QUANTUM GROUPS AND KNOT THEORY LECTURE: WEEK 8

The basic text of this course is [3]. Unexplained notations and the numbering in this syllabus refer by default to [3].

1. The Yang Baxter equation and braid group representations

All the material in this chapter is covered unless stated otherwise.

1. **The Yang Baxter equation.**

Exercise (a). *See* [3, Exercise 1.5(a)].

Exercise (b). *See* [3, Exercise 1.5(b)]. *However, an additional condition is necessary to make sure that the matrix is invertible.*

2. **Artin's braid group.** Most things here were already covered in week 2.

3. **Alternative description of B_n.** We have already seen two definitions of the braid group. The topological braid group (let us denote this by B_n^{top}) was defined as the group consisting of isotopy classes of braids with n strands in $\mathbb{C} \times [0,1]$, with the product defined by composition of braids. On the other hand the algebraic braid group B_n^{alg} was defined in the preceding section by generators and relations. In week 2 it was shown that $B_n^{top} \approx B_n^{alg}$. In the current section there is yet another incarnation of B_n: The fundamental group B_n^{fun} of the configuration space Y_n of n distinct, unordered points in $\mathbb{C}$.

Definition 3.1. *Let $p = (1, \dots, n) \in X_n$ and let $\bar{p} = \pi(p) \in Y_n$. Let $B_n^{fun} = \Pi_1(Y_n, \bar{p})$, and let $P_n^{fun} = \Pi_1(X_n, p)$.*

Proposition 3.2. *The canonical quotient map $\pi : X_n \to Y_n$ is a regular covering map with deck transformation group S_n.*

Proof. This is a standard result in basic algebraic topology: if a finite group G acts freely on a Hausdorff space X which is connected and locally pathwise connected then the quotient map $\pi : X \to G\backslash X$ is a regular covering with deck transformation group G, see [1, Proposition III 7.2, Exercise III 7.1]. □

Corollary 3.3. *We have a fundamental exact sequence*

$$1 \to P_n^{fun} \to B_n^{fun} \to S_n \to 1 \tag{3.1}$$

Proof. This is a standard fact about regular covering maps see e.g. [1, Corollary III 6.9]. □

It is useful to understand the exact sequence (3.1) in more detail. The *monodromy action* $\mu : \Xi \times B_n^{fun} \to \Xi$ of B_n^{fun} on the fiber $\Xi := \pi^{-1}(\bar{p}) = S_n p$ of π above $\bar{p}$ is defined as follows. If $b \in B_n^{fun}$ and $x \in \Xi$ and let l be a loop in Y_n representing b. Then $\mu(x, b)$ is the end point $g(1) \in \Xi$ of the unique lifting $g : [0,1] \to X_n$ of l which starts at x. The monodromy action yields a transitive right action of B_n^{fun} on Ξ. This right action commutes with the (simply transitive) left action of the deck transformation group S_n on Ξ. This defines a surjective homomorphism $\alpha : B_n^{fun} \to S_n$ by requiring that $\alpha(b)p = \mu(p,b)$. By definition of the monodromy action the kernel of α is equal to $\pi_*(P_n^{fun}) \lhd B_n^{fun}$. By the homotopy lifting theorem (see e.g. [1, Theorem III 3.4]) it is clear that $\pi_* : P_n^{fun} \to B_n^{fun}$ is a monomorphism.

Definition 3.4. *The group P_n^{fun} is identified with the normal subgroup $\pi_*(P_n^{fun}) \lhd B_n^{fun}$ and is called the* pure braid group*.*

We now construct a surjective anti-homomorphism $\lambda : B_n^{top} \to B_n^{fun}$. Let $b = [g] \in B_n^{top}$ be represented by a continuous map $g = (g_1, g_2, \ldots, g_n) : [0,1] \to X_n$. Thus the map $g_i : [0,1] \to \mathbb{C}$ represents the i-th strand of the topological braid b. The condition that the strands are disjoint is equivalent to saying that the image of g is contained in X_n. Observe that the path $\pi \circ g$ is a closed path in Y_n which begins and ends at $\bar{p}$, and hence determines an element $\sigma_g \in B_n^{fun}$. It is clear that if $g : [0,1] \to X_n$ is isotopic to h in the sense of braids (i.e. we require at all times of the isotopy that the number of intersection points of the strands with a plane of the form $\mathbb{C} \times \{t\}$ (with $t \in (0,1)$) is equal to n) then the loops $g \circ \pi$ and $h \circ \pi$ are homotopic, hence $\sigma_g = \sigma_h$. (As an aside, it is known (Artin) that the equivalence relation defined on the set of braids by the above notion of braid isotopy is the same as the equivalence relation defined by allowing the more general isotopies of tangles). Therefore we have a well defined map $\lambda : B_n^{top} \to B_n^{fun}$ by putting $\lambda([g]) = \sigma_g$. This is clearly an anti-homomorphism.

Exercise (c). *Check that λ is an anti-homomorphism.*

It is also clear that λ is surjective. Any element of B_n^{fun} is of the form $[l] \in B_n^{fun}$ for a closed loop l in Y_n beginning at $\bar{p}$. Then l has a unique lift $g : [0,1] \to X_n$ starting at $p \in X_n$, which is clearly equivalent to saying that $\lambda([g]) = [l]$.

It was shown by Artin that λ is also injective, in other words: if $g \circ \pi$ and $h \circ \pi$ are *homotopic* loops in Y_n (homotopy with fixed end points) then $[g], [h] \in B_n^{top}$ are *isotopic braids*. We will not show this result here. We summarize the above discussion in the main result of this section.

Theorem 3.5. *(Artin) The map $\lambda : B_n^{top} \to B_n^{fun}$ defined by $\lambda([g]) = \sigma_g := [g \circ \pi]$ is an anti-isomorphism. The pure braid group P_n^{top} is mapped to P_n^{fun} by λ.*

Exercise (d). *(The monodromy action) Recall the homomorphism $f : B_n^{top} \to S_n$ of exercise 2 of week 2. Prove that $\alpha(\lambda(b)) = f(b)^{-1}$, in other words prove that $\mu(p, \lambda(b))_i = g_i(1)$, where $g : [0,1] \to X_n$ is a topological braid representing b.*

Exercise (e). *We use the notation for the generators of the braid group as in* [3, Definition 2.1]*. The following element of the braid group is a central element: $\beta_n = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n \in B_n^{top}$ (it is the same as the central element $(c_n)^2$ introduced in week 2, exercise (b), but represented by a different word. You do not need to prove this fact). Let $\gamma_n \in B_n^{top}$ be the isotopy class of braids represented represented by $g = (g_1, \ldots, g_n) : [0,1] \to \mathbb{C}^n$ given by $g_k(s) = k \exp 2\pi i s$.*

(i) *Show that* $\gamma_n \in B_n^{top}$ *is central.*
(ii) *Draw* γ_4 *and prove that* $\gamma_4 = \beta_4$.
(iii)* *(bonus) Prove that (ii) generalizes to arbitrary* n.

2. Hopf algebras and monoidal categories

A good additional reference for this chapter is [2].

1. **Hopf algebras.** (See also [2, Chapter III, 1-3]). Excluded are Example 1.6 and Exercise 1.7(a).

1.8. *Sweedler's sigma notation.* The Sweedler notation is also denoted e.g. as follows:

$$(\mathrm{id} \otimes \Delta)\Delta(x) = \sum_{(x)} x' \otimes x'' \otimes x''' \tag{1.1}$$

Exercise (f). *Suppose that* A *and* B *are bialgebras.*
(i) *Show that* $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ *determines a unique algebra structure on* $A \otimes B$.
(ii) *Show that the map* $\Delta : A \otimes B \to (A \otimes B) \otimes (A \otimes B)$ *defined by* $\Delta = (\mathrm{id_A} \otimes \tau_{A,B} \otimes \mathrm{id_B}) \circ \Delta_A \otimes \Delta_B$ *together with the map* $\epsilon : A \otimes B \to k$ *defined by* $\epsilon(a \otimes b) := \epsilon_A(a)\epsilon_B(b)$ *defines a bialgebra structure on* $A \otimes B$ *with comultiplication* Δ *and counit* ϵ.

Exercise (g). *Show that* $A \otimes B$ *is a Hopf algebra if* A *and* B *are Hopf algebras.*

Exercise (h). (i) *Given two bialgebras* A, B *define a convolution product* $*$ *on* $\mathrm{Hom}(A, B)$ *generalizing the convolution product on* $\mathrm{End}(A)$ *for a bialgebra* A.
(ii) *Prove that the convolution product defines an algebra structure on* $\mathrm{Hom}(A, B)$ *with unit* $\eta_B \circ \epsilon_A$.

Exercise (i). *Let* $A = (A, \mu, \eta, \Delta, \epsilon)$ *be a bi-algebra. Show that* $A^{op} = (A, \mu^{op}, \eta, \Delta, \epsilon)$ *and* $A^{cop} = (A, \mu, \eta, \Delta^{op}, \epsilon)$ *are also bi-algebras.*

Exercise (j). *Let* H *be an Hopf algebra. A two-sided ideal* $I \subset H$ *is called a Hopf ideal if*

$$\begin{aligned} I &\subset Ker(\epsilon), \\ \Delta(I) &\subset I \otimes H + H \otimes I, \\ S(I) &\subset I. \end{aligned}$$

Show that the quotient algebra H/I *has a unique Hopf algebra structure such that the canonical map* $\pi : H \to H/I$, $\pi(h) = h + I$, *becomes a morhism of Hopf algebras.*

Exercise (k). *Let* H *be a Hopf algebra with antipode* S.
(i) *We equip* $A := \mathrm{Hom}_k(H \otimes H, H)$ *with convolution algebra structure as defined above. Let* $\mu \in \mathrm{Hom}(H \otimes H, H)$ *denote the multiplication and define* $\nu = \mu^{op} \circ (S \otimes S) \in A$ *and* $\rho = S \circ \mu \in A$. *Prove that* $\rho * \mu = \mu * \nu = \eta \circ \epsilon \circ \mu$.
(ii) *Prove that* $S : H \to H^{op}$ *is an algebra homomorphism.*
(iii) *Prove dually that* $S : H \to H^{cop}$ *is a morphism of co-algebras.*
(iv) *Show that* $H^{op,cop}$ *is also a Hopf algebra with antipode* S, *and that* $S : H \to H^{op,cop}$ *is a morphism of Hopf algebras.*

Exercise (l). *Let H be a Hopf algebra with antipode S.*

(i) *Show that the convolution products on* $\mathrm{End}_k(H^{op})$ *and on* $\mathrm{End}_k(H^{cop})$ *are each others opposite.*

(ii) *Show that* $S^2 \in \mathrm{End}_k(H^{op})$ *is is a left and right inverse for S with respect to convolution.*

(iii) *Show that* $S^2 = \mathrm{id}_H$ *iff* H^{op} *(or equivalently* H^{cop}*) is a Hopf algebra with antipode S.*

(iv) *Suppose that S is invertible. Show that H^{op} and H^{cop} are Hopf algebras with antipode S^{-1}.*

References

[1] G.E. Bredon, *Topology and Geometry*, Springer GTM 139 (1993)
[2] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)
[3] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)