QUANTUM INVARIANTS I

Recall that U is the quantized universal enveloping algebra of $\mathfrak{sl}_2$ over $\mathbb{K} = k(q^{\frac{1}{2}})$, where k is an algebraically closed field of characteristic zero. In this text we show that the braided monoidal category Mod_U^{fd} of finite dimensional type 1 U-modules over $\mathbb{K}$ is semisimple, and we describe the simple modules explicitly. We use the results to represent the skein category in Mod_U^{fd}.

Convention: In this syllabus all U-modules are considered over $\mathbb{K}$ unless specified explicitly otherwise. In other words, an U-module will mean a $\mathbb{K}$-module M together with an $\mathbb{K}$-algebra homomorphism $\pi : U \to \mathrm{End}_{\mathbb{K}}(M)$ (or equivalently, a $\mathbb{K}$-module M with a $\mathbb{K}$-bilinear left action $U \times M \to M$, given by $(u, m) \mapsto um (= \pi(u)(m))$).

1. Highest weight modules of U

Recall that U is the associative unital algebra over $\mathbb{K}$ with generators $K^{\pm 1}, E, F$ and relations

$$KK^{-1} = 1 = K^{-1}K, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$
$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F.$$

For a left U-module M and $\lambda \in \mathbb{K}^\times$ we write M_λ for the K-eigenspace with eigenvalue λ,

$$M_\lambda = \{m \in M \mid Km = \lambda m\}.$$

Definition 1.1. *Let M be a left U-module and $\lambda \in \mathbb{K}^\times$.*
(i) *An element $m \in M$ is called a highest weight vector of weight λ if $0 \neq m \in M_\lambda$ and $Em = 0$.*
(ii) *M is called a highest weight module of highest weight λ if $M = Um$ with $m \in M$ a highest weight vector of weight λ.*

Remark 1.2. If M is a highest weight module of highest weight λ, then the spectrum (=set of eigenvalues) $\mathrm{Spec}_M(K)$ of the action of K on M satisfies $\lambda \in \mathrm{Spec}_M(K) \subseteq \lambda q^{2\mathbb{Z}_{\leq 0}}$. Indeed, if $m \in M$ is the highest weight vector of highest weight λ, then M is spanned by $\{F^i m\}_{i \in \mathbb{Z}_{\geq 0}}$ (use here that $\{F^i E^j K^l\}_{i,j \in \mathbb{Z}_{\geq 0}, l \in \mathbb{Z}}$ is a $\mathbb{K}$-basis of U and the fact that $M = Um$) and $K(F^i m) = q^{-2i} \lambda F^i m$. In particular, the highest weight of a highest module is unique.

Let $\lambda \in \mathbb{K}^\times$ and let

$$M(\lambda) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{K} m_n(\lambda)$$

be an infinite dimensional $\mathbb{K}$-vector space with $\mathbb{K}$-linear basis $\{m_n(\lambda)\}_{n\in\mathbb{Z}_{\geq 0}}$. Set $m_{-1}(\lambda) := 0$ in $M(\lambda)$. Recall the q-number

$$(m)_{q^2} := \frac{q^{2m}-1}{q^2-1} \in \mathbb{K}, \qquad m \in \mathbb{Z}_{\geq 0}.$$

Proposition 1.3. **(i)** *The following formulas uniquely define a left U-module structure on $M(\lambda)$,*

$$\begin{aligned} Km_n(\lambda) &= \lambda q^{-2n} m_n(\lambda), \\ Em_n(\lambda) &= \big(n\big)_{q^2}\left(\frac{q^{2-2n}\lambda-\lambda^{-1}}{q-q^{-1}}\right) m_{n-1}(\lambda), \\ Fm_n(\lambda) &= m_{n+1}(\lambda). \end{aligned} \tag{1.1}$$

The resulting left U-module $M(\lambda)$ is a highest weight module of weight λ, with highest weight vector $m_0(\lambda)$.
(ii) *Any highest weight left U-module M of weight λ is a quotient of $M(\lambda)$.*

Proof. **(i)** Consider $I(\lambda) = UE + U(K-\lambda)$, the left ideal of U generated by E and $K-\lambda$. The set

$$\{F^nE^iK^j \mid n, i \in \mathbb{Z}_{\geq 0}, \ \ j \in \mathbb{Z}\}$$

is a $\mathbb{K}$-linear basis of U by the Poincaré-Birkhoff-Witt theorem, hence the elements

$$F^n + I(\lambda), \qquad n \geq 0 \tag{1.2}$$

form a $\mathbb{K}$-linear basis of the quotient space $U/I(\lambda)$. We now identify the $\mathbb{K}$-vector space $M(\lambda)$ with $U/I(\lambda)$ via the $\mathbb{K}$-linear isomorphism $M(\lambda) \xrightarrow{\sim} U/I(\lambda)$ mapping the distinguished $\mathbb{K}$-basis element $m_n(\lambda)$ of $M(\lambda)$ to $F^n + I(\lambda) \in U/I(\lambda)$ for all $n \in \mathbb{Z}_{\geq 0}$. With this particular realization of $M(\lambda)$ as quotient of U by the left ideal $I(\lambda)$, it canonically inherits a left U-module structure. Computing its action explicitly on the basis elements (1.2) we obtain (1.1); the first equation follows from the relation $KF^n = q^{-2n}F^nK$ in U,

$$Km_n(\lambda) = q^{-2n}F^nK + I(\lambda) = q^{-2n}\lambda(F^n + I(\lambda)) = q^{-2n}\lambda m_n(\lambda).$$

The second equation of (1.1) follows from

$$\begin{aligned} Em_n(\lambda) &= EF^n + I(\lambda) \\ &= F^nE + \big(n\big)_{q^2}F^{n-1}\left(\frac{q^{2-2n}K-K^{-1}}{q-q^{-1}}\right) + I(\lambda) \\ &= \big(n\big)_{q^2}\left(\frac{q^{2-2n}\lambda-\lambda^{-1}}{q-q^{-1}}\right) m_{n-1}(\lambda), \end{aligned}$$

with the obvious adjustments for $n = 0$. The third equation of (1.1) is trivial.
(ii) Let M be a highest weight U-module of highest weight λ, with highest weight vector m. Consider U as left U-module by left multiplication in U ($u \cdot u' := uu'$ for $u, u' \in U$). Then the map $U \to M$, defined by $u \mapsto um$, is a surjective U-module morphism. Since

$m \in M$ is a highest weight vector of weight λ, it factors through the ideal $I(\lambda)$, hence it gives rise to a well defined surjective U-module morphism $M(\lambda) = U/I(\lambda) \rightarrow M$. $\square$

Definition 1.4. *$M(\lambda)$ is called the Verma module of highest weight $\lambda \in \mathbb{K}^\times$.*

A left U-module M has two trivial U-submodules, namely $\{0\}$ and M.

Definition 1.5. *A left U-module M is simple if $M \neq \{0\}$ and if $\{0\}$ and M are the only U-submodules of M.*

Under mild conditions we can now describe the finite dimensional simple left U-modules over $\mathbb{K}$ as follows.

Proposition 1.6. *Let $\lambda \in \mathbb{K}^\times$, $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$.*
(i) *$M(\lambda)$ is simple if $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$.*
(ii) *$M(\epsilon q^n)$ has a unique nontrivial U-submodule $N(\epsilon q^n)$, isomorphic to the Verma module $M(\epsilon q^{-2-n})$.*
(iii) *The left U-module $L(\epsilon, n) := M(\epsilon q^n)/N(\epsilon q^n)$ is a finite dimensional simple U-module of dimension $n+1$. Furthermore, $L(\epsilon, n) \simeq L(\epsilon', n')$ if and only if $(\epsilon, n) = (\epsilon', n')$.*
(iv) *Let M be a finite dimensional, simple left U-module. If M has an eigenvector for the action of K, then $M \simeq L(\epsilon, n)$ for a unique $(\epsilon, n) \in \{\pm 1\} \times \mathbb{Z}_{\geq 0}$.*

Proof. We start with some preliminary remarks before embarking on the proof of the four statements. Let $\lambda \in \mathbb{K}^\times$ and let $\{0\} \neq N \subseteq M(\lambda)$ be a U-submodule. Fix $0 \neq m \in N$ and write

$$m = \sum_{j \in J} c_j m_j(\lambda)$$

with $c_j \in \mathbb{K}^\times$ and J a nonempty finite subset of $\mathbb{Z}_{\geq 0}$. For $i \in J$ we then have

$$m_i(\lambda) = c_i^{-1} \prod_{j \in J \setminus i} \left(\frac{K - q^{-2j}\lambda}{q^{-2i}\lambda - q^{-2j}\lambda} \right) m \in N.$$

Furthermore, if $m_i(\lambda) \in N$ for a given $i \geq 1$, then acting by E we conclude from the previous proposition that $m_{i-1}(\lambda) \in N$ if $\lambda \neq \pm q^{i-1}$.
(i) Let $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$ and suppose that $\{0\} \neq N \subseteq M(\lambda)$ is a U-submodule. Let $i \in \mathbb{Z}_{\geq 0}$ such that $m_i(\lambda) \in N$. Since $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$, we conclude from the above remarks that the highest weight vector $m_0(\lambda)$ is also contained in N. Since $M(\lambda) = U m_0(\lambda)$, we conclude that $N = M(\lambda)$, hence $M(\lambda)$ is a simple U-module.
(ii) Suppose that $\lambda = \epsilon q^n$ ($\epsilon \in \{\pm 1\}$, $n \in \mathbb{Z}_{\geq 0}$). Then $E m_{n+1}(\epsilon q^n) = 0$ by the previous proposition, hence $N(\epsilon q^n) := \bigoplus_{j \geq n+1} \mathbb{K} m_j(\epsilon q^n)$ is a U-submodule of $M(\epsilon q^n)$. It is a highest weight module of highest weight ϵq^{-2-n}, with highest weight vector $m_{n+1}(\epsilon q^n)$. It is easily seen to be isomorphic to $M(\epsilon q^{-2-n})$ (hence it is simple by **(i)**).

Let $0 \neq N \subseteq M(\epsilon q^n)$ be an U-submodule. Let i be the smallest nonnegative integer such that $m_i(\epsilon q^n) \in N$. We conclude from the remarks so far that there are two posibilities: $i = 0$ (in which case $N = M(\epsilon q^n)$) or $i = n+1$ (in which case $N = N(\epsilon q^n)$). This completes the proof of **(ii)**.

(iii) Let $\pi : M(\epsilon q^n) \to L(\epsilon, n)$ be the canonical map. It is a surjective U-module morphism. If $M \subseteq L(\epsilon, n)$ is an U-submodule, then $N(\epsilon q^n) \subseteq \pi^{-1}(M) \subseteq M(\epsilon q^n)$ is an U-submodule of $M(\epsilon q^n)$, hence $\pi^{-1}(M)$ equals $N(\epsilon q^n)$ or $M(\epsilon q^n)$ by the observation in the previous paragraph. Thus $M = \{0\}$ or $M = L(\epsilon, n)$, hence $L(\epsilon, n)$ is a simple U-module.

Since $N(\epsilon q^n) = \bigoplus_{j \geq n+1} \mathbb{K} m_j(\epsilon q^n)$ it is clear that $\{m_i(\epsilon q^n) + N(\epsilon q^n)\}_{i=0}^n$ is a $\mathbb{K}$-linear basis of $L(\epsilon, n)$, hence $\dim_{\mathbb{K}}(L(\epsilon, n)) = n + 1$.

If $L(\epsilon, n) \simeq L(\epsilon', n')$ as U-modules then the corresponding sets of eigenvalues for the actions of K on $L(\epsilon, n)$ and $L(\epsilon', n')$ should be the same. This forces $(\epsilon, n) = (\epsilon', n')$.

(iv) Let M be a finite dimensional, simple left U-module with $M_\lambda \neq \{0\}$ for some $\lambda \in \mathbb{K}^\times$. The algebraic sum $\sum_{s=0}^\infty M_{q^{2s}\lambda}$ in M is direct, since the summands are eigenspaces of the action of K on M for different eigenvalues. But M is finite dimensional over $\mathbb{K}$ by assumption, hence $M_{q^{2s}\lambda} \neq \{0\}$ for only finitely many $s \in \mathbb{Z}_{\geq 0}$. Since $E^r M_\lambda \subseteq M_{q^{2r}\lambda}$ there exists a $m' \in M_\lambda$ and $r \in \mathbb{Z}_{\geq 0}$ such that $m := E^r m' \in M_{q^{2r}\lambda}$ is nonzero while $Em = E^{r+1} m' = 0$. Set $\mu = q^{2r}\lambda$. Since M is simple, we have $M = Um$, hence M is a highest weight U-module of highest weight μ, with highest weight vector m. Let $\phi : M(\mu) \to M$ be the corresponding surjective U-module morphism, characterized by $\phi(m_0(\mu)) = m$. Then $\mathrm{Ker}(\phi) \subset M(\mu)$ is a nontrivial left U-submodule of $M(\mu)$ and $M(\mu)/\mathrm{Ker}(\phi) \simeq M$. By the previous observations, $\mu = \epsilon q^n$ for some $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$, and $\mathrm{Ker}(\phi) = N(\epsilon q^n)$. Hence $M \simeq L(\epsilon, n)$. $\square$

Corollary 1.7. *For $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$ choose*

$$\overline{m}_j(\epsilon q^n) := m_j(\epsilon q^n) + N(\epsilon q^n), \qquad (j = 0, \ldots, n)$$

as $\mathbb{K}$-linear basis of $L(\epsilon, n) = M(\epsilon q^n)/N(\epsilon q^n)$. Furthermore set $\overline{m}_{-1}(\epsilon q^n) = 0 = \overline{m}_{n+1}(\epsilon q^n)$. The U-action on $L(\epsilon, n)$ is explicitly given by

$$\begin{aligned}
K\overline{m}_j(\epsilon q^n) &= \epsilon q^{n-2j}\overline{m}_j(\epsilon q^n), \\
E\overline{m}_j(\epsilon q^n) &= \epsilon q^{1-n}\big(j\big)_{q^2}\big(n+1-j\big)_{q^2}\overline{m}_{j-1}(\epsilon q^n), \\
F\overline{m}_j(\epsilon q^n) &= \overline{m}_{j+1}(\epsilon q^n)
\end{aligned}$$

for $j = 0, \ldots, n$.

Proof. Direct computation. $\square$

We now investigate the morphisms between U-modules.

Lemma 1.8. **(i)** *Suppose M and N are non-isomorphic simple left U-modules. Then*

$$\mathrm{Hom}_U(M, N) = \{0\}.$$

(ii) *If M is a highest weight module then* $\mathrm{End}_U(M) = \mathrm{span}_{\mathbb{K}}\{\mathrm{Id}_M\}$.

Proof. **(i)** Let $\phi \in \mathrm{Hom}_U(M, N)$. The kernel of ϕ is an U-submodule of M, hence it is $\{0\}$ or M. If ϕ is injective, then the image of ϕ is a nonzero U-submodule of N, hence it is equal to N. In this case ϕ is an isomorphism, contradicting the fact that $M \not\simeq N$. Thus the kernel of ϕ is M, which means that $\phi = 0$.

(ii) Let $\lambda \in \mathbb{K}^\times$ be the highest weight of M. Choose a corresponding highest weight vector $m \in M$ of highest weight λ. Then $M_\lambda = \mathrm{span}_{\mathbb{K}}\{m\}$ since M is a quotient of $M(\lambda)$, and $M(\lambda)_\lambda$ is one-dimensional.

Fix now $\phi \in \mathrm{End}_U(M)$. By the previous paragraph we have $\phi(m) = \mu m$ for some $\mu \in \mathbb{K}$. Let now $m' \in M$. Since M is a highest weight module with highest weight vector m, there exists an $u \in U$ so that $m' = um$. Then $\phi(m') = \phi(um) = u\phi(m) = \mu m'$. We conclude that $\phi = \mu \mathrm{Id}_M$. □

Exercise 1.9. *Recall that the linear dual $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ of a left U-module V is a left U-module by*

$$(X\phi)(v) := \phi(S(X)v), \qquad \phi \in V^*,\ v \in V,\ X \in U.$$

Show that $L(\epsilon, n)^* \simeq L(\epsilon, n)$ as U-modules for all $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$.

2. The category Mod_U^{fd} again

Recall that a left U-module M is of type 1 if

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^n}.$$

We defined before the braided monoidal category Mod_U^{fd} consisting of the finite dimensional type 1 left U-modules. From the last section we immediately get the following explicit description of the simple modules in Mod_U^{fd}.

Corollary 2.1. *The simple U-modules in Mod_U^{fd} are the $L(n) := L(+1, n)$ ($n \in \mathbb{Z}_{\geq 0}$), up to isomorphism.*

Exercise 2.2. *Let M be a finite dimensional type 1 U-module and consider the subspace*

$$M^e = \{m \in M \mid Em = 0\}.$$

Show that

$$M^e = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M^e_{q^n}$$

with $M^e_{q^n} = M^e \cap M_{q^n}$ the space of highest weight vectors in M of highest weight q^n.

We now proceed to show that a finite dimensional type 1 U-module M is semisimple, i.e. that M is isomorphic to a direct sum of the simple modules $L(n)$ ($n \geq 0$).

Theorem 2.3. *Let $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$. Then*

$$M \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L(n)^{\oplus k_n}$$

as U-modules, with $k_n = \dim_{\mathbb{K}}(M^e_{q^n})$.

We give some preparatory results before embarking on the proof of the theorem.

Lemma 2.4. *Let $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ and let $N \subseteq M$ be a U-submodule. Then $N, M/N \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.*

Proof. Only the statement that N is of type 1 requires proof. Clearly

$$\bigoplus_{n\in\mathbb{Z}} N \cap M_{q^n} \subseteq N \subset M.$$

We have to show that the first inclusion is an equality. Let $m \in N$ and write $m = \sum_{j\in J} m_j$ with $J \subset \mathbb{Z}$ finite and $m_j \in M_{q^j}$. Then

$$m_n = \left(\prod_{j\in J\setminus\{n\}} \frac{K - q^j}{q^n - q^j}\right) m \in N$$

for all $n \in J$, hence $m \in \bigoplus_{n\in\mathbb{Z}} N \cap M_{q^n}$. □

The following basic observation will be of use.

Lemma 2.5. *Let $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$. There exists a finite sequence of finite dimensional type 1 U-submodules*

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{r-1} \subset M_r = M$$

such that, for $1 \leq j \leq r$, the quotient U-module M_j/M_{j-1} is simple (hence isomorphic to $L(n)$ for some $n \in \mathbb{Z}_{\geq 0}$). Such a sequence is called a composition series for M of length r.

Proof. Fix a nonzero module $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$. Using induction to the $\mathbb{K}$-dimension of M, it suffices to show that there exists a U-submodule $N \subsetneq M$ such that M/N is simple.

If M is simple, then we are done ($N = \{0\}$ does the job). If M is not simple, then there exists an U-submodule $\{0\} \neq L \subsetneq M$. The quotient U-module $\overline{M} := M/L$ is of type 1 and of strictly smaller dimension than M. By the induction hypothesis there exists an U-submodule $\overline{N} \subset \overline{M}$ such that $\overline{M}/\overline{N}$ is simple.

Consider the canonical surjective U-module morphism $\pi : M \to \overline{M}$. Then $N := \pi^{-1}(\overline{N}) \subset M$ is a U-submodule, hence N is of type 1, and $M/N \simeq \overline{M}/\overline{N}$ is simple. □

A composition series does not have to be unique, but the subsequent simple quotients are unique up to reshuffling (this is the Jordan-Hölder theorem):

Lemma 2.6. *Let $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ and suppose that*

$$\begin{aligned} &\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M, \\ &\{0\} = M'_0 \subset M'_1 \subset \cdots \subset M'_{s-1} \subset M'_s = M \end{aligned} \tag{2.1}$$

are two composition series of M. Then $r = s$ and for some permutation σ of $\{1, \ldots, r\}$ we have

$$M_j/M_{j-1} \simeq M'_{\sigma(j)}/M'_{\sigma(j)-1}, \qquad j = 1, \ldots, r. \tag{2.2}$$

Proof. We call two composition series (2.1) of M equivalent if $r = s$ and if (2.2) holds true for some permutation σ of $\{1, \ldots, r\}$ (this is indeed an equivalence relation). We are thus required to show that all composition series of M are equivalent.

We prove it by induction to the minimum $\min(r, s)$ of the lengths of the two composition series. If it is zero (respectively one) then $M = \{0\}$ (respectively M is simple), so there is nothing to prove. Let (2.1) be two composition series of M with $r, s \geq 2$ and suppose that the lemma is correct for composition series with one of the two composition series of length $< \min(r, s)$. Without loss of generality we may assume that $r \leq s$.

Case (i): Suppose that $M_{r-1} = M'_{s-1}$. We will denote this U-module by N. Then

$$\begin{aligned}\{0\} &= M_0 \subset M_1 \subset \cdots \subset M_{r-2} \subset M_{r-1} = N,\\ \{0\} &= M'_0 \subset M'_1 \subset \cdots \subset M'_{s-2} \subset M'_{s-1} = N\end{aligned}$$

are composition series for the U-submodule N of M. By the induction hypothesis, they are equivalent, in particular $r = s$. Since $M/M_{r-1} = M/N = M/M'_{r-1}$, the composition series (2.1) for M are equivalent.

Case (ii): Suppose $M_{r-1} \neq M'_{s-1}$. Set $N = M_{r-1} + M'_{s-1}$. Then $M_{r-1} \subsetneq N \subset M$ is a U-submodule. The nonzero U-submodule N/M_{r-1} of M/M_{r-1} equals M/M_{r-1} since M/M_{r-1} is simple, hence $N = M$. We thus obtain isomorphisms

$$\begin{aligned}M/M_{r-1} &= (M_{r-1} + M'_{s-1})/M_{r-1} \simeq M'_{s-1}/(M_{r-1} \cap M'_{s-1}),\\ M/M'_{s-1} &= (M_{r-1} + M'_{s-1})/M'_{s-1} \simeq M_{r-1}/(M_{r-1} \cap M'_{s-1})\end{aligned}$$

of U-modules. We choose now a composition series

$$\{0\} = K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1}$$

for $M_{r-1} \cap M'_{s-1}$, which can be complemented to form a composition series for M in two ways,

$$\begin{aligned}\{0\} &= K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1} \subset M_{r-1} \subset M,\\ \{0\} &= K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1} \subset M'_{s-1} \subset M.\end{aligned} \tag{2.3}$$

Clearly the two composition series (2.3) for M are equivalent.

The first composition series of (2.1), respectively the first composition series of (2.3), contains a composition series for M_{r-1} of length $r - 1$, respectively of length $t + 1$. By the induction hypothesis, these two composition series for M_{r-1} are equivalent. In particular, $t = r - 2$. It follows that the first composition series for M of (2.1) is equivalent to the second composition series for M of (2.3).

The second composition series of (2.3), respectively the second composition series of (2.1), contains a composition series for M'_{s-1} of length $t + 1 = r - 1$, respectively of length $s - 1$. Again by the induction hypothesis, these two composition series for M'_{s-1} are equivalent, in particular $r = s$. Consequently the second composition series for M of (2.3) is equivalent to the second composition series for M of (2.1). By the transitivity of the equivalence relation, we conclude now that the two composition series of (2.1) for M are equivalent. □

We can now prove the theorem in a special case.

Lemma 2.7. *Let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ has a composition series*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

with $M_j/M_{j-1} \simeq L(n)$ for all $j = 1\ldots, r$. Then M is semisimple, i.e. $M \simeq L(n)^{\oplus r}$ as U-modules.

Proof. Using that $L(n)_{q^n}$ is one-dimensional and that $L(n)_{q^{n+2}} = \{0\}$, we claim that $M_{j,q^{n+2}} = \{0\}$, $M_{j,q^n} = M^e_{j,q^n}$ (with the notations from Exercise 2.2) and $\dim_{\mathbb{K}}(M^e_{j,q^n}) = j$. We proceed by induction to j. It is clearly true for $j = 1$. Suppose it is true for $j-1$. Let

$$\pi_j : M_j \longrightarrow M_j/M_{j-1} \simeq L(n)$$

be the canonical surjective morphism of U-modules. If $m \in M_{j,q^{n+2}}$ then $m \in \mathrm{Ker}(\pi_j)$, hence $m \in M_{j-1,q^{n+2}} = \{0\}$. Thus $M_{j,q^{n+2}} = \{0\}$.

Since M_j is of type 1, there exists a nonzero $m \in M_{j,q^n}$ mapping under π_j to a highest weight vector of $L(n)$. In particular $m \notin M_{j-1}$, hence $\mathbb{K}m + M_{j-1,q^n} \subseteq M_{j,q^n}$ is a direct sum. Let $v \in M_{j,q^n}$, then for some $c \in \mathbb{K}$ we have $v - cm \in M_{j,q^n} \cap \mathrm{Ker}(\pi_j) = M_{j-1,q^n}$. We conclude that

$$\mathbb{K}m \oplus M_{j-1,q^n} = M_{j,q^n},$$

hence $\dim_{\mathbb{K}}(M_{j,q^n}) = j$. Since $M_{j-1,q^n} = M^e_{j-1,q^n}$ by the induction hypothesis, it remains to show that $Em = 0$, but this is clear since $Em \in M_{j,q^{n+2}} = \{0\}$.

Fix now a $\mathbb{K}$-basis $\{v_1, \ldots, v_r\}$ of $M^e_{q^n}$. For $j = 1, \ldots, r$ the U-submodule $V_j := Uv_j \subseteq M$ generated by $v_j \in M^e_{q^n}$ is a finite dimensional highest weight U-module of weight q^n, with highest weight vector v_j. Thus $V_j \simeq L(n)$ as U-modules. Write V for the U-submodule $\sum_{j=1}^r V_j$ of M. By construction, $V_{q^n} = \oplus_{i=1}^r \mathbb{K}v_i = M_{q^n}$. Let $\pi : M \to M/V$ be the canonical surjective U-module morphism. Since M is of type 1, $\pi(M_{q^n}) = (M/V)_{q^n}$. Then $\mathrm{Ker}(\pi) \cap M_{q^n} = V \cap M_{q^n} = V_{q^n}$, hence $(M/V)_{q^n} \simeq M_{q^n}/V_{q^n} = \{0\}$. We claim that $M/V = \{0\}$.

Suppose that $M/V \neq \{0\}$. Then we can construct composition series

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_{s-1} \subset N_s = V,$$
$$\{0\} = \overline{M}_0 \subset \cdots \subset \overline{M}_{t-1} \subset \overline{M}_t = M/V$$

with $t \geq 1$. Then the above two composition series can be combined to give a composition series for M,

$$\{0\} = N_0 \subset \cdots \subset N_s = V = \pi^{-1}(\overline{M}_0) \subset \pi^{-1}(\overline{M}_1) \subset \cdots \subset \pi^{-1}(\overline{M}_t) = M.$$

Using the previous lemma, we conclude that $\pi^{-1}(\overline{M}_1)/V$ is a simple U-submodule of M/V, isomorphism to $L(n)$. But this is impossible since $(M/V)_{q^n} = \{0\}$. Thus we conclude that $M = V = \sum_{j=1}^r V_j$. Observe that $\dim_{\mathbb{K}}(M) = (n+1)^r$ equals $\sum_{j=1}^r \dim_{\mathbb{K}}(V_j)$ since

$V_j \simeq L(n)$, hence the sum $M = \sum_{j=1}^r V_j$ is direct. We conclude that

$$M \simeq \bigoplus_{j=1}^r V_j \simeq L(n)^{\otimes r}$$

as U-modules, as desired. $\square$

Denote $Z(U) = \{X \in U \mid XY = YX \quad \forall Y \in U\}$ for the center of U. Let M be a left U-module. The map $m \mapsto Zm$ defines an endomorphism $Z_M \in \mathrm{End}_U(M)$ (since Z is central). This gives a natural transformation $Z : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ for the category $\mathcal{C}$ of left U-modules since

$$f \circ Z_M = Z_N \circ f, \qquad f \in \mathrm{Hom}_U(M, N).$$

Note that Z restricts to a natural transformation for the identity functors of the full subcategory Mod_U^{fd} of $\mathcal{C}$.

Write

$$C = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \in U. \tag{2.4}$$

Proposition 2.8. **(i)** *$C \in Z(U)$.*
(ii) *If M is highest weight U-module of highest weight $\lambda \in \mathbb{K}^\times$, then*

$$C_M = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}\mathrm{Id}_M.$$

In particular,

$$C_{L(n)} = \frac{q^{1+n} + q^{-1-n}}{(q-q^{-1})^2}\mathrm{Id}_{L(n)}$$

for $n \in \mathbb{Z}_{\geq 0}$.

Proof. **(i)** We have to show that C commutes with the algebraic generators $K^{\pm 1}, E$ and F of U. This is verified by direct computations. We show here the computation for E,

$$\begin{aligned} EC &= EFE + \frac{qEK + q^{-1}EK^{-1}}{(q-q^{-1})^2} \\ &= \left(FE + \frac{K - K^{-1}}{q - q^{-1}}\right)E + \left(\frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}\right)E \\ &= \left(FE + \frac{(q-q^{-1})(K-K^{-1}) + q^{-1}K + qK^{-1}}{(q-q^{-1})^2}\right)E \\ &= \left(FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}\right)E = CE. \end{aligned}$$

(ii) Let M be a highest weight U-module of highest weight $\lambda \in \mathbb{K}^\times$. Then $C_M = c_M \mathrm{Id}_M$ for some $c_M \in \mathbb{K}$ by Lemma 1.8. To compute the constant c_M, let $m \in M$ be a highest weight vector of M, so that $Km = \lambda m$ and $Em = 0$. Then

$$C_M(m) = Cm = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} m$$

by the explicit expression for the central element C. Hence $c_M = \frac{q\lambda+q^{-1}\lambda^{-1}}{(q-q^{-1})^2}$. □

Exercise 2.9. *Let $\lambda, \mu \in \mathbb{K}^\times$. Show that*

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} = \frac{q\mu + q^{-1}\mu^{-1}}{(q-q^{-1})^2}$$

if and only if $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$.

The exercise in particular shows that C acts by different scalars on the simple U-modules $L(n)$ $(n \in \mathbb{Z}_{\geq 0})$ in Mod_U^{fd}.

To prove the theorem, we investigate C_M for an arbitrary finite dimensional type 1 U-module M. We begin with the following basic observation.

Lemma 2.10. *Let M be a finite dimensional type 1 U-module. We have*

$$M = \bigoplus_{\mu \in \mathbb{K}} M_{(\mu)}, \qquad M_{(\mu)} = \{m \in M \mid (C_M - \mu \mathrm{Id}_M)^n m = 0 \ \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

Proof. We write

$$c_n = \frac{q^{1+n} + q^{-1-n}}{(q-q^{-1})^2} \in \mathbb{K}, \qquad n \in \mathbb{Z}_{\geq 0}$$

for the scalar such that $C_{L(n)} = c_n \mathrm{Id}_{L(n)}$. Choose a composition series $\{0\} = M_0 \subset \cdots \subset M_{r-1} \subset M_r = M$ of M. Then $M_j/M_{j-1} \simeq L(n_j)$ for some $n_j \in \mathbb{Z}_{\geq 0}$ $(1 \leq j \leq r)$. Choosing a $\mathbb{K}$-basis of M compatible with the composition series (so first fixing a basis of M_0, extending it to a basis of M_1 and continuing this procedure until we have a $\mathbb{K}$-basis of M), we see that the characteristic polynomial of $C_M \in \mathrm{End}_{\mathbb{K}}(M)$ is of the form

$$\det(C_M - \lambda \mathrm{Id}_M) = \prod_{j=1}^r (c_{n_j} - \lambda)^{s_j} \in \mathbb{K}[\lambda]$$

with $s_j = \dim_{\mathbb{K}}(L(n_j))$. The characteristic function of $C_M \in \mathrm{End}_{\mathbb{K}}(M)$ thus decomposes as a product of linear factors over $\mathbb{K}$. A well known result from linear algebra now says that there exists a $\mathbb{K}$-basis of M such that the associated matrix of C_M is in Jordan normal form (see, e.g., [6, XIV, §2]). For a fixed Jordan block J of size $t \times t$ with eigenvalue $\mu \in \mathbb{K}$ on its diagonal we have $(J - \mu \mathrm{I})^t = 0$, with I (resp. 0) the $t \times t$ unit matrix (resp zero matrix). This proves the lemma. □

We can now easily complete the proof of Theorem 2.3. Let M be a finite dimensional type 1 U-module, and write

$$M = \bigoplus_{\mu \in \mathbb{K}} M_{(\mu)}$$

for its generalized eigenspace decomposition under the action of C. Since $C \in Z(U)$ (thus $C_M \in \mathrm{End}_U(M)$), the generalized eigenspaces $M_{(\mu)}$ are U-submodules of M. It thus remains to show that the finite dimensional type 1 U-submodules $M_{(\mu)}$ is semisimple.

Choose a composition series

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_r = M_{(\mu)}$$

for $M_{(\mu)}$. For $1 \leq t \leq r$ we thus have $N_t/N_{t-1} \simeq L(n_t)$ for some $n_t \in \mathbb{Z}_{\geq 0}$. The only eigenvalue of $C_{M_{(\mu)}}$ is μ, hence

$$\mu = \frac{q^{1+n_t} + q^{-1-n_t}}{(q-q^{-1})^2}$$

for $t = 1, \ldots, r$. This implies that $n_t = n$ is independent of t. Lemma 2.7 now shows that $M_{(\mu)}$ is semisimple, which completes the proof of Theorem 2.3.

Exercise 2.11. *Let M be a finite dimensional type 1 U-module. Show that*

$$\dim_{\mathbb{K}}\bigl(\mathrm{End}_U(M)\bigr) = \sum_{n=0}^{\infty} \dim_{\mathbb{K}}\bigl(M^e_{q^n}\bigr)^2.$$

3. A representation of the skein category

In this section we give a representation of the skein category $\mathcal{S}(q^{-\frac{1}{2}})$ over $\mathbb{K}$. Recall that $\mathcal{S}(q^{-\frac{1}{2}})$ is the strict tensor category with objects the nonnegative integers $\mathbb{Z}_{\geq 0}$, and with morphisms $\mathrm{Hom}_{\mathcal{S}}(k,l)$ the linear skein $E_{kl}(q^{-\frac{1}{2}})$ over $\mathbb{K}$.

We call a strict tensor category $\mathcal{M}$ linear over $\mathbb{K}$ if the set $\mathrm{Hom}_{\mathcal{M}}(U,V)$ of morphisms from U to V are vector spaces over $\mathbb{K}$ for all objects U and V of $\mathcal{M}$, and if the tensor product and composition maps

$$\circ : \mathrm{Hom}_{\mathcal{M}}(V,W) \times \mathrm{Hom}_{\mathcal{M}}(U,V) \to \mathrm{Hom}_{\mathcal{M}}(U,W),$$
$$\otimes : \mathrm{Hom}_{\mathcal{M}}(U,V) \times \mathrm{Hom}_{\mathcal{M}}(U',V') \to \mathrm{Hom}_{\mathcal{M}}(U \otimes U', V \otimes V')$$

are $\mathbb{K}$-bilinear for all objects U, V, U', V', W of $\mathcal{M}$. The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is an example of a $\mathbb{K}$-linear strict tensor category.

For $\mathbb{K}$-linear strict tensor categories the constructions of last week on generators and relations can be adjusted to encorporate the $\mathbb{K}$-linearity of the category in a natural way. As a consequence we get the following result.

Proposition 3.1. *Let $\mathcal{M}$ be a $\mathbb{K}$-linear strict tensor category with unit object $\mathbb{I}$. Let V be an object of $\mathcal{M}$ and suppose that $\phi \in \mathrm{Hom}_{\mathcal{M}}(V \otimes V, \mathbb{I})$ and $\psi \in \mathrm{Hom}_{\mathcal{M}}(\mathbb{I}, V \otimes V)$ are* *morphisms satisfying*

$$\phi \circ \psi = -(q+q^{-1})\mathrm{id}_{\mathbb{I}},$$
$$(\phi \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \psi) = \mathrm{id}_V = (\mathrm{id}_V \otimes \phi) \circ (\psi \otimes \mathrm{id}_V).$$

Then there exists a unique strict tensor functor $F : \mathcal{S}(q^{-\frac{1}{2}}) \to \mathcal{M}$ satisfying

1. $F(1) = V$,
2. $F : E_{kl}(q^{-\frac{1}{2}}) \to \mathrm{Hom}_{\mathcal{M}}(V^{\otimes k}, V^{\otimes l})$ *is $\mathbb{K}$-linear,*
3. $F(\cup) = \psi$ *and* $F(\cap) = \phi$.

Proof. See appendix 4 for a detailed proof.

□

Composing the $\mathbb{K}$-linear strict tensor functor $F : \mathcal{S}(q^{-\frac{1}{2}}) \to \mathcal{M}$ from Proposition 3.1 with the strict tensor functor

$$G : \mathcal{T} \to \mathcal{S}(q^{-\frac{1}{2}})$$

from Section 4.4 in the syllabus of week 15, we obtain a strict tensor functor

$$P = F \circ G : \mathcal{T} \to \mathcal{M}.$$

Given an isotopy class $[T]$ of a (k,l)-ribbon tangle T, we thus obtain an *isotopy invariant* $P([T]) \in \mathrm{Hom}_{\mathcal{M}}(V^{\otimes k}, V^{\otimes l})$ of T. The invariant $P([T])$ for a (k,l)-ribbon tangle T can be computed as follows. Represent T by a generic (k,l)-tangle diagram D_T (with the proper number of curls in the diagram to account for the framing of T). Now view D_T as element in $E_{kl}(q^{-\frac{1}{2}})$ and smoothen the crossings by the skein relation in $E_{kl}(q^{-\frac{1}{2}})$. In the resulting $\mathbb{K}$-linear sum of (k,l)-tangle diagrams without crossings we get rid of a closed loop on the cost of a factor $-(q+q^{-1})$. We replace minima (respectively maxima) by the morphism ψ (respectively ϕ) in $\mathcal{M}$ and we replace id_r by $\mathrm{id}_{V^{\otimes r}}$, while respecting the tensor product and composition rules in both the categories $\mathcal{S}(q^{-\frac{1}{2}})$ and $\mathcal{M}$. The resulting morphism in $\mathcal{M}$ is our desired invariant of T.

For ribbon links we immediately re-obtain the Kauffman bracket,

Corollary 3.2. *In the above set-up we have*

$$P([T]) = -(q+q^{-1})\langle T\rangle_{q^{-\frac{1}{2}}}\mathrm{id}_{\mathbb{I}}$$

for a ribbon link T.

We now are going to construct an explicit $\mathbb{K}$-linear strict tensor functor $F_{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \to \widetilde{\mathrm{Mod}}_U^{fd}$, where $\widetilde{\mathrm{Mod}}_U^{fd}$ is the strict monoidal category associated to Mod_U^{fd} by Mac Lane's coherence theorem. In passing from Mod_U^{fd} to $\widetilde{\mathrm{Mod}}_U^{fd}$, we essentially get rid of the associativity constraint by fixing a particular paranthesis order in multiple tensor products. By definition the objects in $\widetilde{\mathrm{Mod}}_U^{fd}$ are the finite sequences $S = (M_1, M_2, \ldots, M_k)$ of objects M_j in Mod_U^{fd} ($k \in \mathbb{Z}_{\geq 0}$). For such a sequence S we define

$$F(S) = ((\cdots(M_1 \otimes M_2) \otimes \cdots) \otimes M_{k-1}) \otimes M_k \in \mathrm{Ob}(\mathrm{Mod}_U^{fd}),$$

which should be read as the unit object $L(0) = \mathbb{K}$ if $k = 0$. The morphisms $S \to T$ for two sequences S and T in $\widetilde{\mathrm{Mod}}_U^{fd}$ are by defininition the morphisms $F(S) \to F(T)$ in Mod_U^{fd}. Clearly $\widetilde{\mathrm{Mod}}_U^{fd}$ becomes a category in this way, and the map F, defined as the identity on morphisms, defines an equivalence $F : \mathrm{Mod}_U^{fd} \to \widetilde{\mathrm{Mod}}_U^{fd}$ of categories. The next step is to turn $\widetilde{\mathrm{Mod}}_U^{fd}$ in a strict tensor category. On objects, the tensor product is concatenation with $\emptyset$ serving as unit object,

$$S * S' = (M_1, \ldots, M_k, M'_1, \ldots, M'_l)$$

for $S = (M_1, \ldots, M_k)$ and $S' = (M'_1, \ldots, M'_l)$. On morphisms, the tensor product is more elaborate to construct: one uses the left/right unit constraint and the associativity constraint to define natural isomorphisms $\varphi(S, S') : F(S) \otimes F(S') \to F(S * S')$ (essentially setting the parantheses in the right order), which then allows one to define

$$f * f' = \varphi(T, T') \circ (f \otimes f') \circ \varphi(S, S')^{-1}$$

for morphisms $f : S \to T$ and $f' : S' \to T'$. Doing it carefully, we obtain a strict monoidal category $\widetilde{\mathrm{Mod}}_U^{fd}$, and F can be extended to an equivalence of tensor categories (see [3, Section XI.5] for details). We abuse notations and write $M_1 \otimes \cdots \otimes M_k$ for $S = (M_1, \ldots, M_k) \in \mathrm{Ob}(\widetilde{\mathrm{Mod}}_U^{fd})$ as well as for $F(S) = ((\cdots (M_1 \otimes M_2) \otimes \cdots \otimes M_{k-1}) \otimes M_k \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$.

Recall the two-dimensional vector representation V of U with basis v_+ and v_- and action defined by

$$Kv_\pm = q^{\pm 1} v_\pm, \qquad Ev_+ = 0 = Fv_-, \qquad Ev_- = v_+, \qquad Fv_+ = v_-.$$

We have $V \simeq L(1)$ as U-modules, with the isomorphism given by the identification $v_+ \leftrightarrow \overline{m}_0(q)$ and $v_- \leftrightarrow \overline{m}_1(q)$ of the associated distinguished bases. The 4-dimensional type 1 U-module $V \otimes V$ thus is isomorphic to a direct sum of simple modules. To find out the simple components we look for highest weight vectors in $V \otimes V$. Observe that

$$\begin{aligned}
(V \otimes V)_{q^2} &= \mathrm{span}_{\mathbb{K}}\{v_+ \otimes v_+\}, \\
(V \otimes V)_1 &= \mathrm{span}_{\mathbb{K}}\{v_+ \otimes v_+, v_- \otimes v_+\}, \\
(V \otimes V)_{q^{-2}} &= \mathrm{span}_{\mathbb{K}}\{v_- \otimes v_-\}.
\end{aligned}$$

It is easy to check that $w_2 = v_+ \otimes v_+$ (respectively $w_0 = v_+ \otimes v_- - qv_- \otimes v_+$) is a highest weight vector in $V \otimes V$ of highest weight q^2 (respectively 1). Thus $V_2 = Uw_2$ (respectively $V_0 = Uv_0$) is a U-submodule of $V \otimes V$, isomorphic to $L(2)$ (respectively $L(0)$). Thus $V_0 \cap V_2 = \{0\}$ and $V_0 + V_2 = V \otimes V$ (by a dimension count), hence $V \otimes V \simeq L(0) \oplus L(2)$ as U-modules.

A basis of V_2 is given by $\{w_2, Fw_2, F^2 w_2\}$. Rescaling, we obtain the concrete $\mathbb{K}$-basis

$$\{u_0, u_1, u_2\} = \{v_+ \otimes v_+, v_+ \otimes v_- + q^{-1} v_- \otimes v_+, v_- \otimes v_-\}.$$

We now fix two morphisms

$$\phi \in \mathrm{Hom}_U(V \otimes V, \mathbb{K}), \qquad \psi \in \mathrm{Hom}_U(\mathbb{K}, V \otimes V)$$

such that $\phi \circ \psi = -(q+q^{-1})\mathrm{Id}_{\mathbb{K}}$. The two morphisms are determined up to a nonzero constant: other possible choices are $(\phi', \psi') = (c\phi, c^{-1}\psi)$ for $c \in \mathbb{K}^\times$. An explicit choice for the morphisms are

$$\psi(1) = -(q+q^{-1})(v_+ \otimes v_- - qv_- \otimes v_+),$$

and

$$\phi|_{V_2} \equiv 0, \qquad \phi(v_+ \otimes v_- - qv_- \otimes v_+) = 1.$$

Corollary 3.3. *We have a $\mathbb{K}$-linear strict tensor functor $F_V^{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \to \widetilde{\mathrm{Mod}}_U^{fd}$ defined by $F_V^{sk}(1) = V$ and*

$$F_V^{sk}(\cup) = \psi, \qquad F_V^{sk}(\cap) = \phi.$$

Proof. By Proposition 3.1 it suffices to show that

$$(\mathrm{Id}_V \otimes \phi) \circ (\psi \otimes \mathrm{Id}_V) = \mathrm{Id}_V = (\phi \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \psi), \tag{3.1}$$

which can be verified by a direct computation. □

Exercise 3.4. **(i)** *Prove* (3.1).
(ii) *Define $t = F_V^{sk}(\cup \circ \cap) = \psi \circ \phi \in \mathrm{End}_U(V^{\otimes 2})$. Show that*

$$\begin{aligned} t(v_+ \otimes v_+) &= 0 = t(v_- \otimes v_-), \\ t(v_+ \otimes v_-) &= -q^{-1} v_+ \otimes v_- + v_- \otimes v_+, \\ t(v_- \otimes v_+) &= v_+ \otimes v_- - qv_- \otimes v_+. \end{aligned}$$

Denote P_V for the associated strict tensor functor

$$P_V := F_V^{sk} \circ G : \mathcal{T} \to \widetilde{\mathrm{Mod}}_U^{fd}.$$

Then the endomorphism $P_V(X^+) \in \mathrm{End}_U(V^{\otimes 2})$ is a solution of the Yang-Baxter equation.

Proposition 3.5. *We have $P_V(X^+) = c_{V,V}$ with respect to the braiding c of Mod_U^{fd}.*

Proof. With the notations from Exercise 3.4 we have

$$P_V(X^+) = q^{-\frac{1}{2}} \mathrm{id}_{V \otimes V} + q^{\frac{1}{2}} t$$

by the skein relation applied to $G(X^+) \in E_{1,1}(q^{-\frac{1}{2}})$. The endomorphism t is explicitly computed in Exercise 3.4. Comparing the resulting formulas for $P_V(X^+)$ with the explicit formula for $c_{V,V}$ constructed before, we see that $P_V(X^+) = c_{V,V}$. □

At a later stage we shall construct for any $M \in \mathrm{Ob}(\mathrm{Mod}_U^{fd})$ a strict tensor functor $\mathcal{T}_{or} \to \widetilde{\mathrm{Mod}}_U^{fd}$, with $\mathcal{T}_{or}$ the category of oriented ribbon tangles, such that X^+ (with the proper orientation) is mapped to $c_{M,M}$. It is called the Reshetikhin-Turaev functor. Only for $M = V$ the restriction of the Reshetikhin-Turaev functor to a suitable full sub-category of $\mathcal{T}_{or}$ factors through the skein category $\mathcal{S}(q^{-\frac{1}{2}})$.

Remark 3.6. Let $\mathcal{C}_V$ be the full sub-category of $\widetilde{\mathrm{Mod}}_U^{fd}$ with objects $V^{\otimes n}$ $(n \geq 0)$, where $V^{\otimes 0}$ is the unit object $\mathbb{K}$. Then $\mathcal{C}_V$ is canonically a $\mathbb{K}$-linear strict tensor category and the functor F_V^{sk} is an equivalence $F_V^{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \xrightarrow{\sim} \mathcal{C}_V$ of $\mathbb{K}$-linear strict tensor categories.

4. Appendix A: generators and relations for the skein category

A strict tensor category $\mathcal{M}$ is called $\mathbb{K}$-linear if for all objects $U, V \in \mathrm{Ob}(\mathcal{M})$ the set $\mathrm{Hom}_{\mathcal{M}}(U,V)$ of morphisms $U \to V$ is a $\mathbb{K}$-vector space and if the composition and tensor product maps

$$\begin{aligned} \circ &: \mathrm{Hom}_{\mathcal{M}}(V,W) \times \mathrm{Hom}_{\mathcal{M}}(U,V) \to \mathrm{Hom}_{\mathcal{M}}(U,W), \\ \otimes &: \mathrm{Hom}_{\mathcal{M}}(U,V) \times \mathrm{Hom}_{\mathcal{M}}(U',V') \to \mathrm{Hom}_{\mathcal{M}}(U \otimes U', V \otimes V') \end{aligned}$$

are $\mathbb{K}$-bilinear for all objects U, U', V, V', W in $\mathcal{M}$. The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is an example of a $\mathbb{K}$-linear strict tensor category.

Suppose we have a $\mathbb{K}$-linear strict monoidal category $\mathcal{M}$ with the objects and morphisms forming sets (the latter requirement is usually refered to by saying that $\mathcal{M}$ is small). Let G be a set of morphisms in $\mathcal{M}$ such that the associated set $\mathcal{A}_G$ of elementary morphisms with respect to G. We write $\mathcal{A}_G^{\mathbb{K}}(U,V)$ for the $\mathbb{K}$-linear span of $\mathcal{A}_G(U,V)$ in $\mathrm{Hom}_{\mathcal{M}}(U,V)$. We say that G generates $\mathcal{M}$ if any morphism in $\mathcal{M}$ can be written as $\mathbb{K}$-linear combination of admissible compositions of elementary morphisms.

For two objects U and V in $\mathcal{M}$ we set $W_G^{\mathbb{K}}(U,V)$ for the $\mathbb{K}$-vector space with basis the set $w = w_1 * w_2 * \cdots * w_m$ of admissible words in the alphabet $\mathcal{A}_G$ such that the source of w_m equals U and the target of w_1 equals V. We now extend this by defining $\mathbb{K}$-bilinear maps

$$* : W_G^{\mathbb{K}}(V,W) \times W_G^{\mathbb{K}}(U,V) \to W_G^{\mathbb{K}}(U,W)$$

by requiring it to be concatenation (with respect to $*$) on admissable words in the alphabet $\mathcal{A}_G$. The $*$ is associative in the usual sense.

Let $W_G^{\mathbb{K}}$ be the set of elements $w \in W_G^{\mathbb{K}}(U,V)$ $(U, V \in \mathrm{Ob}(\mathcal{M}))$. As before we have a canonical map $w \mapsto [w]$ from $W_G^{\mathbb{K}}$ to the set of morphisms of $\mathcal{M}$, which restricts to a $\mathbb{K}$-linear map $W_G^{\mathbb{K}}(U,V) \to \mathrm{Hom}_{\mathcal{M}}(U,V)$ for all objects U and V in $\mathcal{M}$ and which maps an admissible word $w = w_1 * w_2 * \cdots * w_m$ with the w_i in the alphabet $\mathcal{A}_G$ to $[w] = w_1 \circ w_2 \circ \cdots \circ w_m$. Observe that $W_G^{\mathbb{K}}(U,V) \to \mathrm{Hom}_{\mathcal{M}}(U,V)$ is surjective for all objects U and V if G generates $\mathcal{M}$. The equivalence relation $\sim_{\mathcal{M}}$ on $W_G^{\mathbb{K}}$ is now defined as $r \sim_{\mathcal{M}} r'$ iff $[r] = [r']$.

For U an object of $\mathcal{M}$ there are well-defined maps $\mathrm{id}_U \otimes \cdot : W_G^{\mathbb{K}} \to W_G^{\mathbb{K}}$ and $\cdot \otimes \mathrm{id}_U : W_G^{\mathbb{K}} \to W_G^{\mathbb{K}}$, defined as follows: restricted to $W_G^{\mathbb{K}}(V,W)$, they are the $\mathbb{K}$-linear maps

$$\begin{aligned} \mathrm{id}_U \otimes \cdot \, &: W_G^{\mathbb{K}}(V,W) \to W_G^{\mathbb{K}}(U \otimes V, U \otimes W), \\ \cdot \otimes \mathrm{id}_U \, &: W_G^{\mathbb{K}}(V,W) \to W_G^{\mathbb{K}}(V \otimes U, W \otimes U) \end{aligned}$$

determined by

$$\begin{aligned}\mathrm{id}_U \otimes (w_1 * \cdots * w_m) &= (\mathrm{id}_U \otimes w_1) * \cdots * (\mathrm{id}_U \otimes w_m),\\ (w_1 * \cdots * w_m) \otimes \mathrm{id}_U &= (w_1 \otimes \mathrm{id}_U) * \cdots * (w_m \otimes \mathrm{id}_U)\end{aligned} \tag{4.1}$$

where $w = w_1 * \cdots * w_m$ is an admissible word in $W_G^{\mathbb{K}}(V, W)$ in the alphabet $\mathcal{A}_G$ (note that the right hand sides in (4.1) are again admissible words in the alphabet $\mathcal{A}_G$). The above two maps on $W_G^{\mathbb{K}}$ commute.

With these maps at hand, the procedure to incorporate relations on $W_G^{\mathbb{K}}$ is as follows. We start with a set $R \subset W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ of elements (r, r') having the same source and target: $r, r' \in W_G^{\mathbb{K}}(U, V)$ for some objects U and V. We enlarge R to a subset R_1 of $W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ and from R_1 to $\widetilde{R}$ in the same way as in the syllabus of three weeks ago. Then we say that $w \sim_R w'$ for $w, w' \in W_G^{\mathbb{K}}$ iff $w, w' \in W_G^{\mathbb{K}}(U, V)$ for some objects U and V and if w' can be obtained from w by a finite sequence of steps of the following form:

(1) Adding an element of the form $\lambda(w_1 * \mathrm{id}_Y * w_2 - w_1 * w_2)$ with $\lambda \in \mathbb{K}$, $w_1 \in W_G^{\mathbb{K}}(Y, V)$ and $w_2 \in W_G^{\mathbb{K}}(U, Y)$ for some object Y.
(2) Adding an element of the form $\lambda(w_1 * r' * w_2 - w_1 * r * w_2)$ for some $\lambda \in \mathbb{K}$, $w_1 \in W_G^{\mathbb{K}}(Z, V)$, $w_2 \in W_G^{\mathbb{K}}(U, Y)$ and $(r, r') \in \widetilde{R}$ with r, r' having source Y and target Z.

Note: if $[r] = [r']$ for $(r, r') \in R$, then $\sim_R$ is stronger than $\sim_{\mathcal{M}}$.

Definition 4.1. *Let G be a set of morphisms of a $\mathbb{K}$-linear strict tensor category $\mathcal{M}$. Let $R \subset W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ be a set of pairs (r, r') with r, r' having the same source and target. We say that (G, R) is a presentation of $\mathcal{M}$ if G is a set of generators of $\mathcal{M}$ and if $\sim_R$ agrees with $\sim_{\mathcal{M}}$ on $W_G^{\mathbb{K}}$.*

Analogous to Theorem 4.22 in the syllabus of three weeks ago one can now prove

Theorem 4.2. *The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is generated as $\mathbb{K}$-linear strict tensor category by $\{\cup, \cap\}$ with the set of defining relations being given by*

$$R = \{((\mathrm{id}_1 \otimes \cap) * (\cup \otimes \mathrm{id}_1), \mathrm{id}_1), ((\cap \otimes \mathrm{id}_1) * (\mathrm{id}_1 \otimes \cup), \mathrm{id}_1), (\cap * \cup, -(q + q^{-1})\mathrm{id}_0)\}.$$

Proof. We will use the following fact about the $E_{kl}(q^{-\frac{1}{2}})$. Let D_{kl} be the $\mathbb{K}$-vector space with $\mathbb{K}$-basis (representatives of) equivalence classes of planar isotopy classes of smooth (no crossings and no loops) (k, l)-tangle diagrams. It is well known (see the proof of Thm. 3.17 in the syllabus of three weeks ago) that D_{kl} is zero-dimensional if $k + l$ is odd, and its dimension is the nth Catalan number if $k + l = 2n$. We define the (Kauffman) diagram category $\mathcal{D}(q^{-\frac{1}{2}})$ as follows. The objects are $\mathbb{Z}_{\geq 0}$, and $\mathrm{Hom}_{\mathcal{D}(q^{-\frac{1}{2}})}(k, l) = D_{kl}$. The composition is putting smooth tangle diagrams on top of each other, and removing the loops on the cost of a factor $-(q + q^{-1})$ for each removed loop.

It is clear that $\mathcal{D}(q^{-\frac{1}{2}})$ is a $\mathbb{K}$-linear strict tensor category again: unit object is 0 and tensor product is putting tangle diagrams next to each other. Furthermore, we have a $\mathbb{K}$-linear strict tensor functor $F : \mathcal{D}(q^{-\frac{1}{2}}) \to \mathcal{S}(q^{-\frac{1}{2}})$ which is the identity on objects and which maps a smooth (k,l)-tangle diagram to its corresponding class in $E_{kl}(q^{-\frac{1}{2}}) = \mathrm{Hom}_{\mathcal{S}(q^{-\frac{1}{2}})}(k,l)$. It is clear that $F : D_{kl} \to E_{kl}(q^{-\frac{1}{2}})$ is surjective for all $k,l \in \mathbb{Z}_{\geq 0}$. The injectivity of F amounts to proving that the representatives of the planar isotopy classes of the smooth (k,l)-tangle diagrams, viewed as elements in $E_{kl}(q^{-\frac{1}{2}})$, are $\mathbb{K}$-linearly independent. This requires an argument to which we come back in the syllabus of next week (compare also with Theorem 3.17 in the syllabus of three weeks ago). Thus F is an equivalence of $\mathbb{K}$-linear strict tensor categories.

Thus the theorem can be alternatively formulated that (G,R) is a presentation of the diagram category $\mathcal{D}(q^{-\frac{1}{2}})$ as a $\mathbb{K}$-linear strict tensor category. Taking a smooth (k,l)-tangle diagram, it is planar isotopic to a generic smooth (k,l)-tangle diagram, which can be written as $[w]$ for an admissable word w in the elementary morphisms $\mathcal{A}_G$. Thus G generates $\mathcal{D}(q^{-\frac{1}{2}})$.

First note that $\sim_R$ is stronger than $\sim_{\mathcal{D}}$ for $\mathcal{D} = \mathcal{D}(q^{-\frac{1}{2}})$. We show that $w \sim_{\mathcal{D}} w'$ (so $[w]=[w']$) implies $w \sim_R w'$ for $w,w' \in W_G^{\mathbb{K}}$. So suppose that $w,w' \in W_G^{\mathbb{K}}(k,l)$ satisfy $w \sim_{\mathcal{D}} w'$, so $[w]=[w']$ in D_{kl}. Write $\{[u_i]\}_{i\in I}$ for a set of representatives of planar isotopy classes of smooth (k,l)-tangle diagrams (which is a basis of D_{kl}), then we can uniquely write $[w] = \sum_i \lambda_i [u_i] = [w']$ for unique $\lambda_i \in \mathbb{K}$. On the other hand, let $\{v_j\}_{j\in J}$ be the canonical $\mathbb{K}$-basis of $W_G^{\mathbb{K}}(k,l)$ consisting of admissable words in the elementary morphisms $\mathcal{A}_G$. There exists a unique surjective map $\sigma : J \to I$ and unique nonnegative integers $m_j \in \mathbb{Z}_{\geq 0}$ such that

$$[v_j] = d^{m_j}[u_{\sigma(j)}], \qquad j \in J,$$

where $d = -q - q^{-1}$. We now write in $W_G^{\mathbb{K}}(k,l)$,

$$w = \sum_j \mu_j v_j, \qquad w' = \sum_j \mu_j' v_j \tag{4.2}$$

for unique $\mu_j, \mu_j' \in \mathbb{K}$. The assumption is that

$$[w] = \sum_j \mu_j [v_j] = \sum_j \mu_j d^{m_j}[u_{\sigma(j)}]$$

in $\mathrm{Hom}_{\mathcal{D}}(k,l)$ equals

$$[w'] = \sum_j \mu_j' [v_j] = \sum_j \mu_j' d^{m_j}[u_{\sigma(j)}],$$

hence, for all $i \in I$, we have

$$\sum_{j\in\sigma^{-1}(i)} \mu_j d^{m_j} = \sum_{j\in\sigma^{-1}(i)} \mu_j' d^{m_j}. \tag{4.3}$$

So for $w,w' \in W_G^{\mathbb{K}}(k,l)$ given by (4.2), such that the coefficients satisfy (4.3), we need to show that $w \sim_R w'$. For this it suffices to show that

$$\sum_{j\in\sigma^{-1}(i)} \mu_j v_j =: w_i \sim_R w_i' := \sum_{j\in\sigma^{-1}(i)} \mu_j' v_j$$

for all $i \in I$. For this it suffices to show that for $j, j' \in \sigma^{-1}(i)$ we have $d^{-m_j} v_j \sim_R d^{-m_{j'}} v_{j'}$. This is easy and goes by the same (but simpler) argument as for ribbon tangles, as in the proof of Theorem 4.22 in the syllabus of three weeks ago.

$\square$

We call a strict tensor functor $F : \mathcal{M} \to \mathcal{N}$ between $\mathbb{K}$-linear strict tensor categories $\mathcal{M}$ and $\mathcal{N}$ $\mathbb{K}$-linear if $F : \mathrm{Hom}_{\mathcal{M}}(U, V) \to \mathrm{Hom}_{\mathcal{N}}(F(U), F(V))$ is $\mathbb{K}$-linear for $U, V \in \mathrm{Ob}(\mathcal{M})$. We have now the direct analog of Theorem 4.24 in the syllabus of three weeks ago, with the extra assumption that the categories are $\mathbb{K}$-linear and that the strict tensor functor F is $\mathbb{K}$-linear. In other words, we conclude that Proposition 3.1 holds true.

5. Appendix B: tensor product decompositions

For $r, s \in \mathbb{Z}_{\geq 0}$ the tensor product module $L(r) \otimes L(s)$ is finite dimensional and of type 1, hence it decomposes as a direct sum of the simple U-modules $L(n)$ $(n \geq 0)$. The following theorem (quantum Clebsch-Gordon decomposition) gives a more precise statement.

Theorem 5.1. *For $r, s \in \mathbb{Z}_{\geq 0}$ we have*

$$L(r) \otimes L(s) \simeq L(|r-s|) \oplus L(|r-s|+2) \oplus \cdots \oplus L(r+s-2) \oplus L(r+s)$$

as U-modules.

Proof. It suffices to prove the theorem for $r \geq s$ (indeed, $L(r) \otimes L(s) \simeq L(s) \otimes L(r)$ since Mod_U^{fd} is a braided category). In this case we are asked to prove that

$$L(r) \otimes L(s) \simeq \bigoplus_{l=0}^{s} L(r-s+2l).$$

The dimension over $\mathbb{K}$ of the left hand side is $(r+1)(s+1)$, which agrees with the dimension of the right hand side since

$$\sum_{l=0}^{s} (r-s+2l+1) = (r+1)(s+1).$$

To prove the theorem it thus suffices to construct highest weight vectors $w_l \in L(r) \otimes L(s)$ of highest weight $r-s+2l+1$ for $l = 0, \ldots, s$. First note that

$$\big(L(r) \otimes L(s)\big)_{q^{r-s+2l}} = \mathrm{span}_{\mathbb{K}}\{\overline{m}_n(q^r) \otimes \overline{m}_{s-l-n}(q^s)\}_{n=-l}^{s-l}$$

for $l = 0, \ldots, s$. Now a vector

$$w_l = \sum_{n=-l}^{s-l} c_n \big(\overline{m}_n(q^r) \otimes \overline{m}_{s-l-n}(q^s)\big) \in \big(L(r) \otimes L(s)\big)_{q^{r-s+2l}}, \qquad c_n \in \mathbb{K}$$

satisfies $Ew_l = 0$ if and only if

$$c_{n+1} = -q^{n-r-s} \frac{\big(s-l-n\big)_{q^2} \big(l+n+1\big)_{q^2}}{\big(n+1\big)_{q^2} \big(r-n\big)_{q^2}} c_n \qquad (n = -l, \ldots, s-l-1)$$

(note that the denominators are nonzero). There is a nontrivial solution to these recurrence relations, hence there exists a highest weight vector of highest weight q^{r-s+2l} in $L(r) \otimes L(s)$ for all $l \in \{0, \ldots, s\}$. $\square$

The following result should be compared to Theorem 3.17 in the syllabus of three weeks ago.

Corollary 5.2. *For $k, l \in \mathbb{Z}_{\geq 0}$ we have*

$$\mathrm{Dim}_{\mathbb{K}}\big(\mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})\big) = \begin{cases} 0, & \text{if } k+l \text{ odd}, \\ \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}, & \text{if } k+l = 2n \text{ even}. \end{cases}$$

Proof. For $n \in \mathbb{Z}_{\geq 0}$ we have

$$V^{\otimes n} \simeq \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L(n)^{\otimes c_j(n)} \tag{5.1}$$

for unique $c_j(n) \in \mathbb{Z}_{\geq 0}$. For $n = 0$ we read the left hand side as $L(0) = \mathbb{K}$ (the unit object in Mod_U^{fd}), so that $c_0(0) = 1$ and $c_j(0) = 0$ for $j \geq 1$. By the above theorem we have the recurrence relations

$$c_0(n) = c_1(n-1), \qquad c_j(n) = c_{j+1}(n-1) + c_{j-1}(n-1) \quad (j \geq 1) \tag{5.2}$$

for $n \geq 1$. These recurrence relations, together with the initial conditions $c_0(0) = 1$ and $c_j(0) = 0$ ($j \geq 1$), determine the coefficients $c_j(n)$ uniquely. Now define the polynomial $V_j(y)$ of degree $j \geq 0$ by

$$V_j(x + x^{-1}) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}} = \sum_{l=0}^{j} x^{j-2l}.$$

It is the character of the simple $\mathfrak{sl}_2$-module of dimensional $j+1$ or, from the viewpoint of orthogonal polynomials, a Chebyshev polynomial of the second kind (up to a rescaling of the variable). The $V_j(y)$'s satisfy $V_0(y) = 1$ and the recurrence relation

$$yV_j(y) = V_{j+1}(y) + V_{j-1}(y), \qquad j \geq 1.$$

By induction to n we conclude from the recurrence relations for the $c_j(n)$ and the $V_j(y)$ that

$$y^n = \sum_{j=0}^{\infty} c_j(n) V_j(y)$$

(this is in fact the equivalent reformulation of (5.1) for the characters of the associated $\mathfrak{sl}_2$-modules). Since

$$V_j(2\cos(\theta)) = \frac{\sin((j+1)\theta)}{\sin(\theta)}$$

we have the orthogonality relations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} V_i(2\cos(\theta))V_j(2\cos(\theta))\sin^2(\theta)d\theta = \delta_{i,j}.$$

We conclude that

$$\begin{aligned}
\mathrm{Dim}_{\mathbb{K}}\big(\mathrm{Hom}_U(V^{\otimes k}, V^{\otimes l})\big) &= \sum_{j=0}^{\infty} c_j(k)c_j(l) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(2\cos(\theta))^{k+l}\sin^2(\theta)d\theta \\
&= -\frac{1}{4\pi}\sum_{r=0}^{k+l}\binom{k+l}{r}\int_{-\pi}^{\pi} e^{i(2r-k-l)\theta}(e^{2i\theta}+e^{-2i\theta}-2)d\theta.
\end{aligned}$$

The latter expression is zero if $k+l$ is odd. If $k+l=2n$ is even, then it equals

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

the nth Catalan number. □

Combining the results of section 3 in the syllabus of four weeks ago with Theorem 3.17 in the syllabus of three weeks ago, we conclude

Corollary 5.3. *The Temperley-Lieb algebra* $\mathrm{TL}_n(q^{\frac{1}{2}})$ *is isomorphic to* $\mathrm{End}_U(L(1)^{\otimes n})$ *as algebra over* $\mathbb{K}$. *The explicit isomorphism has been constructed in Exercise 3.14 of the syllabus of week 14.*

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