Take home exam **Semisimple Lie algebras**

- Date: June 3, 2013.
- Return date: June 17, 2013. Either electronically (single document, max. 5mb), or mailbox Opdam or Stokman, before 10am.
- Please do not forget to write your student number and email address on your work.
- Formulate carefully all the results you use.
- This is an individual take home exam, collaboration is not permitted.
- Always motivate your answer. Good luck!
- Weights: 1: 24 (a1;b2;c3;d3;e3;f3;g2;h2;i1;j2;k2), 2: 26 (a2;b1;c2;d3;e3;f3;g2;h2;i3;j2;k3), 3:19 (a3;b2;c2;d4;e3;f2;g3) and 4:21 (a1;b3;c2;d4;e3;f2;g2;h4).

Exercise 1. Let F be a field of characteristic unequal to two, in which the equations $x^2+1=0$ and $x^2+2=0$ have roots. Let $n \in \mathbb{N}$ with $n \geq 2$. Let $\mathfrak{gl}(n,F)$ denote the Lie algebra of all n by n matrices with coefficients in F. For $X \in \mathfrak{gl}(n,F)$ we denote by ${}^TX$ its transpose.

(a) Let $g \in \mathrm{GL}(n,F)$ be an invertible n by n matrix. Show that the map $c_g : \mathfrak{gl}(n,F) \to \mathfrak{gl}(n,F)$ given by $c_g(Y) := gYg^{-1}$ is an automorphism.

(b) Let $\mathfrak{g} := \{X \in \mathfrak{gl}(n,F) \mid {}^TX = -X\} \subset \mathfrak{gl}(n,F)$ be the subspace of skew-symmetric matrices. Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,F)$.

(c) Prove that the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n,F)$ defined in (b) is isomorphic to $\mathfrak{o}(n,F)$. (Hint: Distinguish the cases n being even and odd. In both cases, find an element $g \in \mathrm{GL}(n,F)$ such that $s = s^{-1} = g \cdot {}^Tg$, where s is the matrix used by Humphreys, Section 1.2 to define $\mathfrak{o}(n,F)$. Show that $c_g(\mathfrak{g}) = \mathfrak{o}(n,F)$. It may be helpful to solve the cases $n=2$ and $n=3$ first.)

Assume from now on that F is algebraically closed and of characteristic zero, and that $n \geq 3$.

(d) Suppose that $\mathfrak{l} \subset \mathfrak{gl}(n,F)$ is a Lie subalgebra and $\mathfrak{s} \subset \mathfrak{l}$ an ideal. Suppose that $\lambda \in \mathfrak{s}^*$ is a linear functional, and define $V_\lambda := \{v \in F^n \mid Xv = \lambda(X)v \, \forall X \in \mathfrak{s}\}$. Show that $V_\lambda \subset F^n$ is a subrepresentation of F^n for $\mathfrak{l}$.

(e) In the situation of (d), suppose that $\mathfrak{l}$ acts irreducibly on F^n. Show that $Z(\mathfrak{l})$ consists of the subalgebra $\mathfrak{l} \cap F.1_n \subset \mathfrak{l}$ (where $1_n \in \mathrm{GL}(n,F)$ denotes the identity matrix), and that $\mathfrak{l}$ is reductive (i.e. $\mathrm{Rad}(\mathfrak{l}) = Z(\mathfrak{l})$).

(f) Show that the orthogonal subalgebra $\mathfrak{o}(n,F) \subset \mathfrak{gl}(n,F)$ is semisimple. (Hint: Observe that $\mathfrak{l} \subset \mathfrak{gl}(n,F)$ acts irreducibly on F^n if and only if the *associative* subalgebra $L \subset \mathrm{End}(F^n)$ generated by the identity matix 1_n and $\mathfrak{l}$ satisfies $L = \mathrm{End}(F^n)$. Use this to show that $\mathfrak{o}(n,F)$ acts irreducibly on F^n.)

Assume in the rest of this exercise that $n = 2l \geq 6$.

(g) Show that the intersection $\mathfrak{h}$ of the diagonal subalgebra $\delta \subset \mathfrak{gl}(n,F)$ with $\mathfrak{o}(n,F)$ consists of the matrices of the form $\mathrm{Diag}(x_1,x_2,\ldots,x_l,-x_1,-x_2,\ldots,-x_l)$ (with $(x_1,x_2,\ldots,x_l) \in F^l$ arbitrary). Show that $\mathfrak{h}$ a Cartan subalgebra.

(h) Describe the root space decomposition and the root system Φ of $\mathfrak{o}(n, F)$ with respect to its Cartan subalgebra $\mathfrak{h}$ as in (g) explicitly. Conclude that $\mathfrak{o}(n, F)$ (and hence $\mathfrak{g}$) is a simple Lie algebra of rank l.
(i) Describe the weight lattice of Φ explicitly.
(j) Give the weight space decomposition of $V = F^n$ with respect to $\mathfrak{h} \subset \mathfrak{o}(n, F)$.
(k) Show that the set of weights of $V = F^n$ is an orbit under the Weyl group $W(\Phi)$ of the root system Φ of $\mathfrak{o}(n, F)$ with respect to $\mathfrak{h}$.

Exercise 2. In this exercise F denotes an algebraically closed field of characteristic zero. Consider the classical Lie subalgebra $L := \mathfrak{o}(8, F) \subset \mathfrak{gl}(8, F)$ given by

$$L = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\, b,\, c,\, d \in \mathfrak{gl}(4, F) \ \ \& \ \ {}^T a = -d,\ {}^T b = -b,\ {}^T c = -c \}$$

with its Cartan subalgebra $H := \delta \cap \mathfrak{o}(8, F)$ as in exercise 1. Recall that the root system Φ of L with respect to H has type D_4.

(a) Show that the subset $B \subset L$ defined by requiring that a is upper triangular and $c = 0$, is a Borel (i.e. maximal solvable) subalgebra containing H.
(b) Describe Φ explicitly as a set of linear functionals on H, and describe the set $\Phi^+ := \{\alpha \in \Phi \mid L_\alpha \subset B\}$ of positive roots corresponding to B, and its basis $\Delta \subset \Phi^+$, explicitly.
(c) Recall that the defining representation $V = F^8$ of L is irreducible. Show that its highest weight λ is a fundamental weight. Draw the Dynkin diagram $D_L = D(\Phi, \Delta)$ of Φ with respect to Δ, and mark the vertex which corresponds to λ.

Let $E_{\mathbb{Q}} \subset H^*$ be the $\mathbb{Q}$-linear subspace spanned by Φ, and let $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$. We will identify the group $\Gamma(D_L)$ of automorphisms of the Dynkin diagram canonically with the group $\Gamma(\Phi, \Delta) \subset \mathrm{GL}(E)$ of automorphisms of Φ preserving Δ (cf. Section 12.2 of Humphreys' book). The latter group will also be identified with the subgroup of $\mathrm{GL}(H^*)$ preserving Φ and Δ as in Section 14.2 of Humphreys' book.

(d) Show that there exists a subgroup $\Gamma(L)$ of the group $\mathrm{Aut}(L, B, H)$ of automorphisms of L preserving B and H, which is isomorphic to the group $\Gamma(D_L)$ via the homomorphism $\gamma \to {}^t(\gamma|_H)^{-1}$ (where we have denoted the transpose of a linear endomorphism ϕ by ${}^t\phi$).

We fix a subgroup $\Gamma(L) \subset \mathrm{Aut}(L, B, H)$ as in (d) for the remaining part of this exercise, and we identify $\Gamma(L)$ and $\Gamma(D_L)$ via the isomorphism described in (d).

(e) Let $\gamma \in \Gamma(L)$, and let V_γ denote the representation of L obtained from the defining representation V by precomposing with γ^{-1}. In other words, V_γ is the representation of L with underlying vector space V, in which $X \in L$ acts by $\gamma^{-1}(X)$. Prove that V_γ is an irreducible L module with highest weight equal to $\gamma(\lambda)$. Give the explicit description of the sets of weights and the highest weights of all equivalence classes of irreducible representations of the form V_γ, as linear functionals on H.

(f) Let $\mathfrak{g}$ be a semisimple Lie algebra over F with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, and let $\tau \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ be an automorphism of $\mathfrak{g}$ preserving $\mathfrak{b}$ and $\mathfrak{h}$. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(U)$ be a finite dimensional irreducible $\mathfrak{g}$-module over F, with highest weight μ. Suppose that $\tau(\mu) = \mu$. Prove that there exists a $g_\tau \in \mathrm{GL}(U)$ such that for all $X \in \mathfrak{g}$:

$$\pi(\tau^{-1}(X)) = c_{g_\tau}(\pi(X))$$

Show that the canonical image $\overline{g_\tau}$ of g_τ in $\mathrm{PGL}(U)$ is unique.

(g) Let $1 \neq \gamma \in \Gamma(L)$ be such that $\gamma(\lambda) = \lambda$. Find an explicit matrix $g_\gamma \in \mathrm{GL}(V)$ such that $\gamma^{-1}(X) = c_{g_\gamma}(X)$ for all $X \in L$.

(h) Suppose that $\gamma \in \Gamma(L)$ is such that $\gamma(\lambda) \neq \lambda$. Does there exist an element $g \in \mathrm{GL}(V)$ such that $c_g(X) = \gamma(X)$? Explain your answer.

(i) Compute the set of weights of $\Lambda^2(V)$. Prove that $\Lambda^2(V)$ is isomorphic to the adjoint representation of $L = \mathfrak{o}(8, F)$. Show that its highest weight is also a fundamental weight.

(j) Show that the representations $\Lambda^2(V_\gamma)$ are mutually equivalent for al $\gamma \in \Gamma(L)$.

(k) Suppose that $\gamma_1, \gamma_2 \in \Gamma(L)$ are such that $\lambda_1 \neq \lambda_2$, where $\lambda_i := \gamma_i(\lambda)$ for $i \in \{1, 2\}$. Describe the decomposition in irreducibles of $V_{\gamma_1} \otimes V_{\gamma_2}$.

Exercise 3. Let L be a Lie algebra over an algebraically closed field F of characteristic zero. A Lie algebra homomorphism $\chi : L \to F$ is called a linear character of L. Denote $\widehat{L} \subset L^*$ for the subspace of linear characters.

(a) Prove that $\widehat{L} \simeq \big(L/[L, L]\big)^*$ as vector spaces.

(b) Show that $\widehat{L} = \{0\}$ if L is semisimple.

In the remainder of the exercise we assume that L is a semisimple Lie algebra over F. Let $H \subset L$ be a Cartan subalgebra and $\Phi = \Phi(L, H) \subset H^*$ the associated root system. Choose a base Δ of Φ and write Φ^+ for the corresponding set of positive roots. Consider the Lie subalgebra

$$B := H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$$

of L, where $L_\alpha \subset L$ is the root space of L associated to the root $\alpha \in \Phi$.

(c) Show that $\widehat{B} \simeq H^*$ as vector spaces.

For $\lambda \in H^*$ we write $I_B(\lambda)$ for the left ideal of the universal enveloping algebra $\mathcal{U}(B)$ generated by $h - \lambda(h)1$ ($h \in H$) and by L_α ($\alpha \in \Phi^+$), where 1 stands for the unit element of $\mathcal{U}(B)$.

(d) Prove that $\mathrm{Dim}_F\big(\mathcal{U}(B)/I_B(\lambda)\big) = 1$.

Consider now the left ideal $I(\lambda)$ of the universal enveloping algebra $\mathcal{U}(L)$ generated by $h-\lambda(h)1$ ($h \in H$) and by L_α ($\alpha \in \Phi^+$). Let $N_- \subset L$ be the Lie subalgebra generated by the $L_{-\alpha}$ ($\alpha \in \Phi^+$).

(e) Show that there exists a unique linear isomorphism $\mathcal{U}(N_-) \otimes_F I_B(\lambda) \xrightarrow{\sim} I(\lambda)$ mapping $X \otimes_F Y$ to XY for $X \in \mathcal{U}(N_-)$ and $Y \in I_B(\lambda)$ (here we view $\mathcal{U}(N_-)$ and $\mathcal{U}(B)$ as subalgebras of $\mathcal{U}(L)$ in the canonical way).

(f) Prove that the left $\mathcal{U}(L)$-module $Z(\lambda) := \mathcal{U}(L)/I(\lambda)$, considered as L-module, is standard cyclic.

(g) Show that $Z(\lambda) \simeq \mathcal{U}(N_-)$ as vector spaces.

Exercise 4. Let F be an algebraically closed field of characteristic zero and set $A := F[z_1, z_2]$ for the associative F-algebra of polynomials over F in two commuting variables z_1, z_2. If $p \in A$ is given explicitly by $p = \sum_{k,l=0}^{\infty} c_{k,l} z_1^k z_2^l$ ($c_{k,l} \in F$ all but finitely many zero) and $a, b \in A$, then we write $p(a,b) := \sum_{k,l=0}^{\infty} c_{k,l} a^k b^l \in A$. With this notation, we in particular have $p(z_1, z_2) = p$.

Let $\mathcal{D} \subseteq \mathrm{End}_F(A)$ be the unital subalgebra generated by z_i and ∂_i ($i = 1, 2$), where z_i is now interpreted as the linear operator on A defined by $p \mapsto z_i p$ and ∂_i is the linear operator on A satisfying $\partial_1(z_1^k z_2^l) = k z_1^{k-1} z_2^l$ respectively $\partial_2(z_1^k z_2^l) = l z_1^k z_2^{l-1}$ for $k, l \in \mathbb{Z}_{\geq 0}$. We write 1 for the unit element of $\mathcal{D}$, and $[\cdot,\cdot]$ for the Lie bracket of the Lie algebra $L(\mathcal{D})$ associated to $\mathcal{D}$ (the commutator bracket).

(a) Show that $[z_i, z_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, z_j] = \delta_{i,j} 1$, where $\delta_{i,j}$ is the Kronecker delta function ($\delta_{i,j} = 1$ if $i = j$ and $= 0$ otherwise).

Let $\{h, x, y\}$ be the linear basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{sl}(2,F)$. Define a linear map $\rho : \mathfrak{sl}(2,F) \to \mathcal{D}$ by

$$\rho(X) := -(az_1 + bz_2)\partial_1 - (cz_1 + dz_2)\partial_2, \qquad X := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sl}(2,F).$$

(b) Prove that $\rho : \mathfrak{sl}(2,F) \to L(\mathcal{D})$ is a homomorphism of Lie algebras.

We also write ρ for the canonical extension of the Lie algebra homomorphism ρ to a unital algebra homomorphism $\mathcal{U}(\mathfrak{sl}(2,F)) \to \mathcal{D}$. Composing $\rho : \mathcal{U}(\mathfrak{sl}(2,F)) \to \mathcal{D}$ with the canonical embedding $\mathcal{D} \hookrightarrow \mathrm{End}_F(A)$ turns A into an infinite dimensional $\mathcal{U}(\mathfrak{sl}(2,F))$-module. The corresponding representation map $\mathcal{U}(\mathfrak{sl}(2,F)) \to \mathrm{End}_F(A)$ is again denoted by ρ.

(c) Show that A decomposes as a direct sum of finite dimensional $\mathfrak{sl}(2,F)$-modules.

(d) Give the explicit decomposition of A as direct sum of finite dimensional irreducible $\mathfrak{sl}(2,F)$-modules (in other words, if $L(m)$ ($m \in \mathbb{Z}_+$) is the irreducible $\mathfrak{sl}(2,F)$-module of dimension $m+1$, determine the numbers $k_m \in \mathbb{Z}_{\geq 0}$ such that $A \simeq \bigoplus_{m=0}^{\infty} k_m L(m)$, as $\mathfrak{sl}(2, F)$-modules, where $k_m L(m)$ stands for the direct sum of k_m copies of $L(m)$).

Part (c) implies that $\rho(x)$ and $\rho(y)$ are locally nilpotent endomorphisms of A, hence we have well defined F-linear automorphisms $\exp(\rho(x))$ and $\exp(\rho(y))$ of A.

(e) Let $p \in A$. Prove that

$$\exp(\rho(x))p = p(z_1 - z_2, z_2), \qquad \exp(\rho(y))p = p(z_1, z_2 - z_1).$$

Set $\sigma := \exp(\rho(x)) \exp(-\rho(y)) \exp(\rho(x)) \in \mathrm{Aut}_F(A)$.

(f) Prove that $\sigma(p) = p(-z_2, z_1)$ for $p \in A$.

(g) Show that $\sigma \rho(h) \sigma^{-1} = -\rho(h)$.

(h) Let V be a finite dimensional $\mathfrak{sl}(2, F)$-module with representation map

$$\pi : \mathfrak{sl}_2(F) \to \mathrm{End}_F(V).$$

Set $V_n := \{v \in V \mid \pi(h)v = nv\}$. Prove that

$$\exp(\pi(x)) \exp(-\pi(y)) \exp(\pi(x)) V_n = V_{-n}, \qquad \forall\, n \in \mathbb{Z}.$$