QUANTUM GROUPS AND KNOT THEORY: WEEK 5

The basic text of this lecture is [6, Chapter 1 and 2], but we will discuss some foundational material, in particular the relation between the topological and the combinatorial theory of knots, more extensively. The present discussion is partly intended as an introduction to these foundational results, without technical details since these would lead us far beyond the scope of this course.

1. What is a knot?

Definition 0.1. *A knot K is a subset $K \subset \mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$. A link is a finite disjoint union of knots.*

In other words, a knot $K \subset \mathbb{S}^3$ is the image $K = \phi(\mathbb{S}^1)$ of an injective continuous map $\phi : \mathbb{S}^1 \to \mathbb{S}^3$. Indeed, since $\mathbb{S}^1$ is compact and $\mathbb{S}^3$ is a Hausdorff space it follows from basic topology that such ϕ is an embedding, i.e. a homeomorphism onto its image. By a well known result in topology ("invariance of dimension") it follows that knots contain no interior points. It follows easily that the same property holds for links, and in particular that a link is not all of $\mathbb{S}^3$. Therefore we may assume without loss of generality in the definition that a knot or a link is a subset of $\mathbb{R}^3$.

The above definition is natural and elegant, but unfortunately it allows the existence of so called *wild knots* which is unwanted in our context.

Definition 0.2. *A knot K is called* tame *if for every point $p \in K$ there exists a closed neighborhood B of p such that the pair $(B, K \cap B)$ is homeomorphic to the pair $(B_0(1), [N, Z])$ where $B_0(1)$ denotes the unit closed 3-ball in $\mathbb{R}^3$, and $[N, Z]$ is the straight line segment connecting the north pole N and south pole Z. A point $p \in K$ which fails to satisfy this property is called a* wild point. *A knot which is not tame is called* wild.

Consider the subset of $\mathbb{R}^3$ displayed in Figure 1. It is not hard to construct an injective continuous map from $\mathbb{S}^1 \to \mathbb{R}^3$ with this set as its image, in other words this set is a knot

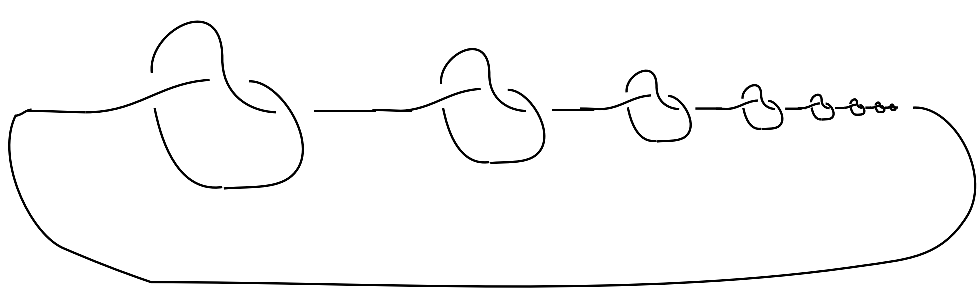

Figure 1. A knot with a wild point

in the above sense. It is not a tame knot, since the "accumulation point of knots" in the upper strand is clearly a wild point.

There are two well known ways to avoid wildness, by requiring additional structure of the embedding. Although equivalent we discuss both because both these approaches have certain merits.

Definition 0.3. *A smooth knot is the image of a smooth embedding of* $\mathbb{S}^1$ *into* $\mathbb{R}^3$ *(or* $\mathbb{S}^3$*).*

A smooth knot is thus the image of a smooth function $\phi : \mathbb{S}^1 \to \mathbb{R}^3$ which is a homeomorphism onto its image and such that its tangent map does not vanish (equivalently, the corresponding 1-periodic smooth function $\mathbb{R} \to \mathbb{R}^3$ has a non-vanishing velocity vector). It follows from a standard result in analysis (the "constant rank theorem") that every point p in the image of such curve admits a coordinate neighborhood with coordinates (x, y, z) centered at p in which the curve look like the straight line $x = y = 0$. Hence smooth knots are tame.

Definition 0.4. *A combinatorial knot is a simple closed polygonal curve in* $\mathbb{R}^3$*.*

By a simple closed polygonal curve we mean a union of finitely many straight line segments which is homeomorphic to $\mathbb{S}^1$. Clearly combinatorial knots are tame.

We are not so much interested in knots themselves but we are interested in knots up to the action of the group of orientation preserving homeomorphisms of $\mathbb{S}^3$ onto $\mathbb{S}^3$ (or diffeomorphisms, when working with smooth knots, or piecewise linear isomorphisms for combinatorial knots). In the world of combinatorial knots this equivalence relation can be formulated in a completely different and very useful alternative form, namely that of combinatorial equivalence. In order to understand the connection between these definitions we need understand a little bit more about piecewise linear maps and spaces. So let us pause the discussion of knots for a moment now and consider these fundamental notions.

2. Piecewise linear structures

Piecewise linear structures are the mathematical abstractions to capture the idea that one can approximate smooth shapes (e.g. a smooth surface in $\mathbb{R}^3$) by gluing small linear pieces or tiles. The fundamental building block is the linear simplex.

Definition 0.1. *A set* $X = \{x_0, x_1, \ldots, x_k\} \subset \mathbb{R}^n$ *is called affine linearly independent if* $\sum_{i=0}^{k} c_i x_i = 0$ *and* $\sum_{i=0}^{k} c_i = 0$ *implies that* $c_i = 0$ *for all* $i = 0, \ldots, k$*. A subset* $\sigma \in \mathbb{R}^n$ *is called a linear* k*-simplex (for some* $-1 \leq k \leq n$*) if* σ *is the convex hull of an affine linearly independent set* $X \subset \mathbb{R}^n$ *of cardinality* $k + 1$*. If* $X = \{x_0, x_1, \ldots, x_k\}$ *we have explicitly*

$$\sigma = \{\sum_{i=0}^{k} c_i x_i \mid c_i \geq 0 \text{ and } \sum_{i=0}^{k} c_i = 1\} \tag{0.1}$$

We use the notation $\sigma = [X] = [x_0, x_1, \ldots, x_k]$. Observe that X is determined by σ as the subset of extremal points (the points which can not be written as a nontrivial convex combination of points of σ). A k-simplex $\sigma = [X] \subset \mathbb{R}^n$ gives rice to a collection of 2^{k+1}

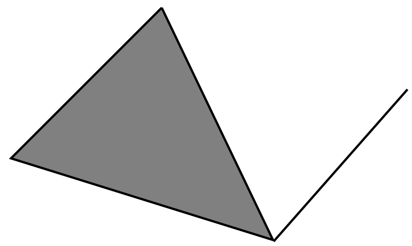

FIGURE 2. A polyhedron in the plane

l-simplices (with $l \leq k$) $\tau \subset \sigma$ of the form $\tau = [Y]$ with $Y \subset X$. These are called the faces of σ. Observe that $\emptyset$ is a face of all simplices.

Definition 0.2. *A locally finite simplicial complex in $\mathbb{R}^n$ is a collection S of simplices in $\mathbb{R}^n$ satisfying the following rules:*

(a) *S is locally finite, i.e. for every $x \in |S| := \cup_{\sigma \in S}\sigma$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x such that U meets only finitely many $\tau \in S$.*
(b) *If $\sigma \in S$ and $\tau \subset \sigma$ is a face of σ then $\tau \in S$.*
(c) *If $\sigma_1, \sigma_2 \in S$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

We call $|S| := \cup_{\sigma \in S}\sigma$ the (Euclidean) polyhedron associated to S, and S a triangulation of $|S|$.

Example 0.3. *Let σ be a k-simplex in $\mathbb{R}^n$, and let S be the collection of its faces. Then S is a finite simplical complex.*

Example 0.4. *The boundary $\partial\sigma$ of an n-simplex $\sigma \subset \mathbb{R}^n$ is a polyhedron with finite triangulation by the set S of* proper *faces of σ.*

Example 0.5. *A standard triangulation of $\mathbb{R}^n$ is given as follows. Let $\mathfrak{S}_n$ denote the group of permutations of the set $\{1, 2, \ldots, n\}$. To each $w \in \mathfrak{S}_n$ we associate a set X_w of $n+1$ vertices of the unit cube $[0,1]^n$ by*

$$X_w := \{0, e_{w(1)}, e_{w(1)} + e_{w(2)}, \ldots, e_1 + e_2 + \cdots + e_n\} \tag{0.2}$$

Clearly X_w consists of $n+1$ affine linearly independent vectors for all $w \in \mathfrak{S}_n$, and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ belongs to $\sigma_w := [X_w]$ if and only if

$$0 \leq x_{w(n)} \leq x_{w(n-1)} \leq \cdots \leq x_{w(1)} \leq 1 \tag{0.3}$$

It follows that the collection S of all translations of the faces of the simplices σ_w ($w \in \mathfrak{S}_n$) over the set $\mathbb{Z}^n$ of integral lattice points is a triangulation of $\mathbb{R}^n$.

Example 0.6. *Let $(a_i)_{i \geq 1}$ be a series of positive real numbers such that $a_i \uparrow 1$. Put $a_0 = 0$. Then the collection $\{\sigma_i \mid i \geq 1\}$ of 1-simplices $\sigma_i := [a_{i-1}, a_i]$ together with the set of their faces (the empty set and the set of points $\{a_i \mid i \geq 0\}$ is a triangulation of the half-open interval $[0,1)$.*

Example 0.7. *A triangulation of the upper half plane in $\mathbb{R}^2$ is shown in figure 3.*

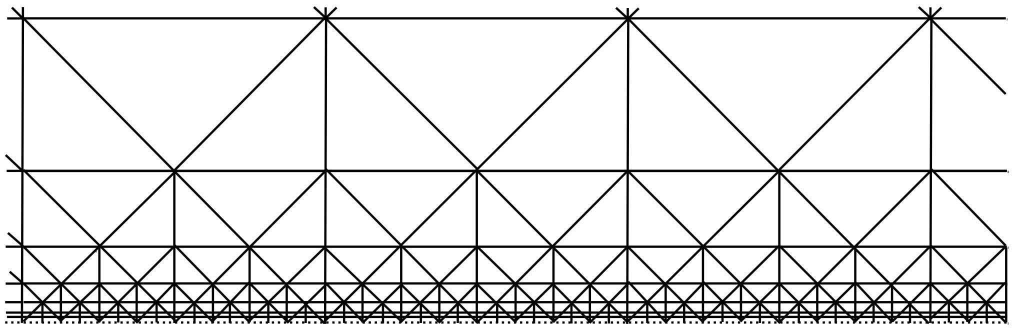

FIGURE 3. A triangulation of the upper half plane

1. Linear subdivisions and piecewise linear maps.

Definition 1.8. *Let $X = |S|$ be a polyhedron with triangulation S. A triangulation S' of X is called a refinement or a linear subdivision of S (notation $S' \lhd S$) if for every simplex $\sigma' \in S'$ there exists a $\sigma \in S$ such that $\sigma' \subset \sigma$.*

Example 1.9. *(Barycentric subdivision of a simplex) Consider the triangulation of a standard k-simplex $\sigma = [x_0, x_1, \ldots, x_k] \subset \mathbb{R}^n$ by its set S of faces. We define a set of k-simplices $\sigma_w \subset \sigma$ which are parameterized by the elements $w \in \mathfrak{S}_{k+1}$ (the group of permutations of $\{0, 1, \ldots, k\}$) as follows: $\sigma_w := [p_0, p_1, \ldots, p_k]$, where the points p_i are defined by*

$$p_i := \frac{1}{i+1} \sum_{j=0}^{i} x_{w(i)} \tag{1.1}$$

The point p_i is called the barycenter of the face $[x_{w(0)}, \ldots, x_{w(i)}]$ of σ. See figure 4. Clearly the collection σ_w (with $w \in \mathfrak{S}_k$) together with the collection of their faces forms a linear subdivision $S' \lhd S$. This is called the barycentric subdivision of S. Let $\tau \in S$ be a face of σ. Observe that the collection S_τ of elements $\sigma' \in S'$ such that $\sigma' \subset \tau$ is the barycentric subdivision of τ.

Example 1.10. *(Barycentric subdivision) Let S be any locally finite simplicial complex. Apply the barycentric subdivision to all of its simplices $\sigma \in S$. This defines a linear subdivision S' of S which is called the barycentric subdivision of S.*

The barycentric subdivision of S is very convenient since it can be applied to any simplicial complex and it has the nice property that the maximum diameter of a simplex (the *mesh*) of S' is at most $k/(k+1)$ times the maximum diameter of the simplices of S, where k is the maximal dimension of a simplex of S. This implies that for any $\epsilon > 0$ we can obtain a refinement $S' \lhd S$ (by iterated barycentric subdivisions) such that the mesh of S' is smaller than ϵ.

2. Piecewise linear maps and spaces.

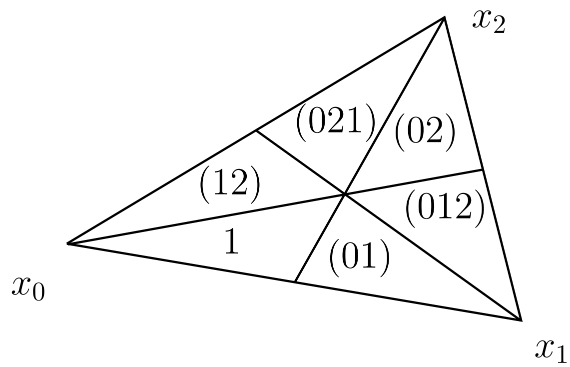

FIGURE 4. Barycentric subdivision

Definition 2.11. *Let S,T be locally finite simplical complexes. A map $f : |S| \to |T|$ is called a piecewise linear map (or PL-map) with respect to the triangulations S of $|S|$ and T of $|T|$ if there exists a refinement $S' \lhd S$ such that for all $\sigma' \in S'$, the restriction $f|_{\sigma'}$ is linear, and there exists a $\tau \in T$ such that $f(\sigma') \subset \tau$. Here linearity means that $f(\sum_{i=0}^k c_i x_i) = \sum_{i=0}^k c_i f(x_i)$ where $\sigma' = [x_0, x_1, \ldots, x_k]$ and where the coefficients c_i satisfy (as usual) $c_i \geq 0$ and $\sum_i c_i = 1$.*

It is clear from this definition that PL-maps are continuous. We now list some basic but nontrivial facts about PL-maps. We do not prove these statements here, since this would lead us too far afield. We refer the interested reader to [7].

(i) Compositions of PL-maps between locally finite simplicial complexes are PL-maps.
(ii) If $f : |S| \to |T|$ is a PL-map between *finite* simplicial complexes then one can find refinements $S' \lhd S$ and $T' \lhd T$ such that f is a simplicial map from S' to T' (this means that f maps simplices of S' linearly to simplices of T').
(iii) If f is a PL-map which is a homeomorphism then f^{-1} is also PL. We say that f is a PL-isomorphism.
(iv) (see [1]) If $U \subset |S|$ is an open subset of a locally finite simplicial complex, then there exists a locally finite triangulation S_U of U such that the inclusion map $U \hookrightarrow |S|$ is a PL-map. Moreover, if S'_U is another such locally finite triangulation then the identity map of U is a PL-isomorphism with regards to these two triangulations.

Notice that property (iii) is not true for smooth maps.

Now we are ready to define PL-structures on topological manifolds.

Definition 2.12. *A topological manifold M of dimension m is a metrizable topological space such that each point $x \in M$ admits an open neighborhood $U \ni x$ which is homeomorphic to an open subset of $\mathbb{R}^m$.*

We will simply speak of manifolds instead of topological manifolds.

A *chart* of M is a pair (U, ϕ) with $U \subset M$ open and $\phi : U \to V = \phi(U) \subset \mathbb{R}^m$ a homeomorphism of U to an open subset V of $\mathbb{R}^m$. A collection of charts covering M is called an atlas of M.

Given two charts (U_1, ϕ_1) and (U_2, ϕ_2) we call the map

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \tag{2.1}$$

a transition map. Observe that the transition map is a homeomorphism between open subsets of $\mathbb{R}^m$ and that its inverse is also a transition map. We say that the two charts are PL-compatible if the transition maps are PL-isomorphisms.

Definition 2.13. *A PL-manifold is a manifold with an atlas of PL-compatible charts.*

This way of dressing a manifold with additional structure is universal. For instance one defines in a similar fashion the notions of a C^k-manifold or a smooth manifold. We may generalize the definitions to manifolds with boundary but we will not do this here.

Definition 2.14. *A map $f : M \to N$ between two PL-manifolds is called a PL-map if f is continuous and if for every chart (U_1, ϕ) of M and (U_2, ψ) of N the map*

$$\psi \circ f \circ \phi^{-1} : \phi(U_1 \cap f^{-1}(U_2)) \to V_2 = \psi(U_2) \tag{2.2}$$

is a PL-map.

Recall that a homeomorphism which is a PL-map is a PL-isomorphism (i.e. its inverse is PL too).

Of course a (locally finite) simplicial complex $X = |S|$ is not necessarily a manifold or a manifold with boundary (see e.g. figure 2). It is easy to see that a sufficient condition to ensure that X is a manifold (closed, i.e. with empty boundary) is the requirement that the triangulation S of X is an m-dimensional combinatorial manifold, which means that it is a triangulated Euclidean polyhedron in which the *link* of every point $x \in X$ is homeomorphic to $\mathbb{S}^{m-1}$. Here the link of $x \in X$ is defined as $\overline{St(x)} - St(x)$, where $St(x)$ denotes the star of x. In turn the star of x is the set of points $y \in X$ such that x is contained in the smallest simplex containing y. See figure 5. In a combinatorial manifold X the star $St(x)$ of a point x is homeomorphic to an open ball of $\mathbb{R}^m$, and it follows easily that X is a PL-manifold. It is not difficult to see that in this context, a PL-map $X \to Y$ between combinatorial manifolds X and Y is nothing but a PL-map in the context of polyhedra as given before. It is a nontrivial fact that all PL-manifolds are PL-isomorphic to a combinatorial manifold.

We mention in passing that triangulation of manifolds is a difficult and delicate subject. Not all manifolds admit a PL-structure, and even if a manifold admits a PL-structure then there may exist distinct PL-structures which are not PL-isomorphic and which may not be combinatorial manifolds. However, for smooth manifolds the situation is simpler if we insist on piecewise smoothness of the triangulations, as witnessed by the following classical result:

Theorem 2.15. *(Whitehead (1940)) If M is a smooth manifold then there exists a homeomorphism $\phi : X \to M$ where X is a combinatorial manifold and where ϕ is smooth on the simplices of X (such a homeomorphism is called a piecewise differentiable triangulation*

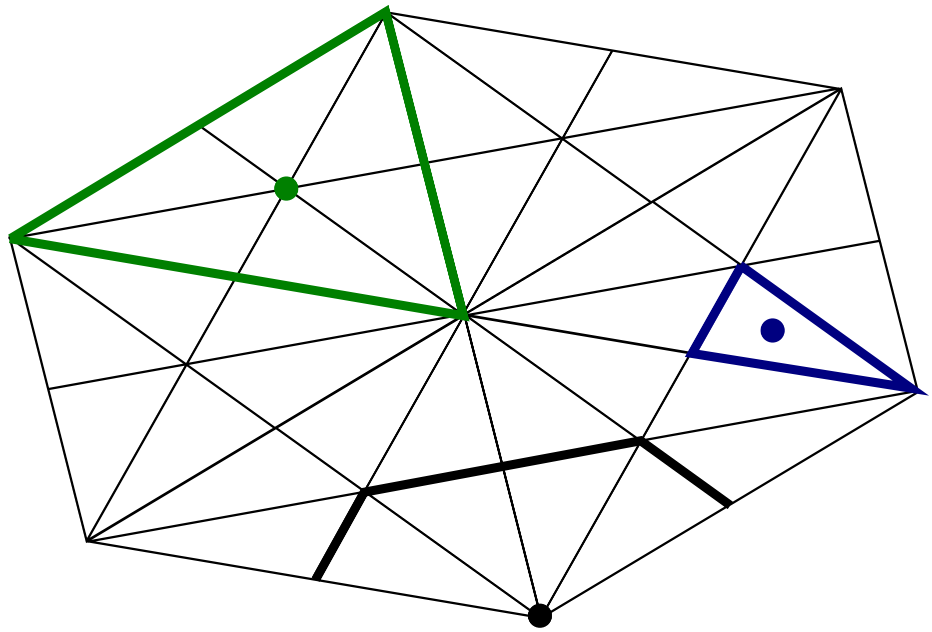

FIGURE 5. Three examples of a link of a point

of M). Moreover, if $\psi : Y \to M$ is any piecewise differentiable homeomorphism from a triangulated polyhedron Y then Y is in fact a combinatorial manifold and $\psi \circ \phi^{-1}$ is a PL-isomorphism.

If we restrict our attention to manifolds of dimension ≤ 3 (as we do in knot theory) the situation is much simpler. The following highly nontrivial result due to Moise (1954) plays a key role for knot theory:

Theorem 2.16. *Let M be a 3-manifold. Then M admits a triangulation which is unique up to PL-isomorphisms.*

Example 2.17. *Consider the standard simplex $\sigma = [e_0, e_1, \dots, e_n]$ in the affine hyperplane $V \subset \mathbb{R}^{n+1}$ given by the equation $\sum_i x_i = 1$, and the sphere $\mathbb{S}^{n-1} = \mathbb{S}^n \cap V$. Let $X = \partial(\sigma)$ be the boundary of σ, with its triangulation as in Example 0.4. Let $\phi : X \to \mathbb{S}^{n-1}$ be the central projection in V with respect to the center $(1/n+1, \dots, 1/n+1)$ of $\mathbb{S}^{n-1}$. This is clearly a bijective continuous map, hence a homeomorphism (since X is compact and $\mathbb{S}^n$ is Hausdorffs). It is easy to see directly that X is a combinatorial manifold here. This triangulation of $\mathbb{S}^{n-1}$ realizes explicitly the (unique) piecewise differentiable PL-structure of $\mathbb{S}^{n-1}$.*

We now return to the discussion of knots. Let us first connect Definition 0.4 of combinatorial knots with the general discussion of PL-structures in the present section.

Proposition 2.18. *A combinatorial knot is the image of an injective PL-map $\alpha : \mathbb{S}^1 \to \mathbb{R}^3$.*

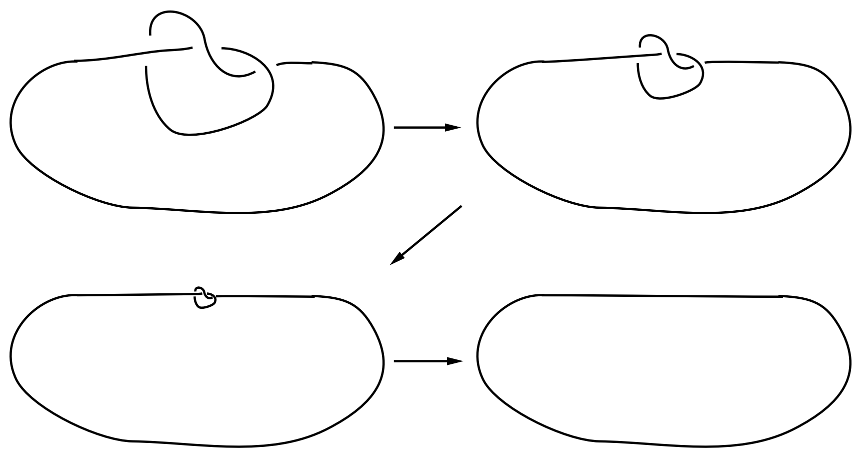

FIGURE 6. An isotopy of embeddings of the circle

Proof. By the above example (for $n = 2$) and definitions we see that a PL-map $\alpha : \mathbb{S}^1 \to \mathbb{R}^3$ gives rise to a PL-map (in the sense of polyhedra) $\beta := \alpha \circ \phi : X \to \mathbb{R}^3$, where X is a triangle in the plane. After a suitable linear subdivision of X we obtain a combinatorial circle (an N-gon for some suitable N) such that β is linear on its 1-simplices. If β is also injective then its image is a simple closed polygonal curve in $\mathbb{R}^3$. □

For this reason we will also refer to combinatorial links as PL-links.

3. Equivalence classes of links

As explained in the first section, we are interested in the equivalence classes of links rather than the links themselves. The equivalence relation is called *ambient isotopy* and is defined as follows. An isotopy of $\mathbb{S}^3$ is a homeomorphism $\alpha : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3 \times [0,1]$ such that for all $t \in [0,1]$ we have $\alpha(s,t) = (\alpha_t(s), t)$ and such that $\alpha_0(s) = s$ for all $s \in \mathbb{S}^3$.

Definition 0.1. *Two links $L_1, L_2 \subset \mathbb{S}^3$ are called equivalent (or ambient isotopic) if there exists an isotopy α of $\mathbb{S}^3$ such that $K_2 = \alpha_1(K_1)$. This definition applies to equivalence in the topological category, the smooth category and the PL-category (where we of course use the appropriate notion of isotopy in the relevant category).*

This notion of equivalence captures the intuitive idea of equivalent knots as being knots that can be deformed into each other. We remark that a more naive idea of deformation, where we do not require that the deformation is in fact a deformation of the ambient space $\mathbb{S}^3$, does not suffice to capture the idea of deformation of knots. Indeed, such an isotopy $\mathbb{S}^1 \times [0,1] \to \mathbb{S}^3 \times [0,1]$ of embeddings could look like shown in figure 6.

Theorem 0.2. *The natural map from the set of equivalence classes of PL-links to the set of equivalence classes of* tame *links is a bijection. Similarly for the set of equivalence classes of smooth knots.*

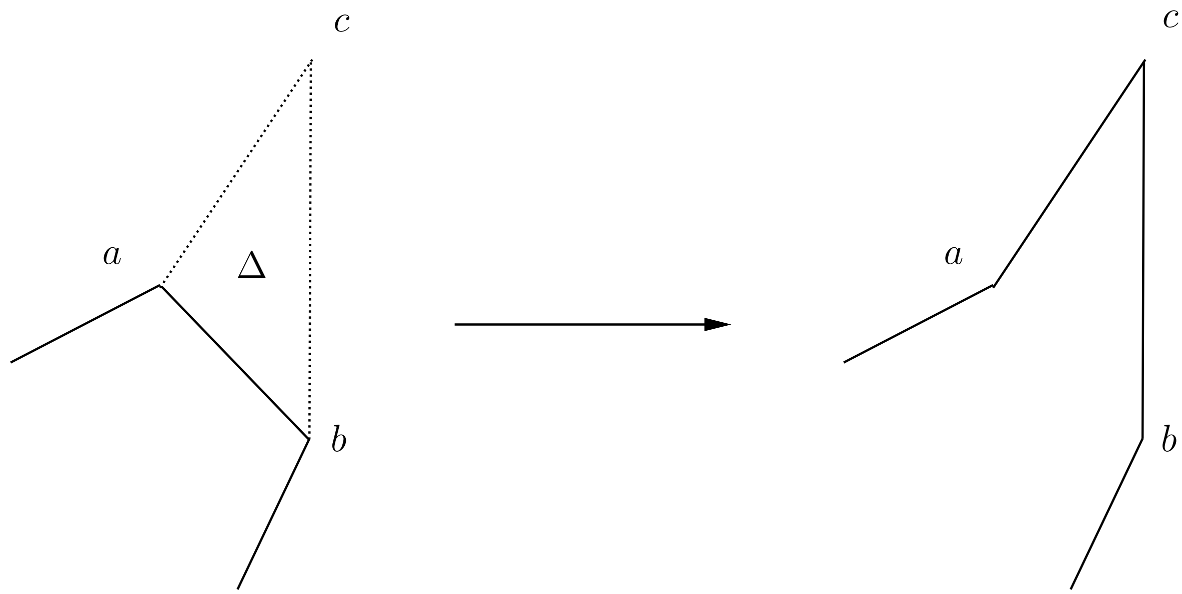

FIGURE 7. A Reidemeister delta move.

The first assertion of this theorem is a classical result of Moise (1954). The second assertion is rather elementary, using Moise's result, by smoothing of PL-knots.

We may define link equivalence in a simpler way, by taking advantage of a special feature of $\mathbb{S}^3$, which states that any orientation preserving homeomorphism of $\mathbb{S}^3$ is isotopic to the identity map (a deep result due to Fisher (1960)). This is true in the topological category, as well as in the PL- and smooth categories. Thus we have:

Theorem 0.3. *Two topological (respectively smooth, PL) links L_1 and L_2 are equivalent if and only if there exists an orientation preserving homeomorphism (respectively diffeomorphism, PL-isomorphism) $\alpha : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\alpha(L_1) = L_2$.*

In the PL-category we can define equivalence of links in yet another way:

Definition 0.4. *Let L be a PL-link in $\mathbb{R}^3$. A Reidemeister Δ-move is a modification of L as follows. Let $\Delta = [a, b, c] \subset \mathbb{R}^3$ be a linear 2-simplex such that $\Delta \cap L = [a, b]$ is a 1-simplex of L. Now replace L by $L' = (L \backslash [a, b]) \cup [a, c] \cup [c, b]$. See figure 7.*

Definition 0.5. *We say that two PL-links are combinatorially equivalent if they can be connected by a chain of Reidemeister Δ-moves and inverse Δ-moves.*

It is easy to see that a Δ-move or its inverse can be realized by application of an orientation preserving PL-homeomorphism of $\mathbb{R}^3$. In fact one can prove:

Theorem 0.6. *Two PL-links L_1 and L_2 in $\mathbb{R}^3$ are equivalent if and only if they are combinatorially equivalent.*

The proof of this important Theorem can be found in [3, Proposition 1.10]. This result opens up the possibility to study knot equivalence by combinatorial means (Reidemeister moves), and has important theoretical applications. This will be subject of next week's lecture. Equivalent knots and links are often identified unless this leads to confusion. The next definition is an example of this habit:

Definition 0.7. *A knot $K \subset \mathbb{R}^3$ is called the* unknot *or* trivial knot *if K is equivalent to the standard unit circle in $\mathbb{R}^2$.*

The Schoenflies Theorem (for embeddings of $\mathbb{S}^1$ in $\mathbb{S}^2$) asserts that any planar knot is the unknot:

Theorem 0.8. *Let $K \subset \mathbb{R}^2$ be a planar knot. Then K is the unknot.*

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