make use of the identities*

$$(\epsilon \otimes \mathrm{Id})(R) = 1 = (\mathrm{Id} \otimes \epsilon)(R)$$

for the universal R-matrix, see [3, Chpt. 2, Prop. 4.2]*.*

2. Quantum $\mathfrak{sl}_2$

Drinfeld's theorem (Theorem 1.7) allows one to produce lots of interesting examples of braided Hopf algebras. Prominent examples of braided Hopf algebras that can be constructed using the quantum double construction, are quantized universal enveloping algebras of semisimple Lie algebras $\mathfrak{g}$, although some extra care needs to be taken since the associated Hopf algebras are infinite dimensional.

In this section we explain this example in case $\mathfrak{g} = \mathfrak{sl}_2$ (the 2×2 traceless matrices). We first introduce it directly as a Hopf algebra. As a second step, we show that it is the quotient Hopf algebra of a quantum double. The (formal) R-matrix of the quantum double then provides it with a R-matrix of its own.

2.1. **Universal enveloping algebras again.** We first return to the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and its canonical cocommutative Hopf algebra structure. Recall that a Lie algebra $\mathfrak{g}$ is a k-vector space together with a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) satisfying

$$
\begin{aligned}
&[X,Y] = -[Y,X],\\
&[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0
\end{aligned}
$$

for all $X,Y,Z \in \mathfrak{g}$. The second equality is called the Jacobi identity. For an associative algebra A the commutator bracket

$$[a,a'] = aa' - a'a \qquad (a,a' \in A)$$

turns A into a Lie algebra, denoted by $\mathfrak{g}(A)$ (check the Jacobi identity!). The general linear Lie algebra $\mathfrak{gl}_k(V)$ associated to a finite dimensional k-vector space V is $\mathfrak{g}(\mathrm{End}_k(V))$. It is isomorphic to the Lie algebra of $\dim_k(V)\times\dim_k(V)$-matrices with entries in k, with Lie bracket the commutator bracket with respect to matrix multiplication. The special linear Lie algebra $\mathfrak{sl}_k(V)$ associated to V is the Lie subalgebra of $\mathfrak{gl}_k(V)$ consisting of endomorphisms $\phi \in \mathfrak{gl}_k(V)$ (cq. matrices) with zero trace. This is indeed a Lie algebra with respect to the commutator bracket, since the commutator of two traceless endomorphisms is traceless again.

We recall now shortly some general facts about universal enveloping algebras, discussed already in previous lectures. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The *universal enveloping algebra* of $\mathfrak{g}$ is the associative algebra $U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g})$ with $I(\mathfrak{g})$ the two-sided ideal in the tensor algebra $T(\mathfrak{g})$ generated by the elements

$$X\otimes Y - Y\otimes X - [X,Y], \qquad \forall\, X,Y \in \mathfrak{g}.$$

In other words, the Lie bracket of $X,Y \in \mathfrak{g}$, viewed as elements in $\mathfrak{g}(T(\mathfrak{g}))$, is forced to equal $[X,Y]$ when projected onto $U(\mathfrak{g})$, where $[X,Y]$ is the Lie bracket of X and Y in the Lie algebra $\mathfrak{g}$.

The tensor algebra $T(\mathfrak{g})$ has a natural cocommutative Hopf algebra structure, characterized by $\epsilon(X) = 0$, $\Delta(X) = X\otimes 1 + 1\otimes X$ and $S(X) = -X$ for all $X \in \mathfrak{g}$; the ideal $I(\mathfrak{g})$ is a Hopf ideal of $T(\mathfrak{g})$, hence $U(\mathfrak{g})$ is a cocommutative Hopf algebra. A fundamental result, which we state here without proof, is the Poincaré-Birkhoff-Witt theorem. It implies in particular that the canonical linear map $\mathfrak{g} \to U(\mathfrak{g})$ is injective.

Theorem 2.1. *If $\{a_1,\ldots,a_r\}$ is a k-linear basis of $\mathfrak{g}$, then the set of monomials*

$$\{a_1^{m_1}a_2^{m_2}\cdots a_r^{m_r} \mid m_j \in \mathbb{Z}_{\geq 0}\}$$

in $U(\mathfrak{g})$ is a k-linear basis of $U(\mathfrak{g})$.

We consider now the case $\mathfrak{g} = \mathfrak{sl}_2 \simeq \mathfrak{sl}(k^{\oplus 2})$ of traceless 2×2 matrices in detail. It is a three dimensional vector space with basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie algebra structure is determined by

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{2.1}$$

The corresponding universal enveloping algebra $U(\mathfrak{sl}_2)$ thus is the associative, unital algebra over k with generators H, E, F and defining relations (2.1). It has for instance $\{H^l F^m E^n \mid l, m, n \in \mathbb{Z}_{\geq 0}\}$ as a linear basis.

Consider the Lie subalgebras $\mathfrak{b}_\pm \subset \mathfrak{sl}_2$ with $\mathfrak{b}_+$ (respectively $\mathfrak{b}_-$) the upper (respectively lower) triangular matrices in $\mathfrak{sl}_2$. Then $\mathfrak{b}_+$ (respectively $\mathfrak{b}_-$) is two-dimensional with k-basis $\{H, E\}$ (respectively $\{H, F\}$). The resulting universal enveloping algebra $U(\mathfrak{b}_+)$ (respectively $U(\mathfrak{b}_-)$) is isomorphic to the Hopf subalgebra of $U(\mathfrak{sl}_2)$ generated by H and E (respectively H and F).

2.2. **The quantized universal enveloping algebra, as a Hopf algebra.** Let $k(q)$ be the field of k-rational functions in the indeterminate q.

Definition 2.2. *We denote U for the unital associative algebra over $k(q)$ with generators K, K^{-1}, E and F with defining relations*

$$\begin{aligned} KE &= q^2 EK, & KF &= q^{-2} FK, \\ KK^{-1} &= 1 = K^{-1}K, & EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \tag{2.2}$$

In other words, U is the quotient algebra $T(V)/J$ where V is the four dimensional $k(q)$-vector space with basis $\{K, K^{-1}, E, F\}$ and J the two-sided ideal of the corresponding tensor algebra $T(V)$ over $k(q)$ generated by the elements $KE - q^2EK$, $KF - q^{-2}FK$, $KK^{-1} - 1$, $K^{-1}K - 1$ and $EF - FE - \frac{K-K^{-1}}{q-q^{-1}}$.

To formally re-obtain $U(\mathfrak{sl}_2)$ from U one views K as q^H and one takes the limit $q \to 1$. This idea is made mathematically more rigorous in the following exercise.

Exercise 2.3. *Let v be a fixed nonzero element in the field k.*

For $v \neq \pm 1$ let U_v be the unital associative algebra over k with generators K, K^{-1}, E, F and defining relations (2.2) *in which q is replaced by v.*

(i) *Define $\widehat{U}_v$ for the unital associative algebra over k with generators K, K^{-1}, E, F, L and with defining relations*

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, & (v - v^{-1})L &= K - K^{-1}, \\ KEK^{-1} &= v^2 E, & LE - EL &= v(EK + K^{-1}E), \\ KFK^{-1} &= v^{-2} F, & LF - FL &= -v^{-1}(FK + K^{-1}F), \\ EF - FE &= L. \end{aligned}$$

Show that $U_v \simeq \widehat{U}_v$ *as algebras when* $v \neq \pm 1$.
(ii) *Show that*

$$U(\mathfrak{sl}_2) \simeq \widehat{U}_1/(K-1)$$

as algebras.

The following lemma gives the Poincaré-Birkhoff-Witt property of U.

Lemma 2.4. *The set* $\{F^l K^m E^n \,|\, m \in \mathbb{Z},\, l,n \in \mathbb{Z}_{\geq 0}\}$ *is a* $k(q)$*-basis of* U.

Proof. We follow the arguments in the proof of [1, Chpt. 1, Thm. 1.5].
By the commutation relations in U it is clear that $\{F^l K^m E^n \,|\, m \in \mathbb{Z},\, l,n \in \mathbb{Z}_{\geq 0}\}$ spans U. We have to show it is a linear independent set.
We claim that there exists a unique algebra homomorphism

$$\pi : U \to \mathrm{End}_{k(q)}\big(k(q)[X,Y,Z^{\pm 1}]\big)$$

satisfying

$$\begin{aligned}
\pi(F)(Y^s Z^n X^r) &= Y^{s+1} Z^n X^r,\\
\pi(E)(Y^s Z^n X^r) &= q^{-2n} Y^s Z^n X^{r+1} + \left(\frac{q^s - q^{-s}}{q-q^{-1}}\right) Y^{s-1} \left(\frac{Zq^{1-s} - Z^{-1}q^{s-1}}{q-q^{-1}}\right) Z^n X^r,\\
\pi(K^{\pm 1})(Y^s Z^n X^r) &= q^{\mp 2s} Y^s Z^{n \pm 1} X^r
\end{aligned}$$

for $r,s \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$ (the second term for the formula of $\pi(E)(Y^s Z^n X^r)$ should be read as zero if $s=0$). Indeed, π is well defined as algebra homomorphism $\pi : T(V) \to \mathrm{End}_{k(q)}\big(k(q)[X,Y,Z^{\pm 1}]\big)$, and $J \subset \mathrm{Ker}(\pi)$ since the generators of J are in the kernel of π (check this!).
The elements $\pi(F^l K^m E^n)(1) = Y^l Z^m X^n$ ($m \in \mathbb{Z}$, $l,n \in \mathbb{Z}_{\geq 0}$) are linear independent in $k(q)[X,Y,Z^{\pm 1}]$, hence so are the elements $F^l K^m E^n$ ($m \in \mathbb{Z}$, $l,n \in \mathbb{Z}_{\geq 0}$) in U. □

In the following proposition we deform the Hopf algebra structure of $U(\mathfrak{sl}_2)$ to turn U into a non commutative and non cocommutative Hopf algebra.

Proposition 2.5. *The algebra* U *is a Hopf algebra with comultiplication characterized by*

$$\begin{aligned}
\Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1},\\
\Delta(E) &= E \otimes 1 + K \otimes E,\\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F,
\end{aligned}$$

counit characterized by

$$\epsilon(K^{\pm 1}) = 1, \qquad \epsilon(E) = 0 = \epsilon(F),$$

and invertible antipode characterized by

$$S(K^{\pm 1}) = K^{\mp 1}, \qquad S(E) = -K^{-1}E, \qquad S(F) = -FK.$$

Proof. The maps Δ, ϵ and S can be uniquely defined as algebra homomorphisms $\Delta : T(V) \to T(V)^{\otimes 2}$, $\epsilon : T(V) \to k(q)$ and $S : T(V) \to T(V)^{op}$ by the above defining formulas. It is a direct verification that the resulting maps turn $T(V)$ into a Hopf algebra. Indeed, the Hopf algebra axioms need only to be verified on the algebraic generators of $T(V)$, in which case they follow by a direct check. For instance,

$$\begin{aligned}\mu(S \otimes \mathrm{Id})\Delta(E) &= S(E)1 + S(K)E \\ &= -K^{-1}E + K^{-1}E \\ &= 0 \\ &= \epsilon(E)1.\end{aligned}$$

As the next step we need to verify that the two-sided ideal $J \subset T(V)$ is a Hopf ideal. Consider the algebra maps $\overline{\Delta} : T(V) \to U \otimes U$ and $\overline{S} : T(V) \to U^{op}$ obtained by $\overline{\Delta} = (\pi \otimes \pi) \circ \Delta$ and $\overline{S} = \pi \circ S$, with $\pi : T(V) \to U = T(V)/J$ the canonical map. It suffices to prove that J is contained in the kernel of ϵ, $\overline{\Delta}$ and $\overline{S}$. These are direct verifications again. For instance,

$$\begin{aligned}\overline{\Delta}(KEK^{-1}) &= (K \otimes K)(E \otimes 1 + K \otimes E)(K^{-1} \otimes K^{-1}) \\ &= KEK^{-1} \otimes 1 + K \otimes KEK^{-1} \\ &= q^2(E \otimes 1 + K \otimes E) = q^2\overline{\Delta}(E),\end{aligned}$$

hence $KEK^{-1} - q^2E \in \mathrm{Ker}(\overline{\Delta})$. Finally, it is easy to produce the inverse of the antipode of U. It is the unique algebra homomorphism $U \to U^{op}$ satisfying $K^{\pm 1} \mapsto K^{\mp 1}$, $E \mapsto -EK^{-1}$ and $F \mapsto -KF$. □

Remark 2.6. The conventions for the Hopf algebra U which we have chosen here are the same as the ones in [3, 1], but they differ from the ones in [2, 4] (in [2, 4] the Hopf algebra U^{cop} is the quantized universal enveloping algebra of $\mathfrak{sl}_2$).

Exercise 2.7. *Fill in the details of the proof of Proposition 2.5.*

As a consequence of Proposition 2.5, the category Mod_U of left U-modules over $k(q)$ is a monoidal category. We will show next week that a suitable full sub-category of Mod_U is a *braided* monoidal category by realizing U as a quotient of a generalized quantum double $\mathcal{D}_\varphi(U_+, U_-)$, where U_+ (resp. U_-) is the Hopf sub-algebra of U generated by $K^{\pm 1}$, E (respectively $K^{\pm 1}$, F), and with $\varphi : U_+ \times U_- \to k(q)$ a suitable nondegenerate Hopf pairing.

Exercise 2.8. *Show that $U_\pm \simeq T(V_\pm)/J_\pm$, where V_+ (resp. V_-) is the three dimensional $k(q)$-vector space with basis $\{K^{\pm 1}, E\}$ (resp. $\{K^{\pm 1}, F\}$) and with $J_+ \subset T(V_+)$ (resp. $J_- \subset T(V_-)$) the two-sided ideal generated by $KK^{-1} - 1$, $K^{-1}K - 1$ and $KE - q^2EK$ (resp. $KK^{-1} - 1$, $K^{-1}K - 1$ and $KF - q^{-2}FK$).*

2.3. **U as a quotient of a generalized quantum double.** We start with the construction of the relevant Hopf pairing.

Lemma 2.9. *There exists a unique Hopf pairing*

$$\varphi : U_+ \times U_- \to k(q)$$

satisfying

$$\varphi(E,F) = \frac{1}{q^{-1}-q}, \qquad \varphi(E,K^\xi) = 0 = \varphi(K^\eta,F), \qquad \varphi(K^\xi,K^\eta) = q^{-2\xi\eta}$$

for $\xi,\eta \in \{\pm 1\}$.

Proof. We use the notations and result of Exercise 2.8. By Lemma 2.8 in the syllabus of last week we have a unique bialgebra pairing $\widetilde{\varphi} : T(V_+) \times T(V_-) \to k(q)$ satisfying the stated conditions, where $T(V_+)$ and $T(V_-)$ are Hopf algebras with comultiplication, counit and antipode given by

$$\begin{aligned} \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, & \Delta(E) &= E \otimes 1 + K \otimes E, \\ \epsilon(K^{\pm 1}) &= 1, & \epsilon(E) &= 0, \\ S(K^{\pm 1}) &= K^{\mp 1}, & S(E) &= -K^{-1}E \end{aligned}$$

for $T(V_+)$ and

$$\begin{aligned} \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, & \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \\ \epsilon(K^{\pm 1}) &= 1, & \epsilon(F) &= 0, \\ S(K^{\pm 1}) &= K^{\mp 1}, & S(F) &= -FK \end{aligned}$$

for $T(V_-)$, cf. Proposition 2.5. Using Exercise 2.5 in the syllabus of last week one verifies that $\widetilde{\varphi}$ is in fact a Hopf pairing. For example, one of the equalities one needs to check is $\widetilde{\varphi}(S(E),F) = \widetilde{\varphi}(E,S^{-1}(F))$, or equivalently $\widetilde{\varphi}(K^{-1}E,F) = \widetilde{\varphi}(E,KF)$. Using the (co)multiplication axiom of $\widetilde{\varphi}$, this is equivalent to verifying that

$$\begin{aligned} &\widetilde{\varphi}(K^{-1},K^{-1})\widetilde{\varphi}(E,F) + \widetilde{\varphi}(K^{-1},F)\widetilde{\varphi}(E,1) \\ &\qquad = \widetilde{\varphi}(E,K)\widetilde{\varphi}(1,F) + \widetilde{\varphi}(K,K)\widetilde{\varphi}(E,F). \end{aligned}$$

By the defining identities of $\widetilde{\varphi}$, both the left and the right hand side equal $q^{-2}(q^{-1}-q)^{-1}$.

It remains to show that the Hopf ideal $J_\pm \subset T(V_\pm)$ is contained in $I_{U_\pm}$, where (recall from last week)

$$\begin{aligned} I_{U_+} &= \{X \in U_+ \mid \widetilde{\varphi}(X,Y) = 0 \quad \forall Y \in U_-\}, \\ I_{U_-} &= \{Y \in U_- \mid \widetilde{\varphi}(X,Y) = 0 \quad \forall X \in U_+\}. \end{aligned}$$

We discuss the argument for J_+. We use the following
Claim: If $\widetilde{\varphi}(X,Y) = 0$ for the generators X of the two-sided ideal J_+ and for $Y \in V_- \oplus k$, then $J_+ \subset I_{U_+}$.
Proof of the claim: If $\widetilde{\varphi}(X,Y) = 0$ for the generators X of the two-sided ideal J_+ and for $Y \in V_- \oplus k$, then $\widetilde{\varphi}(X,Y) = 0$ for all $X \in J_+$ and $Y \in V_- \oplus k$ by the (co)multiplication axiom for $\widetilde{\varphi}$ and the fact that $\Delta(V_- \oplus k) \subset (V_- \oplus k)^{\otimes 2}$. It now also follows that $\widetilde{\varphi}(X,Y) = 0$ for all $X \in J_+$ and $Y \in T(V_-)$, in view of the (co)multiplication axiom for $\widetilde{\varphi}$ and the fact that J_+ is a Hopf ideal. Hence, $J_+ \subset I_{U_+}$.

We now use the claim to show that $J_+ \subset I_{U_+}$. First note that $J_+ \subset \mathrm{Ker}(\epsilon)$, so $\widetilde{\varphi}(X, 1) = 0$ for all $X \in J_+$. The verifications of $\widetilde{\varphi}(X, Y) = 0$ for X a generator of the two-sided ideal J_+ and for $Y \in V_-$ are direct computations. For instance,

$$\begin{aligned}\widetilde{\varphi}(KE - q^2 EK, F) &= \widetilde{\varphi}(K, K^{-1})\widetilde{\varphi}(E, F) + \widetilde{\varphi}(K, F)\widetilde{\varphi}(E, 1) \\ &\quad - q^2\widetilde{\varphi}(E, K^{-1})\widetilde{\varphi}(K, F) - q^2\widetilde{\varphi}(E, F)\widetilde{\varphi}(K, 1) \\ &= \frac{q^2}{q^{-1} - q} + 0 + 0 - \frac{q^2}{q^{-1} - q} = 0.\end{aligned}$$

The Hopf pairing $\widetilde{\varphi}$ thus gives rise to a Hopf pairing $\varphi : U_+ \times U_- \to k(q)$ by the formula $\varphi(X + J_+, Y + J_-) := \widetilde{\varphi}(X, Y)$ for $X + J_+ \in T(V_+)/J_+ = U_+$ and $X + J_- \in T(V_-)/J_- = U_-$. This is the desired Hopf pairing. □

Proposition 2.10. *Let $\mathcal{D}$ be the unital associative algebra over $k(q)$ with generators $E, F, K^{\pm 1}$ and $K'^{\pm 1}$ and with defining relations*

$$\begin{gathered} KK^{-1} = 1 = K^{-1}K, \qquad KE = q^2 EK, \\ K'K'^{-1} = 1 = K'^{-1}K', \qquad K'F = q^{-2}FK', \\ K'E = q^2 EK', \qquad KF = q^{-2}FK, \\ KK' = K'K, \qquad EF - FE = \frac{K - K'^{-1}}{q - q^{-1}}. \end{gathered}$$

Then $\mathcal{D} \simeq \mathcal{D}_\varphi(U_+, U_-)$ as algebras, with the isomorphism $\mathcal{D} \to \mathcal{D}_\varphi(U_+, U_-)$ determined by

$$\begin{aligned} K^{\pm 1} &\mapsto K^{\pm 1} \otimes 1, \qquad & K'^{\pm 1} &\mapsto 1 \otimes K^{\pm 1}, \\ E &\mapsto E \otimes 1, & F &\mapsto 1 \otimes F. \end{aligned} \tag{2.3}$$

Proof. We first compute the commutation relations of the elements $K^{\pm 1} \otimes 1$, $E \otimes 1$ with $1 \otimes K^{\pm 1}$ and $1 \otimes F$ in the generalized quantum double $\mathcal{D}_\varphi(U_+, U_-)$ using the "straightening rule" for the multiplication in $\mathcal{D}_\varphi(U_+, U_-)$. We claim that we get the following list of relations in $\mathcal{D}_\varphi(U_+, U_-)$,

$$\begin{aligned} &(E \otimes 1)(1 \otimes F) - (1 \otimes F)(E \otimes 1) = \frac{K \otimes 1 - 1 \otimes K^{-1}}{q - q^{-1}}, \\ &(E \otimes 1)(1 \otimes K) = q^{-2}(1 \otimes K)(E \otimes 1), \\ &(K \otimes 1)(1 \otimes F) = q^{-2}(1 \otimes F)(K \otimes 1), \\ &(K \otimes 1)(1 \otimes K) = (1 \otimes K)(K \otimes 1). \end{aligned} \tag{2.4}$$

We derive the second relation as an example, the others are left as an important exercise (see below). We first note that

$$\sum_{(K)} K_{(1)} \otimes K_{(2)} \otimes K_{(3)} = (\Delta \otimes \mathrm{Id})(\Delta(K)) = K \otimes K \otimes K$$

in $U_\pm$ and

$$\sum_{(E)} E_{(1)} \otimes E_{(2)} \otimes E_{(3)} = E \otimes 1 \otimes 1 + K \otimes E \otimes 1 + K \otimes K \otimes E$$

in U_+, in view of the explicit definitions of the Hopf algebra maps. By the straightening rule in the generalized quantum double we compute in $\mathcal{D}_\varphi(U_+, U_-)$,

$$\begin{aligned}(1 \otimes K)(E \otimes 1) =& \sum_{(E)} \varphi(E_{(1)}, S(K))\varphi(E_{(3)}, K)(E_{(2)} \otimes 1)(1 \otimes K)\\ =&\varphi(E, K^{-1})\varphi(1, K)(1 \otimes K) + \varphi(K, K^{-1})\varphi(1, K)(E \otimes 1)(1 \otimes K)\\ &+ \varphi(K, K^{-1})\varphi(E, K)(K \otimes 1)(1 \otimes K)\\ =&q^2(E \otimes 1)(1 \otimes K)\end{aligned}$$

in view of the definition of φ.

Now we are ready to prove the proposition. By (2.4) and the fact that the canonical linear embeddings $U_\pm \to \mathcal{D}_\varphi(U_+, U_-)$ are algebra morphisms, we have a unique algebra homomorphism $\phi : \mathcal{D} \to \mathcal{D}_\varphi(U_+, U_-)$ satisfying (2.3). It remains to show that ϕ is an isomorphism.

By the defining relations in $\mathcal{D}$ it is clear that the set $\mathcal{B} = \{E^m K^l F^r K'^n \mid l, n \in \mathbb{Z},\ m, r \in \mathbb{Z}_{\geq 0}\} \subset \mathcal{D}$ of monomials is a spanning set of $\mathcal{D}$ over $k(q)$. Clearly $\phi(\mathcal{B})$ is a $k(q)$-linear basis of $\mathcal{D}_\varphi(U_+, U_-)$, hence $\mathcal{B}$ is a $k(q)$-linear basis of $\mathcal{D}$. The algebra morphism ϕ thus maps a $k(q)$-linear basis of $\mathcal{D}$ to a $k(q)$-linear basis of $\mathcal{D}_\varphi(U_+, U_-)$, hence it is an isomorphism. ☐

Exercise 2.11. *Prove the remaining commutation relations* (2.4) *in* $\mathcal{D}_\varphi(U_+, U_-)$.

By the previous proposition the algebra $\mathcal{D}$ has a unique Hopf algebra structure such that the algebra isomorphism $\mathcal{D} \xrightarrow{\sim} \mathcal{D}_\varphi(U_+, U_-)$ is an isomorphism of Hopf algebras. We have the following description of the quantized universal enveloping algebra U of $\mathfrak{sl}_2$ in terms of the generalized quantum double $\mathcal{D}$.

Corollary 2.12. *The two-sided ideal* $(K - K') \subset \mathcal{D}$ *is a Hopf ideal. Furthermore,*

$$\mathcal{D}/(K - K') \simeq U$$

as Hopf algebras.

Proof. By Proposition 2.10 and the defining relations in U there is a unique algebra homomorphism $U \to \mathcal{D}/(K - K')$ mapping $K^{\pm 1}, E, F$ to their respective classes in the quotient algebra $\mathcal{D}/(K - K')$. It can be easily inverted by the previous proposition. Indeed, it follows from Proposition 2.10 and the defining relations in U that the assignments sending the generators $E, F, K^{\pm 1}, K'^{\pm 1}$ of $\mathcal{D}$ to the elements $E, F, K^{\pm 1}, K^{\pm 1}$ in U respectively, uniquely extend to an algebra homomorphism $\mathcal{D} \to U$. The kernel contains the two-sided ideal $(K - K')$, hence it gives rise to a well defined algebra homomorphism $\mathcal{D}/(K - K') \to U$. This map clearly inverts the earlier constructed algebra homomorphism $U \to \mathcal{D}/(K - K')$.

It is a standard check to verify that the two-sided ideal $(K - K') \subset \mathcal{D}$ is a Hopf ideal. Comparing the explicit formulas for the Hopf algebra maps of U and the quotient Hopf algebra $\mathcal{D}/(K - K')$, one verifies that the constructed algebra isomorphism is in fact an isomorphism of Hopf algebras. □

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