QUANTUM INVARIANTS II

Recall that U is the quantized universal enveloping algebra of $\mathfrak{sl}_2$ over $\mathbb{K} = k(q^{\frac{1}{2}})$, where k is an algebraically closed field of characteristic zero. In this text we show that the braided monoidal category Mod_U^{fd} of finite dimensional type 1 U-modules over $\mathbb{K}$ is a ribbon category. With the associated Reshetikhin-Turaev functor we construct quantum invariants of ribbon-links (the so-called colored Jones polynomial). We show that the HOMFLY polynomial arises as quantum invariant from the Reshetikhin-Turaev construction applied to $U_q(\mathfrak{sl}_n)$. In the appendix we discuss the Clebsch-Gordan decomposition of tensor product U-modules and we give tools to compute the colored Jones polynomial in the skein. An important element in the Temperley-Lieb algebra makes here its appearance, the so-called Jones-Wenzl idempotent.

1. Duality in Mod_U^{fd} and skein functors

Recall that Mod_U^{fd} is the braided monoidal category of finite dimensional type 1 U-modules over $\mathbb{K}$. Recall that $M^* = \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ for $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ is again a finite dimensional type 1 U-module by

$$(Xf)(m) = f(S(X)m), \qquad X \in U, \;\; m \in M, \;\; f \in M^*.$$

For $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ we define $d_M \in \mathrm{Hom}_{\mathbb{K}}(M^* \otimes M, \mathbb{K})$ by

$$d_M(f \otimes m) = f(m), \qquad f \in M^*, \;\; m \in M,$$

and we define $b_M \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, M \otimes M^*)$ by

$$b_M(1) = \sum_i e_i \otimes e_i^*$$

where $\{e_i\}_i$ is a $\mathbb{K}$-linear basis of M and $\{e_i^*\}_i$ its dual basis of M^*. Recall the following result (applied to the Hopf algebra U).

Lemma 1.1. *The assignment $M \mapsto (M^*, b_M, d_M)$ for $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ defines a left duality on Mod_U^{fd}.*

For the category Mod_U^{fd} the left duality directly gives rise to strict tensor functors of the skein category (hence also of the category of ribbon-tangles). We have seen it in detail for the two-dimensional simple U-module $L(1)$ in the syllabus of last week. We now shortly discuss it for the simple U-module $L(n)$ of dimension $n+1$, with canonical basis $\{\overline{m}_j(q^n)\}_{j=0}^n$ (see Corollary 1.7 in the syllabus of last week for $\epsilon = +1$ for the module $L(n)$ and the explicit action of the generators $K^{\pm 1}$, E and F of U on the basis elements). By Exercise

1.9 in the syllabus of last week, the simple U-modules $L(n)$ are self-dual: $L(n)^* \simeq L(n)$. Concretely, an U-module isomorphism $\xi_n \in \mathrm{End}_U(L(n)^*, L(n))$ is given by

$$\xi_n(\overline{m}_j(q^n)^*) = c_j \overline{m}_{n-j}(q^n), \qquad 0 \leq j \leq n,$$

where $\{\overline{m}_j(q^n)^*\}_j$ is the basis of $L(n)^*$ dual to $\{m_j(q^n)\}_j$ and with the constants $c_j \in \mathbb{K}^\times$ uniquely determined by the recurrence relation

$$c_j = -q^{n+2-2j} c_{j-1} \quad (1 \leq j \leq n), \qquad c_n = 1.$$

We now define morphisms

$$\psi_n = (\mathrm{id}_{L(n)} \otimes \xi_n) \circ b_{L(n)} \in \mathrm{Hom}_U(\mathbb{K}, L(n) \otimes L(n)),$$
$$\phi_n = d_{L(n)} \circ (\xi_n^{-1} \otimes \mathrm{id}_{L(n)}) \in \mathrm{Hom}_U(L(n) \otimes L(n), \mathbb{K}).$$

Exercise 1.2. *Show by a direct computation that*

$$(\mathrm{id}_{L(n)} \otimes \phi_n) \circ (\psi_n \otimes \mathrm{id}_{L(n)}) = \mathrm{id}_{L(n)} = (\phi_n \otimes \mathrm{id}_{L(n)}) \circ (\mathrm{id}_{L(n)} \otimes \psi_n).$$

Remark 1.3. Later on we show that the braided monoidal category Mod_U^{fd} with left duality is a ribbon category. Exercise 1.2 can then be proven without computations using the graphical calculus.

Lemma 1.4. *For $\phi_n \circ \psi_n \in \mathrm{End}_U(\mathbb{K})$ we have*

$$(\phi_n \circ \psi_n)(1) = (-1)^n \left(\frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \right).$$

Proof. A direct computation using the definition of ξ_n, the left duality, and the canonical basis elements of $L(n)$ shows that

$$(\phi_n \circ \psi_n)(1) = \sum_{j=0}^{n} \frac{c_j}{c_{n-j}}.$$

Set $d_j = c_j / c_{n-j}$ for $0 \leq j \leq n$. Then $d_j = q^2 d_{j-1}$ for $1 \leq j \leq n$ and $d_0 = c_0$, hence

$$(\phi_n \circ \psi_n)(1) = c_0 \sum_{j=0}^{n} q^{2j} = c_0 \frac{1 - q^{2(n+1)}}{1 - q^2}.$$

Now by the definition of the c_j, we obtain $c_0 = (-1)^n q^{-n}$, which leads to the desired expression. □

If the polynomial

$$X^4 + (-1)^n \left(\frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \right) X^2 + 1 \in \mathbb{K}[X] \tag{1.1}$$

does not have a root in $\mathbb{K}$, then we adjoint a root a_n to $\mathbb{K}$. We now work over the field $\mathbb{K}(a_n)$, both for the skein category as well as for the category Mod_U^{fd}. Note that for $n = 1$ the polynomial has the root $a_1 = q^{-\frac{1}{2}} \in \mathbb{K}$, in which case we can thus stick to our original field $\mathbb{K}$.

Proposition 1.5. *Over the field $\mathbb{K}(a_n)$ we have a unique $\mathbb{K}(a_n)$-linear strict tensor functor $F_n^{sk}: \mathcal{S}(a_n) \to \widetilde{\mathrm{Mod}}_U^{fd}$ satisfying*

(1) $F_n^{sk}(1) = L(n)$,
(2) $F_n^{sk}(\cup) = \psi_n$ *and* $F_n^{sk}(\cap) = \phi_n$.

Proof. In view of Proposition 3.1 of the syllabus of last week and the previous observations, it suffices to verify that $(\phi_n \circ \psi_n)(1) = -(a_n^2 + a_n^{-2})$. But this follows from the fact that a_n is a root of the polynomial (1.1). □

For $n = 1$ this is Corollary 3.3 in the syllabus of last week.

Definition 1.6. *We call $F_1^{sk}: \mathcal{S}(q^{-\frac{1}{2}}) \to \widetilde{\mathrm{Mod}}_U^{fd}$ the skein functor.*

Recall that we have a canonical strict tensor functor $G: \mathcal{T} \to \mathcal{S}(a_n)$. We obtain a strict tensor functor

$$G_n^{sk} = \widetilde{F}_n \circ G : \mathcal{T} \to \widetilde{\mathrm{Mod}}_U^{fd}.$$

The automorphism

$$G_n^{sk}(X^+) = a_n \mathrm{id}_{L(n)\otimes L(n)} + a_n^{-1}\psi_n \circ \phi_n \in \mathrm{Aut}_U(L(n) \otimes L(n))$$

thus is a solution of the Yang-Baxter equation on $L(n)$. For $n = 1$ the solution $G_1^{sk}(X^+)$ coincides with $c_{L(1),L(1)}$, when a_1 is taken to be $q^{-\frac{1}{2}}$, cf. Proposition 3.5 in the syllabus of last week (if $a_1 = q^{\frac{1}{2}}$ is chosen, then $G_1^{sk}(X^+) = c_{L(1),L(1)}^{-1}$). We formalize this in the following corollary.

Corollary 1.7. *The functor $G_1^{sk}: \mathcal{T} \to \widetilde{\mathrm{Mod}}_U^{fd}$ for $a_1 = q^{-\frac{1}{2}}$ is the unique strict tensor functor satisfying $G_1^{sk}(1) = L(1)$ and*

$$G_1^{sk}(X^{\pm}) = c_{L(1),L(1)}^{\pm 1}, \qquad G_1^{sk}(\cup) = \psi_1, \qquad G_1^{sk}(\cap) = \phi_1.$$

Proof. We already remarked that G_1^{sk} is a strict tensor functor satisfying all the required properties. It is uniquely characterized by these properties since $\{X^{\pm}, \cup, \cap\}$ is a set of generators of $\mathcal{T}$. □

We promote in the next section the braided monoidal category Mod_U^{fd} with left duality $(*, b, d)$ and braiding c to a ribbon category. The Reshetikhin-Turaev functor then gives many new examples of oriented ribbon tangle invariants. Only when colouring the ribbon-tangles with $L(1)$ we come back to the oriented ribbon tangle invariants which we have just constructed via the functor G_1^{sk}.

2. Mod_U^{fd} is a ribbon category

We have seen that the braiding on Mod_U^{fd} arises from the fact that U is (formally) a braided Hopf algebra, with universal R-matrix

$$R = \left(\sum_{m=0}^{\infty} \frac{(q^{-1} - q)^m}{(m)_{q^2}!} E^m \otimes F^m \right) q^{-\frac{H\otimes H}{2}}.$$

R does not make sense as element in $U \otimes U$, but $R_{M\otimes N}$ does as $\mathbb{K}$-linear endomorphism of $M \otimes N$ for $M, N \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$. We do now the same for the formal Drinfeld type element $u = \mu(S \otimes \mathrm{id})(R_{21})$ and its alleged inverse $u^{-1} = \mu(S^{-1} \otimes S)(R_{21}) = \mu(\mathrm{id} \otimes S^2)(R_{21})$ (the second equality follows from $(S \otimes S)(R) = R$). We start first with a straightforward lemma.

Lemma 2.1. *We have $S^2(X) = K^{-1}XK$ for $X \in U$.*

Proof. Both $X \mapsto S^2(X)$ and $X \mapsto K^{-1}XK$ are algebra automorphisms of U. Hence it suffices to check equality for $X = K, E$ and F. This is obvious for $X = K$ since $S(K) = K^{-1}$. Furthermore,

$$S^2(E) = S(-K^{-1}E) = -S(E)S(K)^{-1} = K^{-1}EK,$$

and similarly $S^2(F) = S(-FK) = K^{-1}FK$. $\square$

We set

$$\begin{aligned} u &= \sum_{m=0}^{\infty} \frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!} S(F)^m E^m q^{\frac{H^2}{2}}, \\ v &= \sum_{m=0}^{\infty} \frac{(q^{-1}-q)^m}{q^{2m}\left(m\right)_{q^2}!} F^m E^m K^{-m} q^{-\frac{H^2}{2}}. \end{aligned} \tag{2.1}$$

The meaning of the expression of e.g. u is that it represents (in universal form) the family $u_M \in \mathrm{End}_{\mathbb{K}}(M)$ ($M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$) of endomorphisms defined by

$$u_M = \sum_{m=0}^{\infty} \frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!} S(F)^m E^m q^{\frac{H^2}{2}}|_M$$

(which is a finite sum), where the factor $q^{\frac{H^2}{2}}$ acts on $M[n] = M_{q^n}$ by

$$q^{\frac{H^2}{2}}|_{M_{q^n}} = q^{\frac{n^2}{2}} \mathrm{Id}_{M_{q^n}}.$$

Lemma 2.2. *Formally, u is the Drinfeld element $\mu(S \otimes \mathrm{id})(R_{21})$ and v is its inverse.*

Proof. Formally write

$$q^{-\frac{H\otimes H}{2}} = \sum_{k=0}^{\infty} \frac{(-\log(q)/2)^k H^k \otimes H^k}{k!}.$$

Using $S(H) = -H$ (see Exercise 2.7 in the syllabus of four weeks ago), we get

$$\begin{aligned} \mu(S \otimes \mathrm{id})(R_{21}) &= \sum_{k,m\geq 0} \frac{(\log(q)/2)^k}{k!} \frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!} H^k S(F)^m E^m H^k \\ &= \sum_{m\geq 0} \frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!} S(F)^m E^m \sum_{k\geq 0} \frac{(\log(q)/2)^k}{k!} H^{2k} \\ &= \sum_{m\geq 0} \frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!} S(F)^m E^m q^{\frac{H^2}{2}}, \end{aligned}$$

where the second equality is due to the fact that $S(F)^m E^m$ maps M_{q^r} to M_{q^r} for all r. For v, we first observe that

$$\mu(\mathrm{id}\otimes S^2)(R_{21}) = \sum_{k,m\geq 0}\frac{(-\log(q)/2)^k}{k!}\frac{(q^{-1}-q)^m}{\left(m\right)_{q^2}!}F^m H^k K^{-1}E^m K H^k$$

by the previous lemma, hence

$$\begin{aligned}\mu(\mathrm{id}\otimes S^2)(R_{21})|_{M[s]} &= \sum_{m\geq 0}\frac{(q^{-1}-q)^m}{q^{2m}\left(m\right)_{q^2}!}F^m q^{-\frac{sH}{2}}E^m|_{M[s]}\\ &= \sum_{m\geq 0}\frac{(q^{-1}-q)^m}{q^{2m}\left(m\right)_{q^2}!}F^m q^{-\frac{s(s+2m)}{2}}E^m|_{M[s]}\\ &= \sum_{m\geq 0}\frac{(q^{-1}-q)^m}{q^{2m}\left(m\right)_{q^2}!}F^m E^m K^{-m} q^{-\frac{H^2}{2}}|_{M[s]} = v|_{M[s]}.\end{aligned}$$

□

Lemma 2.3. **(i)** *The linear endomorphisms* $\theta_M := (vK^{-1})_M = (K^{-1}v)_M \in \mathrm{End}_{\mathbb{K}}(M)$ *and* $\theta^-_M := (uK)_M = (Ku)_M \in \mathrm{End}_{\mathbb{K}}(M)$ *are* U*-linear for all* $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.
(ii) *We have*

$$\theta_{L(n)} = q^{-\frac{n(n+2)}{2}}\mathrm{id}_{L(n)}, \qquad \theta^-_{L(n)} = q^{\frac{n(n+2)}{2}}\mathrm{id}_{L(n)}$$

for $n \in \mathbb{Z}_{\geq 0}$.
(iii) $\theta : \mathrm{Id}_{\mathrm{Mod}_U^{fd}} \to \mathrm{Id}_{\mathrm{Mod}_U^{fd}}$ *is a natural isomorphism with inverse* θ^-.
(iv) *The natural isomorphism* θ *is compatible with left duality,*

$$(\theta_M \otimes \mathrm{id}_{M^*})b_M = (\mathrm{id}_M \otimes \theta_{M^*})b_M$$

for all $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.

Proof. **(i)** This follows by a direct computation using Lemma 3.2 in the syllabus of four weeks ago. Since we discuss a different conceptual argument in a moment (see Remark 2.5), we do not give the full details.
(ii) We have $\mathrm{End}_U(L(n)) = \mathrm{span}_{\mathbb{K}}\{\mathrm{id}_{L(n)}\}$, see Lemma 1.8 in the syllabus of last week, hence $\theta_{L(n)} = c_n \mathrm{id}_{L(n)}$ for some $c_n \in \mathbb{K}^\times$. When computing $\theta_{L(n)}$ on the highest weight vector of $L(n)$ only the $m = 0$ term of the expression of θ gives a nonzero contribution, hence

$$\theta_{L(n)}(\overline{m}_0(q^n)) = K^{-1}q^{-\frac{H^2}{2}}\overline{m}_0(q^n) = q^{-n}q^{-\frac{n^2}{2}}\overline{m}_0(q^n) = q^{-\frac{n(n+2)}{2}}\overline{m}_0(q^n).$$

Thus $c_n = q^{-\frac{n(n+2)}{2}}$. A similar argument gives the result for θ^-.
(iii) It suffices to show that θ is natural. But the endomorphism $\theta_M \in \mathrm{End}_U(M)$ is given by the action of some element from U on M, hence $\theta_N \circ f = f \circ \theta_M$ for $f \in \mathrm{Hom}_U(M, N)$.

(iv) It suffices to prove it for $M = L(n)$ a simple module (since any module M in Mod_U^{fd} splits as direct sum of simple modules). But for $M = L(n)$ we have $L(n)^* \simeq L(n)$, hence

$$\theta_{L(n)^*} = q^{-\frac{n(n+2)}{2}} \mathrm{id}_{L(n)^*}$$

by the naturality of θ. We conclude that

$$(\theta_{L(n)} \otimes \mathrm{id}_{L(n)^*}) b_{L(n)} = q^{-\frac{n(n+2)}{2}} b_{L(n)} = (\mathrm{id}_{L(n)} \otimes \theta_{L(n)^*}) b_{L(n)},$$

as desired. □

Lemma 2.3 thus turns

$$\theta = vK^{-1} = K^{-1}v = \sum_{m=0}^{\infty} \frac{(q^{-1}-q)^m}{q^{2m}(m)_{q^2}!} F^m E^m K^{-m-1} q^{-\frac{H^2}{2}}$$

into the prime candidate for a twist of Mod_U^{fd}.

Corollary 2.4. **(i)** *$u_M \in \mathrm{End}_{\mathbb{K}}(M)$ is a $\mathbb{K}$-linear isomorphism with inverse v_M for all $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.*
(ii) *$S^2(X)_M = (uXv)_M$ for all $X \in U$ and $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.*

Proof. **(i)** Is immediate.
(ii) By By Lemma 2.1 we have

$$S^2(X)_M = (K^{-1}XK)_M = (K^{-1}\vartheta)_M X_M (\theta K)_M = (uXv)_M,$$

where $\vartheta = Ku$. □

Note that we have obtained two properties of u which we have derived before for the Drinfeld element in any finite dimensional braided monoidal category. In fact, with some care one can show that most identities for the Drinfeld element u hold true in the present situation as long as they are properly interpreted as operator identities on finite dimensional type 1 U-modules. In particular we have

$$u_{M\otimes N} = (c_{N,M} c_{M,N})^{-1}(u_M \otimes u_N) \tag{2.2}$$

for all $M, N \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.

Remark 2.5. The U-linearity of θ_M (see Lemma 2.3**(i)**) follows without computations from the property $S^2(X) = uXv$ ($X \in U$) of the Drinfeld element. Indeed, for $X \in U$ and $m \in M$ we have

$$X\theta_M(m) = XvK^{-1}m = vS^2(X)K^{-1}m = vK^{-1}Xm = \theta_M(Xm),$$

where the third equality is by Lemma 2.1.

Theorem 2.6. *The braided monoidal category Mod_U^{fd} with (standard) left duality is a ribbon category with twist θ.*

Proof. All that remains to be proven is that θ satisfies

$$\theta_{M\otimes N} = c_{N,M}c_{M,N}(\theta_M \otimes \theta_N) \tag{2.3}$$

for $M, N \in \mathrm{Mod}_U^{fd}$. We compute using (2.2),

$$\begin{aligned}
\theta_{M\otimes N} &= u_{M\otimes N}^{-1} K_{M\otimes N}^{-1} \\
&= (u_M^{-1} \otimes u_N^{-1}) c_{N,M} c_{M,N} K_{M\otimes N}^{-1} \\
&= (v_M \otimes v_N) K_{M\otimes N}^{-1} c_{N,M} c_{M,N} \\
&= \big((vK^{-1})_M \otimes (vK^{-1})_N\big) c_{N,M} c_{M,N} \\
&= (\theta_M \otimes \theta_N) c_{N,M} c_{M,N} \\
&= c_{N,M} c_{M,N} (\theta_M \otimes \theta_N),
\end{aligned}$$

where we have used $\Delta(K^{-1}) = K^{-1} \otimes K^{-1}$ in the fourth equality, and the naturality of the braiding in the last equality.

$\square$

As a consequence, the graphical calculus applies to $\widetilde{\mathrm{Mod}}_U^{fd}$. In particular, applying Proposition 1.11 in the syllabus of two weeks ago in the present situation gives for the quantum trace tr_q of Mod_U^{fd} the simple expression

$$\mathrm{tr}_q(f) = \mathrm{Tr}_{\mathbb{K}}(m \mapsto f(K^{-1}m)), \qquad f \in \mathrm{End}_U(M),$$

where $\mathrm{Tr}_{\mathbb{K}}$ is the usual trace.

Example. Consider the simple module $L(n)$ of degree $n+1$. Then its quantum dimension is

$$\begin{aligned}
\dim_q(L(n)) &= \mathrm{tr}_q(\mathrm{id}_M) \\
&= \mathrm{Tr}_{\mathbb{K}}(m \mapsto K^{-1}m) \\
&= \sum_{j=0}^{n} q^{2j-n} \\
&= \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}
\end{aligned}$$

since $\{\overline{m}_j(q^n)\}_{j=0}^n$ is a basis of $L(n)$ satisfying $K^{-1}\overline{m}_j(q^n) = q^{2j-n}\overline{m}_j(q^n)$.

3. The colored Jones polynomial

To simplify notations we write $\mathcal{C} = \widetilde{\mathrm{Mod}}_U^{fd}$. Let

$$F_{RT}^{\mathcal{C}} : \mathcal{T}_{\mathcal{C}}^{G} \to \mathcal{C}$$

be the Reshetikin-Turaev functor, where $\mathcal{T}_{\mathcal{C}}^{G}$ is the category of $\mathcal{C}$-colored oriented ribbon graphs.

Recall the subcategory $\mathcal{T}_{\mathcal{C}}$ of $\mathcal{T}_{\mathcal{C}}^{G}$ with the same objects, but with morphisms the isotopy classes of $\mathcal{C}$-colored oriented ribbon tangles (no coupons). Let $[L] \in \mathrm{End}_{\mathcal{T}_{\mathcal{C}}^{G}}(\emptyset)$, i.e. an isotopy class of a $\mathcal{C}$-colored $(0,0)$-ribbon graph L. Then

$$F_{RT}^{\mathcal{C}}([L]) \in \mathrm{End}_{U}(\mathbb{K}) \simeq \mathbb{K}$$

is an isotopy invariant of L.

Exercise 3.1. *Let L be a $\mathcal{C}$-colored oriented ribbon link (so no coupons). Show by graphical calculus that $F_{RT}^{\mathcal{C}}([L])$ does not dependent on the orientation of L.*
Hint: *Use that $M \simeq M^*$ for all $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.*

For an oriented ribbon link $\mathcal{L}$ and $n \in \mathbb{Z}_{\geq 1}$ denote $\mathcal{L}_n$ for the associated oriented $\mathcal{C}$-colored ribbon link obtained by coloring the components of $\mathcal{L}$ by the simple U-module $L(n)$ of dimension $n+1$. Note that

$$F_{RT}^{\mathcal{C}}([\mathcal{L}_n]) \in \mathrm{End}_{U}(\mathbb{K}) \simeq \mathbb{K} \tag{3.1}$$

is an invariant of $\mathcal{L}$ which does not dependent on the orientation of $\mathcal{L}$ in view of the previous exercise. Furthermore,

$$F_{RT}^{\mathcal{C}}([\mathcal{O}_n^m]) = \dim_q(L(n))^m$$

where $\mathcal{O}^m$ is the unlink (without twists) with m components. We modify the invariant as follows. We write D for the tangle diagram associated to the oriented ribbon link $\mathcal{L}$ (with the proper amount of curls to account for the twists in $\mathcal{L}$). Denote $w(D)$ for the writhe of D. Recall that $w(D)$ only depend on the isotopy class of the oriented ribbon link $\mathcal{L}$. We set

$$P_n(\mathcal{L}) = q^{\frac{n(n+2)w(D)}{2}} \frac{F_{RT}^{\mathcal{C}}([\mathcal{L}_n])}{\dim_q(L(n))} \tag{3.2}$$

for an oriented ribbon link $\mathcal{L}$. Observe that $P_n(\mathcal{L})$ in general does depend on the orientation of $\mathcal{L}$. On the other hand, we have

Lemma 3.2. *$P_n(\mathcal{L})$ only depends on the core of the oriented ribbon link $\mathcal{L}$.*

Proof. We have $F_{RT}^{\mathcal{C}}(\phi_{L(n)}) = \theta_{L(n)} = q^{-n(n+2)/2}\mathrm{id}_{L(n)}$ and $F_{RT}^{\mathcal{C}}(\phi'_{L(n)}) = q^{n(n+2)/2}\mathrm{id}_{L(n)}$. On the other hand, removing a positive twist ϕ in $\mathcal{L}$ decreases the writhe by one while removing a negative twist ϕ' in $\mathcal{L}$ increases the writhe by one. It follows that $P_n(\mathcal{L})$ is invariant under removing positive/negative twists in $\mathcal{L}$, hence it only depends on the core of $\mathcal{L}$. □

We conclude that $P_n(\mathcal{L})$ defines an invariant of the oriented link $\mathcal{L}$ which satisfies

$$P_n(\mathcal{O}^m) = \dim_q(L(n))^{m-1}$$

for the oriented unlink $\mathcal{O}^m$ with m components.

Definition 3.3. *The invariant $P_n(\mathcal{L}) \in \mathbb{K}$ is called the n-colored Jones polynomial of the oriented link $\mathcal{L}$.*

The terminology is motivated by the following observation.

Theorem 3.4. *$P_1(\mathcal{L})$ is the Jones polynomial $V_{\mathcal{L}}(\sqrt{-1}q^{-\frac{1}{2}}) \in k[q, q^{-1}]$ of the oriented link $\mathcal{L}$.*

Proof. Recall that $F_{RT}^{\mathcal{C}}(X_{L(1)}^{\pm}) = c_{L(1),L(1)}^{\pm 1}$. By (3.4) in the syllabus of four weeks ago we thus have

$$q^{-\frac{1}{2}} F_{RT}^{\mathcal{C}}([(\mathcal{L}_+)_1]) - q^{\frac{1}{2}} F_{RT}^{\mathcal{C}}([(\mathcal{L}_-)_1]) = (q^{-1} - q) F_{RT}^{\mathcal{C}}([(\mathcal{L}_0)_1]) \tag{3.3}$$

where $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$ is a Conway triple. We have

$$w(D_{\pm}) = w(D_0) \pm 1$$

for a Conway triple $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$ where (D_+, D_-, D_0) are the corresponding link diagrams. Hence multiplying (3.3) by $q^{\frac{3w(D_0)}{2}}$ we get

$$q^{-2} P_1(\mathcal{L}_+) - q^2 P_1(\mathcal{L}_-) = (q^{-1} - q) P_1(\mathcal{L}_0) \tag{3.4}$$

for a Conway triple $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$. Furthermore, $P_1(\mathcal{O}) = 1$. Comparing with the characterizing properties of the Jones polynomial $V_{\mathcal{L}}(a)$, amongst which the relation

$$a^4 V_{\mathcal{L}_+}(a) - a^{-4} V_{\mathcal{L}_-}(a) = (a^{-2} - a^2) V_{\mathcal{L}_0}(a)$$

for a Conway triple $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$, we conclude that $P_1(\mathcal{L}) = V_{\mathcal{L}}(\sqrt{-1}q^{-\frac{1}{2}})$. □

The Conway skein relation (3.4) gives a convenient tool to compute the Jones polynomial $P_1(\mathcal{L})$ of a given oriented link $\mathcal{L}$. Unfortunately, such a skein relation is not available for the general n-colored Jones polynomial. A useful technique to compute n-colored Jones polynomials will be discussed in the appendix.

4. The HOMFLY polynomial

Recall from Section 4 in the syllabus of four weeks ago the quantized universal enveloping algebra of $\mathfrak{sl}_n$ ($n \geq 2$) over the field $\mathbb{K}$, which we set throughout this section to be $k(q^{\frac{1}{n}})$. Recall that the quantized universal enveloping algebra $U_q(\mathfrak{sl}_n)$ is generated as algebra by elements K_i, E_i, F_i ($1 \leq i \leq n-1$).

Many results derived in detail for $\mathfrak{sl}_2$ have a natural counterpart for $\mathfrak{sl}_n$. In particular, $U_q(\mathfrak{sl}_n)$ is formally a ribbon algebra, with explicit universal R-matrix $\mathcal{R}$ and associated ribbon element given by $K_{2\rho}u$, where u is the Drinfeld element $\mu(S \otimes \mathrm{id})(\mathcal{R}_{21})$ and

$$K_{2\rho} := \prod_{j=1}^{n-1} K_j^{j(n-j)} \in U_q(\mathfrak{sl}_n) \tag{4.1}$$

(in the present situation we have $S^2(X) = K_{2\rho}^{-1} X K_{2\rho}$ for all $X \in \mathcal{U}_q(\mathfrak{sl}_n)$). To make sense of these formal elements we consider the category $\mathrm{Mod}_{U_q(\mathfrak{sl}_n)}^{fd}$ of finite dimensional type

1 $U_q(\mathfrak{sl}_n)$-modules over $\mathbb{K}$. Recall that a finite dimensional $U_q(\mathfrak{sl}_n)$-module M over $\mathbb{K}$ is called of type 1 if

$$M = \bigoplus_{\underline{\mu}\in\mathbb{Z}^{n-1}} M[\underline{\mu}],$$

where for $\underline{\mu} = (\mu_1, \ldots, \mu_{n-1})$,

$$M[\underline{\mu}] = \{m \in M \mid K_i m = q^{\mu_i} m \quad (1 \leq i \leq n-1)\}.$$

Then $\mathrm{Mod}^{fd}_{U_q(\mathfrak{sl}_n)}$ becomes a ribbon category with the usual monoidal structure and left duality, with braiding

$$c_{M,N} = \tau_{M,N} \circ \mathcal{R}|_{M\otimes N}$$

(τ the flip), and with twist

$$\theta_M = (u^{-1}K_{2\rho}^{-1})_M.$$

Furthermore, the category $\mathrm{Mod}^{fd}_{U_q(\mathfrak{sl}_n)}$ is semisimple, and the simple finite dimensional type 1 $U_q(\mathfrak{sl}_n)$-modules are described as follows. The isomorphism classes of the simples are parametrized by $\mathbb{Z}^{m-1}_{\geq 0}$. Concretely, for $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}_{\geq 0}$ the associated finite dimensional simple $U_q(\mathfrak{sl}_n)$-module $L(\underline{\lambda})$ over $\mathbb{K}$ is the unique simple quotient of the Verma module $U_q(\mathfrak{sl}_n)/I(\underline{\lambda})$, where $I(\underline{\lambda})$ is the left $U_q(\mathfrak{sl}_n)$-ideal generated by E_i and $K_i - q^{\lambda_i}$ for $1 \leq i \leq n-1$. It follows that $L(\underline{\lambda})[\underline{\lambda}]$ is one-dimensional. A nonzero $v_{\underline{\lambda}} \in L(\underline{\lambda})[\underline{\lambda}]$ is called a highest weight vector of $L(\underline{\lambda})$. Up to a nonzero multiplicative constant it is uniquely characterized by the properties

$$E_i v_{\underline{\lambda}} = 0, \qquad K_i v_{\underline{\lambda}} = q^{\lambda_i} v_{\underline{\lambda}}$$

for all $1 \leq i \leq n-1$.

Recall the vector representation V of $U_q(\mathfrak{sl}_n)$ from Section 4 in the syllabus of four weeks ago. It is the n-dimensional $\mathbb{K}$-vector space $V = \bigoplus_{i=1}^n \mathbb{K}v_i$ with $U_q(\mathfrak{sl}_n)$-action determined by

$$\begin{aligned} K_j^{\pm 1} v_i &= q^{\mp\epsilon_i(H_j)} v_i, \\ E_j v_i &= \delta_{i,j} v_{i+1}, \\ F_j v_{i+1} &= \delta_{j,i+1} v_i. \end{aligned}$$

For $1 \leq i \leq n-1$ set $\omega_i \in \mathbb{Z}^{n-1}$ for the vector with 1 on the ith entry and zeroes everywhere else.

Exercise 4.1. *Show that $V \simeq L(\omega_{n-1})$ with highest weight vector $v_n \in V$.*

Remark 4.2. In general $M \not\simeq M^*$ in $\mathrm{Mod}^{fd}_{U_q(\mathfrak{sl}_n)}$ if $n > 2$. For instance, $L(\omega_{n-1})^* \simeq L(\omega_1)$.

We remark that

$$\theta_{L(\underline{\lambda})} = q^{-\langle\underline{\lambda},\underline{\lambda}+2\rho\rangle}\mathrm{id}_{L(\underline{\lambda})} \tag{4.2}$$

(see e.g. [1, §6.7] but be aware that different conventions for the Hopf algebra structure of $U_q(\mathfrak{sl}_n)$ are used), where

$$\rho = \omega_1 + \omega_2 + \cdots + \omega_{n-1}$$

and with $\langle \cdot, \cdot \rangle : \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \to \mathbb{Z}$ the bilinear pairing defined by

$$\langle \omega_i, \omega_j \rangle = \begin{cases} \frac{i(n-j)}{n}, & i \leq j, \\ \frac{j(n-i)}{n}, & i \geq j. \end{cases} \tag{4.3}$$

Remark 4.3. To explain the strange looking formula for the bilinear form, consider $\mathbb{R}^n$ with standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$ and set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$. Then we realize $\mathbb{Z}^{n-1}$ in $\mathbb{R}^n$ as the lattice spanned by

$$\omega_i = \frac{(n-i)}{n}(\epsilon_1 + \cdots + \epsilon_i) - \frac{i}{n}(\epsilon_{i+1} + \cdots + \epsilon_n)$$

for $1 \leq i \leq n$. Note that the ω_i lies in the hyperplane $(\epsilon_1 + \cdots + \epsilon_n)^\perp$ of $\mathbb{R}^n$ (as do the elements α_i). Furthermore, $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$ for $1 \leq i, j \leq n-1$. Computing $\langle \omega_i, \omega_j \rangle$ we obtain (4.3). With this realization of the ω_i's and α_i's in $\mathbb{R}^n$ we also have

$$2\rho = \sum_{j=1}^{n} j(n-j)\alpha_j,$$

which justifies the notation $K_{2\rho}$ for the expression (4.1).

Exercise 4.4. *Check that*

$$\theta_V = q^{-n+\frac{1}{n}} \mathrm{id}_V.$$

Note that the quantum trace tr_q for $\mathrm{Mod}^{fd}_{U_q(\mathfrak{sl}_n)}$ is given by

$$\mathrm{tr}_q(f) = \mathrm{Tr}_{\mathbb{K}}(m \mapsto f(K_{2\rho}^{-1}m)), \qquad f \in \mathrm{End}_{U_q(\mathfrak{sl}_n)}(M).$$

Exercise 4.5. *Show that*

$$\dim_q(V) = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for the vector representation $V \simeq L(\omega_{n-1})$.

We are now all set to apply the Reshetikhin-Turaev functor with colors from the strict ribbon category

$$\mathcal{C} := \widetilde{\mathrm{Mod}}^{fd}_{U_q(\mathfrak{sl}_n)}.$$

We start with the analogue of the colored Jones polynomial in the present set-up. Let $\mathcal{L}$ be an oriented ribbon link. For $\underline{\lambda} \in \mathbb{Z}^{n-1}_{\geq 0}$ define the quantum invariant

$$H_{\underline{\lambda}}(\mathcal{L}) = q^{\langle \underline{\lambda}, \underline{\lambda} + 2\rho \rangle w(D)} \frac{F^{\mathcal{C}}_{RT}\big([\mathcal{L}_{\underline{\lambda}}]\big)}{\dim_q(L(\underline{\lambda}))} \in \mathbb{K} \tag{4.4}$$

of $\mathcal{L}$, where D is the tangle diagram associated to $\mathcal{L}$ and $\mathcal{L}_{\underline{\lambda}}$ is the $\mathcal{C}$-colored oriented ribbon link obtained by coloring each component of $\mathcal{L}$ by $L(\underline{\lambda})$. For $n = 2$ it reduces to the colored

Jones polynomial of the previous section. Observe that the quantum invariant $H_{\underline{\lambda}}(\mathcal{L})$ only depends on the core of the oriented ribbon link $\mathcal{L}$. We furthermore have $H_{\underline{\lambda}}(\mathcal{O}) = 1$, where $\mathcal{O}$ is the unknot.

It is a difficult task to compute the quantum invariant $H_{\underline{\lambda}}(\mathcal{L})$ explicitly for a given oriented link $\mathcal{L}$. The skein theoretic methods for computing the colored Jones polynomial (see the appendix) are not applicable here. An exception is $\lambda = \omega_{n-1}$, in which case $H_{\omega_{n-1}}(\mathcal{L})$ is expressible in terms of the HOMFLY polynomial.

Recall (see [4, X.4]) that the HOMFLY polynomials $H(\mathcal{L}) = H(\mathcal{L})(x, y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of oriented links $\mathcal{L}$ are uniquely characterized by the properties

(1) If $\mathcal{L}$ and $\mathcal{L}'$ are isotopic then $H(\mathcal{L}) = H(\mathcal{L}')$,
(2) $H(\mathcal{O}) = 1$,
(3) If $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$ is a Conway triple, then

$$xH(\mathcal{L}_+) - x^{-1}H(\mathcal{L}_-) = yH(\mathcal{L}_0).$$

Theorem 4.6. *We have*

$$H_{\omega_{n-1}}(\mathcal{L}) = H(\mathcal{L})(q^{-n}, q^{-1} - q)$$

for an oriented link $\mathcal{L}$.

Proof. It suffices to show that $H_{\omega_{n-1}}(\mathcal{L})$ satisfies the skein relation

$$q^{-n}H_{\omega_{n-1}}(\mathcal{L}_+) - q^{n}H_{\omega_{n-1}}(\mathcal{L}_-) = (q^{-1} - q)H_{\omega_{n-1}}(\mathcal{L}_0) \tag{4.5}$$

for a Conway triple $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$. Since $V \simeq L(\omega_{n-1})$ is the vector representation, formula (4.1) in the syllabus of four weeks ago shows that

$$q^{-\frac{1}{n}}F_{RT}^{\mathcal{C}}([(\mathcal{L}_+)_{\omega_{n-1}}]) - q^{\frac{1}{n}}F_{RT}^{\mathcal{C}}([(\mathcal{L}_-)_{\omega_{n-1}}]) = (q^{-1} - q)F_{RT}^{\mathcal{C}}([(\mathcal{L}_0)_{\omega_{n-1}}]).$$

Now multiplying both sides by

$$q^{\langle \omega_{n-1}, \omega_{n-1}+2\rho\rangle w(D_0)} = q^{(n-\frac{1}{n})w(D_0)}$$

we get (4.5) by straightforward manipulations (compare with the proof of Theorem 3.4). □

5. Appendix : colored Jones polynomial and skein theory

We return in this appendix to the quantized universal enveloping algebra U of $\mathfrak{sl}_2$ and we relate the colored Jones polynomial to skein theory. We write $\mathcal{C}$ for the strict ribbon category $\widetilde{\mathrm{Mod}}_U^{fd}$ of finite dimensional type 1 U-modules over $\mathbb{K}$.

5.1. **Quantum Clebsch-Gordan decomposition.** We first need to study the tensor product of U-modules in some detail. For $r, s \in \mathbb{Z}_{\geq 0}$ the tensor product module $L(r) \otimes L(s)$ of U is finite dimensional and of type 1, hence it decomposes as a direct sum of the simple U-modules $L(n)$ $(n \geq 0)$. The following theorem (quantum Clebsch-Gordan decomposition) gives a more precise statement.

Theorem 5.1. *For $r, s \in \mathbb{Z}_{\geq 0}$ we have*

$$L(r) \otimes L(s) \simeq L(|r-s|) \oplus L(|r-s|+2) \oplus \cdots \oplus L(r+s-2) \oplus L(r+s)$$

as U-modules.

Proof. It suffices to prove the theorem for $r \geq s$ (indeed, $L(r) \otimes L(s) \simeq L(s) \otimes L(r)$ since Mod_U^{fd} is a braided category). In this case we are asked to prove that

$$L(r) \otimes L(s) \simeq \bigoplus_{l=0}^{s} L(r-s+2l).$$

The dimension over $\mathbb{K}$ of the left hand side is $(r+1)(s+1)$, which agrees with the dimension of the right hand side since

$$\sum_{l=0}^{s} (r-s+2l+1) = (r+1)(s+1).$$

To prove the theorem it thus suffices to construct highest weight vectors $w_l \in L(r) \otimes L(s)$ of highest weight $r-s+2l+1$ for $l = 0, \ldots, s$. First note that

$$\big(L(r) \otimes L(s)\big)_{q^{r-s+2l}} = \mathrm{span}_{\mathbb{K}}\{\overline{m}_n(q^r) \otimes \overline{m}_{s-l-n}(q^s)\}_{n=-l}^{s-l}$$

for $l = 0, \ldots, s$. Now a vector

$$w_l = \sum_{n=-l}^{s-l} k_n\big(\overline{m}_n(q^r) \otimes \overline{m}_{s-l-n}(q^s)\big) \in \big(L(r) \otimes L(s)\big)_{q^{r-s+2l}}, \qquad k_n \in \mathbb{K}$$

satisfies $Ew_l = 0$ if and only if

$$k_{n+1} = -q^{n-r-s} \frac{\big(s-l-n\big)_{q^2}\big(l+n+1\big)_{q^2}}{\big(n+1\big)_{q^2}\big(r-n\big)_{q^2}} k_n \qquad (n = -l, \ldots, s-l-1)$$

(note that the denominators are nonzero). There is a nontrivial solution to these recurrence relations, hence there exists a highest weight vector of highest weight q^{r-s+2l} in $L(r) \otimes L(s)$ for all $l \in \{0, \ldots, s\}$. $\square$

Remark 5.2. We thus have two natural basis of $L(r) \otimes L(s)$, namely (for $r \geq s$)

$$\{\overline{m}_i(q^r) \otimes \overline{m}_j(q^s) \mid 0 \leq i \leq r, \quad 0 \leq j \leq s\}$$

and

$$\{F^k w_l \mid 0 \leq k \leq r-s+2l, \quad 0 \leq l \leq s\}$$

with the notations from the proof of Theorem 5.1. Thus we have an expansion

$$F^k w_l = \sum_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s}} \begin{pmatrix} r & s & r+s-2l \\ i & j & k \end{pmatrix} \overline{m}_i(q^r) \otimes \overline{m}_j(q^s)$$

for certain coefficients

$$\begin{pmatrix} r & s & r+s-2l \\ i & j & k \end{pmatrix} \in \mathbb{K}$$

called quantum $3j$-symbols or quantum Clebsch-Gordan coefficients. Explicit expressions for quantum $3j$-symbols are known (they can be expressed in terms of a family of orthogonal polynomials called q-Hahn polynomials). Such formulas can be used to explicitly compute colored Jones polynomials of particular oriented links, although the computations soon become rather involved.

Exercise 5.3. *Let $L(n)$ be the simple $n+1$ dimensional U-module.*
(i) *Verify the following formulas.*

$$\begin{aligned}
F^{\mathcal{C}}_{RT}(\cup_{L(n)}) &= b_{L(n)} : 1 \mapsto \sum_{j=0}^{n} \overline{m}_j(q^n) \otimes \overline{m}_j(q^n)^*, \\
F^{\mathcal{C}}_{RT}(\cap_{L(n)}) &= d_{L(n)} : \phi \otimes m \mapsto \phi(m), \\
F^{\mathcal{C}}_{RT}(\cup^-_{L(n)}) &= b^-_{L(n)} : 1 \mapsto \sum_{j=0}^{n} q^{n-2j}(\overline{m}_j(q^n)^* \otimes \overline{m}_j(q^n)), \\
F^{\mathcal{C}}_{RT}(\cap^-_{L(n)}) &= d^-_{L(n)} : m \otimes \phi \mapsto \phi(K^{-1}m)
\end{aligned}$$

for $m \in L(n)$ and $\phi \in L(n)^*$.
Hint: *By the previous theorem* $\mathrm{Hom}_U(\mathbb{K}, L(n)^* \otimes L(n))$ *and* $\mathrm{Hom}_U(L(n) \otimes L(n)^*, \mathbb{K})$ *are one-dimensional. Use this fact to compute $F^{\mathcal{C}}_{RT}(\cup^-_{L(n)})$ and $F^{\mathcal{C}}_{RT}(\cap^-_{L(n)})$.*
(ii) *Recall the isomorphism $\alpha_{L(n)} : L(n) \to L(n)^{**}$ of Proposition 2.11 in the syllabus of two weeks ago. Show that*

$$\alpha_{L(n)}(\overline{m}_i(q^n)) = q^{2i-n}\overline{m}_i(q^n)^{**}$$

for $1 \leq i \leq n$.
(iii) *Show that the isomorphism $\xi_n : L(n)^* \to L(n)$ of section 1 satisfies*

$$\xi_n^* = (-1)^n \alpha_{L(n)} \circ \xi_n.$$

(iv) *Use graphical calculus and* **(iii)** *to show that*

$$\phi_n \circ \psi_n = (-1)^n \dim_q(L(n)),$$

which thus reproves Lemma 1.4.

Corollary 5.4. *For $k, l \in \mathbb{Z}_{\geq 0}$ we have*

$$\mathrm{Dim}_{\mathbb{K}}\big(\mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})\big) = \begin{cases} 0, & \text{if } k+l \text{ odd}, \\ \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}, & \text{if } k+l = 2n \text{ even}. \end{cases}$$

Proof. For $n \in \mathbb{Z}_{\geq 0}$ we have

$$V^{\otimes n} \simeq \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L(n)^{\otimes k_j(n)} \tag{5.1}$$

for unique $k_j(n) \in \mathbb{Z}_{\geq 0}$. For $n = 0$ we read the left hand side as $L(0) = \mathbb{K}$ (the unit object in $\mathcal{C} = \mathrm{Mod}_U^{fd}$), so that $k_0(0) = 1$ and $k_j(0) = 0$ for $j \geq 1$. By the above theorem we have the recurrence relations

$$k_0(n) = k_1(n-1), \qquad k_j(n) = k_{j+1}(n-1) + k_{j-1}(n-1) \quad (j \geq 1) \tag{5.2}$$

for $n \geq 1$. These recurrence relations, together with the initial conditions $k_0(0) = 1$ and $k_j(0) = 0$ ($j \geq 1$), determine the coefficients $k_j(n)$ uniquely. Now define the polynomial $V_j(y)$ of degree $j \geq 0$ by

$$V_j(x + x^{-1}) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}} = \sum_{l=0}^{j} x^{j-2l}.$$

It is the character of the simple $\mathfrak{sl}_2$-module of dimensional $j+1$ or, from the viewpoint of orthogonal polynomials, a Chebyshev polynomial of the second kind (up to a rescaling of the variable). The $V_j(y)$'s satisfy $V_0(y) = 1$ and the recurrence relation

$$yV_j(y) = V_{j+1}(y) + V_{j-1}(y), \qquad j \geq 1.$$

By induction to n we conclude from the recurrence relations for the $k_j(n)$ and the $V_j(y)$ that

$$y^n = \sum_{j=0}^{\infty} k_j(n) V_j(y)$$

(this is in fact the equivalent reformulation of (5.1) for the characters of the associated $\mathfrak{sl}_2$-modules). Since

$$V_j(2\cos(\theta)) = \frac{\sin((j+1)\theta)}{\sin(\theta)}$$

we have the orthogonality relations

$$\frac{1}{\pi} \int_{-\pi}^{\pi} V_i(2\cos(\theta)) V_j(2\cos(\theta)) \sin^2(\theta) d\theta = \delta_{i,j}.$$

We conclude that

$$\begin{aligned}
\mathrm{Dim}_{\mathbb{K}}\big(\mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})\big) &= \sum_{j=0}^{\infty} k_j(k) k_j(l) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (2\cos(\theta))^{k+l} \sin^2(\theta) d\theta \\
&= -\frac{1}{4\pi} \sum_{r=0}^{k+l} \binom{k+l}{r} \int_{-\pi}^{\pi} e^{i(2r-k-l)\theta} (e^{2i\theta} + e^{-2i\theta} - 2) d\theta.
\end{aligned}$$

The latter integral is zero if $k+l$ is odd. If $k+l=2n$ is even, then it equals

$$\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}$$

the nth Catalan number. $\square$

5.2. **The skein functor revisited.** We write $V=L(1)$ for the two-dimensional simple U-module. Let $\mathcal{C}_V$ be the full braided monoidal subcategory of $\widetilde{\mathrm{Mod}}_U^{fd}$ with objects $V^{\otimes m}$ ($m\in\mathbb{Z}_{\geq 0}$). Recall the strict tensor functor F_1^{sk} from the first section.

Theorem 5.5. *The skein functor F_1^{sk} defines an equivalence*

$$F_1^{sk}:\mathcal{S}(q^{-\frac{1}{2}})\xrightarrow{\sim}\mathcal{C}_V$$

of strict $\mathbb{K}$-linear braided monoidal categories.

Proof. Clearly F_1^{sk} defines a bijection on objects, mapping $m\in\mathbb{Z}_{\geq 0}$ to $V^{\otimes m}$. We also remarked in the first section that F_1^{sk} respects the braidings. It thus remains to show that the $\mathbb{K}$-linear maps

$$F_1^{sk}:E_{kl}(q^{-\frac{1}{2}})\rightarrow \mathrm{Hom}_U(V^{\otimes k},V^{\otimes l})$$

($k,l\in\mathbb{Z}_{\geq 0}$) are linear isomorphisms.

We may assume that $k+l$ is even (otherwise the vector spaces are zero). We write D_{kl} for the $\mathbb{K}$-vector space with $\mathbb{K}$-basis the isotopy classes of smooth (k,l)-tangle diagrams (no crossings and no loops). Note that

$$\dim_{\mathbb{K}}(E_{kl}(q^{-\frac{1}{2}}))\leq \dim_{\mathbb{K}}(D_{kl})=\dim_{\mathbb{K}}(\mathrm{Hom}_U(V^{\otimes k},V^{\otimes l}))$$

(see Corollary 5.4 and the proof of Theorem 3.17 in the syllabus of three weeks ago). We have canonical linear maps

$$D_{kl}\rightarrow E_{kl}(q^{-\frac{1}{2}})\rightarrow \mathrm{Hom}_U(V^{\otimes k},V^{\otimes l}),$$

the second map being F_1^{sk}. We show that these maps are isomorphisms. We denote the composition of the two maps by π.

We define a $\mathbb{K}$-linear map

$$*:D_{kl}\rightarrow D_{lk}$$

sending a smooth (k,l)-tangle diagram D to its reflection D^* in the line $y=\frac{1}{2}$ (where y is the height coordinate). For two smooth (k,l)-tangle diagrams D and D' we define the link diagram

$$\mathrm{Clos}(D^*\circ D')$$

as the closure of the (k,k)-tangle diagram $D^*\circ D'$ (connecting the endpoints $(i,0)$ to $(i,1)$ by nonintersecting arcs for $1\leq i\leq k$). Finally, we define a $\mathbb{K}$-bilinear form

$$\big(\cdot,\cdot\big):D_{kl}\times D_{kl}\rightarrow\mathbb{K}$$

defined on smooth (k,l)-tangle diagrams D and D' by

$$\big(D,D'\big)=\langle\mathrm{Clos}(D^*\circ D')\rangle(q^{-\frac{1}{2}})$$

with $\langle\cdot\rangle(q^{-\frac{1}{2}})$ the Kauffman bracket with parameter $a = q^{-\frac{1}{2}}$.

For smooth (k,l)-tangle diagrams D and D' we have

$$\big(D, D'\big) = (-q - q^{-1})^{m(D,D')}$$

with $m(D,D')$ the number of the components of the link diagram $\mathrm{Clos}(D^* \circ D')$. We have

$$m(D,D') \leq \frac{k+l}{2}$$

for smooth (k,l)-tangle diagrams D and D' with equality if and only if $D' = D$. It follows that the assignment

$$D \mapsto \big(D, \cdot\big)$$

defines a linear isomorphism $\phi : D_{kl} \to D_{kl}^*$.

Now we use that F_1^{sk} is a strict tensor functor, which implies that $\phi(D)$ for $D \in D_{kl}$ only depends on $\pi(D) \in \mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})$. Hence $\pi : D_{kl} \to \mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})$ is injective, thus (by dimension count) it is a linear isomorphism. This concludes the proof of the theorem. □

Corollary 5.6. *The endomorphism algebra* $\mathrm{End}_U(V^{\otimes n})$ *is isomorphic to the Temperley-Lieb algebra* $E_{nn}(q^{-\frac{1}{2}})$*. The algebra isomorphism* $E_{nn}(q^{-\frac{1}{2}}) \xrightarrow{\sim} \mathrm{End}_U(V^{\otimes n})$ *is the skein functor* F_1^{sk} *restricted to* $E_{nn}(q^{-\frac{1}{2}})$*.*

5.3. **The colored Jones polynomial revisited.** We show in this subsection that the n-colored Jones polynomial of an oriented link $\mathcal{L}$ can be expressed as the Reshetikhin-Turaev invariant of a closely related $\mathcal{C}$-colored $(0,0)$-ribbon graph $\widetilde{\mathcal{L}}$ in which all oriented arcs have color $L(1)$. Beside the coupons in the ribbon graph $\widetilde{\mathcal{L}}$, which we investigate in more detail in the next section, we thus have the Conway skein relation of the ordinary Jones polynomial to our disposal to compute the colored Jones polynomial.

We use the notation V for the simple 2-dimensional U-module $L(1)$ for the remainder of the section. We fix $n \in \mathbb{Z}_{>0}$. By Theorem 5.1, and by Lemma 1.8 and Exercise 2.11 in the syllabus of last week we have

$$\dim_{\mathbb{K}}\big(\mathrm{Hom}_U(L(n), V^{\otimes n})\big) = 1 = \dim_{\mathbb{K}}\big(\mathrm{Hom}_U(V^{\otimes n}, L(n))\big)$$

and, given two nonzero morphisms

$$\iota_n \in \mathrm{Hom}_U(L(n), V^{\otimes n}), \qquad \pi_n \in \mathrm{Hom}_U(V^{\otimes n}, L(n)),$$

we have $0 \neq \pi_n \circ \iota_n \in \mathrm{End}_U(L(n)) = \mathrm{span}_{\mathbb{K}}\{\mathrm{id}_{L(n)}\}$. We normalize the morphisms such that $\pi_n \circ \iota_n = \mathrm{id}_{L(n)}$ and we set

$$\widehat{f}_n := \iota_n \circ \pi_n \in \mathrm{End}_U(V^{\otimes n}).$$

Let now $\mathcal{L}$ be an oriented ribbon link. Recall that $\mathcal{L}_n$ is the associated $\mathcal{C}$-colored ribbon link in which all components have color $L(n)$.

Proposition 5.7. *We have*
$$F_{RT}^{\mathcal{C}}([\mathcal{L}_n]) = F_{RT}^{\mathcal{C}}([\widetilde{\mathcal{L}}_n]),$$
where $\widetilde{\mathcal{L}}_n$ is the $(0,0)$-ribbon graph obtained from $\mathcal{L}$ by splitting each ribbon component in n parallel oriented ribbon knots, labelling each ribbon knot by V, and inserting in each band of n parallel ribbon knots the coupon $\widehat{f}_n \in \mathrm{End}_U(V^{\otimes n})$.

Proof. Let $\mathcal{L}_n' \in \mathrm{End}_{\mathcal{T}_{\mathcal{C}}^G}(\emptyset)$ be obtained from $\mathcal{L}_n$ by inserting the coupon $\mathrm{id}_{L(n)} = \pi_n \circ \iota_n$ in each component. Obviously $\mathcal{L}_n \doteq \mathcal{L}_n'$. For each component in $\mathcal{L}_n'$ we take the coupon ι_n and move it along the component in the orientational direction until it meets the coupon π_n again. Compared to $\mathcal{L}_n'$ we have split each component in n parallel oriented ribbon knots, colored them by V, and replaced the original coupon $\mathrm{id}_{L(n)}$ by the coupon $\widehat{f}_n = \iota_n \circ \pi_n$. It follows that
$$\mathcal{L}_n' \doteq \widetilde{\mathcal{L}}_n,$$
hence $\mathcal{L}_n \doteq \widetilde{\mathcal{L}}_n$. □

Corollary 5.8. *Let $\mathcal{L}$ be an oriented ribbon link. Then*
$$P_n(\mathcal{L}) = \frac{q^{\frac{n(n+2)w(D)}{2}}}{\dim_q(L(n))} F_{RT}^{\mathcal{C}}\big([\widetilde{\mathcal{L}}_n]\big)$$
where D is the oriented link diagram associated to $\mathcal{L}$ in the usual way.

5.4. **The Jones-Wenzl idempotents.** Our next aim is to compute $F_{RT}^{\mathcal{C}}\big([\widetilde{\mathcal{L}}_n]\big)$ in the skein. We start with a useful exercise.

Exercise 5.9. *Let $\mathcal{D}_n$ be the set of isotopy classes of smooth (n,n)-tangle diagrams, viewed as $\mathbb{K}$-basis of the Temperley-Lieb algebra $E_{nn}(q^{-\frac{1}{2}})$. It contains id_n (the (n,n)-tangle diagram with all strands going straight down). For $X = \sum_{D \in \mathcal{D}_n} \lambda_D D \in E_{nn}(q^{-\frac{1}{2}})$ with $\lambda_D \in \mathbb{K}$ set*
$$e(X) := \lambda_{\mathrm{id}_n}.$$
Show that $e : E_{nn}(q^{-\frac{1}{2}}) \to \mathbb{K}$ is an algebra homomorphism.

Write $f_n \in E_{nn}(q^{-\frac{1}{2}})$ for the pre-image of $\widehat{f}_n$ under the algebra isomorphism
$$F_1^{sk} : E_{nn}(q^{-\frac{1}{2}}) \xrightarrow{\sim} \mathrm{End}_U(V^{\otimes n}).$$
Note that $f_1 = \mathrm{id}_1 \in E_{11}(q^{-\frac{1}{2}})$. When computing the colored Jones polynomial in the skein, it is convenient to have some tools to deal with f_n. The basis properties are given in the following proposition.

For $1 \leq i < n$ denote $\cap_i \in E_{n,n-2}(q^{-\frac{1}{2}})$ and $\cup_i \in E_{n-2,n}(q^{-\frac{1}{2}})$ for the smooth tangle diagrams
$$\cap_i = \mathrm{id}_{i-1} \otimes \cap \otimes \mathrm{id}_{n-i-1}, \qquad \cup_i = \mathrm{id}_{i-1} \otimes \cup \otimes \mathrm{id}_{n-i-1},$$
viewed as morphisms in the skein category $\mathcal{S}(q^{-\frac{1}{2}})$.

Proposition 5.10. *The element $f_n \in E_{nn}(q^{-\frac{1}{2}})$ satisfies:*

(1) f_n *is an idempotent,* $f_n^2 = f_n$.
(2) $\cap_i f_n = 0$ *and* $f_n \cup_i = 0$ *for* $1 \leq i < n$.
(3) $f_n X = X f_n = e(X) f_n$ *for* $X \in E_{nn}(q^{-\frac{1}{2}})$ *with* e *the map from Exercise 5.9.*
(4) f_n *is the only nonzero element in* $E_{nn}(q^{-\frac{1}{2}})$ *satisfying (1), (2) and (3).*

These statements should be interpreted in the obvious way when $n = 1$.

Proof. It suffices to prove all statements in the image of the equivalence $F_1^{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \xrightarrow{\sim} \mathcal{C}_V$ of strict monoidal categories.
(1) We have
$$\widehat{f_n} \circ \widehat{f_n} = \iota_n \circ (\pi_n \circ \iota_n) \circ \pi_n = \iota_n \circ \pi_n = \widehat{f_n}.$$
(2) Note that $F_1^{sk}(\cap_i) \circ \iota_n = 0$ since $\mathrm{Hom}_U(L(n), V^{\otimes(n-2)}) = \{0\}$ by Theorem 5.1. Hence
$$F_1^{sk}(\cap_i) \circ \widehat{f_n} = (F_1^{sk}(\cap_i) \circ \iota_n) \circ \pi_n = 0.$$
Similarly, $\pi_n \circ F_1^{sk}(\cup_i) = 0$ since $\mathrm{Hom}_U(V^{\otimes(n-2)}, L(n)) = \{0\}$. Hence
$$\widehat{f_n} \circ F_1^{sk}(\cup_i) = \iota_n \circ (\pi_n \circ F_1^{sk}(\cup_i)) = 0.$$
(3) Let $D \in \mathcal{D}_n$ (for the notation see Exercise 5.9). If $D \neq \mathrm{id}_{L(n)}$ then there exists a smooth $(n, n-2)$-tangle diagram D_+ and a smooth $(n-2, n)$-tangle diagram D_- such that
$$\cup_i D_+ = D = D_- \cap_j$$
in the skein category $\mathcal{S}(q^{-\frac{1}{2}})$ for some $1 \leq i, j < n$, hence
$$D f_n = D_-(\cap_j f_n) = 0, \qquad f_n D = (f_n \cup_i) D_+ = 0$$
by (2). This directly leads to the desired result.
(4) Let $g_n \in E_{nn}(q^{-\frac{1}{2}})$ be a nonzero element satisfying properties (1), (2) and (3). Then
$$e(g_n) g_n = g_n^2 = g_n,$$
hence $e(g_n) = 1$. In particular, $e(f_n) = 1$. Then
$$g_n = e(f_n) g_n = f_n g_n = e(g_n) f_n = f_n.$$
□

Definition 5.11. *The element* $f_n \in E_{nn}(q^{-\frac{1}{2}})$ *is called the Jones-Wenzl idempotent.*

We need the following lemma in a moment.

Lemma 5.12. *Let* $\mathcal{L}$ *be an oriented link represented as the closure of a braid with* m *strands. Let* D *be its associated oriented link diagram. Then*
$$(-1)^{\mathrm{comp}(\mathcal{L}) + w(D)} = (-1)^m$$
where $w(D)$ *is the writhe of* D *and* $\mathrm{comp}(\mathcal{L})$ *is the number of components of* $\mathcal{L}$.

Proof. Let $\mathcal{L}'$ be the link obtained from $\mathcal{L}$ by forgetting its orientation. Let $B \in B_m$ be the braid such that $\mathcal{L}'$ is the closure $\mathrm{Clos}(B)$ of B. Let $s \in S_m$ be the element of the symmetric group to which B is mapped under the natural group epimorphism $B_m \to S_m$. Let $c(B)$ be the number of crossings in B. Then

$$(-1)^{w(D)} = (-1)^{c(B)} = (-1)^{l(s)} = \det(s),$$

where

$$l(s) = \{1 \leq i < j \leq m \mid s(i) > s(j)\}$$

is the length of s and $\det(s)$ is the determinant of s (viewed as $m \times m$ permutation matrix with on row i zeroes everywhere expect a one in column $s(i)$). The lemma thus is equivalent to

$$(-1)^{\mathrm{comp}(\mathcal{L}')} = \det(s)(-1)^m, \tag{5.3}$$

in which case all factors are obviously independent of the orientation of $\mathcal{L}$. Now write

$$s = \tau_1 \tau_2 \cdots \tau_r$$

as product of disjoint cykels $\tau_j \in S_m$ (including the cykels of length one). We denote l_j for the cykel-length of τ_j, and m_e (respectively m_o) for the number of cykels with even (respectively odd) cykel-lengths. Then $r = \mathrm{comp}(\mathcal{L}')$ hence

$$(-1)^{\mathrm{comp}(\mathcal{L}')} = (-1)^{m_e + m_o}.$$

On the other hand,

$$\det(s) = \prod_{j=1}^{r} (-1)^{l_j - 1} = (-1)^{m_e}$$

and

$$(-1)^m = (-1)^{m_o}$$

since $m = l_1 + l_2 + \cdots + l_r \equiv m_0$ modulo 2. This proves (5.3). □

Theorem 5.13. *Let $\mathcal{L}$ be an oriented ribbon link and D its associated link diagram. Let $n \in \mathbb{Z}_{>0}$ and let $D_n \in E_{00}(q^{-\frac{1}{2}})$ be the element in the skein obtained from the oriented $\mathcal{C}$-colored $(0,0)$-ribbon graph $\widetilde{\mathcal{L}_n}$ by forgetting its color and orientation and replacing the coupons labelled by $\widehat{f_n} \in \mathrm{End}_U(V^{\otimes n})$ by the Jones-Wenzl idempotent $f_n \in E_{nn}(q^{-\frac{1}{2}})$. Then*

$$F_{RT}^{\mathcal{C}}([\mathcal{L}_n]) = (-1)^{(\mathrm{comp}(\mathcal{L}) + w(D))n} F_1^{sk}(D_n). \tag{5.4}$$

Proof. The key observation is the following relation between the Reskehtikin-Turaev functor $F_{RT}^{\mathcal{C}}$ and the skein functor F_1^{sk} (cf. Corollary 1.5, Corollary 1.7 and Exercise 5.3),

$$\begin{aligned}
&F_{RT}^{\mathcal{C}}(X^{\pm}) = c_{V,V}^{\pm 1} = F_1^{sk}(X^{\pm}), && \\
&F_{RT}^{\mathcal{C}}(\cap_V) = F_1^{sk}(\cap) \circ (\xi_1 \otimes \mathrm{id}_V), && F_{RT}^{\mathcal{C}}(\cup_V) = (\mathrm{id}_V \otimes \xi_1^{-1}) \circ F_1^{sk}(\cup), \\
&F_{RT}^{\mathcal{C}}(\cap_V^-) = F_1^{sk}(\cap) \circ (\mathrm{id}_V \otimes (-\xi_1)), && F_{RT}^{\mathcal{C}}(\cup_V^-) = ((-\xi_1^{-1}) \otimes \mathrm{id}_V) \circ F_1^{sk}(\cup).
\end{aligned} \tag{5.5}$$

In the right hand side of the formulas $X^{\pm}, \cup$ and $\cap$ in $\mathcal{S}(q^{-\frac{1}{2}})$ are interpreted as $G(X^{\pm}), G(\cup)$ and $G(\cap)$ with G the usual strict tensor functor $G : \mathcal{T} \to \mathcal{S}(q^{-\frac{1}{2}})$ and with $X^{\pm}, \cup$ and $\cap$ the generators of the category $\mathcal{T}$ of tangles (so then they represent unoriented tangle diagrams).

Recall that the category $\mathcal{T}$ of ribbon tangles is braided. We denote $\vartheta_n \in \mathrm{End}_{\mathcal{T}}(n)$ for the braiding of $n \in \mathbb{Z}_{>0}$ and we use the same notation for its image $G(\vartheta_n)$ in the Temperley-Lieb algebra $E_{nn}(q^{-\frac{1}{2}})$.

Observe that (5.4) is agreeable with adding (respectively removing) a full twist in one of the arcs of $\mathcal{L}_n$. This amounts to showing that

$$F_{RT}^{\mathcal{C}}(\phi_{L(n)}) = (-1)^n \pi_n F_1^{sk}(\vartheta_n)\iota_n$$

which follows from the graphical calculus by adding the coupon $\pi_n\iota_n$ on top of $\phi_{L(n)}$, moving the coupon ι_n through the twist, applying $F_{RT}^{\mathcal{C}}$ and using (5.5).

Since furthermore both sides of (5.4) are independent of the orientation of $\mathcal{L}$, it suffices to prove the theorem for an oriented ribbon link $\mathcal{L}$ represented as the closure of a braid B of m strands without twists and with orientation such that the m closing arcs are oriented from the top towards the bottom boundary points of the braid. In view of Lemma 5.12 we are then required to prove

$$F_{RT}^{\mathcal{C}}([\mathcal{L}_n]) = (-1)^{mn} F_1^{sk}(D_n).$$

This is immediate from Proposition 5.7 and (5.4). □

The previous (proof of the) theorem can be very convenient in computing in the skein or in computing invariants. We end with giving some examples.

Corollary 5.14. **(i)** *Let $\overline{f}_n \in E_{00}(q^{-\frac{1}{2}})$ be the closure of the Jones-Wenzl idempotent $f_n \in E_{nn}(q^{-\frac{1}{2}})$ (i.e. we connect the top boundary points to the bottom boundary points of f_n by disjoint arcs and view it as element in the skein $E_{00}(q^{-\frac{1}{2}})$). Then*

$$\overline{f}_n = (-1)^n \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}.$$

(ii) *With the notations from the proof of Theorem 5.13 we have*

$$\vartheta_n f_n = (-1)^n q^{-\frac{n(n+2)}{2}} f_n$$

in the Temperley-Lieb algebra $E_{nn}(q^{-\frac{1}{2}})$.

Proof. **(i)** We have by Theorem 5.13,

$$\overline{f}_n = (-1)^n F_{RT}^{\mathcal{C}}([\mathcal{O}_n]) = (-1)^n \dim_q(L(n)) = (-1)^n \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}.$$

For a direct proof in the skein, see [7, XII, Lemma 4.4.2].
(ii) It can be proved by a rather elaborate inductive proof in the skein itself using Proposition 5.10, but here is the conceptual proof by graphical calculus:

$$\begin{aligned}
F_1^{sk}(\vartheta_n f_n) &= F_1^{sk}(\vartheta_n f_n^2)\\
&= F_1^{sk}(f_n \vartheta_n f_n)\\
&= \widehat{f}_n F_1^{sk}(\vartheta_n) \widehat{f}_n\\
&= \iota_n(\pi_n F_1^{sk}(\vartheta_n)\iota_n)\pi_n\\
&= (-1)^n \iota_n F_{RT}^{\mathcal{C}}(\phi_{L(n)})\pi_n\\
&= (-1)^n q^{-\frac{n(n+2)}{2}} \iota_n \pi_n\\
&= (-1)^n q^{-\frac{n(n+2)}{2}} F_1^{sk}(f_n).
\end{aligned}$$

The fifth equality is shown by graphical calculus in the proof of Theorem 5.13 and the sixth equality is by Lemma 2.3. ☐

Corollary 5.15. *The Jones polynomial satisfies*

$$V_{\mathcal{L}}(-q) = (-1)^{\mathrm{comp}(\mathcal{L})-1} V_{\mathcal{L}}(q)$$

for an oriented ribbon link $\mathcal{L}$.

Proof. We compute

$$\begin{aligned}
V_{\mathcal{L}}(-q) &= q^{\frac{3w(D)}{2}} \frac{F_{RT}^{\mathcal{C}}([\mathcal{L}_1])}{q+q^{-1}}\\
&= (-1)^{\mathrm{comp}(\mathcal{L})}(-q^{\frac{3}{2}})^{w(D)} \frac{F_1^{sk}(D)}{q+q^{-1}}\\
&= (-1)^{\mathrm{comp}(\mathcal{L})-1} V_{\mathcal{L}}(q).
\end{aligned}$$

Here the first equality is by Theorem 3.4, the second by Theorem 5.13 and the third equality by Corollary 3.2 in the syllabus of last week.

The corollary can also be proved by verifying that the right hand side satisfies the characterizing properties of the Jones polynomial $V_{\mathcal{L}}(-q)$. ☐

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