Monadic Second-Order Logic is closed for product update.

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We consider a monadic second-order propositional modal language $\mathcal{L}_{\exists,U}$. This is a notational variant of monadic second-order logic, so it is a very expressive extension of the basic modal language $\mathcal{L}_{\Box}$. Thus it extends strictly the language $\mathcal{L}_{\mu}$ of the modal μ-calculus. The basic modal language is known to be closed for relativisation [Pla89] and for product update [BMS99, Ger99]. $\mathcal{L}_{\mu}$ has also been shown to be closed for both relativisation and for product update [BI08]. We will show that this stronger language $\mathcal{L}_{\exists,U}$ is also closed for relativisation (Proposition 2) and for product update (Theorem 5). We obtain as a corollary the result in [BI08] that $\mathcal{L}_{\mu}$ is closed for product update. We also show how the technique we use, of using 'action nominals', can be applied to obtain more closure results in hybrid logic.

Closures

Assuming familiarity with modal logic, we will define in this Section what we mean by 'closure for relativisation' and 'closure for product update'. These two 'closures' of a language have to do with two semantic operations on models.

Given a modal model $\mathcal{M} = (\Omega, R, V)$, and any set $A \subseteq \Omega$, we write $\mathcal{M}|A$ for the ***relativisation*** of $\mathcal{M}$ to A, defined as $(A, R|A, V|A)$, where $R|A \stackrel{\text{df}}{=} R \cap (A \times A)$ and $(V|A)(p) \stackrel{\text{df}}{=} V(p) \cap A$. Then we say that a language $\mathcal{L}$ is ***closed for relativisation*** just when for any pair of formulae $\{\varphi, A\} \in \mathcal{L}$, there is a formula $\psi \in \mathcal{L}$ such that for any model $\mathcal{M}$, $[\![\psi]\!]^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}|[\![A]\!]^{\mathcal{M}}}$. To spell this definition out: for any formula φ, there is a formula ψ that is true in model $\mathcal{M}$ just when φ 'will be' true in the relativised model $\mathcal{M}|[\![A]\!]^{\mathcal{M}}$. A natural interpretation of relativisation in modal models where the accessibility relation is taken to be *epistemic* is of an action of public announcement. Public announcements were introduced and discussed in this context in [Pla89, GG97, Ben03]. To repeat the temporal idiom, ψ can be thought of as saying that φ *will be* the case *after* announcement of A.

To say that a language $\mathcal{L}$ is closed for relativisation is equivalent to saying that if we were to enrich $\mathcal{L}$ with 'relativisation modalities' then we would obtain a language with exactly the same expressive power as $\mathcal{L}$. That is, consider another language $\mathcal{L}'$, obtained from $\mathcal{L}$ by addition of a family of modalities $\langle !A \rangle$ for each $A \in \mathcal{L}$, endowed with the following semantics:

$$\mathcal{M}, \omega \vDash \langle !\psi \rangle \varphi \stackrel{\text{df}}{\Leftrightarrow} \mathcal{M}, \omega \vDash \psi \text{ and } \mathcal{M}|[\![\psi]\!]^{\mathcal{M}}, \omega \vDash \varphi$$

Then an alternative but equivalent characterisation of relativisation closure is as follows: $\mathcal{L}$ is closed for relativisation iff $\mathcal{L}'$ has precisely the same expressive power as

$\mathcal{L}$.

One can show by a 'compositional analysis' of its semantics that, for example, the basic modal language $\mathcal{L}_\Box$ is closed for relativisation. That is, one gives so-called 'reduction axioms', validities of the form $\langle !\psi\rangle O(\varphi) \equiv Q(\langle !\psi\rangle\varphi)$, for each connective O of the language. This technique, introduced for modal logic in [Pla89], shows not only that a language is closed for relativisation but, when Q is computable from O, that it is *computably* so closed.

A similar technique can be employed to show that a language is closed for 'product update'. Product update was developed in the context of epistemic logic in [Ger99, BMS99]. Semantically, it is a more complicated operation than relativisation. In order to define product update we need to define 'event models'. Event models are defined relative to a language $\mathcal{L}$. An ***event model*** is a relational structure $(\Sigma, \sim, PRE)$, where Σ is a finite non-empty set of 'events', $\sim\subseteq \Sigma \times \Sigma$ is a relation over Σ and $PRE_\_ : \Sigma \to \mathcal{L}$ gives the 'precondition' formula for each 'event'. Product update $\otimes$ is then a function that, given a model and an event model, returns a new model. It is defined as follows:

$$\mathcal{M} \otimes \mathcal{A} \overset{\text{df}}{=} \begin{pmatrix} \{(\omega, \alpha) \in \Omega \times A \mid \mathcal{M}, \omega \vDash Pre_\alpha\}, \\ \{((\omega, \alpha), (\omega', \alpha')) \mid \omega \sim \omega' \; \& \; \alpha \sim \alpha'\}, \\ \{(p, \{(\omega, \alpha) \mid \omega \in V(p)\}) \mid p \in PROP\} \end{pmatrix}$$

Then for any event model $\mathcal{A} = (\Sigma, \sim, PRE)$, we could expand the language $\mathcal{L}$ to a language $\mathcal{L}$' by adding a collection of operators $\langle\alpha\rangle$ for each event $\alpha \in \Sigma$, with the following semantics:

$$\mathcal{M}, \omega \vDash \langle\alpha\rangle\varphi \overset{\text{df}}{\Leftrightarrow} \mathcal{M}, \omega \vDash Pre_\alpha \text{ and } \mathcal{M} \otimes \mathcal{A}, (\omega, \alpha) \vDash \varphi,$$

Then we say that a language $\mathcal{L}$ is ***closed for product update*** just if any language $\mathcal{L}$' formed in this way has the same expressive power as $\mathcal{L}$.

MSO

We define monadic second-order propositional modal logic with a global modality, $\mathcal{L}_{\exists,U}$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \exists p.\varphi \mid U\varphi$$

The semantics of this language are as follows, where $\mathcal{M} = (\Omega, R, V)$ is a model:

- $\mathcal{M}, \omega \vDash p \overset{\text{df}}{\Leftrightarrow} \omega \in V(p)$
- $\mathcal{M}, \omega \vDash \neg\varphi \overset{\text{df}}{\Leftrightarrow} \omega \nvDash \varphi$
- $\mathcal{M}, \omega \vDash \varphi \wedge \psi \overset{\text{df}}{\Leftrightarrow} \omega \vDash \varphi$ and $\omega \vDash \psi$
- $\mathcal{M}, \omega \vDash \Box\varphi \Leftrightarrow \forall\omega'(\omega R\omega' \Rightarrow \omega' \vDash \varphi)$
- $\mathcal{M}, \omega \vDash \exists p.\varphi \overset{\text{df}}{\Leftrightarrow} \exists X \subseteq \Omega : \mathcal{M}[p \mapsto X], \omega \vDash \varphi$
- $\mathcal{M}, \omega \vDash U\varphi \overset{\text{df}}{\Leftrightarrow} \forall\omega' \in \Omega\omega' \vDash \varphi$

(Here and in what follows, '$\mathcal{M}[p \mapsto X]$' denotes the model (Ω, R, V'), where $V'(p) = X$ and for all $q \neq p$, $V'(q) = V(q)$.) The same language but without the universal modality U was first studied in [Fin70]. The language we are studying $\mathcal{L}_{\exists,U}$ which does have the universal modality is a notational variant of monadic second-order logic $\mathcal{L}_{MSO}$:

$$\varphi ::= Px \mid xRy \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \exists P\varphi$$

The semantics of the first-order part of $\mathcal{L}_{MSO}$ are standard, and the propositional quantifier $\exists P$ has the natural interpretation, as quantifying unrestrictedly over subsets of the domain. There is a simple linear translation between the $\mathcal{L}_{\exists,U}$ and $\mathcal{L}_{MSO}$; the only slightly tricky clause is the following:

$$TR(\exists x\varphi(x)) = \exists p.(Ep \wedge \forall q.(U(q \supset p) \supset U(p \supset q)) \wedge \varphi(p))$$

Thus what we prove about the expressivity of $\exists, \mathcal{U}$ can equivalently be read as being about $\mathcal{L}_{MSO}$, and refer to $\mathcal{L}_{\exists,U}$ as "MSO".

MSO is product closed

To warm up we will remark that $\mathcal{L}_{\exists,U}$ is closed under relativisation. In order to do this we want a 'reduction axiom' for the quantifier $\exists p$. The following Fact states such a reduction axiom:

Fact 1 *If p does not occur in A, then:*

$$\vDash \langle !A\rangle\exists p.\varphi \equiv \exists p.(U(p \supset A) \wedge \langle !A\rangle\varphi)$$

Fact 1 plays the central part in the proof of Proposition 2.

Proposition 2 *$\mathcal{L}_{\exists,U}$ is computably closed for relativisation.*

(Interesting proofs are in the Appendix. Proposition 2 is a corollary of Theorem 5 below.)

In the rest of this section we will work towards Theorem 5, which states that $\mathcal{L}_{\exists,U}$ is also (computably) closed for product update. Fix some event model $(\Sigma, \sim, PRE)$ with $\Sigma = \{\alpha_0, \ldots, \alpha_{n-1}\}$. We will show that the language $\mathcal{L}_{\exists,U,\alpha}$ obtained by adding event modalities $\langle\alpha\rangle$ for each $\alpha \in \Sigma$ has precisely the same expressive power as the original language $\mathcal{L}_{\exists,U}$.

In order to show that $\mathcal{L}_{\exists,U}$ is closed for relativisation we needed simply to give a reduction axiom for the quantifier $\exists p$. The case for product update will be a little bit more subtle. We *will* give a reduction axiom for the quantifier $\exists p$, but it will actually be in the context of a richer language. We will need to consider two languages $\mathcal{L}_{\exists,U,j}$ and $\mathcal{L}_{\exists,U,\alpha,j}$. These are obtained from the languages $\mathcal{L}_{\exists,U}$ and $\mathcal{L}_{\exists,U,\alpha}$ by adding n new nullary symbols $j_0, \ldots, j_{n-1}$ which, intuitively speaking, will say, in the model $\mathcal{M} \times \mathcal{A}$, that action α_i has **just occurred**. We will call these j_i's '***action nominals***'. Formally, the semantics of action nominals:

$$\mathcal{M}, \omega \vDash j_i \overset{\text{df}}{\Leftrightarrow} \exists\mathcal{M}'\exists\omega' \in |\mathcal{M}'| : \mathcal{M} = \mathcal{M}' \otimes \mathcal{A} \;\&\; \omega = (\omega', \alpha_i)$$

Notice that this has the following consequence, which justifies our intuitive reading of action nominals as names for the action which has just occured:

$$\mathcal{M} \otimes \mathcal{A}, (\omega, \alpha) \vDash j_i \Leftrightarrow \alpha = \alpha_i$$

The reader who finds that the semantics of action nominals is a bit odd should not fear, for they will be eliminated soon anyway. For now, notice that there are simple reduction axioms:

Remark 3

$$\vDash \langle \alpha_i \rangle j_i \equiv Pre_{\alpha_i}$$

$$\vDash \langle \alpha_j \rangle j_i \equiv \bot \text{ for } i \neq j$$

More importantly, it is now not difficult to write down a reduction axiom for the quantifier $\exists p$:

Lemma 4 *If none of $p_0, \ldots, p_{n-1}$ occur in φ or any Pre_{α_i}, then:*

$$\vDash \langle \alpha \rangle \exists p.\varphi \equiv \exists p_0 \ldots \exists p_{n-1}.(\bigwedge_{i \in n} U(p_i \supset Pre_{\alpha_i}) \wedge \langle \alpha \rangle \varphi(\bigvee_{i \in n} (p_i \wedge j_i)))$$

That is, we could show straightforwardly that $\mathcal{L}_{\exists,U,\mathcal{A},j}$ can be reduced to $\mathcal{L}_{\exists,U,j}$. However, as promised, we will also get rid of the intermediary action nominals, and show that $\mathcal{L}_{\exists,U,\mathcal{A}}$ can be reduced to $\mathcal{L}_{\exists,U}$.

Theorem 5 *$\mathcal{L}_{\exists,U}$ is computably closed for product update. I.e.:*

$$\forall \alpha \in \Sigma, \forall \psi \in \mathcal{L}_{\exists,U}, \exists \chi \in \mathcal{L}_{\exists,U} : \ \vDash \chi \equiv \langle \alpha \rangle \psi,$$

where χ is effectively computable from ψ and (Σ, α).

This is an immediate corollary of the following Lemma.

Lemma 6

$$\forall \alpha \in \Sigma, \forall \psi \in \mathcal{L}_{\exists,U,j}, \exists \chi \in \mathcal{L}_{\exists,U} : \ \vDash \chi \equiv \langle \alpha \rangle \psi$$

A corollary for $\mathcal{L}_\mu$

In this section we observe that the relativisation closure of the modal fixpoint language $\mathcal{L}_\mu$ is a corollary of our Theorem 5. The closure of $\mathcal{L}_\mu$ is already obtained directly in [BI08]. $\mathcal{L}_\mu$ is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \nu p.\varphi,$$

where crucially in $\nu p.\varphi$, φ must be *positive* in p (i.e. all occurrences of p must be under the scope of an even number of negations $\neg$). The new semantic clause for ν is as follows:

$$[\![\nu p.\varphi]\!] = \bigcup \{X \subseteq \Omega \mid \forall \omega \in X, \mathcal{M}[p \mapsto X], \omega \vDash \varphi\}$$

The ν operator expresses *greatest fixpoints*. It is easy to see that $\mathcal{L}_\mu$ is a strict fragment of $\mathcal{L}_{\exists,U}$. Theorem 7 below is a more remarkable result, precisely characterising $\mathcal{L}_\mu$ in terms of $\mathcal{L}_{\exists,U}$ is proved in [JW96]. We say that a modal formula φ is ***bisimulation-invariant*** just if for any bisimilar points models $(\mathcal{M}, \omega)$ and $(\mathcal{M}', \omega')$, $\mathcal{M}, \omega \vDash \varphi \Leftrightarrow \mathcal{M}', \omega' \vDash \varphi$.[1]

[1] For a definition of bisimulations see [BdRV01].

Theorem 7 *([JW96]) A formula $\varphi \in \mathcal{L}_{\exists,U}$ is equivalent to an $\mathcal{L}_\mu$-formula iff it is bisimulation-invariant.*

That elegant result is what leads us to see that $\mathcal{L}_\mu$ is also closed for product update. We just need one additional lemma, which says that adding product-update modalities does not break bisimulation-invariance:

Lemma 8 *If φ and all of $PRE_{\alpha_1}, \ldots, PRE_{\alpha_n}$ are bisimulation-invariant then $\langle\alpha\rangle\varphi$ is bisimulation-invariant.*

Corollary 9

$$\forall \alpha \in \Sigma, \forall \psi \in \mathcal{L}_\mu, \exists \chi \in \mathcal{L}_\mu : \ \vDash \chi \equiv \langle\alpha\rangle\psi,$$

(Proof: Take $\varphi \in \mathcal{L}_\mu$. Then by Theorem 7, φ is bisimulation-invariant. Furthermore, $\varphi \in \mathcal{L}_{\exists,U}$, so by Theorem 5, $\langle\alpha\rangle\varphi \in \mathcal{L}_{\exists,U}$. But by Lemma 8, $\langle\alpha\rangle\varphi$ is bisimulation-invariant and every precondition formula is in $\mathcal{L}_\mu$, so invoking the other direction of Theorem 7, we have $\langle\alpha\rangle\varphi \in \mathcal{L}_\mu$.)

Dynamic Hybrid Languages

Our main result is Theorem 5. There is perhaps another proof, without passing via action nominals, but while the result in itself is not surprising, the use of action nominals yields a concise and smooth proof. They also work for other languages; in this section we will briefly look at how they can be applied to hybrid modal languages. Hybrid modal languages are studied in [Cat05]. They are called 'hybrid' because (presumably) they can talk not just about sets, using the proposition letters $p, q, \ldots$, but also about points, by using 'nominals' $i, j, \ldots$. Hybrid languages are built using two sets, $PROP$ and NOM. In models for hybrid logic, $V : PROP \cup NOM \to 2^\Omega$, with the stipulation that $\forall i \in NOM, V(i)$ is a singleton.

If we change the model, by relativisation or product update, there is no guarantee that $V'(i)$ will still be a singleton: in relativisation we might find that $V(i)$ is no longer in the model, and so $V'(i) = \emptyset$, and with products additionally $V'(i)$ might be larger than a singleton, since there may be more than one α, α' such that $[\![PRE_\alpha]\!] \cap [\![PRE_{\alpha'}]\!] \neq \emptyset$. Both phenomena entail that the *necessitation rule* $\vdash \varphi \Rightarrow \vdash \Box\varphi$, which we have for normal modalities, is not admissible for the modality $[!\varphi]$ in the presence of nominals. In particular, since $\vdash Ei$, we would have $\vdash [!\varphi]Ei$, which the first phenomenon of points disappearing shows would not be sound; similarly the second phenomenon shows that although $\vdash E(i \wedge \varphi) \supset U(i \supset \varphi)$, we do not want $\vdash [!\psi](E(i \wedge \varphi) \supset U(i \supset \varphi))$. ([Roy08] uses a modal language with nominals and relativisation, and points out that the completeness proof remains entirely straightforward and is not complicated by the presence of nominals.)

The hybrid language $\mathcal{L}_\downarrow$ includes the '*binder*': formulae of the form $\downarrow i.\varphi$ are permitted, with the following interpretation:

$$[\![\downarrow i.\varphi]\!] = \{\omega \in \Omega \mid \mathcal{M}[i \mapsto \{\omega\}], \omega \vDash \varphi\}$$

This language $\mathcal{L}_\downarrow$ is precisely the *bounded fragment* of first-order logic [ABM99]. Using the reduction axiom in Fact 10, we can see that it too is closed for product update, stated in Proposition 11.

Fact 10 *If i does not occur in any Pre_{α_i}, then:*

$$\models \langle\alpha\rangle \downarrow i.\phi \equiv \downarrow i.\langle\alpha\rangle\phi(i \wedge j_i)$$

Proposition 11 *$\mathcal{L}_{\downarrow}$ is computably closed for product update. I.e.:*

$$\forall\alpha \in \Sigma, \forall\psi \in \mathcal{L}_{\downarrow}, \exists\chi \in \mathcal{L}_{\downarrow} : \ \models \chi \equiv \langle\alpha\rangle\psi,$$

where χ is effectively computable from ψ and (Σ, α).

Conclusion

We have shown that monadic second-order logic is closed for product update. We did this using 'action nominals', and showed that a similar approach works for the bounded fragment.

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Proof of Lemma 4

Statement of Lemma: If none of $p_0, \ldots, p_{n-1}$ occur in φ or any Pre_{α_i}, then:

$$\vDash \langle\alpha\rangle\exists p.\varphi \equiv \exists p_0 \ldots \exists p_{n-1}.(\bigwedge_{i\in n} U(p_i \supset Pre_{\alpha_i}) \wedge \langle\alpha\rangle\varphi(\bigvee_{i\in n}(p_i \wedge j_i)))$$

We can prove this equivalence directly by the following string of equivalences:

$$\mathcal{M}, \omega \vDash \langle\alpha\rangle\exists p.\varphi(p)$$

iff

$$\mathcal{M} \otimes \mathcal{A}, (\omega, \alpha) \vDash \exists p.\varphi(p)$$

iff

$$\exists X \subseteq \{(\omega', \alpha') \mid \mathcal{M}, \omega' \vDash Pre_{\alpha'}\}, \mathcal{M} \otimes \mathcal{A}[p \mapsto X], (\omega, \alpha) \vDash \varphi(p)$$

iff

$$\exists X \subseteq \bigcup_{i\in n}(\{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_i}\} \times \{\alpha_i\}) : \mathcal{M} \otimes \mathcal{A}[p \mapsto X], (\omega, \alpha) \vDash \varphi(p)$$

iff

$$\exists X_0 \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_0}\}, \ldots, X_{n-1} \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_{n-1}}\} :$$

$$\mathcal{M} \otimes \mathcal{A}[p \mapsto \bigcup X_i \times \{\alpha_i\}], (\omega, \alpha) \vDash \varphi(p)$$

iff

$$\exists X_0 \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_0}\}, \ldots, X_{n-1} \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_{n-1}}\} :$$

$$\mathcal{M} \otimes \mathcal{A}[p_0 \mapsto X_i \times \{\alpha_i\}, \ldots, p_{n-1} \mapsto X_{n-1} \times \{x_{n-1}\}], (\omega, \alpha) \vDash \varphi(\bigvee_{i\in n} p_i)$$

(where $p_0, \ldots, p_{n-1}$ are distinct and do not occur in φ or any Pre_{α_i}), iff

$$\exists X_0 \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_0}\}, \ldots, X_{n-1} \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_{n-1}}\} :$$

$$\mathcal{M} \otimes \mathcal{A}[p_0 \mapsto X_i \times \Sigma, \ldots, p_{n-1} \mapsto X_{n-1} \times \Sigma], (\omega, \alpha) \vDash \varphi(\bigvee_{i\in n}(p_i \wedge j_i))$$

iff

$$\exists X_0 \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_0}\}, \ldots, X_{n-1} \in \{\omega' \in \Omega \mid \omega' \vDash Pre_{\alpha_{n-1}}\} :$$

$$\mathcal{M}[p_0 \mapsto X_i, \ldots, p_{n-1} \mapsto X_{n-1}], \omega \vDash \langle\alpha\rangle\varphi(\bigvee_{i\in n}(p_i \wedge j_i))$$

iff

$$\exists X_0, \ldots, X_{n-1} \subseteq \Omega : \mathcal{M}, \omega \vDash \bigwedge_{i\in n} U(p_i \supset Pre_{\alpha_i}) \wedge \langle\alpha\rangle\varphi(\bigvee_{i\in n}(p_i \wedge j_i))$$

iff

$$\mathcal{M}, \omega, \vDash \exists p_0, \ldots, \exists p_{n-1}.(\bigwedge_{i\in n} U(p_i \supset Pre_{\alpha_i}) \wedge \langle\alpha\rangle\varphi(\bigvee_{i\in n}(p_i \wedge j_i)))$$

Proof of Lemma 6

Statement of Lemma: $\psi \in \mathcal{L}_{\exists,U,j} \Rightarrow \exists \chi \in \mathcal{L}_{\exists,U}: \ \models \chi \equiv \langle\alpha\rangle\psi$.

We would like to prove this by induction on ψ, by using our induction axioms, and using the following inductive hypothesis, where γ is less complex than ψ:

$$\exists \delta \in \mathcal{L}_{\exists,U}: \ \models \delta \equiv \langle\alpha\rangle\gamma \tag{1}$$

Unfortunately, the new case to be treated at the inductive stage, the propositional quantifier case, *increases* the complexity of the formula by the standard definition. Therefore we must proceed with a little more caution than usual. Let $\varepsilon(\psi)$ be the number of propositional existential quantifiers $\exists$ in ψ.

Notice that we can immediately prove (1) for the case when $\varepsilon(\psi) = 0$, for then there are no existential quantifiers to reduce, and so the new cases are just (a) $\psi := j_i$ with $\alpha = \alpha_i$ and (b) $\psi := j_i$ with $\alpha \neq \alpha_i$. Here we use the 'reduction axioms' from Remark 3: for (a), $\models \langle\alpha\rangle j_i \equiv \top$, and for (b) $\models \langle\alpha\rangle j_i \equiv \bot$. (For the 'old' cases see e.g. [BMS99].)

So it will suffice, to prove the Lemma, to suppose that:

$$\text{(1) holds for } \varepsilon(\psi) < k \tag{2}$$

holds, and then to show that (1) holds for $\varepsilon(\psi) = k$ ($k \in \mathbb{N}^+$).

Again, the 'old' cases are known, so we go straight to the new case:

Let $\psi := \exists p.\gamma$. Here we will choose $p_0, \ldots, p_{n-1}$ not occurring in either γ or any of the PRE_α's, to obtain:

$$\psi' := \exists q_0 \ldots \exists q_{n-1}.(\bigwedge_{i\in n} U(q_i \supset Pre_{\alpha_i}) \wedge \langle\alpha\rangle\gamma(\bigvee_{i\in n}(q_i \wedge j_i))).$$

Then by Lemma 4, $\models \psi' \equiv \langle\alpha\rangle\psi$. Furthermore, notice that $\varepsilon(\gamma) = \varepsilon(\psi) - 1 < \varepsilon(\psi)$, so by (2), there exists $\delta \in \mathcal{L}_{\exists,U}$ such that:

$$\models \delta \equiv \langle\alpha\rangle\gamma(\bigvee_{i\in n}(q_i \wedge j_i)). \tag{3}$$

We now claim that $\models \delta' \equiv \psi'$, where:

$$\delta' := \exists q_0 \ldots \exists q_{n-1}.(\bigwedge_{i\in n} U(q_i \supset Pre_{\alpha_i}) \wedge \delta)$$

Then since $\delta' \in \mathcal{L}_{\exists,U}$ we would be done because also $\models \delta' \equiv \langle\alpha\rangle\psi$, as required.

It only remains, then, to show the claim that $\models \delta' \equiv \psi'$. This follows from (3). To see this, notice that:

$$\begin{array}{ll} & \mathcal{M}, \omega \models \delta' \\ \text{iff} & \exists X_0 \subseteq [\![Pre_{\alpha_0}]\!]^{\mathcal{M}}, \ldots, X_{n-1} \subseteq [\![Pre_{\alpha_{n-1}}]\!]^{\mathcal{M}} \text{ such that} \\ & \mathcal{M}[p_0 \mapsto X_0, \ldots, p_{n-1} \mapsto X_{n-1}], \omega \models \delta \\ \text{iff} & \exists X_0 \subseteq [\![Pre_{\alpha_0}]\!]^{\mathcal{M}}, \ldots, X_{n-1} \subseteq [\![Pre_{\alpha_{n-1}}]\!]^{\mathcal{M}} \text{ such that} \\ & \mathcal{M}[p_0 \mapsto X_0, \ldots, p_{n-1} \mapsto X_{n-1}], \omega \models \langle\alpha\rangle\gamma(\bigvee_{i\in n}(q_i \wedge j_i)) \\ \text{iff} & \mathcal{M}, \omega \models \psi' \end{array}$$

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