QUANTIZED UNIVERSAL ENVELOPING ALGEBRA OF $\mathfrak{sl}_2$ AND THE ASSOCIATED BRAIDING

In this text we use the quantum double construction to show that a suitable category of finite dimensional left U-modules is braided, where U is the quantized universal enveloping algebra of $\mathfrak{sl}_2(k)$. We produce the expression of the corresponding braiding as the action of a formal universal R-matrix for U. We indicate the generalization to $\mathfrak{sl}_n(k)$ and the relation to Hecke algebras and Temperley-Lieb algebras. This text complements [4, Chpt. 4].

Convention: As before, k is assumed to be an algebraically closed field of characteristic zero.

We give several exercises in the text.
The **homework exercises** are: Exercise 2.3 and 2.7.

1. Towards a universal R-matrix for U

Recall from last week the Hopf pairing $\varphi : U_+ \times U_- \to k(q)$ between the Hopf subalgebras $U_+ = k(q)\langle K^{\pm 1}, E\rangle$ and $U_- = k(q)\langle K^{\pm 1}, F\rangle$ of the quantized universal enveloping algebra U of $\mathfrak{sl}_2(k)$. It is characterized by the formulas

$$\varphi(E, F) = \frac{1}{q^{-1} - q}, \qquad \varphi(K^{\pm 1}, F) = 0 = \varphi(E, K^{\pm 1}), \qquad \varphi(K^\xi, K^\eta) = q^{-2\xi\eta}$$

for $\xi, \eta \in \{\pm 1\}$. We now exploit Corollary 2.12 in the syllabus of last week, which says that U is the quotient Hopf algebra $\mathcal{D}_\varphi(U_+, U_-)/(K \otimes 1 - 1 \otimes K)$ of the generalized quantum double $\mathcal{D} = \mathcal{D}_\varphi(U_+, U_-)$, to construct braidings on a suitable full monoidal subcategory of Mod_U.

We first show that the Hopf pairing φ is nondegenerate. Formally it will allow us to define a universal R-matrix for $\mathcal{D}$ as in the notes of last week, but it does not make sense as element in $\mathcal{D} \otimes \mathcal{D}$ (since the underlying Hopf algebras $U_\pm$ are not finite dimensional). There are several ways to overcome this technical problem: one can work with topological Hopf algebras, one can consider the theory for q specialized to a root of unity, or one can make sense of the formal universal R-matrix when applied to tensor products for a restrictive class of U-modules. We will follow here the third approach, which suffices for our purposes (the construction of quantum invariants of ribbons and links). The second approach is of vital importance in the construction of quantum invariants of 3-manifolds.

In the following exercise we ask you to prove a noncommutative version of Newton's binomium theorem. For this we introduce some notations. Define the following elements

in $k(q)^\times = k(q) \setminus \{0\}$,

(1.1)

$$(m)_{q^2} = \frac{q^{2m} - 1}{q^2 - 1},$$

$$(m)_{q^2}! = (m)_{q^2}(m-1)_{q^2}\cdots(1)_{q^2}, \qquad \binom{n}{m}_{q^2} = \frac{(n)_{q^2}!}{(n-m)_{q^2}!(m)_{q^2}!}$$

for $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$. A direct computation gives the generalized Pascal identity

(1.2)

$$\binom{m}{k}_{q^2} = \binom{m-1}{k-1}_{q^2} + q^{2k}\binom{m-1}{k}_{q^2}, \qquad 0 \leq k \leq m,$$

where $\binom{s}{r}_{q^2}$ is set to be zero unless r, s are integers satisfying $0 \leq r \leq s$.

Exercise 1.1. *Let A be an unital associative algebra over $k(q)$. Suppose that $x, y \in A$ satisfies $xy = q^{-2}yx$ in A. Show that*

$$(x + y)^n = \sum_{m=0}^{n} \binom{n}{m}_{q^2} x^m y^{n-m}, \qquad \forall\, n \in \mathbb{Z}_{\geq 0}.$$

As a consequence of the previous exercise we obtain

Corollary 1.2. *In $U \otimes U$ we have*

$$\Delta(E^n) = \sum_{m=0}^{n} \binom{n}{m}_{q^2} E^m K^{n-m} \otimes E^{n-m},$$

$$\Delta(F^n) = \sum_{m=0}^{n} \binom{n}{m}_{q^2} F^{n-m} \otimes F^m K^{m-n}$$

for $n \in \mathbb{Z}_{\geq 0}$.

Proof. For $\Delta(E^n)$ (respectively $\Delta(F^n)$) use the previous exercise with $A = U^{\otimes 2}$ and $x = E \otimes 1$, $y = K \otimes E$ (respectively $x = 1 \otimes F$, $y = F \otimes K^{-1}$). $\qquad\square$

Proposition 1.3. *For $r, s \in \mathbb{Z}$ and $m, n \in \mathbb{Z}_{\geq 0}$ we have*

(1.3)

$$\varphi\big(E^m K^r, F^n K^s\big) = q^{-2rs}\frac{(m)_{q^2}!}{(q^{-1} - q)^m}\delta_{m,n}.$$

Proof. This is a rather elaborate inductive proof. First note that the formula is correct for $m = n = 0$ (using the (co)multiplication axiom for φ). We next verify it if either $m = 0$ or $n = 0$: we consider here the case $n = 0$. We then have to show that

$$\varphi(E^m K^r, K^s) = q^{-2rs}\delta_{m,0}.$$

It is ok for $m = 0$. We proceed by induction to m. First note that $\varphi(E, K^s) = 0$ (which follows by induction to s by the (co)multiplication axiom for φ and its validity for $s = -1, 0, 1$). The (co)multiplication axiom for φ now implies the induction step,

$$\varphi\big(E^m K^r, K^s\big) = \varphi(E, K^s)\varphi(E^{m-1} K^r, K^s) = 0$$

for $m > 0$.

We now proceed to prove the validity of (1.3) by induction to $m + n$. Suppose the formula is correct if $m + n < \xi$ for $\xi \in \mathbb{Z}_{>0}$. Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m + n = \xi$. By the previous paragraph we may assume without loss of generality that $m, n > 0$.

We first show that if (1.3) is valid for $r = s = 0$, then it is valid in general. Indeed, under these assumptions, by the previous lemma,

$$\varphi(E^m K^r, F^n) = \sum_{l=0}^{n} \binom{n}{l}_{q^2} \varphi(E^m, F^l K^{l-n})\varphi(K^r, F^{n-l})$$
$$= \varphi(E^m, F^n)\varphi(K^r, 1) = \varphi(E^m, F^n),$$

where the second identity holds by the induction hypothesis. Hence (1.3) is valid for $s = 0$ if it is valid for $r = s = 0$. Similarly,

$$\varphi(E^m K^r, F^n K^s) = \sum_{l=0}^{m} \binom{m}{l}_{q^2} \varphi(E^l K^{r+m-l}, F^n)\varphi(E^{m-l} K^r, K^s)$$
$$= \varphi(E^m K^r, F^n)\varphi(K^r, K^s) = q^{-2rs}\varphi(E^m K^r, F^n),$$

hence (1.3) is valid if it is valid for $s = 0$.

So it remains to show that (1.3) is correct for $r = s = 0$. We compute, using the previous lemma and the induction hypothesis,

$$\varphi(E^m, F^n) = \sum_{l=0}^{m} \binom{m}{l}_{q^2} \varphi(E^l K^{m-l}, F)\varphi(E^{m-l}, F^{n-1})$$
$$= \delta_{m,n}\varphi(E^{m-1}, F^{m-1})\varphi(EK^{m-1}, F) \sum_{l=0}^{m} \binom{m}{l}_{q^2} \delta_{l,1}$$
$$= \delta_{m,n}\frac{(m-1)_{q^2}!}{(q^{-1} - q)^{m-1}}\frac{1}{(q^{-1} - q)}(m)_{q^2}$$
$$= \frac{(m)_{q^2}!}{(q^{-1} - q)^m}\delta_{m,n}.$$

$\square$

Corollary 1.4. *The Hopf pairing* $\varphi : U_+ \times U_- \to k(q)$ *is nondegenerate.*

Proof. Let $U_0 \subset U_+ \cap U_-$ be the subalgebra generated by $K^{\pm 1}$. The previous proposition gives

$$\varphi\big(E^m X, F^n Y\big) = \varphi(X,Y)\frac{(m)_{q^2}!}{(q^{-1}-q)^m}\,\delta_{m,n}$$

for $m,n \in \mathbb{Z}_{\geq 0}$. It thus suffices to show that the restriction of φ to $U_0 \times U_0$ is nondegenerate. Let $X = \sum_{l=L}^{M} c_l K^l \in U_0$ with $c_l \in k(q)$, $L < M$ and suppose that $X \in I_{U_+}$, i.e. that $\varphi(X,Y) = 0$ for all $Y \in U_0$. We show that $c_l = 0$ for all l. Since $X \in I_{U_+}$ if and only if $XK^{-L} \in I_{U_+}$, we may assume without loss of generality that $L = 0$. Then for all $m = 0, \ldots, M$,

$$0 = \varphi(X, K^m) = \sum_{l=0}^{M} c_l q^{-2lm}.$$

The $(M+1) \times (M+1)$-matrix $\big(q^{-2lm}\big)_{0 \leq l,m \leq M}$ with coefficients in $k(q)$ is invertible, since

$$\mathrm{Det}\big(q^{-2lm}\big)_{0 \leq l,m \leq M} = \prod_{0 \leq l < m \leq M} (q^{-2m} - q^{-2l})$$

(Vandermonde determinant evaluation). Hence the c_l's are all zero. $\qquad\square$

Ignoring for the moment the possible complications due to the fact that the $U_\pm$ are infinite dimensional, we can formally write down an universal R-matrix for the generalized quantum double $\mathcal{D} \simeq \mathcal{D}_\varphi(U_+, U_-)$. We subsequently map it to a formal universal R-matrix for U using the fact that $U \simeq \mathcal{D}/(K - K')$.

To get a grip on the specific form of the resulting formal universal R-matrix of U we fix $\{E^m K^r\}_{m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}}$ as basis of U_+ and we consider its dual basis with respect to φ. Since U_+ and U_- are infinite dimensional, this becomes a subtle matter; we again content ourselves for the moment with a formal argument in order to get a useful ansatz for the form of the universal R-matrix.

By Proposition 1.3 the basis of U_- dual to $\{E^m K^r\}_{m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}}$ with respect to φ should formally be of the form

$$\left\{\frac{(q^{-1}-q)^m}{(m)_{q^2}!}F^m f_r(K) \mid m \in \mathbb{Z}_{\geq 0},\ r \in \mathbb{Z}\right\}$$

with $f_r(K) = \sum_{m \in \mathbb{Z}} d_r(m)K^m$ $(d_r(m) \in k(q))$ formal infinite sums such that

$$\varphi(K^r, f_s(K)) := \sum_{m \in \mathbb{Z}} d_s(m)\varphi(K^r, K^m) = \delta_{r,s}, \qquad \forall\, r,s \in \mathbb{Z}.$$

This is a troublesome formula: the middle term is the infinite sum $\sum_{m \in \mathbb{Z}} d_s(m)q^{-2rm}$, which in general will not make sense in $k(q)$; in other words, it is even not possible to interpret $f_s(K)$ as element in U_0^*. Ignoring this obstacle for the moment, we get an ansatz

for a universal R-matrix of $\mathcal{D}$ by formally applying Theorem 1.7 of the syllabus of week 13 to the *infinite* dimensional Hopf algebras U_+ and U_- with nondegenerate Hopf pairing φ:

$$\sum_{m\in\mathbb{Z}_{\geq 0},\, r\in\mathbb{Z}} \frac{(q^{-1}-q)^m}{(m)_{q^2}!} E^m K^r \otimes F^m f_r(K'),$$

while for $U \simeq \mathcal{D}/(K - K^{-1})$ the ansatz for a universal R-matrix becomes

(1.4)
$$R = \overline{R} R_0$$

with

(1.5)
$$\overline{R} = \sum_{m\in\mathbb{Z}_{\geq 0}} \frac{(q^{-1}-q)^m}{(m)_{q^2}!} E^m \otimes F^m$$

and with

(1.6)
$$R_0 = \sum_{r\in\mathbb{Z}} K^r \otimes f_r(K).$$

For the last statement, we have formally used the following remark (its proof being straightfoward).

Remark 1.5. If $\pi : H \to H'$ is a surjective bialgebra morphism and if H is braided with universal R-matrix $R \in H\otimes H$, then H' is braided with universal R-matrix $R' := (\pi\otimes\pi)(R)$.

Since formally $\sum_r K^r \varphi(K^s, f_r(K)) = K^s$ for all $s \in \mathbb{Z}$, we may think of R_0 as the formal reproducing kernel of the linear space U_0 endowed with the nondegenerate symmetric bilinear form $\varphi|_{U_0\times U_0}$.

2. Finite dimensional type 1 modules

We seem still far away from a rigorous braiding for U, but at least we have arrived at a reasonable ansatz for its explicit form ((1.4)-(1.6)). We will use this ansatz to indeed produce a braiding and to give an explicit expression of, and meaning for, R_0. We first need to introduce a suitable sub-category of U-modules to give a rigorous meaning to the formulas.

It is convenient for the sequel to enlarge the ground field $k(q)$ by adjoining a square root of q; we denote it by $\mathbb{K} = k(q^{\frac{1}{2}})$. We extend the base field to $\mathbb{K}$ in the objects we defined so far over $k(q)$, without changing the notations. In particular we write U for the quantized universal enveloping algebra of $\mathfrak{sl}_2$ over $\mathbb{K}$, similarly we write $U_\pm$ for the Hopf subalgebras over $\mathbb{K}$, and we write $\varphi : U_+ \times U_- \to \mathbb{K}$ for the $\mathbb{K}$-bilinear extension of the nondegenerate Hopf pairing φ.

Definition 2.1. *We denote* Mod_U^{fd} *for the full subcategory of finite dimensional left U-modules M over $\mathbb{K}$ satisfying $M = \bigoplus_{n\in\mathbb{Z}} M[n]$ with $M[n] := \{m \in M \,|\, Km = q^n m\}$.*

Modules M such that $M = \bigoplus_{m \in \mathbb{Z}} M[n]$ are called of type 1. We will show at a later stage during the course that the category Mod_U^{fd} is semisimple, and we will describe the simple finite dimensional type 1 U-modules explicitly (up to isomorphism they are parametrized by $N \in \mathbb{Z}_{>0}$). For the moment we content ourselves with the following observation.

Lemma 2.2. *The category Mod_U^{fd} is monoidal with respect to the tensor product of the monoidal category of left U-modules.*

Proof. The trivial U-module $\mathbb{K}$ (with U-action given by the counit) is clearly op type 1. We have to show that the tensor product U-module $M \otimes N$ for finite dimensional type 1 U-modules M and N is of type 1 again. But $\Delta(K) = K \otimes K$ implies that

$$(M \otimes N)[m] = \bigoplus_{s \in \mathbb{Z}} M[s] \otimes N[m - s],$$

hence

$$M \otimes N = \bigoplus_{m \in \mathbb{Z}} (M \otimes N)[m].$$

$\square$

Exercise 2.3. *Let M be a finite dimensional type 1 U-module. Denote $\pi : U \to \mathrm{End}_{\mathbb{K}}(M)$ for the corresponding representation map.*
(i) *Show that $\pi(E)(M[n]) \subset M[n + 2]$ and $\pi(F)(M[n]) \subset M[n - 2]$ for all $n \in \mathbb{Z}$.*
(ii) *Prove that $\pi(E)$ and $\pi(F)$ are nilpotent, meaning that there exists a $N \in \mathbb{N}$ such that $\pi(E)^N m = 0 = \pi(F)^N m$ for all $m \in M$.*

By this exercise the element $\overline{R}$ (see (1.5)) makes sense as linear operator on the tensor product $M \otimes N$ of two finite dimensional type 1 U-modules M and N.

For M a finite dimensional type 1 U-module we define $H_M \in \mathrm{End}_{\mathbb{K}}(M)$ by

$$H_M|_{M[n]} = n \mathrm{Id}_{M[n]} \qquad \forall\, m \in M[n], \ \forall n \in \mathbb{Z}.$$

We simply write H when discussing properties of the linear endomorphisms H_M which are valid for all modules M.

Lemma 2.4. *Let M and N be finite dimensional type 1 U-modules. For all $f \in \mathrm{Hom}_U(M, N)$ we have*

$$H_N \circ f = f \circ H_M.$$

Proof. Since f is U-linear we have $f(M[n]) \subseteq N[n]$ for all $n \in \mathbb{Z}$. $\square$

Remark 2.5. **(i)** Note that H is *not* a natural transformation $H : \mathrm{Id} \to \mathrm{Id}$ for the identity functor Id of Mod_U^{fd}, since H_M is in general not a morphism of U-modules.
(ii) Formally $K = q^H$ as operators on finite dimensional type 1 U-modules.

Lemma 2.6. *Let M and N be finite dimensional type 1 U-modules. Then*

$$H_{M \otimes N} = H_M \otimes \mathrm{Id}_N + \mathrm{Id}_M \otimes H_N,$$
$$H_{\mathbb{K}} = 0$$

where $\mathbb{K}$ is the trivial U-module. We will write these identities as

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \epsilon(H) = 0.$$

Proof. $H_{\mathbb{K}} = 0$ is clear. The first identity follows from the fact that

$$H_{M \otimes N}|_{M[s] \otimes N[t]} = (s+t)\mathrm{Id}_{M[s] \otimes N[t]} = (H_M \otimes \mathrm{Id}_N + \mathrm{Id}_M \otimes H_N)|_{M[s] \otimes N[t]}.$$

$\square$

Exercise 2.7. *Let M be a finite dimensional left U-module.*
(i) *Show that*

$$(X\phi)(m) := \phi(S(X)m), \qquad \phi \in M^*, \ m \in M$$

turns the linear dual $M^* = Hom_{\mathbb{K}}(M, \mathbb{K})$ of M into a left U-module.
(ii) *Show that M^* is of type 1 if M is of type 1.*
(iii) *Prove that*

$$(H_{M^*}(\phi))(m) = \phi(-H_M(m)), \qquad \phi \in M^*, \ m \in M$$

if M is of type 1 (for which we simply write $S(H) = -H$).

3. The braiding on Mod_U^{fd}

In this section we define an operator R_0, given explicitly as power series in $H \otimes H$, such that the linear operators

$$c_{M,N} := \tau_{M,N} \circ \overline{R}R_0 : M \otimes N \to N \otimes M, \qquad \forall M, N \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$$

define a braiding on the monoidal category Mod_U^{fd}.

Exercise 3.1. *Show that there exists a unique $\mathbb{K}$-algebra automorphism $\sigma : U \to U$ satisfying*

$$\sigma(K^{\pm 1}) = K^{\mp 1}, \qquad \sigma(E) = F, \qquad \sigma(F) = E.$$

The following lemma will be of use later on.

Lemma 3.2. *Let $m \in \mathbb{Z}_{>0}$. In U we have the commutation relations*

$$EF^m = F^m E + \left(m\right)_{q^2} F^{m-1}\left(\frac{q^{2-2m}K - K^{-1}}{q - q^{-1}}\right),$$

$$FE^m = E^m F - \left(m\right)_{q^2} E^{m-1}\left(\frac{K - q^{2-2m}K^{-1}}{q - q^{-1}}\right).$$

Proof. The first formula is proved by induction on m (the case $m = 1$ is one of the defining relations of U). The second formula can be proved by applying the algebra automorphism σ of U (see Exercise 3.1) to the first identity. $\square$

It is easy to verify that there exists a unique algebra homomomrphism $\overline{\Delta} : U \to U \otimes U$ satisfying

$$\overline{\Delta}(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \qquad \overline{\Delta}(E) = E \otimes K^{-1} + 1 \otimes E, \qquad \overline{\Delta}(F) = K \otimes F + F \otimes 1.$$

Recall the element $\overline{R}$ (see (1.5)), interpreted as linear endomorphism of $M \otimes N$ for finite dimensional type 1 U-modules M and N.

Lemma 3.3. *Let M and N be finite dimensional type 1 U-modules. Then*

$$\overline{R}\,\overline{\Delta}(X) = \Delta^{op}(X)\overline{R} \qquad \forall X \in U,$$

as linear endomorphisms of $M \otimes N$.

Proof. It suffices to prove the identities for $X = K^{\pm 1}, E, F$, the algebraic generators of U. This is a direct computation using the explicit expression for $\overline{R}$ and the previous lemma. We give as example the computation for $X = E$. Denote

$$F_m = \frac{(q^{-1} - q)^m}{(m)_{q^2}!} F^m \in U,$$

then $\overline{R} = \sum_{m \geq 0} E^m \otimes F_m$ (cf. (1.5)). It follows that

$$\Delta^{op}(E)\overline{R} = 1 \otimes E + \sum_{m \geq 1} E^m \otimes (EF_m + F_{m-1}q^{2-2m}K).$$

By Lemma 3.2,

$$EF_m + F_{m-1}q^{2-2m}K = F_m E + F_{m-1}K^{-1}, \qquad m \geq 1,$$

hence we obtain

$$\Delta^{op}(E)\overline{R} = 1 \otimes E + \sum_{m \geq 1} E^m \otimes (F_m E + F_{m-1}K^{-1})$$

$$= \overline{R}(1 \otimes E + E \otimes K^{-1}) = \overline{R}\,\overline{\Delta}(E),$$

as desired. $\qquad\qquad\square$

We now define

(3.1) $$R_0 = q^{-\frac{H \otimes H}{2}}.$$

The meaning of this formula is as follows: if M and N are finite dimensional type 1 U-modules, then R_0 is the linear endomorphism of $M \otimes N$ defined by

$$R_0(m \otimes n) = q^{-\frac{rs}{2}} m \otimes n, \qquad m \in M[r],\ n \in N[s].$$

We furthermore set $R = \overline{R}R_0$, which makes sense as linear endomorphism of $M \otimes N$ for all finite dimensional type 1 U-modules M and N.

Lemma 3.4. *Let M and N be finite dimensional type 1 U-modules. We have*

$$R\Delta(X) = \Delta^{op}(X)R \qquad \forall X \in U,$$

as linear endomorphisms of $M \otimes N$.

Proof. In view of the previous lemma it suffices to show that

$$R_0 \Delta(X) = \overline{\Delta}(X) R_0$$

for $X = K^{\pm 1}, E, F$. As an example we give the proof for $X = E$. Let M and N be finite dimensional type 1 U-modules and fix $m \in M[r]$ and $n \in N[s]$. In view of Exercise 2.3 we have

$$\begin{aligned}
R_0 \Delta(E)(m \otimes n) &= R_0(E \otimes 1 + K \otimes E)(m \otimes n) \\
&= (q^{-s} q^{-\frac{rs}{2}} E \otimes 1 + q^{-\frac{rs}{2}} 1 \otimes E)(m \otimes n) \\
&= (q^{-s} E \otimes 1 + 1 \otimes E) R_0(m \otimes n) \\
&= (E \otimes K^{-1} + 1 \otimes E) R_0(m \otimes n) = \overline{\Delta}(E) R_0(m \otimes n),
\end{aligned}$$

as desired. $\qquad\square$

Theorem 3.5. *The monoidal category* Mod_U^{fd} *is braided with braiding* $c_{M,N} : M \otimes N \to N \otimes M$ *given by*

$$c_{M,N} = \tau_{M,N} \circ R$$

where τ *is the flip and* R *is given by*

$$R = \overline{R} R_0 = \left(\sum_{m=0}^{\infty} \frac{(q^{-1} - q)^m}{(m)_{q^2}!} E^m \otimes F^m \right) q^{-\frac{H \otimes H}{2}}.$$

The remainder of this subsection is reserved for the proof of the theorem. The previous lemma shows that $c_{M,N} \in \mathrm{Hom}_U(M \otimes N, N \otimes M)$. Translating the other required properties for the braiding c in terms of the formal universal R-matrix R, we see that we still have to prove the following three properties of R.

1. For all finite dimensional type 1 U-modules M and N, the corresponding linear endomorphism $R \in \mathrm{End}_{\mathbb{K}}(M \otimes N)$ is an isomorphism.
2. For all finite dimensional type 1 U-modules M, N and P we have

$$R|_{(M \otimes N) \otimes P} = (R_{13} R_{23})|_{M \otimes N \otimes P},$$

   i.e. $(\Delta \otimes \mathrm{Id})(R) = R_{13} R_{23}$.
3. For all finite dimensional type 1 U-modules M, N and P we have

$$R|_{M \otimes (N \otimes P)} = (R_{13} R_{12})|_{M \otimes N \otimes P},$$

   i.e. $(\mathrm{Id} \otimes \Delta)(R) = R_{13} R_{12}$.

For **1** we use the following exercise.

Exercise 3.6. *For* $m \in \mathbb{Z}_{\geq 0}$ *we have*

$$\sum_{k=0}^{m} \binom{m}{k}_{q^2} (-1)^k q^{k(k-1)} = \delta_{m,0}.$$

Hint: *Use* (1.2).

We now define for finite dimensional type 1 U-modules M and N a linear endomorphism $T \in \operatorname{End}_{\mathbb{K}}(M \otimes N)$ by

$$T|_{M \otimes N[s]} = \sum_{m \geq 0} \frac{(q^{-1} - q)^m}{(m)_{q^2}!} (q^{\frac{sH}{2}} S(E^m) \otimes F^m)|_{M \otimes N[s]}$$

for all $s \in \mathbb{Z}$. Formally T is equal to $(S \otimes \operatorname{Id})(R)$. The following lemma takes care of the first requirement **1**.

Lemma 3.7. *Let M and N be finite dimensional type 1 U-modules. The linear endomorphism $T \in \operatorname{End}_{\mathbb{K}}(M \otimes N)$ is the two-sided inverse of $R \in \operatorname{End}_{\mathbb{K}}(M \otimes N)$.*

Proof. We again use the notation

$$F_m = \frac{(q^{-1} - q)^m}{(m)_{q^2}!} F^m,$$

so that $R = \sum_{m \geq 0}(E^m \otimes F_m)q^{-\frac{H \otimes H}{2}}$ and $T|_{M \otimes N[s]} = \sum_{m \geq 0}(q^{\frac{sH}{2}} S(E)^m \otimes F_m)|_{M \otimes N[s]}$. We show that T is a left inverse of R. The proof that T is the right inverse of R is similar.

It follows from Exercise 2.3 that

$$TR = \sum_{k,l \geq 0} K^{-l} S(E)^k E^l \otimes F_k F_l K^{k+l}$$

as endomorphisms of $M \otimes N$ (compute both sides carefully on $M \otimes N[s]$ to verify the formula). Since $S(E)^k = (-K^{-1}E)^k = (-1)^k q^{k(k-1)} K^{-k} E^k$ and $F_k F_l = \binom{k+l}{k}_{q^2} F_{k+l}$ we now obtain

$$TR = \sum_{k,l \geq 0} (-1)^k q^{k(k-1)} K^{-k-l} E^{k+l} \otimes F_k F_l K^{k+l}$$

$$= \sum_{m \geq 0} \left(\sum_{k=0}^{m} \binom{m}{k}_{q^2} (-1)^k q^{k(k-1)} \right) K^{-m} E^m \otimes F_m K^m$$

$$= 1 \otimes 1,$$

where the last equality follows from the previous exercise. $\qquad\square$

Exercise 3.8. (i) *Let H be a braided Hopf algebra with invertible antipode and with universal R-matrix $R \in H \otimes H$. Show that H^{cop} is a braided Hopf algebra with universal R-matrix R^{-1}.*
(ii) *For $H = U$, show that*

$$R^{-1} = q^{\frac{H \otimes H}{2}} \sum_{m \geq 0} \frac{q^{m^2 - m}(q - q^{-1})^m}{(m)_{q^2}!} E^m \otimes F^m$$

as $\mathbb{K}$-*linear automorphisms of* $M \otimes N$ *(with M and N finite dimensional type 1 U-modules).*
Hint: *Compute* $R^{-1}|_{M[r] \otimes N[s]}$ *for the claimed explicit expression for R^{-1} and show that it coincides with* $T|_{M[r] \otimes N[s]}$.

The proofs of **2** and **3** are similar, we discuss here only the proof of **2**.

By Lemma 2.6 we have $(\Delta \otimes \mathrm{Id})(R_0) = (R_0)_{13}(R_0)_{23}$. It thus suffices to show that

$$(3.2) \qquad\qquad (\Delta \otimes \mathrm{Id})(\overline{R}) = \overline{R}_{13}(R_0)_{13}\overline{R}_{23}(R_0^{-1})_{13}.$$

We use the notations from the proof of the previous lemma, so that $\overline{R} = \sum_{m \geq 0} E^m \otimes F_m$. By Exercise 2.3 we have

$$R_0(1 \otimes F_m)R_0^{-1} = q^{mH} \otimes F_m = K^m \otimes F_m$$

when acting on $M \otimes N$, where M and N are finite dimensional type 1 U-modules, so that

$$
\begin{aligned}
\overline{R}_{13}(R_0)_{13}\overline{R}_{23}(R_0^{-1})_{13} &= \sum_{r,s \geq 0} E^r K^s \otimes E^s \otimes F_r F_s \\
&= \sum_{r,s \geq 0} \binom{r+s}{r}_{q^2} E^r K^s \otimes E^s \otimes F_{r+s} \\
&= \sum_{m \geq 0} \Delta(E^m) \otimes F_m = (\Delta \otimes \mathrm{Id})(\overline{R})
\end{aligned}
$$

as desired, where the third equality is a consequence of Lemma 1.2.

3.1. **An example.** Let V be a two-dimensional $\mathbb{K}$-vector space with basis $\{v_+, v_-\}$. We turn V into a type 1 U-module by the unique algebra homomorphism $\pi : U \to \mathrm{End}_{\mathbb{K}}(V)$ satisfying

$$Kv_{\pm} = q^{\pm 1}v_{\pm}, \qquad Ev_+ = 0 = Fv_-, \qquad Ev_- = v_+, \qquad Fv_+ = v_-.$$

We denote $c = c_{V,V}$ for the associated braiding. We have $c \in \mathrm{Aut}_U(V \otimes V)$, i.e. c is an automorphism of $V \otimes V$ in the category Mod_U^{fd}. Furthermore, c solves the Yang-Baxter equation

$$(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c),$$

hence we can construct a group homomorphism $B_m \to \mathrm{Aut}_U(V^{\otimes m})$ $(m \geq 2)$ of the Artin braid group B_m by

$$\sigma_i \mapsto \mathrm{Id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{Id}_{V^{\otimes(m-i-1)}}, \qquad i = 1, \ldots, m-1$$

(with the proper interpretations for $i = 1$ and $i = m - 1$). Recall here that $\sigma_1, \ldots, \sigma_{m-1}$ are the group generators of B_m satisfying the (characterizing) relations

$$
\begin{aligned}
\sigma_i \sigma_j = \sigma_j \sigma_i, &\qquad \text{if } |i - j| > 1, \\
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, &\qquad \text{if } |i - j| = 1.
\end{aligned}
$$

A direct computation shows that

$$c(v_+ \otimes v_+) = q^{-\frac{1}{2}} v_+ \otimes v_+,$$
$$c(v_+ \otimes v_-) = q^{\frac{1}{2}} v_- \otimes v_+,$$
$$c(v_- \otimes v_+) = q^{\frac{1}{2}} v_+ \otimes v_- + q^{\frac{1}{2}}(q^{-1} - q) v_- \otimes v_+,$$
$$c(v_- \otimes v_-) = q^{-\frac{1}{2}} v_- \otimes v_-.$$

In other words, with respect to the ordered basis $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$ of $V \otimes V$, the braiding c is represented by the 4×4-matrix

(3.3)
$$c = q^{-\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1 - q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3.9. *Show that $c \in \operatorname{End}_U(V \otimes V)$ satisfies the quadratic relation*

$$(c - q^{-\frac{1}{2}})(c + q^{\frac{3}{2}}) = 0.$$

Conclude that $c \in \operatorname{Aut}_U(V \otimes V)$ is an isomorphism satisfying

(3.4)
$$q^{-\frac{1}{2}} c - q^{\frac{1}{2}} c^{-1} = (q^{-1} - q) \operatorname{Id}_{V \otimes V}$$

Compare formula (3.4) to the skein relations for the HOMFLY polynomial! It shows that we are on the right track to reproduce special cases of the HOMFLY polynomial (in fact, the Jones polynomial) from the braid group representation associated to the two-dimensional representation V of U.

By Exercise 3.9 the braid group representation associated to c factors through the Hecke algebra.

Definition 3.10. *Let $\mu \in \mathbb{K}^\times$. The Hecke algebra $H_m(\mu)$ over $\mathbb{K}$ is the quotient $\mathbb{K}[B_m]/I$ of the group algebra of B_m by the two-sided ideal I generated by $(\sigma_i - \mu^{-1})(\sigma_i + \mu)$ for $i = 1, \ldots, m - 1$.*

Exercise 3.9 now has the following immediate consequence.

Corollary 3.11. *We have an algebra homomorphism $\rho : H_m(q) \to \operatorname{End}_U(V^{\otimes m})$ defined by*

$$\rho(\sigma_i) = q^{-\frac{1}{2}} \left(\operatorname{Id}_{V^{\otimes(i-1)}} \otimes c \otimes \operatorname{Id}_{V^{\otimes(m-i-1)}} \right)$$

for $i = 1, \ldots, m - 1$.

Remark 3.12. It turns out that ρ is surjective (this is part of the quantum Schur-Weyl duality, see e.g. [4, Chpt. 7, §2] and references therein).

Definition 3.13. *Let* $\lambda \in \mathbb{K}^\times$ *and* $m \geq 2$. *The Temperley-Lieb algebra* $TL_m(\lambda)$ *is the unital associative algebra over* $\mathbb{K}$ *generated by* $t_1, \ldots, t_{m-1}$ *with defining relations*

$$t_i^2 = -(\lambda^2 + \lambda^{-2})t_i,$$
$$t_i t_j t_i = t_i, \qquad |i - j| = 1,$$
$$t_i t_j = t_j t_i, \qquad |i - j| > 1.$$

Exercise 3.14. *Let* $\mu \in \mathbb{K}^\times$.
(i) *Show that there exists a unique surjective algebra homomorphism* $\pi_\mu : H_m(\mu^2) \to TL_m(\mu)$ *satisfying*

$$\pi_\mu(\sigma_i) = \mu^{-2} + t_i, \qquad i = 1, \ldots, m - 1.$$

(ii) *Show that there exists a unique algebra homomorphism* $\overline{\rho} : TL_m(q^{\frac{1}{2}}) \to \operatorname{End}_\mathbb{K}(V^{\otimes m})$ *satisfying*

$$\overline{\rho}(t_i) = \operatorname{Id}_{V^{\otimes(i-1)}} \otimes t \otimes \operatorname{Id}_{V^{\otimes(m-i-1)}}, \qquad i = 1, \ldots, m - 1$$

with $t : V^{\otimes 2} \to V^{\otimes 2}$ *the* $\mathbb{K}$-*linear map defined by*

$$t(v_+ \otimes v_+) = 0 = t(v_- \otimes v_-),$$
$$t(v_+ \otimes v_-) = -q^{-1}v_+ \otimes v_- + v_- \otimes v_+,$$
$$t(v_- \otimes v_+) = v_+ \otimes v_- - qv_- \otimes v_+.$$

(iii) *Prove that* $\rho = \overline{\rho} \circ \pi_{q^{\frac{1}{2}}}$.

4. The quantized universal enveloping algebra of $\mathfrak{sl}_n(k)$

The results for the quantized universal enveloping algebra of $\mathfrak{sl}_2(k)$ generalize to arbitrary semisimple Lie algebras $\mathfrak{g}$. We discuss some of the key features for the extension of the above results in case $\mathfrak{g} := \mathfrak{sl}_n(k)$ ($n \geq 2$). Detailed proofs can be found in e.g. [2], [5, Chpt. 6 and 8] and [4, Chpt. 4].

We first consider the Lie algebra $\mathfrak{sl}_n(k)$ in more detail. A k-linear basis of $\mathfrak{sl}_n(k)$ is given by

$$\{H_1, \ldots, H_{n-1}\} \cup \{E_{ij}\}_{1 \leq i \neq j \leq n},$$

where $H_i = E_{ii} - E_{i+1,i+1}$ and E_{ij} is the $n \times n$-matrix with all entries zero expect the entry in row i and column j, which is one. Note that

$$[E_{ij}, E_{jk}] = E_{ik}, \qquad [E_{kj}, E_{ji}] = E_{ki}$$

for $1 \leq i < j < k \leq n$, hence $\mathfrak{sl}_n(k)$ is generated as Lie algebra by H_i, E_i, F_i ($i = 1, \ldots, n - 1$) where

$$E_i := E_{i,i+1}, \qquad F_i := E_{i+1,i}.$$

Set $\mathfrak{h} \subset \mathfrak{sl}_n(k)$ for the diagonal matrices in $\mathfrak{sl}_n(k)$. It is spanned by the H_i ($i = 1, \ldots, n-1$). Define $\alpha_i \in \mathfrak{h}^*$ by $\alpha_i(H_i) = 2$, $\alpha_i(H_{i \pm 1}) = -1$ and $\alpha_i(H_j) = 0$ otherwise. Then $\{\alpha_i\}_{i=1}^{n-1}$ is a k-linear basis of $\mathfrak{h}^*$. In fact, let $\widetilde{\mathfrak{h}}$ be the space consisting of diagonal matrices in $\operatorname{End}_k(k^{\oplus n})$. It is spanned by the matrix units E_{ii} ($i = 1 \ldots, n$). It contains $\mathfrak{h}$ as a co-dimension one

subspace. Let $\{\epsilon_i\}_i$ be the dual basis of $\widetilde{\mathfrak{h}}^*$ with respect to the basis $\{E_{ii}\}_i$ of $\widetilde{\mathfrak{h}}$. Then $\alpha_i = (\epsilon_i - \epsilon_{i+1})|_{\mathfrak{h}}$ for $i = 1, \dots n - 1$.

It is now easy to verify that the following relations are satisfied in the Lie algebra $\mathfrak{sl}_n(k)$,

$$[H_i, H_j] = 0,$$
$$[H_i, E_j] = \alpha_j(H_i)E_j,$$
$$[H_i, F_j] = -\alpha_j(H_i)F_j,$$
$$[E_i, F_j] = \delta_{i,j}H_i$$
$$[E_i, E_j] = 0 = [F_i, F_j], \qquad |i - j| > 1,$$
$$[E_i, [E_i, E_{i\pm1}]] = 0 = [F_i, [F_i, F_{i\pm1}]].$$

In the following exercise we ask you to verify some elementary properties of the adjoint representation of a Lie algebra $\mathfrak{g}$.

Exercise 4.1. *Define* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_k(\mathfrak{g})$ *by* $\mathrm{ad}(X)(Y) := [X, Y]$.
(i) *Show that*
$$[\mathrm{ad}(X), \mathrm{ad}(Y)] = \mathrm{ad}([X, Y]), \qquad \forall\, X, Y \in \mathfrak{g},$$
where the left hand side is the commutator bracket in the Lie algebra $\mathfrak{g}(\mathrm{End}(\mathfrak{g}))$.
(ii) *Let* $j \in \mathbb{Z}_{\geq 0}$ *and* $X, Y \in \mathfrak{g}$. *Show that* $\mathrm{ad}(X)^j(Y) \in \mathfrak{g}$ *is equal to*

$$\sum_{i=0}^{r}(-1)^i \binom{r}{i} X^{r-i}YX^i$$

when regarded as element in the universal enveloping algebra $U(\mathfrak{g})$. *Here* $\binom{r}{i}$ *denotes the usual binomial coefficient.*

These considerations lead to the following presentation of $U(\mathfrak{sl}_n(k))$, due to Serre.

Theorem 4.2. *The universal enveloping algebra* $U(\mathfrak{sl}_n(k))$ *is the unital associative algebra over* k *with generators* H_i, E_i, F_i $(i = 1, \dots, n - 1)$ *and defining relations*

$$[H_i, H_j] = 0,$$
$$[H_i, E_j] = \alpha_j(H_i)E_j,$$
$$[H_i, F_j] = -\alpha_j(H_i)F_j,$$
$$[E_i, F_j] = \delta_{i,j}H_i$$
$$[E_i, E_j] = 0 = [F_i, F_j], \qquad |i - j| > 1,$$
$$E_i^2 E_{i\pm1} - 2E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0,$$
$$F_i^2 F_{i\pm1} - 2F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0.$$

Proof. The above remarks show that $U(\mathfrak{sl}_n(k))$ is generated as algebra by the elements H_i, E_i, F_i $(i = 1, \dots, n - 1)$, and that these generators satisfy the above list of relations in $U(\mathfrak{sl}_n(k))$. We refer to [1] for a proof that this gives in fact a presentation of $U(\mathfrak{sl}_n(k))$. $\square$

Denote $\mathbb{K} = k(q^{\frac{1}{n}})$.

Definition 4.3. *The quantized universal enveloping algebra $U_q(\mathfrak{sl}_n)$ is the unital associative algebra over $\mathbb{K}$ with generators $K_i^{\pm 1}, E_i, F_i$ ($i = 1, \ldots, n-1$) and defining relations*

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i,$$

$$K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q^{\alpha_j(H_i)} E_j,$$

$$K_i F_j K_i^{-1} = q^{-\alpha_j(H_i)} F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$[E_i, E_j] = 0 = [F_i, F_j], \qquad |i - j| > 1,$$

$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0,$$

$$F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0.$$

By direct verifications one shows that $U_q(\mathfrak{sl}_n)$ is a Hopf algebra with comultiplication, counit and antipode uniquely determined by the formulas

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \qquad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\epsilon(K_i^{\pm 1}) = 1, \qquad\qquad \epsilon(E_i) = 0 = \epsilon(F_i),$$

$$S(E_i) = -K_i^{-1} E_i, \qquad\qquad S(F_i) = -F_i K_i, \qquad\qquad S(K_i^{\pm 1}) = K_i^{\mp 1},$$

The results that we have established for the quantized universal enveloping algebra U of $\mathfrak{sl}_2(k)$ can be generalized to $\mathfrak{sl}_n(k)$ (and, in fact to any semisimple Lie algebra $\mathfrak{g}$). One can establish a Poincaré-Birkhoff-Witt Theorem for $U_q(\mathfrak{sl}_n)$ and give an explicit nondegenerate Hopf pairing between the Hopf subalgebra $U_q(\mathfrak{b}_+)$ algebraically generated by $K_i^{\pm 1}, E_i$ ($1 \leq i \leq n-1$) and the Hopf subalgebra $U_q(\mathfrak{b}_-)$ algebraically generated by $K_i^{\pm 1}, F_i$ ($1 \leq i \leq n-1$), see [4, Chpt. 4, Thm. 2.2]. It results in the realization of $U_q(\mathfrak{sl}_n)$ as a quotient Hopf algebra of the associated generalized quantum double, and an explicit ansatz for a (formal) universal R-matrix R for $U_q(\mathfrak{sl}_n)$.

The formal universal R-matrix R of $U_q(\mathfrak{sl}_n)$ can subsequently be made into a rigorous braiding for the monoidal category $\mathrm{Mod}_{U_q(\mathfrak{sl}_n)}^{fd}$ of finite-dimensional type 1 $U_q(\mathfrak{sl}_n)$-modules. A $U_q(\mathfrak{sl}_n)$-module M is called of type 1 if

$$M = \bigoplus_{\underline{r} \in \mathbb{Z}^{n-1}} M[\underline{r}], \qquad M[\underline{r}] := \{m \in M \mid K_i m = q^{r_i} m \quad \forall i = 1, \ldots, n-1\},$$

where $\underline{r} = (r_1, \ldots, r_{n-1})$.

The vector representation of $U_q(\mathfrak{sl}_n)$ is the n-dimensional $\mathbb{K}$-vector space $V = \bigoplus_{i=1}^n \mathbb{K}v_i$ with $U_q(\mathfrak{sl}_n)$-action determined by the formulas

$$K_j^{\pm 1}v_i = q^{\mp \epsilon_i(H_j)}v_i,$$
$$E_jv_i = \delta_{i,j}v_{i+1},$$
$$F_jv_{i+1} = \delta_{j,i+1}v_i.$$

It is a direct verification that this indeed defines a type 1 $U_q(\mathfrak{sl}_n)$-module. For $n = 2$ it reduces to the two-dimensional representation V of U from the previous section, with $v_+ = v_2$ and $v_- = v_1$.

The braiding $c = c_{V,V} = \tau_{V,V} \circ R|_{V \otimes V} \in \mathrm{Aut}_{U_q(\mathfrak{sl}_n)}(V \otimes V)$ associated to V can be explicitly computed, see [5, §8.4.2] (note though that $U_q(\mathfrak{sl}_n)$ corresponds to $U_q(\mathfrak{sl}_n)^{cop}$ in [5], hence one has to invert the universal R-matrix from [5] to obtain ours, cf. Exercise 3.8). It leads to the following explicit formulas for the braiding $c \in \mathrm{Aut}_{U_q(\mathfrak{sl}_n)}(V \otimes V)$:

$$c(v_i \otimes v_i) = q^{\frac{1}{n}-1}v_i \otimes v_i, \qquad\qquad 1 \le i \le n,$$
$$c(v_i \otimes v_j) = q^{\frac{1}{n}}(q^{-1} - q)v_i \otimes v_j + q^{\frac{1}{n}}v_j \otimes v_i, \qquad 1 \le i < j \le n,$$
$$c(v_i \otimes v_j) = q^{\frac{1}{n}}v_j \otimes v_i, \qquad\qquad 1 \le j < i \le n.$$

We get for free that it is a solution of the Yang-Baxter equation. The braiding c thus gives rise to a group homomorphism $B_m \to \mathrm{Aut}_{U_q(\mathfrak{sl}_n)}(V^{\otimes m})$ of the Artin braid group B_m $(m \ge 2)$.

The inverse of c is the linear map defined by

$$c^{-1}(v_i \otimes v_i) = q^{-\frac{1}{n}+1}v_i \otimes v_i, \qquad\qquad 1 \le i \le n,$$
$$c^{-1}(v_i \otimes v_j) = q^{-\frac{1}{n}}v_j \otimes v_i, \qquad\qquad 1 \le i < j \le n,$$
$$c^{-1}(v_i \otimes v_j) = q^{-\frac{1}{n}}(q - q^{-1})v_i \otimes v_j + q^{-\frac{1}{n}}v_j \otimes v_i, \qquad 1 \le j < i \le n,$$

compare with [4, Example 1.3]. It follows that

(4.1) $$q^{-\frac{1}{n}}c - q^{\frac{1}{n}}c^{-1} = (q^{-1} - q)\mathrm{Id}_{V \otimes V}.$$

We obtain the following generalization of Corollary 3.11.

Corollary 4.4. *There exists a unique algebra homomorphism* $\rho : H_m(q) \to \mathrm{End}_{U_q(\mathfrak{sl}_n)}(V^{\otimes m})$ *satisfying*

$$\rho(\sigma_i) = q^{-\frac{1}{n}}\big(\mathrm{Id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{Id}_{V^{\otimes(m-i-1)}}\big)$$

for $i = 1, \ldots, m - 1$.

Remark 4.5. Quantum Schur-Weyl duality implies that ρ is surjective, see [4, Chpt. 7, §2].

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