Extended Conformal Symmetry in

Non-Critical String Theory

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Uitgebreide Conforme Symmetrie in Niet-Kritische String Theorie

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

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Introduction and Outline

The ingenious step that led Einstein from the special to the general theory of relativity, was to extend the invariance under Poincaré transformations by invariance under general co-ordinate transformations. Later, similar ideas have been successfully applied to other theories. In Maxwell theory, we have the freedom to add the divergence of an arbitrary space-time dependent phase factor to the vector potential. If we replace the phase factor by an arbitrary group, this leads to Yang-Mills theory with a local non-abelian gauge invariance. Any matter theory with a rigid invariance can be made into a gauge theory, by replacing the rigid invariance by a local one. A somewhat more involved example is supergravity, which results from the passage from global to local supersymmetry.

As an example, consider the action for a free complex scalar field,

$$S \sim \int d^4x \, \partial_\mu \phi \partial^\mu \phi^* \tag{1.1}$$

The action is invariant under the rigid $U(1)$ transformation $\phi \to e^{i\theta}\phi$. To make it invariant under local $U(1)$ transformations, *i.e.* with θ an arbitrary function of x^μ, one can use the principle of 'minimal coupling'. Introduce a real gauge field $A_\mu(x)$, and modify the action to

$$S \sim \int d^4x \, (D_\mu \phi)(D^\mu \phi)^*, \tag{1.2}$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative. If the gauge field transforms under a local $U(1)$ transformation as $A_\mu \to A_\mu - \partial_\mu \theta$, the modified action has a local $U(1)$ invariance. Associated to every rigid invariance is a conserved current or Noether current. The integral of this conserved current is a conserved quantity, that can be identified with the generator of the rigid symmetry. An important feature of the extended action (1.2) is that the term linear in A_μ couples to the Noether current of the rigid symmetry. Indeed, (1.2) contains the term

$$i \int d^4x \, A_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \equiv \int d^4x \, A_\mu J^\mu, \tag{1.3}$$

where J^μ is the conserved current associated to the rigid $U(1)$ invariance.

The action (1.1) has another invariance, namely Poincaré invariance. To make this invariance local, we also have to introduce a gauge field, but now minimal coupling does not work. The role of the gauge field is played by a metric on space-time, and the gauged action reads

$$S \sim \int d^4x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* \tag{1.4}$$

If we write the metric as $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, where $\eta^{\mu\nu}$ is the usual flat space metric, we can expand the action in powers of h. The term linear in h is proportional to the Noether current associated to the global translational invariance, which is the energy-momentum tensor of (1.1). Thus, there is a close analogy between the coupling to a metric and the coupling to a gauge field.

In both (1.2) and (1.4) the gauge fields are non-dynamical. To provide them with some dynamics, a special Lagrangian has to be chosen, that does not break the gauge invariance, and gives a kinetic term for them. For the $U(1)$ gauge field this is the usual Maxwell action

$$S_{\rm MW} \sim \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu), \tag{1.5}$$

whereas for gravity it is the Einstein-Hilbert action

$$S_{\rm EH} \sim \int d^4x \sqrt{-g} R. \tag{1.6}$$

The Maxwell action is renormalizable, and the quantization of the $U(1)$ gauge field can proceed without problems. In contrast, the Einstein-Hilbert action is non-renormalizable, and it does not define a consistent quantum theory of gravity.

In this thesis we investigate the transition from rigid to local invariance in two-dimensional field theory, treating all symmetries, including gauge symmetries and general co-ordinate transformations, on an equal footing. The class of two-dimensional field theories we will discuss have the special property that they are invariant under local rescalings or Weyl transformations of the metric. These theories are called conformal field theories. An example is the two-dimensional version of (1.4),

$$S \sim \int d^2x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^*. \tag{1.7}$$

Usually conformal field theories are not presented in a covariant form, but in a form where the symmetries (general co-ordinate invariance and Weyl invariance) have been used to bring the metric in some fixed form. On a flat two-dimensional space, the metric can always be brought in the form $ds^2 = dt^2 - dx^2$. On a non-trivial curved two-dimensional manifold, the space of metrics modulo general co-ordinate and Weyl transformations is a finite dimensional space, called the moduli space of the manifold.

This gives to rise to all kinds of extra complications, but we will ignore these for the time being.

If we put the metric in the form $ds^2 = dt^2 - dx^2$, there still is an infinite-dimensional residual group of co-ordinate transformations under which the metric transforms into a scale factor times itself. Combined with a Weyl transformation, these leave the metric unaltered. This infinite-dimensional group is the group of conformal transformations. In terms of the light-cone co-ordinates $x^{\pm} = t \pm x$, it consists of the co-ordinate transformations of the type $x^+ \to f(x^+)$, and similar for x^-. Thus, it contains two-commuting copies of the same algebra. Remarkably, the same holds for more general symmetry algebras in two-dimensional conformal field theory: they consist of two-commuting copies of the same algebra, one copy depending only on x^+, and one only on x^-. The first half is called the chiral or holomorphic part of the symmetry algebra, the other the anti-chiral or anti-holomorphic part. By a transition from rigid to local symmetries in two dimensions we mean that we pass from the chiral and anti-chiral symmetry algebra (of the conformal field theory in a fixed background metric) to the symmetry algebra where the parameters may have arbitrary x^+, x^- dependence. An example is the passage from the conformal transformations to general co-ordinate transformations. This can be accomplished by restoring the explicit metric dependence.

The specification of a kinetic term for the metric in two dimensions is a special problem. In two dimensions, the graviton has no degrees of freedom, and this is reflected in the fact that the Einstein-Hilbert action in two-dimensions is a topological invariant. However, it turns out that the Weyl invariance is generically broken at the quantum level, and that it can be restored for any conformal field theory by adding an appropriate multiple of the non-local action

$$\int d^2x \sqrt{-g} R \frac{1}{\sqrt{-g}\Box} \sqrt{-g} R \tag{1.8}$$

to the conformal field theory. This action appears as the induced action for the metric when integrating out the matter degrees of freedom from the conformal field theory, and it can serve as the kinetic term for the metric. That has important implications for string theory, which can be seen as follows. Conformal field theories can be seen as theories of strings propagating in some target space, if we identify the two-dimensional manifold on which the conformal field theory was defined with the surface swept out by the strings as it moves in time. For example, (1.7) corresponds to a string moving in a one-dimensional complex target space. The standard bosonic string moving in d real dimensions is described by $(i = 1, \ldots, d)$

$$S \sim \int d^2x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i. \tag{1.9}$$

String theories are only consistent if the Weyl invariance is not broken at the quantum level, leading in the case of the bosonic string to the well known constraint $d = 26$. However, by including the induced action (1.8) as kinetic term for the metric, we can circumvent this constraint on the number of dimensions and obtain a consistent theory starting with any bosonic string theory, in particular with a lower number of dimensions.

Unfortunately, new problems appear in the presence of a propagating metric. If the number of dimensions in which the bosonic string propagates is larger than one, tachyons appear in the spectrum, and the theory is ill-defined. The main motivation for the work described in this thesis, is that it is possible to make sense of theories in more than one dimension, provided the maximal symmetry algebra of the conformal field theory is an extension of the conformal group. In that case on can introduce additional gauge fields besides the metric and construct kinetic terms for these extra gauge fields as well.

An example of an extension of the conformal group is the so-called W_3 algebra. This algebra is non-linear, and the transformation rules for the fields in the theory under these symmetries contain typically second- and higher-derivatives of the fields. This is one of the reasons why theories invariant under W_3 transformations exhibit intriguing complications. For an infinitesimal transformation $\delta\phi = \epsilon\partial X$ with ϵ constant, one can immediately write down a global version, $\phi' = \exp(\epsilon\partial)\phi(x) = \phi(x + \epsilon)$, and observe that it is a co-ordinate transformation. For transformation rules like $\delta\phi = \epsilon\partial^2\phi$, the corresponding global transformation can formally be written down, it is just $\phi' = \exp(\epsilon\partial^2)\phi$, but these transformations have no obvious geometrical interpretation. Nevertheless one can deal with W_3 transformations in an elegant way by exploiting the fact that W_3 transformations can be described by field dependent gauge transformations based on the group $SL(3)$. This property does not only hold for the W_3 algebra, but for a much larger class of extensions of the conformal group, sometimes denoted by W algebras.

Ultimately, one would like to have a complete understanding of these extended conformal algebras, and exactly solve the conformal field theories that possess a W algebra as a symmetry algebra, especially in the presence of the extra gauge fields. In the subsequent chapters we present a detailed study of these conformal field theories and their properties.

This thesis is organized as follows. In chapter 2 we discuss some of the basic features of two-dimensional conformal field theories and discuss a few examples. Subsequently we show that one can construct an extended conformal algebra given any Lie algebra and an sl_2 that is embedded in it. We give explicit expressions for the Poisson brackets of the classical versions of these algebras, and discuss the importance of the sl_2 embedding. The proof that these algebras exist on the quantum level is somewhat more technical and uses BRST cohomology and spectral sequence techniques. We compute the central charge of these algebras, show that they admit free field realizations and proof that they admit an $N = 2$ supersymmetric extension.

In chapter 3 we show how by introducing gauge fields, the transition from global to local invariance can be made for the chiral or the anti-chiral part of the symmetry algebra. The coupling to the gauge fields is simply linear and proportional to the Noether currents as in (1.3). At the quantum level, the local invariance suffers from anomalies, and the invariance can be restored by adding an action for the gauge fields. Although we do not have an explicit expression for this 'induced' action for the gauge fields, we can nevertheless quantize it to all orders in perturbation theory, and obtain an explicit answer for the quantum effective gauge field action.

The obvious next step is to gauge the full symmetry algebra, both the chiral and anti-chiral part. The coupling to the gauge fields is no longer linear, and can only be realized in case the conformal field theory is a free field theory or a so-called WZNW theory. The resulting gauged action contains auxiliary fields, and these cannot be integrated out unless the symmetry algebra is linear. Furthermore, we give an expression for the classical induced action for the gauge fields, which reduces to (1.8) is case the symmetry algebra is just the conformal group. We discuss what is meant by W gravity, and present a result for the covariant action in the conformal gauge, which is valid to all orders. This quantum action for W gravity coupled to W matter is the main result of sections 3 and 4.

For W algebras, there is an analogue of the moduli space (the space of metrics modulo co-ordinate and Weyl transformations). One cannot gauge away the gauge fields completely, and the gauge-fixed path integral contains an integral over the relevant moduli space. In chapter 5 we compute this moduli space for arbitrary W algebras, and give the structure of W algebras on non-trivial manifolds. The induced actions on non-trivial manifolds involve the WZNW action for non-trivial principal fiber bundles. We show how such a generalized action can be constructed, and that it is regularization dependent, but that the covariant action is independent of the chosen regulator. The moduli space is closely connected to the theory of Higgs bundles, leading to an interesting connection between the moduli space and the self-duality equations in four dimensions. We also discuss the moduli space in the presence of marked points.

Finally, in chapter six we analyze the spectrum of the theories obtained so far, and their correlation functions. For gravity, there are three approaches that give useful information. We briefly review these, and comment on their extension to W gravity. We demonstrate how the sum of two W algebras can be gauged, although W algebras are non-linear, and use this to explain why non-critical W gravity coupled to W matter has a BRST invariance. Furthermore, we use the insight obtained in chapters 3 and 4 to shed some light on the relation with $N = 2$ models and topological G/G models. We use this to speculate about the generalized intersection theories that are relevant for W gravity. In the last part of this chapter, we argue that the genus-zero correlation functions for W gravity can be obtained from a matrix model, and conjecture that the correlation functions for genus larger than zero can be obtained from the same matrix model. The integrable hierarchy that describes these correlation functions is given.

With the results in this thesis a more unified algebraic and geometrical framework emerges, leading to a whole new class of exactly integrable field theories. Many exciting open questions remain, which are still the subject of intense study.

This main part of the thesis is based on the work described in [12, 46-55], but several new results have been included. Related material is presented in the PhD theses of J. Goeree and T. Tjin [160, 301].

2

W Algebras

2.1. Conformal Field Theory

2.1.1. Why Conformal Field Theory?

Conformal field theories have many applications in physics, some of which we will review in just a moment. Apart from these applications, the research interest in conformal field theories has been stimulated by the following fact: conformal fields theories have a rich structure, which can be examined very deeply due to the infinite-dimensional symmetry groups they posses, and have relations with many different branches of mathematics, which enables one quite often to express answers in a elegant and simple way. But, as promised, we start by discussing some of the physical motivations to study conformal field theories.

- Two-dimensional statistical systems are, in the neighborhood of a second-order phase transition, described by conformal field theories [76]. The conformal invariance, which arises owing to the fact that the correlation length diverges near a second-order phase transition, can for instance be used to compute critical exponents.

- Under certain conditions, the perturbative vacua of string theory are described by conformal field theories. If one views string theory (or rather, string field theory) as a candidate theory for the unification of the fundamental forces of nature, it should allow a background independent formulation, because the structure of space-time as we know it should somehow dynamically emerge from the theory, rather than be put in from the start. Although a background independent formulation of string theory has not yet been found, there are strong indications that the relevant phase space for such a theory is the space of all two-dimensional field theories, and that the solutions of its classical equations of motion correspond to the conformally invariant ones. The first indication in this direction came from the study of non-linear sigma models, which can describe strings coupled to certain background fields, that parametrize a subset of the space of two-dimensional field theories. In [71, 289, 132] it was shown that if the equations of motion for the background fields, when derived from their effective action, are satisfied, the beta-functions for the background fields vanish. This, in turn, implies that the sigma model is conformally invariant. It is not necessarily true that conformal invariance implies vanishing of the beta-functions [179], because sometimes, in the presence of non-zero beta-functions, the conformal invariance can be restored by adding a local counterterm to the original sigma model action. These counterterms look as if the sigma model were coupled to extra background fields. Therefore, the equivalence of the equations for conformal invariance and for the vanishing of the beta-functions is presumably restored if one includes all possible local couplings to background fields in the sigma model action from the start. Further support for the idea that conformal field theories correspond to perturbative vacua comes from the fact that conformal invariance implies vanishing of the vacuum expectation value of any operator of non-zero dimension, one of the conditions that one is dealing with a proper perturbative vacuum. In subsequent developments it was proposed that the equations of motion of the string field coincide with renormalization group equations [22], see also [228, 69, 68], and that the beta functions are capable of reproducing the string theory S-matrix [72, 178] at tree-level (at the loop level these relations have been studied in [303]). Zamolodchikov's c-theorem is also worth mentioning in this context [332], which states that on the space of renormalizable two-dimensional field theories, there is a function c that strictly decreases along renormalization group trajectories and takes its minimal values at the points corresponding to conformal field theories, where it equals the central charge. One of the main problems of this framework is the lack of a proper description of the space of two-dimensional field theories that does not refer to any particular background. A recent attempt at such a description for open string field theories is given in [324], where the solutions of the equations of motion are the BRST-invariant theories, corresponding to gauge fixed conformal field theories. In any case, from this point of view the classification of conformal invariant field theories corresponds to the classification of the perturbative vacua of string theory.

- In view of the above it is important to classify all conformal field theories. To find the particular superstring theory that describes the perturbative vacuum we live in, is a different problem. In fact, the only contact with experiment can presently be made via superstring inspired standard models. There are by now many candidate conformal field theories that have three generations of chiral fermions and a realistic symmetry breaking pattern. A detailed quantitative comparison of these conformal field theories with experimental data is very difficult, but for some parameters the results are encouraging. For example, extrapolations of recent LEP measurements of low energy gauge couplings show that they are unified at a scale $\sim \mathcal{O}(10^{16}\text{GeV})$ in the minimal supersymmetric standard model [112]. There is a discrepancy between this value and the unification scale $\sim \mathcal{O}(10^{18}\text{GeV})$ predicted by string theory [194], but that can be partially explained by string treshold corrections [9] or by considering extensions of the minimal supersymmetric standard model [113]. A lot of work in this direction remains to be done before a definite answer can be given to the question whether a string theory exists that give a description of the forces of nature consistent with experiment.

- Conformal field theories are useful in describing physical phenomena that take place in effectively two dimensions. An example is the fractional quantum Hall effect, which takes place in two-dimensional electron gases (see [142] and references therein).
- Four-dimensional string theory has been used to develop new computational methods in perturbative QCD [33].
- Several four-dimensional field theories can in an appropriate approximation be reduced to two-dimensional conformal field theories. The exact solvability of the latter may provide useful insights for the corresponding four-dimensional physics. An example is the two-dimensional description of four-dimensional black holes restricted to the radial and time co-ordinate [325, 73, 165].

2.1.2. Conformal Field Theory

We will be rather brief in this section, as many reviews of conformal field theory have appeared in the literature. For more details, see [75, 76, 77, 157, 29, 140].

Consider a generally covariant Euclidean two-dimensional field theory defined on some two-dimensional surface Σ with co-ordinates x_1, x_2. Unless specified otherwise, we will assume that Σ is the complex plane. General covariance implies that the stress-energy tensor is conserved, $\nabla^\mu T_{\mu\nu} = 0$. Conformal field theory deals with those field theories that in addition to the invariance under general co-ordinate transformations are also invariant under (Weyl) rescalings of the two-dimensional metric, $g^{\mu\nu} \to \lambda(x_1, x_2)\, g^{\mu\nu}$. This invariance implies that the energy-momentum tensor is traceless. The invariance under co-ordinate transformations can be used to put the metric in the form

$$ds^2 = e^{-2\varphi}\, dz\, d\bar{z}, \tag{2.1.1}$$

where $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$. The equations for the energy momentum tensor read, in terms of these complex co-ordinates, $T_{z\bar{z}} = 0$, $\bar{\partial} T_{zz} = 0$ and $\partial T_{\bar{z}\bar{z}} = 0$. From now on we will denote T_{zz} simply by $T(z)$ and $T_{\bar{z}\bar{z}}$ by $\bar{T}(\bar{z})$, indicating that T is a holomorphic function of $z, \bar{z}$ and that $\bar{T}$ is an anti-holomorphic function of the complex co-ordinates. If a function f is neither holomorphic nor anti-holomorphic, we will indicate this by writing $f(z, \bar{z})$. The residual group of co-ordinate transformations that preserve the form (2.1.1) of the metric are the so-called conformal transformations. They are the co-ordinate transformations (plus their anti-holomorphic counterparts) of the form $z \to f(z)$, with f holomorphic, and are generated by $T(z)$. Already at this level we see the important role of complex co-ordinates, to which string theory owes part of its beauty.

We can use the conformal transformations to distinguish a special class of fields, the primary fields. Primary fields ϕ of conformal weights or dimensions $(h, \bar{h})$ transform under conformal transformations as

$$\phi(z, \bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w, \bar{w}). \tag{2.1.2}$$

An important subset of the space of primary fields is the set of holomorphic primary fields, *i.e.* the fields with $\bar{h} = 0$. These fields generate the chiral algebra $\mathcal{A}$ of the theory. The holomorphic primary fields are conserved currents by virtue of the equation $\bar{\partial}(\text{field}) = 0$ they satisfy, and as such they generate symmetries of the theory. The chiral algebra is the maximally extended symmetry algebra of the theory. The stress-energy tensor, being the generator of the conformal transformations, has weight $(2, 0)$ and is one of the generators of the chiral algebra. In the same way one defines the anti-chiral algebra $\bar{\mathcal{A}}$ as the set of anti-holomorphic primary fields

So far, we took the complex plane as the world-sheet of the two-dimensional field theory. In terms of string theory, this corresponds to an infinitely long open string propagating in time. To make contact with the theory of closed strings, we can start with the complex plane without the origin. This is topologically equivalent to an infinitely long cylinder, which we view as a closed string propagating in time. Going back to the punctured complex plane, this means that time goes radially outwards, and that the origin corresponds to $t = -\infty$. Thus we need radial quantization to describe closed strings. In particular this means that the charges associated to a conserved current, that are given by an integral along constant time surfaces, are in this case given by a contour integral around the origin. This has certain implications for the symmetry algebra of the theory. Given two fields $A_1(z)$ and $A_2(z)$ of the chiral algebra, the charges that generate the corresponding symmetries with parameters $\epsilon_1(z)$ and $\epsilon_2(z)$ are given by

$$Q_i = \oint_0 \frac{dz}{2\pi i} \epsilon_i(z) A_i(z). \tag{2.1.3}$$

The commutator of the two charges is easily seen to be equal to

$$[Q_1, Q_2] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i}\, \epsilon_1(z)\, \epsilon_2(w) A_1(z) A_2(w). \tag{2.1.4}$$

This shows that the commutator is completely determined by the singular terms in the short-distance operator product expansion (OPE) of A_1 and A_2,

$$A_1(z)A_2(w) \sim \sum_{r\geq 0} \frac{\{A_1 A_2\}_r(w)}{(z-w)^r} + \text{regular terms} \tag{2.1.5}$$

In the same way we find that

$$[Q_1, A_2(w)] = \oint_w \frac{dz}{2\pi i} \epsilon_1(z) A_1(z) A_2(w) \tag{2.1.6}$$

can also be evaluated using (2.1.5). Therefore, specifying the commutation relations (2.1.4), specifying the OPE's (2.1.5) or specifying the variations of the currents (2.1.6) are three equivalent ways to describe the structure of the chiral algebra, and one can use the formulas given above to go from one to the other. The commutation relations (2.1.3) are usually given for the (Fourier) modes of A_i defined by

$$A_m = \oint_0 \frac{dz}{2\pi i} z^{m+h_A-1} A(z), \qquad A(z) = \sum_m A_m z^{-m-h_A}. \tag{2.1.7}$$

Operator product expansions can be defined for all local fields in the theory, not just for those that are in the chiral algebra. The operator product expansion will in general not only exhibit poles when $z \to w$, but also when $\bar{z} \to \bar{w}$. To given an example of an operator product expansion, consider an infinitesimal co-ordinate transformation $z \to z + \epsilon(z)$ generated by $Q = \oint_0 \frac{dz}{2\pi i} \epsilon(z) T(z)$. From (2.1.2) one derives that

$$\delta_Q \phi(w, \bar{w}) = h\phi\partial\epsilon + \epsilon\partial\phi. \tag{2.1.8}$$

This leads to the operator product expansion

$$T(z)\,\phi(w,\bar{w}) \sim \frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)} + \text{regular terms.} \tag{2.1.9}$$

Naively, the same result is expected to hold for $\phi = T$ as well. However, one can argue [75] that in general on the quantum level the OPE of T with itself is modified to

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{regular,} \tag{2.1.10}$$

where c is the so-called central charge of the conformal field theory. For the modes $L_m = \oint \frac{dz}{2\pi i} z^{-m-1} T(z)$ of T we find the following commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{2.1.11}$$

This is the Virasoro algebra.

We conclude this section with some remarks on the Hilbert space of the theory. There is a one-to-one correspondence between local fields and states in the Hilbert space as follows:

$$|\phi\rangle \leftrightarrow \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{2.1.12}$$

The Hilbert space can be decomposed in representations of the maximal symmetry algebra of the theory, which is the product of the chiral algebra and the anti-chiral algebra, $\mathcal{A} \times \bar{\mathcal{A}}$. In case the Hilbert space is a finite sum of irreducible representations of $\mathcal{A} \times \bar{\mathcal{A}}$, we are dealing with a rational conformal field theory (RCFT). The name is justified by the fact that the central charge and conformal dimensions of primary fields in such theories are always rational [8]. These theories have features which resemble the theory of compact groups. For instance, in a RCFT all unitary representations of $\mathcal{A}$ occur with multiplicity one*, one can extract the analogue of the representation ring, in this case called the fusion rules, one has analogues of the Clebsch-Gordan coefficients, etc. For these results and more, see [308, 88, 243, 46]

The simplest case is when $\mathcal{A}$ is the Virasoro algebra with central charge c. The generator of time translations, $L_0 + \bar{L}_0$, is to be identified with the Hamiltonian. Any representation must have a state with lowest energy, say $|h,c\rangle$ with energy $L_0|h,c\rangle = h|h,c\rangle$. Because there are no states with lower energy, $L_n|h,c\rangle = 0$ for $n > 0$. These are precisely the conditions for a highest weight module of the Virasoro algebra. Such a highest weight module is spanned by the states

$$L_{-k_1} L_{-k_2} \ldots L_{-k_j} |h,c\rangle, \tag{2.1.13}$$

with all $k_i > 0$, and denoted by $M(h,c)$. On M one can define a symmetric bilinear form, compatible with the two conditions $L_n^\dagger = L_{-n}$ and $\langle h,c|h,c\rangle = 1$. If M is not irreducible, it has null states that are orthogonal to every other state with respect to this bilinear form. In that case one can prove that $L(h,c) = M(h,c)/\text{nullstates}$ is an irreducible

*This should be compared with the Peter-Weyl theorem for compact groups G, which states that $L^2(G)$, which is a representation of $G \times G$ via $(h_1,h_2)\cdot f(g) = f(h_1 g h_2^{-1})$, is isomorphic as $G \times G$ representation to $\oplus_R R \times R^*$, where the sum is over all inequivalent unitary irreducible representations of G with multiplicity one. The correspondence with an RCFT is Hilbert space $\leftrightarrow L^2(G)$ and $G \times G \leftrightarrow \mathcal{A} \times \bar{\mathcal{A}}$.

representation of the Virasoro algebra. Usually, M is called a Verma module, and L an irreducible highest-weight module. If one demands in addition that L is unitary, *i.e.* that there are no states of negative norm, one finds the following conditions on h and c [141, 159]: either $c \geq 1$ and $h \geq 0$, or

$$\begin{aligned} c &= 1 - \frac{6(q-p)^2}{pq}, \\ h &= \frac{(qr-ps)^2-(q-p)^2}{4pq}, \qquad 0<r<p, \quad 0<s<q, \end{aligned} \tag{2.1.14}$$

for integers r, s, and $q = p + 1 = 3, 4, \ldots$. The conformal field theories associated to these series are called the unitary minimal models. There is also a set of nonunitary minimal models. The expressions for c and h for these models are given by (2.1.14) with p and q subject to $1 < p + 1 < q = 3, 4, \ldots$. In the sequel both the unitary and nonunitary minimal models will be referred to as the (p, q) minimal models.

2.1.3. Operator Product Algebras and Field Algebras

The chiral algebra is an example of a field algebra or operator product algebra. Such an algebra $\mathcal{A}$ is the vector space spanned by a collection of meromorphic fields $A_\alpha(z)$, that satisfies the following properties

1. If $A_\alpha, A_\beta \in \mathcal{A}$, then also $\partial A_\alpha, \partial A_\beta, (A_\alpha A_\beta) \in \mathcal{A}$, where $(A_\alpha A_\beta)$ denotes the normal ordered product of A_α and A_β, defined by point-splitting regularization [14],

$$(A_\alpha A_\beta)(w) \equiv \oint_w \frac{dz}{2\pi i} \frac{A_\alpha(z)A_\beta(w)}{(z-w)}. \tag{2.1.15}$$

2. Associated to every pair of elements A_α, A_β is an operator product expansion, whose singular part is denoted by $A_\alpha\underbracket{(z)A_\beta}(w)$, which is a finite sum

$$A_\alpha\underbracket{(z)A_\beta}(w) = \sum_{r>0} \frac{\{A_\alpha A_\beta\}_r(w)}{(z-w)^r}. \tag{2.1.16}$$

   Furthermore, $\{A_\alpha A_\beta\}_r \in \mathcal{A}$.

3. Operator product expansions satisfy $A_\alpha\underbracket{(z)A_\beta}(w) = A_\beta\underbracket{(w)A_\alpha}(z)$ modulo regular terms, which implies that

$$\{A_\alpha A_\beta\}_r = \sum_{t\geq 0} \frac{(-1)^{t+r}}{t!} \partial^t \{A_\beta A_\alpha\}_{t+r}. \tag{2.1.17}$$

4. The operator product expansion behaves as follows under ∂ and normal ordering

$$\begin{aligned}
\underbracket{\partial A_\alpha(z)\, A_\beta(w)} &= \partial_z(\underbracket{A_\alpha(z)\, A_\beta(w)}),\\
\underbracket{A_\alpha(z)\, \partial A_\beta(w)} &= \partial_w(\underbracket{A_\alpha(z)\, A_\beta(w)}),\\
\underbracket{A_\alpha(z)\, (A_\beta A_\gamma)(w)} &= \frac{1}{2\pi i}\oint_w \frac{dx}{(x-w)}(\underbracket{A_\alpha(z)A_\beta(x)}A_\gamma(w) + A_\beta(x)\underbracket{A_\alpha(z)A_\gamma(w)}).
\end{aligned} \tag{2.1.18}$$

The latter is precisely Wick's theorem. Notice that the first line implies that $\{\partial A_\alpha A_\beta\}_1 = 0$.

5. The normal ordering is neither commutative nor associative. However, the following two relations are valid

$$\begin{aligned}
(A_\alpha A_\beta) - (A_\beta A_\alpha) &= \sum_{r\geq 0} \frac{(-1)^{r+1}}{r!}\partial^r\{A_\alpha A_\beta\}_r,\\
(A_\alpha(A_\beta A_\gamma)) - (A_\beta(A_\alpha A_\gamma)) &= ((A_\alpha A_\beta)A_\gamma) - ((A_\beta A_\alpha)A_\gamma).
\end{aligned} \tag{2.1.19}$$

6. Associativity of the operator product algebra. This states that for any function $f(x,z,w)$ the following identity holds

$$\begin{aligned}
&\oint_0 \frac{dx}{2\pi i}\oint_x \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}(\underbracket{\underbracket{A_\alpha(z)A_\beta(w)}A_\gamma(x)}f(x,z,w)) +\\
&\oint_0 \frac{dz}{2\pi i}\oint_z \frac{dx}{2\pi i}\oint_x \frac{dw}{2\pi i}(\underbracket{\underbracket{A_\beta(w)A_\gamma(x)}A_\alpha(z)}f(x,z,w)) +\\
&\oint_0 \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}\oint_z \frac{dx}{2\pi i}(\underbracket{\underbracket{A_\gamma(x)A_\alpha(z)}A_\beta(w)}f(x,z,w)) = 0.
\end{aligned} \tag{2.1.20}$$

Associativity of the operator product algebra is equivalent to crossing symmetry of the four-point function (but see [65]).

A field algebra is a conformal field algebra if it contains a field T with OPE given by (2.1.10), and it is generated by a set of primary fields that have an OPE of type (2.1.9) with T. A field algebra is a quantum generalization of a Poisson algebra. Given any Poisson algebra whose fields live on a circle, one can write down a set of OPE's via the correspondence†

$$\begin{aligned}
\underbracket{A_\alpha(z)\, A_\beta(w)} &= \sum_{r>0}\frac{\{A_\alpha A_\beta\}_r(w)}{(z-w)^r} \Leftrightarrow\\
\{A_\alpha(z), A_\beta(w)\}_{\mathrm{PB}} &= \sum_{r>0}\frac{1}{(r-1)!}\{A_\alpha A_\beta\}_r(w)\,\partial_w^{r-1}\delta(z-w).
\end{aligned} \tag{2.1.21}$$

†On the level of charges the equivalence is $\oint \frac{dz}{2\pi i}\epsilon(z)A(z) \leftrightarrow \int dz\, \epsilon(z)A(z)$.

These OPE's do not satisfy the definition of a field algebra. In particular, the algebra is associative and commutative, and Wick's theorem is replaced by

$$\underbrace{A_\alpha(z)\,(A_\beta A_\gamma)}(w) = \underbrace{A_\alpha(z)A_\beta}(w)A_\gamma(w) + A_\beta(w)\underbrace{A_\alpha(z)A_\gamma}(w). \tag{2.1.22}$$

A set of fields and OPE's that are obtained from a Poisson algebra via (2.1.21) will be called a classical operator product of field algebra. In many cases it is possible to introduce a parameter $\hbar$ in the field algebra, such that in the limit where $\hbar \to 0$, a classical field algebra is recovered. In that case we call the field algebra a quantization of the underlying Poisson algebra. Later on we will see plenty of examples of conformal field algebras. We finish this rather abstract exposition by taking a look at the derivations of field algebras. This knowledge is important if we want to use cohomological techniques for field algebras. Given any $Q \in \mathcal{A}$, we can associate a derivation δ_Q to it, given by

$$\delta_Q(A(z)) = [\oint_0 \frac{dw}{2\pi i} Q(w), A(z)] = \{QA\}_1(z). \tag{2.1.23}$$

It follows from (2.1.18) that this is indeed a derivation, meaning that $\delta_Q(\partial A) = \partial\delta_Q A$ and that δ_Q satisfies the Leibnitz rule $\delta_Q(A_\alpha A_\beta) = ((\delta_Q A_\alpha)A_\beta) + (-1)^{\epsilon_Q\epsilon_\alpha}(A_\alpha(\delta_Q A_\beta))$, where $\epsilon_X = 0(1)$ if X is bosonic (fermionic). When is a derivation nilpotent? From the associativity condition (2.1.20) one can derive that

$$\{A_\alpha\{A_\beta A_\gamma\}_1\}_1 - (-1)^{\epsilon_\alpha\epsilon_\beta}\{A_\beta\{A_\alpha A_\gamma\}_1\}_1 = \{\{A_\alpha A_\beta\}_1 A_\gamma\}_1. \tag{2.1.24}$$

If $A_\alpha = A_\beta = Q$ and Q is fermionic, then (2.1.24) reads $\delta_Q^2 = \delta_{\frac{1}{2}\{QQ\}}$, which vanishes if $\{QQ\}_1$ is a total derivative. Thus, to check for example whether a BRST operator is nilpotent or not, we must compute $\{QQ\}_1$ and check whether it is a total derivative or not.

The reader may wonder why we introduced the concept of a field algebra, because one can also work with the Fourier modes of the fields. In terms of these modes, the associativity condition is just the Jacobi identity. However, the full algebra contains certain infinite sums of products of modes, due to the normal ordered products of fields that occur, and it is difficult to characterize which infinite power series are allowed and which are not. In terms of fields, one always works with finite expressions. For this reason it is much easier to prove things rigorously at the level of field algebras than at the level of modes.

2.1.4. The Free Scalar Field

The first example of a conformal field theory is the free scalar field described by the action

$$S = \frac{1}{8\pi} \int d^2x \, \sqrt{g} g^{\alpha\beta} \, \partial_\alpha X \, \partial_\beta X. \tag{2.1.25}$$

The factor in front has been chosen for convenience. In the conformal gauge (2.1.1) this action reads

$$S = \frac{1}{2\pi} \int d^2z \, \partial X \, \bar{\partial} X, \tag{2.1.26}$$

where d^2z is the metric associated to $dx_1 \wedge dx_2$ rather than $dz \wedge d\bar{z}$, to avoid the introduction of factors of i. The chiral algebra is generated by ∂X, the anti-chiral algebra by $\bar{\partial} X$. The corresponding symmetries are $X(z, \bar{z}) \to X(z, \bar{z}) + \epsilon(z)$ and $X(z, \bar{z}) \to X(z, \bar{z}) + \epsilon(\bar{z})$. Under a co-ordinate transformation $z \to z + \epsilon(z)$ the field X transforms as $X \to X + \epsilon \partial X$. This is just a special case of the symmetry generated by ∂X, showing that the energy-momentum tensor is in this case not a new, independent element of the chiral algebra but can be expressed in terms of ∂X. Of course, this follows also immediately from (2.1.25), using the standard definition of the energy momentum tensor $T_{ab} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}}$. The two-point function of the field X is easily derived from (2.1.26). It is given by

$$\langle X(z, \bar{z}) \, X(w, \bar{w}) \rangle = -\pi G(z, w), \tag{2.1.27}$$

where $G(z, w)$ is the Green's function for the operator $\partial \bar{\partial}$. It can be computed from the fundamental identity

$$\bar{\partial}_{\bar{z}} \frac{1}{z - w} = \pi \delta^2(z - w). \tag{2.1.28}$$

Here, the delta function is with respect to the measure d^2z. Of course, (2.1.28) should be understood in the sense of distributions. Integrating this equation yields $G(z, w) = \pi^{-1} \log |z - w|$, showing that X itself is not a well-defined local field. For the two-point function of ∂X we now immediately find

$$\langle \partial X(z) \, \partial X(w) \rangle = \frac{-1}{(z - w)^2} \tag{2.1.29}$$

from which we find the OPE of ∂X with itself,

$$\underbrace{\partial X(z) \, \partial X(w)} = \frac{-1}{(z - w)^2}. \tag{2.1.30}$$

The energy momentum tensor is given by

$$T = -\tfrac{1}{2}(\partial X \partial X), \tag{2.1.31}$$

and from Wick's theorem (2.1.18) we find the OPE's

$$\begin{aligned} \underbrace{T(z)\,\partial X}(w) &= \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{(z-w)}, \\ \underbrace{T(z)\,T}(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \end{aligned} \tag{2.1.32}$$

Thus, ∂X is a primary field of conformal weight $(1,0)$, and T generates a Virasoro algebra with central charge $c = 1$. The Hilbert space consists of the representations of $\mathcal{A} \times \bar{\mathcal{A}}$ that are obtained by acting on the states $|p, \bar{p}\rangle$ with the elements of $\mathcal{A} \times \bar{\mathcal{A}}$. The states $|p, \bar{p}\rangle$ are created from the vacuum $|0\rangle$ by the vertex operators

$$V_{p,\bar{p}} = \exp(ipX + i\bar{p}\bar{X}), \tag{2.1.33}$$

where we decomposed $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$. The vertex operators $V_{p,\bar{p}}$ are primary fields of conformal weight $(p^2/2, \bar{p}^2/2)$. Using the two-point function of X one can in principle compute the correlation functions of the vertex operators [310]. This is more difficult than to compute correlation functions of elements of the chiral algebra. Because the latter consists of conserved currents, the correlation functions are completely fixed by Ward identities. However, when we compute correlation functions of arbitrary states in the Hilbert space, these are only partially fixed by Ward-identities. Their computation is, in a sense, the central problem of conformal field theory.

Any local primary field V with weights (h, h) can be added to the action without affecting general covariance, if we include a suitable power of the metric:

$$\Delta S = \lambda \int d^2 z (\sqrt{g})^{1-h} V. \tag{2.1.34}$$

Clearly, this term spoils conformal invariance unless $h = 1$, in which case the corresponding perturbations are called marginal. Perturbations with $h > 1$ are called irrelevant, perturbations with $h < 1$ are called relevant‡. An example of the latter is the addition

‡This terminology stems from the behavior of the perturbation under the renormalization group flow, the fixed points of which correspond to conformal field theories. A conformal field theory with an irrelevant perturbation added to it flows back to itself, whereas relevant perturbations push the theory away from the original fixed point to another conformal field theory.

of a mass term for the scalar field X in (2.1.25), although strictly speaking the field X is not a primary field. An example of a marginal operator is

$$V = \cos(\sqrt{2}\, X(z,\bar{z})) \tag{2.1.35}$$

A priori it is not clear that the resulting theory will be conformal for noninfinitesimal values of the coupling λ. An additional condition has to be satisfied [193, 90], namely that the operator product expansion of V with itself may not contain any other primary fields with $h = \bar{h} = 1$, as this would imply that in the perturbed conformal field theory V becomes a linear combination of fields with different weights. This condition is easily verified for the perturbation (2.1.35) using momentum conservation that exists because the field X has a zero mode. Therefore we get a one-parameter family of conformal field theories

$$S = \frac{1}{2\pi}\int d^2z\, \partial X\, \bar{\partial} X + \lambda \int d^2z\, \cos(\sqrt{2}\, X(z,\bar{z})). \tag{2.1.36}$$

What happens to the chiral algebra when we add a perturbation to the theory? To investigate this, we take $A \in \mathcal{A}_{\text{unperturbed}}$ and consider how the perturbation transforms under a transformation generated by A

$$\begin{aligned} \delta_A \int d^2w V(w,\bar{w}) &= \int d^2w \oint_w \frac{dz}{2\pi i}\epsilon(z) A(z) V(w,\bar{w}) \\ &= \int d^2w \oint_w \frac{dz}{2\pi i} \sum_{r>0} \frac{\epsilon(z)\{AV\}_r(w,\bar{w})}{(z-w)^r} \\ &= \int d^2w \sum_{r>0} \frac{\partial^{r-1}\epsilon(w)}{(r-1)!}\{AV\}_r(w,\bar{w}) \\ &= \int d^2w\, \epsilon(w) \sum_{r>0} \frac{(-1)^{r-1}}{(r-1)!}\partial^{r-1}\{AV\}_r(w,\bar{w}) \\ &= -\int d^2w\, \epsilon(w)\{VA\}_1(w,\bar{w}), \end{aligned} \tag{2.1.37}$$

where in the last line we used (2.1.17). From this last expression we see that A also generates a symmetry of the perturbed action if $\{VA\}_1 = 0$, or in other words, $A \in \ker(\delta_V)$. The effect of the perturbation is that it has broken part of the symmetry of the original action. The derivation δ_V is a derivation of the full operator algebra, and need not be a derivation of the chiral algebra. Nevertheless, it is easy to see that $\ker(\delta_V)|_{\mathcal{A}}$ is a closed subalgebra of $\mathcal{A}$, that contains the Virasoro algebra. This subalgebra is usually called the centralizer of the vertex operator V. For the free scalar field with perturbation (2.1.35), the centralizer is precisely the Virasoro algebra generated by $T = -\frac{1}{2}(\partial\phi\partial\phi)$. In general, it may happen that at certain finite values of the coupling λ new symmetries will appear, that were not present in the original theory. We will ignore this possibility in the remainder.

The observation that the Virasoro algebra is the centralizer of a vertex operator of a free field algebra, is a very important one. If one can realize a chiral algebra as the centralizer of a free field algebra, this can be extremely useful for the computation of correlation functions [310, 124, 150, 115]. Later on we will see other examples of this phenomenon.

Using the same principle, it is also possible to obtain the Virasoro algebra with $c \neq 1$ from one free scalar field, which is important for the computation of correlators in the minimal models (2.1.14), by introducing a background charge for the field X [119, 140, 310]. At the level of actions, it amounts to adding to the the free field action (2.1.25) a term which couples the free field to the curvature of the two-dimensional surface[§]

$$S = \frac{i\alpha_0}{4\pi} \int d^2x \sqrt{g} R X. \tag{2.1.38}$$

With this term, ∂X is strictly speaking no longer holomorphic, but satisfies $\bar{\partial}\partial X = i\alpha_0\sqrt{g}R/4$, and is no longer part of the chiral algebra. However, if we imagine the curvature R to be localized at infinity, ∂X is still holomorphic and the extra term manifests itself in two ways: it imposes a selection rule on the correlation functions of the theory, and the energy-momentum tensor is modified to

$$T = -\tfrac{1}{2}(\partial X \partial X) - i\alpha_0 \partial^2 X, \tag{2.1.39}$$

with central charge

$$c = 1 - 12\alpha_0^2. \tag{2.1.40}$$

The parameter α_0 is called the background charge of the field X. The background charge term modifies the conformal weight of the vertex operators (2.1.33) from $p^2/2$ to $p^2/2 + \alpha_0 p$. In the presence of a background charge, the condition for the correlation function of a product of vertex operators with momenta p_i to be non-vanishing reads

$$\sum_i p_i = -\alpha_0(2 - 2h), \tag{2.1.41}$$

where h is the number of handles of the two-dimensional surface.

2.1.5. First-Order Systems

First-order systems are described by actions with a first-order kinetic term. A standard example of a conformal field theory of this type is the free Majorana fermion in

[§]Our convention for the curvature is $R = g^{ab} R^{e}{}_{abe}$.

two dimensions. Two other examples that we will encounter later are the fermionic b,c system and the bosonic β,γ system [140]. The fermionic b,c system is described by the action

$$S = \frac{1}{2\pi} \int d^2z \, (b\bar{\partial}c + c\bar{\partial}b) + \text{c.c.}, \tag{2.1.42}$$

where c has conformal weight $(j,0)$ and b has conformal weight $(1-j,0)$. The chiral algebra is generated by b and c. Associated to the invariance $c \to \exp(i\alpha(z))c$ and $b \to \exp(-i\alpha(z))b$ is the ghost-number current $J = -(bc)$. The ghost-number operator $N = \oint_0 \frac{dz}{2\pi i} J(z)$ satisfies $[N, c(w)] = c(w)$ and $[N, b(w)] = -b(w)$. The operator product expansion of b and c is

$$\underbrace{b(z)\, c}(w) = \frac{1}{z-w}, \tag{2.1.43}$$

and the energy-momentum tensor is given by

$$T = (j-1)(b\partial c) - j(c\partial b), \tag{2.1.44}$$

with central charge $c = -2(6j(j-1)+1)$. These b,c systems frequently arise as the Faddeev-Popov ghosts that are needed when gauge-fixing a bosonic symmetry. To gauge-fix a fermionic symmetry, we need bosonic β,γ systems. These are described by the action

$$S = \frac{-1}{2\pi} \int d^2z \, (\beta\bar{\partial}\gamma - \gamma\bar{\partial}\beta) + \text{c.c.}, \tag{2.1.45}$$

where again γ has weight $(j,0)$ and β has weight $(1-j,0)$. The operator product expansion of β and γ is

$$\underbrace{\beta(z)\gamma}(w) = \frac{1}{z-w}, \tag{2.1.46}$$

and the energy-momentum tensor is given by

$$T = (1-j)(\beta\partial\gamma) - j(\gamma\partial\beta), \tag{2.1.47}$$

with central charge $c = 2(6j(j-1)+1)$.

2.1.6. The Wess-Zumino-Novikov-Witten Model

The third example of a conformal field theory we consider is the Wess-Zumino-Novikov-Witten (WZNW) [313, 255, 320] theory [261, 148, 208, 123]. It is a non-linear sigma model with as target space a group manifold and its action is given by

$$kS^{\pm}_{wznw}(g) = -\frac{k}{8\pi} \int_\Sigma d^2x \, \sqrt{g} g^{\alpha\beta} \, \mathrm{Tr}(g^{-1}\partial_\alpha g \, g^{-1}\partial_\beta g) \mp \frac{ik}{12\pi} \int_{B_3} \mathrm{Tr}(g^{-1}dg)^3, \tag{2.1.48}$$

where B_3 is a three-manifold with boundary $\partial B_3 = \Sigma$, g is a group-valued field with values in a compact, simply connected semisimple Lie group and Tr is an invariant form on the Lie algebra $\mathbf{g}$ of G normalized such that the length squared of the longest root of $\mathbf{g}$ is -2. It is related to the Killing form via $\mathrm{Tr}(XY) = (2h)^{-1}\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$, where h is the dual Coxeter number of the Lie algebra. For the Lie groups A_l and C_l it is equal to the trace in the fundamental representation, for B_l and D_l it is one half of that. In (2.1.48) the parameter k is called the level, and must be an integer. The latter condition arises due to the fact that one has the freedom to choose a three-manifold B_3 and an extension of $g : \Sigma \to G$ to $g : B_3 \to G$. If we choose two extensions B_3^1, g^1 and B_3^2, g^2, with $\partial B_3^1 = \partial B_3^2 = \Sigma$, the difference between the WZNW action evaluated for these two choices is

$$\mp\frac{ik}{12\pi}\int_B \mathrm{Tr}(g^{-1}dg)^3, \tag{2.1.49}$$

where B is the closed three-manifold $B = B_3^1 \cup (-B_3^2)$ and $g = g^1 \cup g^2$. We can rewrite (2.1.49) as

$$\mp\frac{ik}{4\pi}\int_{g(B)} \omega, \tag{2.1.50}$$

where ω is the three-form on G defined by $\omega(X, Y, Z) = \mathrm{Tr}(g^{-1}X[g^{-1}Y, g^{-1}Z])$ for $X, Y, Z \in T_gG$. Now $-\omega/8\pi^2$ is the generator of $H^3(G, \mathbb{Z})$ [135], and therefore $\int_{g(B)}\omega$ equals $-8\pi^2$ times the winding number n of $g : B \to G$, from which we see that (2.1.50) equals $\pm 2\pi i k n$. In the Euclidean path integral this exponentiates to one, and the path integral is therefore independent of the choice of extension.

The WZNW action (2.1.48) satisfies an important identity, the Polyakov-Wiegmann identity

$$\begin{aligned} S^-_{wznw}(gh) &= S^-_{wznw}(g) + S^-_{wznw}(h) - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(g^{-1}\partial g\bar\partial h h^{-1}), \\ S^+_{wznw}(gh) &= S^+_{wznw}(g) + S^+_{wznw}(h) - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(g^{-1}\bar\partial g\partial h h^{-1}) \end{aligned} \tag{2.1.51}$$

which can be derived from

$$\mathrm{Tr}((gh)^{-1}d(gh))^3 = \mathrm{Tr}(g^{-1}dg)^3 + \mathrm{Tr}(h^{-1}dh)^3 - 3d\,\mathrm{Tr}(g^{-1}dg \wedge dhh^{-1}). \tag{2.1.52}$$

The Polyakov-Wiegmann identities tell us that S^-_{wznw} is invariant under the infinitesimal symmetries $\delta g = \epsilon(\bar z)g + g\epsilon'(z)$. These are generated by the currents $\mathcal{J} = k\, g^{-1}\partial g$ and $\bar{\mathcal{J}} = k\,\bar\partial g g^{-1}$, that are the generators of the chiral and anti-chiral algebra respectively (notice that the equations of motion imply $\bar\partial\mathcal{J} = 0$). The Poisson bracket of $\mathcal{J}$ with itself is

$$\{J^a(z)J^b(w)\}_{\mathrm{PB}} = k\eta^{ab}\partial_w\delta(z-w) + f^{ab}_{\ \ c}J^c(w)\delta(z-w), \tag{2.1.53}$$

where we introduced a basis $\{T_a\}$ of $\mathbf{g}$, $\eta_{ab} = \mathrm{Tr}(T_a T_b)$, η^{ab} is the inverse of η_{ab}, $f_{ab}{}^c T_c = [T_a, T_b]$ and indices are raised and lowered using η. Using (2.1.21) the corresponding classical OPE is

$$J^a(z)J^b(w) = \frac{k\eta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w}. \tag{2.1.54}$$

One can show that this OPE does not get renormalized at the quantum level, so that it defines a quantum operator product algebra, which is the chiral algebra of the WZNW model. As was the case for the free scalar field, the energy-momentum tensor is not an independent element of the chiral algebra, but a composite field given by

$$T = \frac{1}{2(k+h)}\eta_{ab}(J^a J^b) \tag{2.1.55}$$

with central charge

$$c = \frac{k\dim(G)}{k+h}. \tag{2.1.56}$$

The normalization factor $(2(k+h))^{-1}$ in the definition of the energy-momentum tensor is fixed by the requirement that the operator product expansion of T with itself is of the form (2.1.10). The modes of the currents J^a form an affine Lie algebra; these have been studied extensively in the mathematics literature (see *e.g.* [271, 192]). The unitary representations are labeled by the positive weights $\lambda \in P_+$ of $\mathbf{g}$ that satisfy $\lambda\cdot\psi \leq k$, where ψ is the highest root of $\mathbf{g}$. There are only finitely many such representations, and therefore WZNW theories are examples of rational conformal field theories. The representations are obtained by acting with the chiral algebra on the states $\lim_{z,\bar{z}\to 0} r_\lambda(g(z,\bar{z}))|0\rangle$ where r_λ is the representation of G corresponding to the highest weight λ.

For non-compact groups we use the same definition of the WZNW action, the only difference is that for non-compact groups we require that Tr is such that the length squared of the longest root is $+2$. This means that for A_l and C_l we can still use the trace in the fundamental representation, and that Tr is still $(2h)^{-1}$ times the Killing form. For non-compact groups the kinetic term is unbounded from below, whether one takes k or minus k. Nevertheless we will frequently use the WZNW action for noncompact groups in the sequel, and quantize it without worrying about the unboundedness of the action. That the quantum theory makes sense lies in the fact that we never consider the pure WZNW action for non-compact groups in itself, but only gauged and constrained versions of it. The extra symmetries of these gauged and constrained WZNW theories restore the boundedness of the spectrum, as in [146]. Also, there is no necessity to restrict k to be integral in the case of non-compact real Lie algebras.

We have seen in section 2.1.4 that the Virasoro algebra could be obtained as the centralizer of a vertex operator acting on the chiral algebra of one free scalar field. It turns out that a similar statement holds in the case of affine Lie algebras: all affine Lie algebras are isomorphic to the centralizer of a collection of vertex operators acting on the chiral algebra of a free field theory. The affine Lie algebra is the chiral algebra of the free field theory perturbed by the vertex operators. The free field theory contains in general both free scalar fields X as in section 2.1.4, with chiral algebra generated by ∂X, and first-order systems as in section 2.1.5, with chiral algebra generated by b, c or β, γ. In particular this implies that all currents of the WZNW model can be expressed in terms of free fields in such a way that the free field expressions have the same operator product expansions as the currents. An expression of the currents in terms of free fields is called a free field realization of the underlying affine Lie algebra. The existence of such free field realizations was first shown by Wakimoto in the case of $G = SL_2$ [311], and generalized to arbitrary G in [150, 245, 59, 62, 122]. To express an affine Lie algebra in terms of free fields, one needs $r_G = \text{rank}(G)$ free scalar fields, and a bosonic β, γ system with $j = 0$ for every positive root of $\mathbf{g}$. The affine Lie algebra is the centralizer of r_G vertex operators. Let us, without proof, give a free field realization for the case $G = SL_3(\mathbb{R})$. Working in the fundamental representation we decompose the current as

$$\mathcal{J} = J^a T_a = \begin{pmatrix} H^0 & J^1 & J^3 \\ K^1 & H^1 - H^0 & J^2 \\ K^3 & K^2 & -H^1 \end{pmatrix}. \tag{2.1.57}$$

For $SL_3(\mathbb{R})$ we need two free scalar fields $\phi_i, i = 1, 2$ and three β, γ systems, $\beta_i, \gamma_i, i = 1, 2, 3$. The OPE's are

$$\begin{aligned} \underbrace{\partial\phi_i(z)\,\partial\phi_j(w)} &= \frac{-\delta_{ij}}{(z-w)^2}, \\ \underbrace{\beta_i(z)\,\gamma_j(w)} &= \frac{\delta_{ij}}{z-w}. \end{aligned} \tag{2.1.58}$$

To give the expressions for the currents in a more compact form, we combine (ϕ_1, ϕ_2) into one vector $\boldsymbol{\phi}$, and introduce two vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$ satisfying

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j = -\frac{(\delta_{ij} + 1)(k+3)}{3}. \tag{2.1.59}$$

The free field expressions for the currents read

$$\begin{aligned} J^1 &= \beta_1 + (\beta_2\gamma_3), \\ J^2 &= \beta_3, \\ J^3 &= \beta_2, \\ H^1 &= (\beta_1\gamma_1) + (\beta_2\gamma_2) + \boldsymbol{a}_1 \cdot \partial\boldsymbol{\phi}, \end{aligned}$$

$$
\begin{aligned}
H^2 &= (\beta_2\gamma_2) + (\beta_3\gamma_3) + \boldsymbol{a}_2 \cdot \partial\boldsymbol{\phi}, \\
K^1 &= -(\beta_1(\gamma_1\gamma_1)) + (\beta_3\gamma_2) + (k+1)\partial\gamma_1 + (\gamma_1(\boldsymbol{a}_2 - 2\boldsymbol{a}_1)\cdot\boldsymbol{\phi}), \\
K^2 &= (\beta_1(\gamma_1\gamma_3)) - (\beta_2(\gamma_2\gamma_3)) - (\beta_3(\gamma_3\gamma_3)) - (\beta_1\gamma_2) + k\partial\gamma_3 + (\gamma_3(\boldsymbol{a}_1 - 2\boldsymbol{a}_2)\cdot\boldsymbol{\phi}), \\
K^3 &= -(\beta_1(\gamma_1\gamma_2)) - (\beta_2(\gamma_2\gamma_2)) - (\beta_3(\gamma_3\gamma_2)) + (\beta_1(\gamma_1(\gamma_1\gamma_3))) - (k+1)(\gamma_3\partial\gamma_1) \\
&\quad + k\partial\gamma_2 + (\gamma_1(\gamma_3(2\boldsymbol{a}_1 - \boldsymbol{a}_2)\cdot\boldsymbol{\phi})) - (\gamma_2(\boldsymbol{a}_1 + \boldsymbol{a}_2)\cdot\boldsymbol{\phi}) \qquad (2.1.60)
\end{aligned}
$$

They generate the centralizer of the weight 1 vertex operators

$$
\begin{aligned}
V_1 &= (\beta_1 \exp((2\boldsymbol{a}_1 - \boldsymbol{a}_2)\cdot\boldsymbol{\phi}/(k+3))), \\
V_2 &= ((\beta_3 + (\gamma_1\beta_2)) \exp((2\boldsymbol{a}_2 - \boldsymbol{a}_1)\cdot\boldsymbol{\phi}/(k+3))), \qquad (2.1.61)
\end{aligned}
$$

acting on the chiral algebra generated by $\partial\phi_i, \beta_i\gamma_i$. Free field realizations like (2.1.59) have been used to compute correlation functions in the WZNW model, see [115].

2.2. W Algebras

Historically, W algebras were first introduced in the work of Zamolodchikov [333], who worked out the consistency conditions for an operator algebra that is generated by a energy-momentum tensor T and a spin three field W. He found that there is precisely one such algebra for every value of the central charge. Before continuing the general discussion of W algebras, let us first demonstrate how one can encounter such algebras when considering centralizers of vertex operators.

2.2.1. The $c = 2$ W_3 algebra

We have seen that the Virasoro algebra is the centralizer of a vertex operator acting on the chiral algebra generated by a free scalar field. The next natural step is to take the chiral algebra generated by two scalar fields, and to see whether any interesting extensions of the Virasoro algebra appear as the centralizer of vertex operators acting on that chiral algebra. Thus we take two scalar fields with OPE

$$\underbrace{\partial\phi_i(z)\partial\phi_j}(w) = -\frac{\delta_{ij}}{(z-w)^2}, \tag{2.2.1}$$

and consider the action of vertex operators

$$V_{\boldsymbol{a}_1} = \exp(i(\boldsymbol{a}_1 \cdot \boldsymbol{\phi})\sqrt{2}). \tag{2.2.2}$$

To ensure that the centralizer of this vertex operator contains the Virasoro algebra, $\boldsymbol{a}_1$ must be a unit vector, so that $V_{\boldsymbol{a}_1}$ has the proper conformal weight (1,0). (Henceforth we will restrict our attention to the holomorphic part of the vertex operators). It is straightforward to compute the centralizer of $V_{\boldsymbol{a}_1}$ acting on the chiral algebra generated by $\partial\phi_i$. It contains in addition to the energy momentum tensor $T = -\frac{1}{2}(\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi})$ the spin one field $U = \boldsymbol{a}_1^\perp \cdot \partial\boldsymbol{\phi}$, where $\boldsymbol{a}_1^\perp$ is some vector perpendicular to $\boldsymbol{a}_1$. What happens if we take another vertex operator $V_{\boldsymbol{a}_2}$ simultaneously with $V_{\boldsymbol{a}_1}$?. In that case one can show that in general only the Virasoro algebra survives, unless $(\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)^2 = 1/4$. If the latter condition holds (so that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ make an angle of $\pi/3$ or $2\pi/3$) we find that the centralizer of the chiral algebra is no longer generated by the energy-momentum tensor alone, but that there is also a spin three field that survives, namely

$$W = 2(\boldsymbol{a}_2\cdot\boldsymbol{\phi}(\boldsymbol{a}_2\cdot\boldsymbol{\phi}\boldsymbol{a}_2\cdot\boldsymbol{\phi}))-3(\boldsymbol{a}_2\cdot\boldsymbol{\phi}(\boldsymbol{a}_2\cdot\boldsymbol{\phi}\boldsymbol{a}_1\cdot\boldsymbol{\phi}))-3(\boldsymbol{a}_2\cdot\boldsymbol{\phi}(\boldsymbol{a}_1\cdot\boldsymbol{\phi}\boldsymbol{a}_1\cdot\boldsymbol{\phi}))+2(\boldsymbol{a}_1\cdot\boldsymbol{\phi}(\boldsymbol{a}_1\cdot\boldsymbol{\phi}\boldsymbol{a}_1\cdot\boldsymbol{\phi})). \tag{2.2.3}$$

The operator algebra (for $(\boldsymbol{a}_1 \cdot \boldsymbol{a}_2) = +1/2$) of T and $W_3 = 2iW/9\sqrt{3}$ is equal to the $c = 2$ version of the W_3 algebra found by Zamolodchikov. The latter reads, for generic central charge,

$$\begin{aligned}
\underbrace{T(z)T(w)} &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
\underbrace{T(z)W_3(w)} &= \frac{3W_3(w)}{(z-w)^2} + \frac{\partial W_3(w)}{z-w}, \\
\underbrace{W_3(z)W_3(w)} &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
&\quad + \frac{1}{(z-w)^2}(\frac{3}{10}\partial^2 T(w) + 2\beta^2 \Lambda(w)) \\
&\quad + \frac{1}{z-w}(\frac{1}{15}\partial^3 T(w) + \beta^2 \partial\Lambda(w)),
\end{aligned} \tag{2.2.4}$$

where $\Lambda = (TT) - \frac{3}{10}\partial^2 T$ and

$$\beta^2 = \frac{16}{5c+22}. \tag{2.2.5}$$

2.2.2. General W Algebras

A conspicuous feature of the W_3 algebra is its nonlinearity. This makes the analysis of the W_3 algebra much more difficult than that of the Virasoro algebra, or an affine Lie algebra. Soon after its discovery the W_3 algebra was generalized to the so-called W_N algebras, which are nonlinear algebras containing fields of spin $2, \ldots, N$ [116]. In fact, many more algebras of W type exist; for a recent overview and an extensive list of references see [57]. There seems to be no consensus as to which operator algebras should be referred to as W algebras. One might take the presence of fields with spin larger than two or the nonlinearity as the definition of a W algebra, but is seems to be more natural to call any conformal operator algebra a W algebra. The W algebras that we will be dealing with for a considerable part of this thesis, form a subset of the set of conformal operator algebras that contains the 'standard' W_N algebras. This subset consists of the W algebras that are associated to an embedding of sl_2 in a Lie algebra. The specific properties of these W algebras greatly facilitate their analysis; we return to their discussion in a moment.

2.2.3. Why W Algebras?

In addition to the general motivation to study conformal field theory, there are a some more specific reasons to study W algebras, a few of which we want to mention.

- Knowing the maximal symmetry algebra of a theory facilitates its analysis. In particular, there are extra constraints on the correlation functions of the theory via the Ward identities. For example, the 3-state Potts model [100, 141] has the W_3 algebra as its maximal symmetry algebra [114].

- W algebras are closely related to (new) integrable hierarchies [162]. In the mathematics literature, W algebras first arose in this way in the work of Drinfeld and Sokolov and of Gelfand and Dickey [102, 147], who discovered the classical version of the W_N algebra.

- They are an interesting playground for theories with non-linear symmetries, or for theories with a symmetry algebra with field dependent structure constants. Such symmetry algebras play an important role in, for instance, theories of supergravity [254], and one might hope to learn something from the two-dimensional techniques.

- They are relevant for non-conformal perturbations of conformal field theories. Recall our analysis in (2.1.37). The same derivation still holds if V has conformal weight $h \neq 1$, but in that case the perturbed theory will no longer be conformal invariant. Nevertheless, part of the chiral algebra might survive in the perturbed theory. Another possibility is that $\{VA\}_1$ is nonzero, but equal to a total derivative. If this happens $Q = \oint_0 \frac{dz}{2\pi i} A(z)$ still generates a symmetry, but only a global one (thus with $\epsilon =$ constant), and provides us with a conservation law for the perturbed theory. For more on perturbed conformal field theory, see [331, 125, 34, 195, 334, 203, 247, 126].

2.2.4. W Algebras From sl_2 Embeddings

There are basically three ways to study W algebras. The first one is a direct generalization of the approach of Zamolodchikov, where one postulates a number of generators, and tries to form a closed associative operator algebra [45, 196]. In the second approach one takes any conformal field theory and tries to find the chiral algebra, or a subalgebra thereof, to see whether there are higher spin generators. Typical examples are the Casimir algebras [14] and the coset construction [58, 312]. In the latter approach, one studies the centralizer of an affine subalgebra of some affine Lie algebra.

The third approach is the Drinfeld-Sokolov approach. This approach relies heavily on the connection with Lie algebras. The idea is to start with an affine Lie algebra, and to impose certain constraints on the currents. Using Hamiltonian reduction [1] (or, more precisely, Poisson reduction [258]) one obtains a (classical) reduced algebra. Consider for example the affine Lie algebra based on sl_2 with current

$$\mathcal{J} = J^a T_a \equiv J^- T_- + J^0 T_0 + J^+ T_+ = \begin{pmatrix} J^0 & J^+ \\ J^- & -J^0 \end{pmatrix}. \tag{2.2.6}$$

Recall that Poisson brackets and OPE's are related via (2.1.21), and that the currents have the OPE's given by (2.1.54). These OPE's generate both a classical and a quantum operator algebra. These two differ only in their OPE's of normal ordered products of currents. If we speak about the Poisson algebra of currents, we always mean the Poisson brackets corresponding to the classical operator product algebra defined by (2.1.54). What happens when we impose the constraints $c_1 = J^+ - \xi = 0$ and $c_2 = J^0 = 0$ on the Poisson algebra generated by the currents (2.2.6)? (ξ is an arbitrary nonzero constant). These constraints are second class, and therefore the reduced Poisson algebra is given by the Dirac bracket

$$\{J^-(z), J^-(w)\}_{\text{red}} = \{J^-(z), J^-(w)\} - \int dx \int dy\, \{J^-(z), c_i(x)\}\Delta^{ij}(x,y)\{c_j(y), J^-(w)\}, \tag{2.2.7}$$

where Δ^{ij} is the inverse of $\Delta_{ij} = \{c_i, c_j\}$. For Δ^{ij} we find

$$\Delta^{ij}(x,y) = \begin{bmatrix} 0 & \frac{1}{J^+(x)}\delta(x-y) \\ -\frac{1}{J^+(x)}\delta(x-y) & \frac{k/2}{J^+(x)J^+(y)}\partial_y \delta(x-y) \end{bmatrix}, \tag{2.2.8}$$

and the reduced bracket is

$$\{J^-(z), J^-(w)\} = -\frac{k^3}{2\xi^2}\partial_w^3 \delta(z-w) + \frac{2k}{\xi} J^-(w)\partial_w \delta(z-w) + \frac{k}{\xi} J^-(w)\delta(z-w). \tag{2.2.9}$$

Replacing J^- by kT/ξ this bracket is exactly equal to the Virasoro algebra with $c = -6k$. This is the prototype of the more general construction. We impose constraints on an affine Lie algebra in such a way that there is an sl_2 subalgebra on which the constraints take the same form as for the example above. This will then guarantee that the reduced algebra contains a Virasoro subalgebra, and is a candidate W algebra. Historically, the connection between constrained current algebra and W algebras was first fully employed in [36]. Before that, it had been realized that there is a close connection between the Virasoro algebra and the second Hamiltonian structure of the KdV equation [152, 202, 17], and between classical W_N algebras and generalized KdV hierarchies [18, 42, 329, 134]. The relation between the second Hamiltonian structure of generalized KdV hierarchies and constrained current algebra was already explained in the work of Drinfeld and

Sokolov, but the physics community was not really aware of this fact until [36] appeared. There, the quantization of W algebras is studied from the constrained current algebra point of view. Previously, the quantization of W algebras had been achieved by directly quantizing the second Hamiltonian structure of the generalized KdV hierarchies [116], but this is a less general method [57]. The quantization of the standard W_N algebras using constrained current algebra goes under the name 'quantum Drinfeld Sokolov reduction' and was developed in more detail in [121, 120, 137, 139]. This is a very powerful approach, as it allows one quite easily to construct a free field realization for W algebras, and to reduce the representation theory of W algebras to that of affine Lie algebras [138]. quantization of W algebras.

A different way to connect W algebras to constrained current algebra is via Toda theory [222, 221, 220]. An example of a Toda theory is a theory of two free scalar fields, perturbed by two vertex operators as in section 2.2.1. We saw that the chiral algebra is reduced to a W algebra. The study W algebras as algebras of conserved currents in Toda theories was pursued in [42] and later in [21, 19]. In the latter papers the connection between Toda theories, and constrained current algebras was made. For a review, see [20].

2.2.5. Why sl_2 Embeddings?

Many of the papers on W algebras deal exclusively with the standard W_N algebras. However, it turns out that there are many more W algebras, which are on equal footing with the W_N algebras in that they can all be obtained from an embedding of sl_2 in a Lie algebra $\mathbf{g}$. The standard W_N algebras correspond to the principal embedding of sl_2 in $\mathbf{g}$, but in general there are many more possible inequivalent embeddings. The first such nonstandard W algebra, called $W_3^{(2)}$, was discovered by Polyakov and Bershadsky [264, 38]. The more general case was first studied in [15] and later in [274]. The quantization of these W algebras is performed in [54]. What is so special about sl_2 embeddings that they always give a W algebra? Above we indicated one reason, namely that the Drinfeld-Sokolov reduction of sl_2 itself is the Virasoro algebra, and that the sl_2 embedding is responsible for the occurrence of a Virasoro subalgebra in the final reduced algebra. A second reason for the occurrence of an sl_2 embedding was given in [52], where it was shown that if we demand that the components of a constrained current transform under some special field-dependent gauge transformation as primary fields of fixed conformal weights, then the constraints are always related to an sl_2 embedding. From this point of view, the constraints we impose on the current are only preserved if we add to the standard co-ordinate transformations some extra compensating gauge transformation, a procedure sometimes called 'soldering' [263], and the sl_2 is intimately related to the $sl_2(\mathbb{C})$ invariance of string theory. For the example of sl_2 for which we computed the Dirac bracket this works as follows. If $\mathcal{J}_{\mathrm{constr}}$ denotes the current with the constraints c_0 and c_1 imposed on it

$$\mathcal{J}_{\text{constr}} = \begin{pmatrix} 0 & \xi \\ J^- & 0 \end{pmatrix}, \tag{2.2.10}$$

then we can look for gauge transformations that preserve the constraints, *i.e.* we look for a Lie algebra valued X such that

$$\delta \mathcal{J}_{\text{constr}} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = k\,\partial X + [\mathcal{J}_{\text{constr}}, X]. \tag{2.2.11}$$

This equation can be used to express the T_0 and T_- components of X in terms of its T_+ component, which we denote by ϵ. The most general solution of (2.2.11) reads

$$X = \begin{pmatrix} \frac{k}{2\xi}\partial\epsilon & \epsilon \\ -\frac{k^2}{2\xi^2}\partial^2\epsilon + \frac{1}{\xi}J^-\epsilon & -\frac{k}{2\xi}\partial\epsilon \end{pmatrix} = \frac{\epsilon}{\xi}\mathcal{J}_{\text{constr}} + X'(\epsilon), \tag{2.2.12}$$

where we separated X in a part proportional to $\mathcal{J}_{\text{constr}}$ and a residual part $X'(\epsilon)$. Taking X proportional to $\mathcal{J}_{\text{constr}}$ gives the standard transformation rule ($\delta\mathcal{J} \sim \partial(\epsilon\mathcal{J})$) for a current under a co-ordinate transformation. This transformations does not preserve the constraints, and we need the extra residual 'soldering' part $X'(\epsilon)$ to bring the current back into the constrained form. With X equal to (2.2.12) we find for δJ^-

$$\delta J^- = -\frac{k^3}{2\xi^2}\partial^3\epsilon + \frac{2k}{\xi}J^-\partial\epsilon + \frac{k}{\xi}\partial J^-\epsilon. \tag{2.2.13}$$

Thus, the 'soldering' term $X'(\epsilon)$ modifies the transformation rule for J^- under co-ordinate transformations from the standard one to a Virasoro transformation, and effectively shifts the spin of J^- from one to two.

As an important side remark, we note that (2.2.13) is precisely equal to

$$\frac{k}{\xi}\int dz\,\{\epsilon(z)J^-(z), J^-(w)\}_{\text{dirac}}, \tag{2.2.14}$$

and that therefore the Dirac bracket is completely determined by the field-dependent gauge transformations that preserve the constraints. These are much easier to compute than the Dirac bracket. This (nontrivial) fact holds in fact for all W algebras that can be obtained from sl_2 embeddings [19]: the W transformations can be realized as field-dependent gauge transformations. We will frequently make use of this fact later on.

Finally, a third reason why sl_2 embeddings are necessary, and describe in some sense a generic situation, was given in [64]. There it is shown that under certain mild assumptions one can associate a Lie algebra and an sl_2 embedding to any quantum W algebra. For this to happen it is necessary that the W algebra exists for generic values of the central charge (the W algebra is 'deformable'), and is reductive (for the definition of a reductive W algebra, see [64]).

In the recent paper [118] it is shown that an sl_2 embedding can be associated to every Drinfeld Sokolov reduction. Furthermore, evidence is given that the W algebras corresponding to these sl_2 embeddings exhaust the W algebras that may be obtained from reductions of affine Lie algebras. This provides further motivation for considering the W algebras associated to sl_2 embeddings.

2.2.6. Gauge Invariant Polynomials and the Miura Transformation

Besides the two methods we have described so far to obtain the structure of the reduced sl_2 algebra, namely via a computation of the Dirac bracket and via a computation of the gauge transformations that preserve the constraints, there is a third method. This method works as follows. Instead of starting with two second-class constraints, we can also start with the constraint $c_1 = J^+ - \xi$ only, and define a reduced algebra by means of Hamiltonian reduction. Since c_1 is a first-class constraint, the constrained phase space $\mathcal{M}_{\rm constr}$, defined as the space of polynomials in $\{J^+, J^0, J^-\}$ and their derivatives modulo the constraint c_1, has a gauge invariance generated by the constraint. The reduced algebra obtained from Hamiltonian reduction is then isomorphic to the subalgebra of gauge invariant polynomials of $\mathcal{M}_{\rm constr}$. The second constraint c_2 plays the role of a gauge fixing condition. For the example of sl_2, these gauge invariant polynomials can easily be determined. The gauge transformations generated by c_1 are the gauge transformations by lower triangular matrices. Now consider the current subject to the constraint c_1 and bring it into gauge-fixed form using a lower-triangular gauge transformation

$$n^{-1}(J^0,J^-)\begin{pmatrix} J^0 & \xi \\ J^- & -J^0 \end{pmatrix} n(J^0,J^-) + k\, n^{-1}(J^0,J^-)\partial n(J^0,J^-) = \begin{pmatrix} 0 & \xi \\ T(J^0,J^-) & 0 \end{pmatrix}. \tag{2.2.15}$$

This uniquely fixes $n(J^0, J^-)$ and $T(J^0, J^-)$. It is easy to see that $T(J^0, J^-)$ is gauge invariant by construction: on the gauge orbit passing through $\begin{pmatrix} J^0 & \xi \\ J^- & -J^0 \end{pmatrix}$ there is only one point where $J^0 = 0$, and this point defines $T(J^0, J^-)$. Therefore T is only a function of the orbit, and must be gauge invariant. We find

$$n = \begin{pmatrix} 1 & 0 \\ -J^0/\xi & 1 \end{pmatrix}, \tag{2.2.16}$$

and

$$T(J^-, J^0) = \frac{1}{\xi}(J^0)^2 + J^- - \frac{k}{\xi}\partial J^0. \tag{2.2.17}$$

Using the ordinary brackets for the currents we find once more that $\xi T/k$ generates a Virasoro algebra with $c = -6k$. The same remains true if we drop the term proportional to J^- from (2.2.17). The expression for T without J^- can also be obtained from the differential operator identity

$$(k\partial - J^0)(k\partial + J^0) = (k^2\partial - \xi T). \tag{2.2.18}$$

This identity gives an algebra homomorphism from the Poisson algebra generated by J^0 to the Poisson algebra generated by T, which is known as the classical Miura transformation [116]. It can be obtained from (2.2.15) by the following procedure. Two matrix differential operators L_1 and L_2 are identical if and only if for all vectors ψ the equation $L_1\psi = 0$ is equivalent to $L_2\psi = 0$. To apply this to (2.2.15) we introduce a two-component vector (ψ_1, ψ_2) and rewrite (2.2.15) with $J^- = 0$ as

$$\left(k\partial + \begin{pmatrix} 0 & \xi \\ T(J^0) & 0 \end{pmatrix}\right)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \Leftrightarrow \left(k\partial + \begin{pmatrix} J^0 & \xi \\ 0 & -J^0 \end{pmatrix}\right) n(J^0) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{2.2.19}$$

Since n does not alter ψ_1, we can rewrite the linear differential equations (2.2.19) as second-order differential equations for ψ_1,

$$(k\partial - J^0)(k\partial + J^0)\psi_1 = 0 \Leftrightarrow (k^2\partial - \xi T)\psi_1 = 0, \tag{2.2.20}$$

and we have reobtained the Miura transformation (2.2.18). A free field realization of T can be obtained if we replace J^0 by a free field with the same OPE, for example $J^0 = i\sqrt{\frac{k}{2}}\partial\phi$ where ϕ is a free scalar field.

2.2.7. The Structure of the W Algebra

Everything we have said so far for the Virasoro algebra also holds, modulo some changes, for W algebras related to sl_2 embeddings. In this section we briefly describe the corresponding statements for the general case. All our computations have so far been restricted to classical W algebras. The quantization of these W algebras will be the topic of the next sections.

The starting point of the construction is an embedding i of sl_2 in a Lie algebra $\mathbf{g}$. The generators of sl_2 are called $\{t_-, t_0, t_+\}$ with commutation relations $[t_0, t_+] = t_+$, $[t_0, t_-] = -t_-$ and $[t_+, t_-] = t_0$. The images $i(t_r)$ are denoted by Λ^r, or sometimes also by t_r.

Associated to an embedding of sl_2 in $\mathbf{g}$ is a decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$, with $\mathbf{g}_\pm$ nilpotent. If $\mathrm{ad}\Lambda^0$ has only integral eigenvalues, then the decomposition is with respect to the sign of the eigenvalue. If $\mathrm{ad}\Lambda^0$ has half-integral eigenvalues, then the situation is somewhat more complicated. Let $\mathbf{g}_{\frac{1}{2}}$ be the subspace of $\mathbf{g}$ of Λ^0-eigenvalue $+\frac{1}{2}$. On $\mathbf{g}_{\frac{1}{2}}$ there is a non-degenerate skew-form $\omega(X, Y) = \mathrm{Tr}(\Lambda^-[X, Y])$. Thus we can decompose $\mathbf{g}_{\frac{1}{2}} = \mathcal{I}_+ \oplus \mathcal{I}_-$ into two maximally isotropic subspaces. Now there is a gradation of $\mathbf{g}$ such that $\mathcal{I}_\pm$ has degree $\pm\frac{1}{2}$, and $\Lambda^{-,0,+}$ has degree 0. The sum of this gradation and the gradation given by Λ^0 defines a new gradation of $\mathbf{g}$ which is integral, and the decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$ is with respect to this new gradation, which we denote by δ.

The W algebra is the result of imposing certain first-class constraints on $\mathcal{J}$. If we decompose $\mathcal{J} = J^a T_a$ and $\Lambda^+ = l^a T_a$ in terms of a basis T_a of $\mathbf{g}$, the constraints are

$$J^a \equiv \chi(J^a) = \xi l^a, T_a \in \mathbf{g}_+, \tag{2.2.21}$$

where we introduced the one-dimensional representation χ of $\mathbf{g}_+$, and we assume that the basis $\{T_a\}$ is such that every basis element has a well-defined degree with respect to δ and Λ^0. The constraints (2.2.21) are first-class. They impose the constraint c_1 on the sl_2 subalgebra $i(sl_2)$. If we would have taken $\mathbf{g}_+$ to be the positive degree part of $\mathbf{g}$ with respect to the Λ^0 gradation, (2.2.21) will not necessarily be first-class, and that is the reason why we introduced a new gradation. The first-class constraints generate gauge invariance, which in this case are the gauge transformations with parameter in $\mathbf{g}_-$. These gauge transformations can be used to put $\mathcal{J}$ in the form

$$\mathcal{J} = \xi\Lambda^+ + W, W \in \ker \mathrm{ad}(\Lambda^-). \tag{2.2.22}$$

There is a one-to-one correspondence between the independent components of W and the representations of sl_2 in which $\mathbf{g}$ decomposes under the embedding. The components of W correspond to the lowest weights of these representations. For this reason, (2.2.22) is sometimes referred to as the lowest weight gauge. It is possible to gauge fix the $\mathbf{g}_-$ symmetry differently, leading to other gauge choices for W. The advantage of this gauge choice is that the components of W will automatically correspond to primary fields.

At this stage one can employ any of the three techniques discussed previously to compute the structure of the W algebra. The simplest one is to use the relation between the W algebra and gauge transformations that preserve the constraints (2.2.22). The parameter X of this gauge transformation must therefore satisfy

$$k\partial X + [\xi\Lambda^+ + W, X] \in \ker \mathrm{ad}(\Lambda^-). \tag{2.2.23}$$

To solve this equation we introduce a linear operator $L : \mathbf{g} \to \mathbf{g}$ that is the inverse of $\mathrm{ad}(\Lambda^+) : \mathrm{im}(\mathrm{ad}(\Lambda^-)) \to \mathrm{im}(\mathrm{ad}(\Lambda^+))$, extended by 0 to the whole of $\mathbf{g}$. In the next chapter we show that L has a geometrical interpretation as an homotopy operator that contracts gauge transformations into W transformations. Two basic properties of L are

$$L \circ \mathrm{ad}(\Lambda^+) = \Pi_{\mathrm{im}\,\mathrm{ad}(\Lambda^-)}, \qquad \mathrm{ad}(\Lambda^+) \circ L = \Pi_{\mathrm{im}\,\mathrm{ad}(\Lambda^+)} \tag{2.2.24}$$

where Π_V denotes the orthogonal projection on the subspace V. The first of these operators, when applied to (2.2.23), yields

$$\mathrm{ad}(\Lambda^+)(\xi + L(k\partial + \mathrm{ad}(W)))(X) = 0, \tag{2.2.25}$$

from which it follows that the general solution for X is

$$X = \frac{1}{1 + \xi^{-1}L(k\partial + \mathrm{ad}(W))}F, \tag{2.2.26}$$

where F has values in $\ker(\mathrm{ad}(\Lambda^+))$, and contains the parameters for the W transformations. The denominator in (2.2.26) can be expanded in a power series. This power series truncates after a finite number of steps, because L lowers the degree with respect to the δ gradation by -1, and all the components of W have degree ≤ 0. Analogous to the Virasoro case we can now establish the following general expression for the Dirac brackets of the W algebra

$$\int dz\, \{\mathrm{Tr}(F(z)W(z)), W^a(w)\}_{\mathrm{dirac}} T_a = (k\partial + \xi \mathrm{ad}(\Lambda^+) + \mathrm{ad}(W)) \frac{1}{1 + \xi^{-1}L(k\partial + \mathrm{ad}(W))}F, \tag{2.2.27}$$

with $W = W^a T_a$. The stress-energy tensor of the W algebra is given by

$$T = \frac{1}{2k}\mathrm{Tr}(\xi\Lambda^+ + W)^2. \tag{2.2.28}$$

The parameter F for general co-ordinate transformations with parameter $\epsilon(z)$ can be found from $\int dz\, \mathrm{Tr}(F(z)W(z)) = \int dz\, \epsilon(z)T(z)$, and equals

$$F_{\mathrm{coord}}(z) = (\frac{\xi}{k}\Lambda^+ + \frac{1}{k}\Pi_0 W)\epsilon(z), \tag{2.2.29}$$

where Π_0 is the projection on the centralizer of the sl_2 embedding, so that $\mathrm{Tr}(WW) = \mathrm{Tr}(\Pi_0 W \Pi_0 W)$. Inserting (2.2.29) into (2.2.27) yields the transformation rule

$$\int dz\, \{\epsilon(z)T(z), W^a(w)\}_{\mathrm{dirac}} T_a = -\frac{k^2}{\xi}\Lambda^- \partial^3\epsilon + (1 - \mathrm{ad}\Lambda^0)(W)\partial\epsilon + \partial W \epsilon, \tag{2.2.30}$$

which shows that all components of W except the one proportional to Λ^- transform as primary fields with weight given by the eigenvalue of $1 - \mathrm{ad}\Lambda^0$, and that T defined in (2.2.28) generates a Virasoro algebra with $c = -12k\mathrm{Tr}(\Lambda^+\Lambda^-)$.

The gauge invariant polynomials corresponding to the independent components of W can be computed in the same way as in (2.2.15). The resulting gauge invariant polynomials contain the currents J^a with $T_a \in \mathbf{g}_0 \oplus \mathbf{g}_+$. The Miura transformation is obtained by putting the J^a with $T_a \in \mathbf{g}_-$ equal to zero. This does not change the algebra generated by the gauge invariant polynomials. The Miura map is therefore an algebra homomorphism from the W algebra into the affine Lie algebra based on $\mathbf{g}_0$. A free field realization of the W algebra can subsequently be obtained by replacing the currents by a free field realization of the affine Lie algebra based on $\mathbf{g}_0$. For the standard W_N algebras, $\mathbf{g}_0$ is just the Cartan subalgebra, and the corresponding abelian affine Lie algebra can be expressed in terms of $\mathrm{rank}(G) = N - 1$ free scalar fields. This expressions for the W fields in terms of these $N - 1$ free scalar fields can be given as an identity between two N^{th}-order differential operators as in (2.2.20), which is the standard way to give the Miura transformation. In general, the Miura transformation can be expressed as an identity between two non-linear matrix differential operators of dimension $n_i \times n_i$, where n_i is the number of different representations in which the fundamental representation of $\mathbf{g}$ decomposes under the action of $i(sl_2)$.

2.2.8. Quantization of the W Algebra

So far we have only considered classical W algebras, *i.e.* they are given by Poisson brackets. The corresponding OPE's (using (2.1.21)) do not satisfy the criteria for an operator product algebra. Let us call a collection of OPE's that correspond to Poisson brackets a 'classical operator product algebra'. Then a quantization of a classical operator product algebra $\mathcal{A}_{\mathrm{class}}$ is an operator product algebra $\mathcal{A}$ depending on a free parameter $\hbar$ satisfying

(i) $\mathcal{A}$ is a free $\mathbb{C}[[\hbar]]$ module,

(ii) $\mathcal{A}/\hbar\mathcal{A} \simeq \mathcal{A}_{\text{class}}$,

(iii) if $\pi : \mathcal{A} \to \mathcal{A}/\hbar\mathcal{A} \simeq \mathcal{A}_{\text{class}}$ denotes the natural projection, then for all $A_\alpha, A_\beta \in \mathcal{A}$

$$\pi(A_\alpha)\underbrace{(z)\pi(A_\beta)}(w) = \pi(\frac{1}{\hbar}A_\alpha\underbrace{(z)A_\beta}(w)). \tag{2.2.31}$$

When we display a set of generators of $\mathcal{A}$ and their OPE's, we usually put $\hbar = 1$, but it can sometimes be instructive to restore the $\hbar$ dependence in $\mathcal{A}$.

The classical W algebra was obtained by imposing constraints on the Poisson algebra generated by some currents. Rather than trying to quantize the W algebra directly, one can also first quantize the Poisson algebra generated by the currents (which is precisely the affine Lie algebra based on $\mathbf{g}$), and then impose the constraints. The constraints (2.2.21) are first-class and can be imposed using the BRST formalism [210]. For simplicity we will take $\xi = 1$ in the remainder. The ξ dependence can always easily be restored by the sl_2 automorphism $\Lambda^\pm \to \xi^{\pm 1}\Lambda^\pm$. From now on we let latin indices $a, b, \ldots$ run over the entire basis of $\mathbf{g}$, Greek indices $\alpha, \beta, \ldots$ over a basis of $\mathbf{g}_+$ and barred Greek indices $\bar\alpha, \bar\beta, \ldots$ over a basis of $\mathbf{g}_0 \oplus \mathbf{g}_-$, so that $\lambda^\alpha T_\alpha + \lambda^{\bar\alpha} T_{\bar\alpha} = \lambda^a T_a$.

The BRST formalism requires that we introduce in addition to the currents J^a a b^α, c_α system for each constraint $\chi(J^\alpha)$. Let $F(\Omega_k)$ denote the operator product algebra generated by these fields, where k refers to the level of the affine Lie algebra, *i.e.*

$$\begin{aligned} J^a\underbrace{(z)J^b}(w) &= \frac{\hbar k \eta^{ab}}{(z-w)^2} + \frac{\hbar f^{ab}_{\ \ c}}{z-w}, \\ b^\alpha\underbrace{(z)c_\beta}(w) &= \frac{\hbar \delta^\alpha_\beta}{z-w} \end{aligned} \tag{2.2.32}$$

The BRST operator is then [210] $D(.) = [Q, .]$ where $Q = \oint \frac{dz}{2\pi i} J_{\text{BRST}}(z)$ and

$$J_{\text{BRST}}(z) = (J^\alpha(z) - \chi(J^\alpha(z)))c_\alpha(z) - \frac{1}{2} f^{\alpha\beta}_{\ \ \gamma}(b^\gamma(c_\alpha c_\beta))(z) \tag{2.2.33}$$

D is of degree 1 (i.e. $D(F(\Omega_k)^{(l)}) \subset F(\Omega_k)^{(l+1)}$) and $D^2 = 0$ which means that $F(\Omega_k)$ is a complex. One is then interested in calculating the cohomology (or Hecke algebra) of this complex because the zeroth cohomology is nothing but the quantization of the classical W algebra [210, 121, 139]. This problem has been solved for the so called 'finite W algebras' in [54].

The first step is to split the BRST current into two pieces [121]

$$J_{\mathrm{BRST},0}(z) = -\chi(J^{\alpha}(z))c_{\alpha}(z) \tag{2.2.34}$$

$$J_{\mathrm{BRST},1}(z) = J^{\alpha}(z)c_{\alpha}(z) - \frac{1}{2}f^{\alpha\beta}_{\gamma}(b^{\gamma}(c_{\alpha}c_{\beta}))(z) \tag{2.2.35}$$

and to make $F(\Omega_k)$ into a double complex $F(\Omega_k) = \bigoplus_{rs} F(\Omega_k)^{(r,s)}$ by assigning the following (bi)grades to its generators

$$\begin{aligned} \deg(J^{a}(z)) &= (-k,k) \quad \text{if } T_a \in \mathbf{g}_k \\ \deg(c_{\alpha}(z)) &= (k,1-k) \quad \text{if } T_\alpha \in \mathbf{g}_k \\ \deg(b^{\alpha}(z)) &= (-k,k-1) \quad \text{if } T_\alpha \in \mathbf{g}_k \end{aligned} \tag{2.2.36}$$

Here, $\mathbf{g}_k$ is the subspace of $\mathbf{g}$ of elements of degree k with respect to the grading δ that was defined in the beginning of section 2.2.7. The operators $D_0 : F(\Omega_k)^{(r,s)} \to F(\Omega_k)^{(r+1,s)}$ and $D_1 : F(\Omega_k)^{(r,s)} \to F(\Omega_k)^{(r,s+1)}$ associated in the obvious way to $J_{\mathrm{BRST},0}$ and $J_{\mathrm{BRST},1}$ satisfy $D_0^2 = D_1^2 = D_0D_1 + D_1D_0 = 0$ verifying that we have obtained a double complex.

Let us now calculate the action of the operators D_0 and D_1 on the generators of $F(\Omega_k)$. For this it is convenient to introduce $\hat{J}^a(z) = J^a(z) + f^{a\beta}_{\gamma}(b^{\gamma}c_{\beta})(z)$. One then finds by explicit calculation

$$\begin{aligned} D_0(\hat{J}^a(z)) &= -\hbar f^{a\beta}_{\gamma}\chi(J^{\gamma}(z))c_{\beta}(z) \\ D_0(c_{\alpha}(z)) &= 0 \\ D_0(b^{\alpha}(z)) &= -\hbar\chi(J^{\alpha}(z)) \\ D_1(\hat{J}^a(z)) &= \hbar f^{\alpha a}_{\bar{\beta}}\hat{J}^{\bar{\beta}}(z)c_{\alpha}(z) + k\hbar\eta^{a\alpha}\partial c_{\alpha}(z) - \hbar^2 f^{\alpha e}_{\beta}f^{\beta a}_{e}\partial c_{\alpha}(z) \\ D_1(c_{\alpha}(z)) &= -\frac{\hbar}{2}f^{\beta\gamma}_{\alpha}(c_{\beta}c_{\gamma})(z) \\ D_1(b^{\alpha}(z)) &= \hbar\hat{J}^{\alpha}(z). \end{aligned}$$

From these formulas it immediately follows that $D(\hat{J}^{\alpha}(z)) = 0$ and $D(b^{\alpha}(z)) = \hbar(\hat{J}^{\alpha}(z) - \chi(J^{\alpha}(z)))$. This means that the subspace $F^{\alpha}(\Omega_k)$ of $F(\Omega_k)$ generated by $J^{\alpha}(z)$ and $b^{\alpha}(z)$ is actually a subcomplex. The cohomology of this complex can easily be calculated and one finds $H^*(F^{\alpha}(\Omega_k); D) = \mathbb{C}[[\hbar]]$. Note also that due to the Poincare-Birkhoff-Witt theorem for field algebras (which follows immediately from the relations (2.1.18)) the normal ordering map

$$(\ldots) : F_{red}(\Omega_k) \otimes \bigotimes_{\alpha} F^{\alpha}(\Omega_k) \to F(\Omega_k) \tag{2.2.37}$$

defined by $A_1(z) \otimes \ldots \otimes A_l(z) \mapsto (A_1 \ldots A_l)(z)$ (where we always use the convention $(ABC)(z) = (A(BC))(z)$) is an isomorphism of vector spaces. Due to this and the fact that the BRST operator acts as a derivation on $F(\Omega_k)$ we have

$$\begin{aligned} H^*(F(\Omega_k);D) &\simeq H^*(F_{red}(\Omega_k);D)\otimes\bigotimes_{\alpha}H^*(F^{\alpha}(\Omega_k);D) \\ &\simeq H^*(F_{red}(\Omega_k);D) \end{aligned} \tag{2.2.38}$$

Where in the first step we used a Kunneth like theorem given in [54].

In order to calculate $H^*(F_{red}(\Omega_k);D)$ one uses the fact that $F_{red}(\Omega_k)$ is actually a double complex which makes calculation of the cohomology possible via a spectral sequence argument [238, 121, 54]. The first term E_1 of the spectral sequence is the D_0 cohomology of $F_{red}(\Omega_k)$. Note that we can write $D_0(\hat{J}^{\bar{\alpha}}(z)) = -\hbar\mathrm{Tr}([t_+,T^{\bar{\alpha}}]T^{\beta}c_{\beta}(z))$. Therefore $D_1(\hat{J}^{\bar{\alpha}}(z)) = 0$ iff $T_{\bar{\alpha}}\in\mathbf{g}_{lw}$ where $\mathbf{g}_{lw}$ is the set of elements of $\mathbf{g}$ that are annihilated by ad_{t_-} (the lowest weight vectors of the sl_2 multiplets) and where we used the fact [54] that $T_{\bar{\alpha}}\in\mathrm{Ker}(\mathrm{ad}_{t_-})$ iff $T^{\bar{\alpha}}\in\mathrm{Ker}(\mathrm{ad}_{t_+})$. It can also easily be seen that for all β there exists a linear combination $a(\beta)_{\bar{\alpha}}\hat{J}^{\bar{\alpha}}(z)$ such that $D_0(a(\beta)_{\bar{\alpha}}\hat{J}^{\bar{\alpha}}(z)) = \hbar c_{\beta}(z)$. From this it follows [54] that purely on the level of vector spaces we have

$$H^n(F_{red}(\Omega_k);D_0)\simeq F_{lw}(\Omega_k)\delta_{k,0} \tag{2.2.39}$$

where $F_{lw}(\Omega_k)$ is the subspace of $F(\Omega_k)$ generated by the fields $\{J^{\bar{\alpha}}(z)\}_{T_{\bar{\alpha}}\in\mathbf{g}_{lw}}$. Since the only cohomology that is nonzero is of degree 0 the spectral sequence degenerates at the first term, i.e. $E_\infty = E_1$ and we find the end result

$$H^n(F_{red}(\Omega_k);D)\simeq F_{lw}(\Omega_k)\delta_{k,0} \tag{2.2.40}$$

Having calculated the BRST cohomology at the level of vector spaces one now can construct the cohomology (or W algebra) generators and their OPEs via a procedure called the tic tac toe construction [56]. Consider a generator $\hat{J}^{\bar{\alpha}}(z)$ of degree $(p,-p)$ of the field algebra $F_{lw}(\Omega_k)$ (i.e. $T_{\bar{\alpha}}\in\mathbf{g}_{lw}$) then the generator of cohomology associated to this element is given by

$$W^{\bar{\alpha}}(z) = \sum_{l=0}^{p}(-1)^l W_l^{\bar{\alpha}}(z) \tag{2.2.41}$$

where $W_0^{\bar{\alpha}}(z)\equiv J^{\bar{\alpha}}(z)$ and $W_l^{\bar{\alpha}}(z)$ can be determined inductively by

$$D_1(W_l^{\bar{\alpha}}(z)) = D_0(W_{l+1}^{\bar{\alpha}}(z)) \tag{2.2.42}$$

It is easy to check, using the fact that $D_0(J^{\bar{\alpha}}(z)) = 0$ for $T_{\bar{\alpha}}\in\mathbf{g}_{lw}$ that indeed $D(W^{\bar{\alpha}}(z)) = 0$.

The formalism presented above provides us with a completely algorithmic procedure of calculating the W algebra associated to a certain sl_2 embedding: First determine the space $\mathbf{g}_{lw}$. Then take a current $\hat{J}^{\bar{\alpha}}(z)$ with $T_{\bar{\alpha}} \in \mathbf{g}_{lw}$ and inductively calculate the fields $W_l^{\bar{\alpha}}(z)$ using relations (2.2.42). The field (2.2.41) is then the corresponding W generator and the relations in the W algebra are then just the OPEs between the fields $\{W^{\bar{\alpha}}(z)\}_{T_{\bar{\alpha}} \in \mathbf{g}_{lw}}$ calculated using the OPEs in $F(\Omega_k)$.

In principle this algebra closes only modulo D-exact terms. But since we computed the D cohomology on a reduced complex generated by $\hat{J}^{\bar{\alpha}}(z)$ and $c_\alpha(z)$, and this reduced complex is zero at negative ghost number, there simply are no D exact terms at ghost number zero. Thus the algebra generated by $\{W^{\bar{\alpha}}(z)\}_{T_{\bar{\alpha}} \in \bar{g}_{lw}}$ closes in itself.

As was shown in [54] for finite W algebras, the operator product algebra generated by the fields $W^{\bar{\alpha}}(z)$ is isomorphic to the operator product algebra generated by their (bi)grade (0,0) components $W_p^{\bar{\alpha}}(z)$ (the proof in the infinite-dimensional case is completely analogous and will not be repeated here). The fields $W_p^{\bar{\alpha}}(z)$ are of course elements of the field algebra generated by the currents $\{\hat{J}^{\bar{\alpha}}(z)\}_{T_{\bar{\alpha}} \in \bar{g}_0}$. The relations (*i.e.* the OPEs) satisfied by these currents are almost identical to the relations satisfied by the unhatted currents

$$\underbrace{\hat{J}^{\bar{\alpha}}(z)\,\hat{J}^{\bar{\beta}}}(w) = \frac{\hbar k \eta^{\bar{\alpha}\bar{\beta}} + \hbar^2 k^{\bar{\alpha}\bar{\beta}}}{(z-w)^2} + \frac{\hbar f^{\bar{\alpha}\bar{\beta}}_{\ \ \bar{\gamma}} \hat{J}^{\bar{\gamma}}(w)}{z-w} \tag{2.2.43}$$

where $k^{\bar{\alpha}\bar{\beta}} = f^{\bar{\alpha}\lambda}_{\ \ \gamma} f^{\bar{\beta}\gamma}_{\ \ \lambda}$. Now $\mathbf{g}_0$ is a direct sum of simple subalgebras and $u(1)$ subalgebras,

$$\mathbf{g}_0 \simeq \bigoplus_j \mathbf{g}_{0,j}. \tag{2.2.44}$$

Within the $\mathbf{g}_{0,j}$ component of $\mathbf{g}_0$ we have the identity

$$k^{\bar{\alpha}\bar{\beta}} = \eta^{\bar{\alpha}\bar{\beta}}(h - h_j) \tag{2.2.45}$$

where h is the dual Coxeter number of $\mathbf{g}$ and h_j is the dual Coxeter number of $\mathbf{g}_{0,j}$. We therefore find that the field algebra generated by the currents $\{\hat{J}^{\bar{\alpha}}(z)\}_{T_{\bar{\alpha}} \in \mathbf{g}_0}$, denoted from now on by $\hat{F}_0$, is nothing but the operator product algebra associated to a affine Lie algebra. This affine Lie algebra is not simply $\mathbf{g}_0$ (whose operator product algebra is generated by the unhatted currents) however, because in g_0 all components have the same level while in $\hat{F}_0$ the level varies from component to component as follows from equation (2.2.45). This is just a result of the ghost contributions $k^{\bar{\alpha}\bar{\beta}}$ in the OPEs of the hatted currents.

From the above we find that the map

$$W^{\bar{\alpha}}(z) \mapsto (-1)^p W_p^{\bar{\alpha}}(z) \tag{2.2.46}$$

is an embedding of the W algebra into $\hat{F}_0$. This provides the quantization and generalization to arbitrary sl_2 embeddings of the Miura map. The generalized Miura transformations for a certain special class of sl_2 embeddings were also recently given in [83].

As a result of the generalized quantum Miura transformation *any* representation or realization of $\hat{F}_0$ gives rise to a representation or realization of the W algebra. In particular one obtains a free field realization of the W algebra by choosing a free field realization of $\hat{F}_0$. Given our formalism it is therefore straightforward to construct free field realizations for any W algebra that can be obtained by Drinfeld-Sokolov reduction.

2.2.9. The Stress-Energy Tensor

It is possible to give a general expression for the stress-energy tensor of a W algebra related to an arbitrary sl_2 embedding. For this purpose we write t_0 as $t_0 = s^a T_a$, where the s^a is only nonzero if T_a lies in the Cartan subalgebra. Furthermore, let δ_α be the eigenvalue of ad_{t_0} acting on T_α, thus $[t_0, T_\alpha] = \delta_\alpha T_\alpha$. From this it is easy to see that $\delta_\alpha = s_a f^{\alpha a}_{\alpha}$. Then the stress-energy tensor is

$$T = \frac{1}{2(k+\hbar h)}\left(\eta_{a_0 b_0}(\hat{J}^{a_0}\hat{J}^{b_0}) + 2\eta_{b\alpha}\hat{J}^b\chi(J^\alpha) - 2(k+\hbar h)s_a\partial\hat{J}^a + \hbar\eta_{b\alpha}f^{b\alpha}_{\ \ e}\partial\hat{J}^e\right), \tag{2.2.47}$$

where the indices a_0, b_0 run only over $\mathbf{g}_0$, and h is again the dual Coxeter number. By adding a D-exact term $D(R)$ to (2.2.47), where

$$R = \frac{1}{k+\hbar h}\eta_{b\alpha}(J^b J^\alpha) + \frac{1}{2(k+\hbar h)}\eta_{e\alpha}f^{e\beta}_{\ \ \gamma}(b^\alpha(b^\gamma c_\beta)), \tag{2.2.48}$$

we can rewrite it as

$$T = \frac{1}{2(k+\hbar h)}\eta_{ab}(J^a J^b) - s_a\partial J^a + (\delta_\alpha - 1)b^\alpha\partial c_\alpha + \delta_\alpha\partial b^\alpha c_\alpha, \tag{2.2.49}$$

which has the familiar form of improved Sugawara stress-energy tensor plus the stress-energy tensors of a set of free $b-c$ systems. The other generators of the W algebra cannot in general be written as the sum of a current piece plus a ghost piece. Actually, (2.2.49) is precisely what one would expect to get from a constrained WZNW model.

Notice that δ_α is the degree of T_α with respect to t_0, whereas α in (2.2.49) runs over $\mathbf{g}_+$ which was defined with respect to a new, different, integral grading of the Lie algebra.

In terms of the level k and the Cartan element of the sl_2 embedding t_0 (called the 'defining vector' since it determines the whole sl_2 subalgebra up to inner automorphisms) the central charge of the W algebra is given by [166, 20]

$$c(k;t_0) = d_0 - \frac{1}{2}\dim(\mathbf{g}_{\frac{1}{2}}) - 12\mathrm{Tr}\left(\frac{\rho}{\sqrt{k+h}} - t_0\sqrt{k+h}\right)^2 \tag{2.2.50}$$

where d_0 is the dimension of the subspace of $\mathbf{g}$ of elements of degree 0 with respect to the grading defined by t_0, $\mathbf{g}_{\frac{1}{2}}$ is the subspace of $\mathbf{g}$ of elements of degree $1/2$ with respect to that grading, and ρ is half the sum of the positive roots, $\rho = \frac{1}{2}\sum_{a\in\Delta^+} f^{b_0 a}_{\ \ \ a} T_{b_0}$.

2.2.10. The Virasoro Algebra

The Virasoro algebra is now easily quantized. Using the same conventions as in the previous sections for sl_2, we find for the BRST operator $Q = \oint_0 \frac{dz}{2\pi i}(c_+(z)(J^+(z) - 1))$, for the hatted currents $\hat{J}^\pm = J^\pm$ and $\hat{J}^0 = J^0 - (b^+c_+)$, and for the properly normalized generator of the BRST cohomology

$$T = \frac{1}{k+2\hbar}(\hat{J}^- + (\hat{J}^0\hat{J}^0) - (k+\hbar)\partial\hat{J}^0). \tag{2.2.51}$$

This is the generator of a Virasoro algebra with $c = 13\hbar - 6\hbar^2/(k+2\hbar) - 6(k+2\hbar)$, a result first found by Bershadsky and Ooguri [36].

2.2.11. The W_3 Algebra

The Zamolodchikov W_3 algebra (with $\hbar = 1$) is given in (2.2.4). It is obtained from the principal sl_2 embedding in sl_3 (see section 2.2.14). In terms of the following basis

$$J^aT_a = \begin{pmatrix} J^4+J^5 & J^2+J^3 & J^1 \\ J^6+J^7 & -2J^4 & J^2-J^3 \\ J^8 & J^7-J^6 & J^4-J^5 \end{pmatrix} \tag{2.2.52}$$

the sl_2 embedding is $i(t_+) = T_2$, $i(t_0) = T_5$ and $i(t_-) = T_7$. The generators of the cohomology (with $\hbar = 1$) and of the W_3 algebra with central charge $c = 50 - 24(k+$

3) $- 24/(k+3)$ are

$$
\begin{aligned}
T &= \frac{1}{2(k+3)}(\hat{J}^7 + \frac{3}{2}(\hat{J}^4\hat{J}^4) + \frac{1}{2}(\hat{J}^5\hat{J}^5) - (k+2)\partial\hat{J}^5), \\
W &= \left(\frac{48}{(5c+22)(k+3)^3}\right)^{\frac{1}{2}} (\hat{J}^8 + 2(\hat{J}^5\hat{J}^6) - 2(\hat{J}^4\hat{J}^7) - (k+2)\partial\hat{J}^6 - 2(\hat{J}^4(\hat{J}^4\hat{J}^4)) \\
&\quad +2(\hat{J}^4(\hat{J}^5\hat{J}^5)) - (k+2)(\hat{J}^4\partial\hat{J}^5) - 3(k+2)(\hat{J}^5\partial\hat{J}^4) + (k+2)^2\partial^2\hat{J}^4). \quad (2.2.53)
\end{aligned}
$$

The Miura transformation is obtained by putting $\hat{J}^6 = \hat{J}^7 = \hat{J}^8 = 0$ in (2.2.53). A free field realization follows from the Miura transformation by replacing $\hat{J}^4$ and $\hat{J}^5$ by expressions in terms of free fields, *i.e.* $\hat{J}^4 = i\sqrt{(k+3)/6}\partial\phi_1$ and $\hat{J}^5 = i\sqrt{(k+3)/2}\partial\phi_2$.

2.2.12. The $W_3^{(2)}$ Algebra

The final example we discuss is the $W_3^{(2)}$ algebra [38, 264]. It is convenient to pick a slightly different basis for sl_3, namely

$$
J^aT_a = \begin{pmatrix} J^4+J^5 & J^2 & J^1 \\ J^6 & -2J^4 & J^3 \\ J^8 & J^7 & J^4-J^5 \end{pmatrix} \quad (2.2.54)
$$

The sl_2 embedding is similar to the embedding of isospin in $SU(3)$ under which the octet decomposes into the four isospin representations ($\Lambda, \Xi^{0,-}, (p,n)$ and $\Sigma^{-,0,+}$). Thus, the W algebra will be generated by four fields. The sl_2 embedding is $i(t_+) = T_1, i(t_0) = \frac{1}{2}T_5$ and $i(t_-) = \frac{1}{2}T_8$. This is an example where the grading given by Λ^0 is non-integral, and $\mathbf{g}_{\frac{1}{2}}$ is spanned by T_2 and T_3. To get an integral gradation, we add to Λ^0 the gradation given by $-T_4/6$, which assigns degree $-\frac{1}{2}$ to T_2 and degree $+\frac{1}{2}$ to T_3. Then we get the integral gradation δ

$$
\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad (2.2.55)
$$

which also shows what the decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$ looks like. The correctly normalized generators ($\hbar = 1$) of the BRST cohomology that form a $W_3^{(2)}$ algebra [38] are

$$
\begin{aligned}
H &= 2\hat{J}^4, \\
G^+ &= \hat{J}^6,
\end{aligned}
$$

$$\begin{aligned} G^- &= -\hat{J}^7 + (\hat{J}^2\hat{J}^5) + 3(\hat{J}^2\hat{J}^4) + (k+1)\partial\hat{J}^2, \\ T &= \frac{1}{k+3}(\hat{J}^8 + (\hat{J}^5\hat{J}^5) + 3(\hat{J}^4\hat{J}^4) + (\hat{J}^2\hat{J}^6) + (k+1)\partial\hat{J}^5), \end{aligned} \quad (2.2.56)$$

with central charge $c = 25 - 6(k+3) - 24/(k+3)$. In the case at hand the subalgebra $\mathbf{g}_0$ is spanned by T_2, T_4, T_5 and T_6. Obviously $\mathbf{g}_0 \simeq sl_2 \oplus u(1)$. Therefore $\hat{F}_0$ is the direct sum of an affine sl_2 and an affine $u(1)$ field algebra, and using (2.2.45) the levels of these can be calculated to be $k+1$ and $k+3$ respectively. A free field realization of the $W_3^{(2)}$ algebra can now be found by putting $\hat{J}^7 = \hat{J}^8 = 0$ in (2.2.56), and by replacing the remaining hatted currents, that generate $\hat{F}^0$, by Wakimoto free field realizations. Actually, given a free field realization of an affine Lie algebra, one can always obtain another free field realization ('conjugate Wakimoto realization' [276]) by applying the Chevalley automorphism $H_\alpha \to -H_\alpha$ and $E_{\pm\alpha} \to -E_{\mp\alpha}$ to the free field realization. This procedure gives two different free field realizations for $W_3^{(2)}$. These are bosonic versions of the two free field realizations of the $N=2$ superconformal algebra, of which the $W_3^{(2)}$ algebra is a bosonic version, that were recently discussed in [39]. Thus, several free field realizations naturally fit into this framework.

2.2.13. Free Field Expressions for W Algebras

We have seen in some examples how a free field realization of the W algebra follows through the Miura transformation. What we did not consider so far, was whether these W algebras are the centralizer of a set of vertex operators acting on the field algebra generated by the free fields. Here, we construct these vertex operators. The idea is simply to first take a free field realization of the affine Lie algebra based on $\mathbf{g}$. We denote the field algebra generated by the free fields by $F_\phi \otimes F_{\beta,\gamma}$. The affine Lie algebra $\bar{\mathbf{g}}$ is the centralizer of vertex operators $V_i, i = 1 \ldots r_{\mathbf{g}}$ acting on the free field algebra,

$$\bar{\mathbf{g}} \simeq \cap_i \ker \delta_{V_i}|_{F_\phi \otimes F_{\beta,\gamma}}. \quad (2.2.57)$$

Clearly, the BRST operator (2.2.33) commutes with the action of the vertex operators, because the BRST operator is build up from currents that commute, by definition, with the vertex operators V_i. It is then easy to see that the W algebra can also be obtained as the centralizer of the vertex operators action on $H_Q^*(F_\phi \otimes F_{\beta,\gamma} \otimes F_{b,c})$, where Q is equal to (2.2.33), with the currents replaced by free field expressions. If we decompose $F_{\beta,\gamma} = F_{\beta_0,\gamma_0} \otimes F_{\beta_\alpha,\gamma_\alpha}$, where β_0, γ_0 correspond to the roots that have degree zero with respect to the integral grading defined by δ, and $\beta_\alpha, \gamma_\alpha$ are the β, γ systems corresponding to roots with positive degree, then the following 'quartet confinement' holds (for a suitable free field realization of $\bar{\mathbf{g}}$)

$$H_Q^*(F_\phi \otimes F_{\beta_0,\gamma_0} \otimes F_{\beta_\alpha,\gamma_\alpha} \otimes F_{b,c}) \simeq F_\phi \otimes F_{\beta_0,\gamma_0}. \quad (2.2.58)$$

For sl_2, this was proven in [36], for W_3 in [129] and for W_N in [128]. In the general case the proof is as follows: there are many possible different free field realizations of affine Lie algebras. They are constructed through Gauss decompositions [150]. In particular, a free field realization is obtained via (after appropriate normal ordering)

$$\mathcal{J} = kn^{-1}\partial n + n^{-1}(\mathcal{J}_0 + \beta^\alpha T_\alpha)n, \tag{2.2.59}$$

where $\mathcal{J}_0$ contains an arbitrary free field realization of $\mathbf{g}_0$, and $n \in \exp \mathbf{g}_-$ is the lower triangular matrix $n = \mathbf{1} + \gamma_\alpha T^\alpha$. With this particular free field realization, the constraints $J^\alpha = \chi(J^\alpha)$ are linearly equivalent to the set of constraints $\beta^\alpha - \chi(J^\alpha)$. Therefore, there exists a field redefinition of $\{\beta^\alpha, \gamma_\alpha, c_\alpha, b^\alpha\}$ such that the BRST operator takes the form

$$Q = \oint_0 \frac{dz}{2\pi i} c_\alpha(\beta^\alpha - \chi(J^\alpha)). \tag{2.2.60}$$

This major simplification allows us to compute

$$\begin{aligned} H_Q^*(F_\phi \otimes F_{\beta_0,\gamma_0} \otimes F_{\beta_\alpha,\gamma_\alpha} \otimes F_{b,c}) &\simeq H_Q^*(F_\phi \otimes F_{\beta_0,\gamma_0}) \otimes \bigotimes_\alpha H_Q^*(F_{\beta^\alpha,\gamma_\alpha} \otimes F_{b^\alpha,c_\alpha}) \\ &\simeq H_Q^*(F_\phi \otimes F_{\beta_0,\gamma_0}) \\ &\simeq F_\phi \otimes F_{\beta_0,\gamma_0} \end{aligned} \tag{2.2.61}$$

where in the first line we used a Kunneth like theorem from [53], the second line follows from a simple cohomology calculation on the subcomplex generated by $\beta^\alpha, b^\alpha, \gamma_\alpha, c_\alpha$, and the third line from the fact that $Q = 0$ on $F_\phi \otimes F_{\beta_0,\gamma_0}$. The vertex operators V_i are BRST equivalent to certain vertex operators $\tilde{V}_i$ acting on $F_\phi \otimes F_{\beta_0,\gamma_0}$. This proves that the W algebra is equal to

$$\cap_i \ker \delta_{\tilde{V}_i}|_{F_\phi \otimes F_{\beta_0,\gamma_0}}. \tag{2.2.62}$$

To do this in practice, one need not always construct a new free field realization of the type (2.2.59), but this procedure works for any free field realization as long as the constraints can be brought in the form $\beta^\alpha - \chi(J^\alpha)$. For example, if we take the free field realization (2.1.60) the constraints for the W_3 algebra are $\beta_1 = \beta_3 = 1$ and $\beta_2 = 0$. It follows that the W_3 algebra is the centralizer of

$$\begin{aligned} \tilde{V}_1 &= \exp((2\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot \boldsymbol{\phi}/(k+3)), \\ \tilde{V}_2 &= \exp((2\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \boldsymbol{\phi}/(k+3)) \end{aligned} \tag{2.2.63}$$

acting on F_{ϕ_1,ϕ_2}. These free fields are free fields with a background charge. The background charge vanishes for $k = -2$, and precisely in that case this realization of the W_3 algebra reduces to the one discussed in section 2.2.1.

The same free field realization (2.1.60) can also be used to describe $W_3^{(2)}$ in this way. The constraints are $\beta_3 = 0$ and $\beta_2 = 1$. The $W_3^{(2)}$ algebra is the centralizer of

$$\begin{aligned} \tilde{V}_1 &= (\beta_1 \exp((2\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot \boldsymbol{\phi}/(k+3))), \\ \tilde{V}_2 &= (\gamma_1 \exp((2\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \boldsymbol{\phi}/(k+3))), \end{aligned} \tag{2.2.64}$$

acting on $F_{\phi_1,\phi_2} \otimes F_{\beta_1,\gamma_1}$.

This procedure can also be used to construct free field resolutions for irreducible representations of W algebras. (for the relation between quantum Hamiltonian reduction and the representation theory of W_N algebras, see [138]) However, rigorous proofs are only available for the case sl_2 [36, 35]. The structure of the resolutions for W_N algebras is conjectured in [60, 138].

2.2.14. Some Explicit Results for sl_n

The sl_2 embeddings into sl_n are in one to one correspondence with the partitions of n [106]. Let $(n_1, n_2 \ldots)$ be a partition of n with $n_1 \geq n_2 \geq \ldots$, then one can define a different partition $(m_1, m_2, \ldots)$ of n by letting m_k be the number of i for which $n_i \geq k$. Furthermore let $s_t = \sum_{i=1}^{t} m_i$. Then the sl_2 embedding associated to the partition $(n_1, n_2, \ldots)$ is given by

$$\begin{aligned} t_+ &= \sum_{l\geq 1} \sum_{k=1}^{n_l - 1} E_{l+s_{k-1},l+s_k}, \\ t_0 &= \sum_{l\geq 1} \sum_{k=1}^{n_l} (\frac{n_l+1}{2} - k) E_{l+s_{k-1},l+s_{k-1}}, \\ t_- &= \sum_{l\geq 1} \sum_{k=1}^{n_l - 1} \frac{k(n_l-k)}{2} E_{l+s_k,l+s_{k-1}} \end{aligned}$$

where E_{ij} is as usual the $n \times n$ matrix with zeros everywhere except for the matrix element (i,j) which is equal to one. The element δ which defines the grading on sl_n that we use to impose the constraints is given by

$$\delta = \sum_{k\geq 1} \sum_{j=1}^{m_k} \left(\frac{\sum_l l m_l}{\sum_l m_l} - k \right) E_{s_{k-1}+j,s_{k-1}+j}. \tag{2.2.65}$$

One can check that in case the grading provided by t_0 is integer then $\delta = t_0$.

The fundamental representation of sl_n decomposes into irreducible sl_2 multiplets. This we denote by $\underline{n} \rightarrow \oplus_l \underline{n_l} \equiv \oplus_i p_i \underline{i}$, where $\underline{i}$ is the i-dimensional representation of sl_2.

We then have the following identities that come in useful when calculating the central charge for a certain specific case.

$$
\begin{aligned}
\frac{1}{2}\dim(\bar{g}_{\frac{1}{2}}) &= \sum_{i>0,k\geq 0} i p_i p_{i+2k+1}, \\
d_0 &= -1+\sum_{i>0} i p_i^2 + 2\sum_{i>0,k>0} i p_i p_{i+2k}, \\
\mathrm{Tr}(\rho)^2 &= \frac{1}{12}(n^3-n), \\
\mathrm{Tr}(t_0)^2 &= \frac{1}{12}\sum_i p_i(i^3-i), \\
\mathrm{Tr}(t_0\rho) &= \frac{1}{12}\left(\sum_i p_i^2(i^3-i)+\sum_{i<r} i(i+1)(3r-i-2)p_i p_r\right).
\end{aligned} \tag{2.2.66}
$$

2.2.15. Extensions of W Algebras

W algebras are extensions of the Virasoro algebra. One can also pose the question: when does a certain W algebra $\mathcal{A}_1$ contain another W algebra $\mathcal{A}_2$? A sufficient condition is:
If $\mathcal{A}_1$ is obtained from the Drinfeld-Sokolov reduction associated to the algebra homomorphism $i_1 : sl_2 \to \mathbf{g}_1$, and $\mathcal{A}_2$ from $i_2 : sl_2 \to \mathbf{g}_2$, then $\mathcal{A}_1 \subset \mathcal{A}_2$ if (i) $\mathbf{g}_2 = \mathbf{g}_1 \oplus \mathbf{g}'$ and $i_2|_{\mathbf{g}_1} = i_1$, or (ii) if there is an algebra homomorphism $j : \mathbf{g}_1 \to \mathbf{g}_2$ such that $i_2 = j \circ i_1$.

We want to illustrate this by proving that every W algebra that is obtained from an embedding of sl_2 in sl_n as in the previous section, admits an $N = 2$ supersymmetric extension. Given a partition $(n_1, n_2, \ldots)$ of n, define $n'_i = n_i - 1$ and $n' = \sum_i n'_i$. Also define m'_k and s'_t as in the previous section, for the integers n'_i. Now consider the Lie superalgebra $\mathbf{g} = sl(n|n')$, with the following sl_2 embedding:

$$
\begin{aligned}
t_+ &= \sum_{l\geq 1}\sum_{k=1}^{n_l-1} E_{l+s_{k-1},l+s_k} + \sum_{l\geq 1}\sum_{k=1}^{n'_l-1} E_{n+l+s'_{k-1},n+l+s'_k} \\
t_0 &= \sum_{l\geq 1}\sum_{k=1}^{n_l}\left(\frac{n_l+1}{2}-k\right)E_{l+s_{k-1},l+s_{k-1}} + \sum_{l\geq 1}\sum_{k=1}^{n'_l}\left(\frac{n'_l+1}{2}-k\right)E_{n+l+s'_{k-1},n+l+s'_{k-1}} \\
t_- &= \sum_{l\geq 1}\sum_{k=1}^{n_l-1}\frac{k(n_l-k)}{2}E_{l+s_k,l+s_{k-1}} + \sum_{l\geq 1}\sum_{k=1}^{n'_l-1}\frac{k(n'_l-k)}{2}E_{n+l+s'_k,n+l+s'_{k-1}}.
\end{aligned} \tag{2.2.67}
$$

This embedding factors via $sl_2 \to sl_n \oplus sl'_n \to sl(n|n')$, and since the embedding in the first component of $sl_n \oplus sl'_n$ is identical to the one in the previous subsection, we conclude that the super W algebra obtained from Hamiltonian reduction of $sl(n|n')$ contains the W algebra associated to $(n_1, n_2, \ldots)$. To prove that it contains the $N=2$ superconformal algebra, we use the fact that the $N=2$ superconformal algebra can also be obtained via Hamiltonian reduction, namely of $sl(2|1)$ [37]. In view of the conditions mentioned in the beginning of this section, it is sufficient to show that the embedding (2.2.67) factors via $sl(2|1)$. This can be shown by explicit calculation. In terms of the basis

$$\mathcal{J} = \begin{pmatrix} \frac{t_0}{2} + \frac{\hat{t}_0}{3} & t_+ & s_{++} \\ \frac{t_-}{2} & -\frac{t_0}{2} + \frac{\hat{t}_0}{3} & s_+ \\ s_{--} & s_- & \frac{2\hat{t}_0}{3} \end{pmatrix} \tag{2.2.68}$$

the remaining components of $sl(2|1)$ are embedded in $sl(n|n')$ via

$$\begin{aligned} s_+ &= \sum_{l\geq 1}\sum_{k=1} n'_l k E_{s_k+l,n+s'_{k-1}+l} \\ s_{++} &= \sum_{l\geq 1}\sum_{k=1} n'_l E_{l+s_{k-1},n+l+s'_{k-1}} \\ s_- &= \sum_{l\geq 1}\sum_{k=1} n'_l E_{n+s'_{k-1}+l,s_k+l} \\ s_{--} &= \sum_{l\geq 1}\sum_{k=1} n'_l (n_l - k) E_{n+s'_{k-1}+l,s_{k-1}+l} \\ \hat{t}_0 &= \sum_{l=1}^{n} \frac{n'}{n+n'} E_{l,l} + \sum_{l=1}^{n'} \frac{n}{n+n'} E_{n+l,n+l}. \end{aligned} \tag{2.2.69}$$

Closer inspection reveals that all original W fields are now the top components of an $N=2$ multiplet. The lowest components of these multiplets also form a W algebra, associated to the partition $(n'_1, n'_2, \ldots)$ of n'. The same procedure should in principle allow one to construct more general extensions of W algebras, *e.g.* such that they contain an extended superconformal algebra. For the standard W_N algebras, the procedure given here says that $N=2$ extended W_N algebras can be obtained via Hamiltonian reduction of $sl(n|n-1)$. This has been shown more explicitly in [188, 253]. Curiously enough, $N=2$ algebras also show up in computations of the BRST cohomology of reductions of $sl(n_1+n_2)$, associated to the sl_2 embedding $\underline{n_1+n_2} \to \underline{n_1} \oplus \underline{n_2}$ [197]. It is not clear whether this is related to the $N=2$ algebras constructed in this section.

3

Chiral W gravity

3.1. Gauging the Chiral Algebra

Consider the action $S(\phi)$ for some conformal field theory with a corresponding chiral algebra. Any element A of the chiral algebra generates a symmetry of the action. These are characterized by parameters ϵ that are holomorphic, and the associated symmetry generated by $\oint_0 \frac{dz}{2\pi i}\epsilon(z)A(z)$. These symmetries are independent of $\bar{z}$, and one might say that they are 'global' with respect to $\bar{z}$. Gauging the A symmetry means that we want to make the action invariant under A symmetries that are 'local' with respect to $\bar{z}$, so that it is invariant under the action of $Q = \oint_0 \frac{dz}{2\pi i}\,\epsilon(z,\bar{z})A(z)$. The variation of the action under δ_Q reads

$$\delta_Q S(\phi) = -\frac{1}{\pi}\int d^2z\, A(z,\bar{z})\bar{\partial}\epsilon(z,\bar{z}). \tag{3.1.1}$$

To compensate for this variation, we introduce a gauge field μ, which transforms as $\delta_Q\mu = \bar{\partial}\epsilon(z,\bar{z}) + \ldots$, and add a term to the action in which the gauge field couples linearly to the current A,

$$S'(\phi,\mu) = S(\phi) + \frac{1}{\pi}\int d^2z\mu A. \tag{3.1.2}$$

The variation of the extra term in (3.1.2) under δ_Q is

$$\delta_Q\left(\frac{1}{\pi}\int d^2z\,\mu A\right) = \frac{1}{\pi}\int d^2z\,((\delta_Q\mu)A + \mu\sum_{r>0}\frac{\partial^{r-1}\epsilon}{(r-1)!}\{AA\}_r), \tag{3.1.3}$$

and we see that the action can only be made invariant by adding extra terms to the μ transformation rule if $\{AA\}_r$ contains A for each r. When the latter is not the case, we have to introduce new gauge fields corresponding to all the extra fields that appear in the OPE of A with itself. This does not come as a surprise: of course we can only gauge a closed subalgebra of the chiral algebra. Suppose this subalgebra is generated by $\{A_\alpha\}$, then the gauged action simply reads

$$S'(\phi, \{\mu_\alpha\}) = S(\phi) + \frac{1}{\pi}\int d^2z \sum_\alpha \mu_\alpha A_\alpha. \tag{3.1.4}$$

Under a general variation generated by $Q = \oint_0 \frac{dz}{2\pi i}\sum_\alpha \epsilon_\alpha(z,\bar{z})A_\alpha(z,\bar{z})$ the variations of the gauge fields read

$$\delta_Q \mu_\alpha = \bar{\partial}\epsilon_\alpha - \frac{D}{DA_\alpha}\left(\sum_{\beta,\gamma,r>0} \mu_\beta \frac{\partial^{r-1}\epsilon_\gamma}{(r-1)!}\{A_\gamma A_\beta\}_r\right), \tag{3.1.5}$$

where the 'derivatives' D/DA_α are defined quite arbitrarily, as long as for all X

$$\int d^2z\, X = \int d^2z \sum_\alpha A_\alpha \frac{DX}{DA_\alpha}. \tag{3.1.6}$$

This is a quite cumbersome way of writing down the transformation rules. In practise one just performs a transformation and directly reads of the transformation rules for the μ_α. There is an ambiguity in doing this, however. If for example the transformation rule for A_α contains $\epsilon A_\beta A_\gamma$, one can compensate this by by adding to the μ_β transformation rule $-\epsilon\mu_\alpha A_\gamma$, but also by adding to the μ_γ transformation rule the term $-\epsilon\mu_\alpha A_\beta$. The freedom in the choice of 'derivatives' D/DA_α is a reflection of this ambiguity. Put differently, there are several different ways to rewrite a non-linear algebra as a linear algebra with field-dependent structure constants.

That (3.1.4) is sufficient to gauge the chiral algebra was first shown for the W_3 algebra in [181] and later for arbitrary chiral algebras in [182]. Note that the identity operator must also be gauged if any of the OPE's has a central term. This can sometimes happen, for example for a free boson (as is illustrated in the next subsection), although the discussion here uses only the classical OPE's. Of course if we gauge the identity operator, the theory will be invariant under any transformation, by adding a suitable transformation rule for the identity gauge field. To avoid problems of this kind, we will assume that we use the gauge field of the identity operator only to absorb the central terms in the OPE's. We will see later that we can sometimes replace the identity gauge field by a complicated function of the remaining gauge fields, which would spoil the simple linear form of the gauge couplings in (3.1.4). Before proceeding with the general discussion, let us first discuss the example of the free scalar field.

3.1.1. The Free Scalar Field

The chiral algebra for the action

$$S = \frac{1}{2\pi} \int d^2z\, \partial X \bar{\partial} X \tag{3.1.7}$$

is generated by $J = \partial X$. Let us try to gauge fix the symmetry generated by J. Under the transformations generated by $\oint_0 \frac{dz}{2\pi i}\, \epsilon J$ the field X transforms as $\delta X = -\epsilon$. According to the general scheme outlined above, the gauged action is

$$S' = \frac{1}{2\pi} \int d^2z\, \partial X (\bar{\partial} X + 2\mu_J) \tag{3.1.8}$$

where μ transforms as $\delta\mu = \bar{\partial}\epsilon$. However, we expect a problem due to the central term in the OPE of J with itself. Indeed, S' is not invariant but we have

$$\delta S' = -\frac{1}{\pi} \int d^2z\, \partial\epsilon \mu_J. \tag{3.1.9}$$

This reflects the fact that we also need to include a gauge field for the identity operator. With this extra gauge field the gauged action is

$$S' = \frac{1}{2\pi} \int d^2z\, \partial X (\bar{\partial} X + 2\mu_J) + \frac{1}{\pi} \int d^2z \mu_{\mathbf{1}}. \tag{3.1.10}$$

Supplemented with the transformation rule $\delta\mu_{\mathbf{1}} = \mu_J \partial\epsilon$ for the identity gauge field, this action is invariant. Notice that none of the transformation rules contains $\mu_{\mathbf{1}}$, suggesting that it may be possible to express $\mu_{\mathbf{1}}$ in terms of the other fields in the theory. In the case at hand, it is easy to write down such an expression for $\mu_{\mathbf{1}}$, $\mu_{\mathbf{1}} = \frac{1}{2}\mu_J \frac{\partial}{\bar{\partial}} \mu_J$. Using this the gauged action simply reads

$$S' = \frac{1}{2\pi} \int d^2z\, \partial (X + \frac{1}{\bar{\partial}}\mu_J)\, \bar{\partial} (X + \frac{1}{\bar{\partial}}\mu_J) \tag{3.1.11}$$

which is manifestly invariant. Unfortunately, it is a nonlocal action, because of the presence of the inverse of the derivative $\bar{\partial}$.

The complication with the gauge field for the identity operator does not exist if we gauge the Virasoro algebra, since (classically) the energy momentum tensor $T(X) = -\frac{1}{2}(\partial X \partial X)$ has $c = 0$. The gauged action reads

$$S = \frac{1}{2\pi} \int d^2z\, \partial X \bar{\partial} X + \frac{1}{\pi} \int d^2z\, \mu T(X) = \frac{1}{2\pi} \int d^2z\, \partial X (\bar{\partial} - \mu\partial) X, \tag{3.1.12}$$

with transformation rules $\delta X = \epsilon \partial X$ and $\delta \mu = \bar{\partial}\epsilon - \mu\partial\epsilon + \epsilon\partial\mu$. The interesting observation is that the right hand side of (3.1.12) is precisely the 'covariant' action for a free scalar field (2.1.25) in the metric

$$ds^2 = e^{-2\varphi}(dz + \mu d\bar{z})d\bar{z}. \tag{3.1.13}$$

The $(-1, 1)$ differential μ in (3.1.13) is known as the Beltrami differential, and the gauge field can be identified with this particular component of the metric. From now on we will call the gauge choice (3.1.13) for the metric the chiral gauge. By analogy we hope that the gauge fields for possible other spin s fields of the chiral algebra are components of some tensor field $A_{\mu_1 \cdots \mu_s}$, in such a way that the gauged action is equal to a fully covariant action in some generalized chiral gauge. Such an identification would make it easier to write down actions that are manifestly invariant under some chiral algebra or under some W algebra, and shed light on the geometrical structures that underly the chiral algebra. We will come back to this problem later, when we discuss the gauging of the full symmetry algebra $\mathcal{A} \times \bar{\mathcal{A}}$.

3.1.2. The Identity Gauge Field

One might wonder whether it is always possible to express the identity gauge field $\mu_{\mathbf{1}}$ in terms of the other fields, since it does not occur in any of the transformation rules. If $\mu_{\mathbf{1}}$ is a function of the other fields, it must satisfy the following equation,

$$\int d^2z \left(\sum_\alpha \left(\frac{\delta \mu_{\mathbf{1}}}{\delta \mu_\alpha} \delta_Q \mu_\alpha + \frac{\delta \mu_{\mathbf{1}}}{\delta A_\alpha} \delta_Q A_\alpha \right) \right) = \int d^2z \sum_\gamma \epsilon_\gamma \sum_\beta \frac{(-1)^{h_\beta + h_\gamma} \partial^{h_\beta + h_\gamma - 1}}{(h_\beta + h_\gamma - 1)!} \mu_\beta \{A_\gamma A_\beta\}_{h_\gamma + h_\beta}. \tag{3.1.14}$$

The left hand side is just $\delta_Q \int d^2z\, \mu_{\mathbf{1}}$, and the right hand side contains all the central terms of the algebra we are gauging; h_β and h_γ are the conformal weights of A_β and A_γ. Upon partial integrating the left hand side of this equation, we obtain a set of partial differential equations for $\delta\mu_{\mathbf{1}}/\delta\mu_\alpha$ and $\delta\mu_{\mathbf{1}}/\delta A_\alpha$. These can be solved order by order in μ_α and A_α. However, since we will never need the explicit form of the solution, we do not pursue the study of the identity gauge field any further.

3.1.3. The Chiral Induced Action

We now gauge a (classical) subalgebra $\mathcal{A}^{\rm cl}_{\rm sub}$ of the full classical chiral algebra $\mathcal{A}^{\rm cl}$, and assume that this subalgebra has no central terms in its OPE's, so that the identity gauge field is not needed. In that case the coupling to the gauge fields is simply linear. Since the number of gauge fields is equal to the number of symmetries, we would naively expect that the induced action $\Gamma[\mu_\alpha]$ defined by (where we restored $\hbar$ dependence)

$$e^{-\frac{1}{\hbar}\Gamma[\mu_\alpha]} = \frac{\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}(S(\phi)+\frac{1}{\pi}\int d^2z\, \mu_\alpha A_\alpha)}}{\int \mathcal{D}\phi e^{-\frac{1}{\hbar}S(\phi)}} = \left\langle e^{-\frac{1}{\pi\hbar}\int d^2z\, \mu_\alpha A_\alpha}\right\rangle_{S(\phi)} \tag{3.1.15}$$

is independent of the gauge fields μ_α. However, when we quantize the theory, the classical algebra $\mathcal{A}^{\rm cl}$ is replaced by a quantum operator product algebra $\mathcal{A}$, and it may happen that the quantization $\mathcal{A}_{\rm sub}$ of $\mathcal{A}^{\rm cl}_{\rm sub}$ has central terms. These central terms spoil the gauge invariance of $\Gamma[\mu_\alpha]$, and make it a non-trivial function of the gauge fields*. The derivatives of Γ with respect to μ_α generate the insertion of A_α in the path integral. The precise relation between the quantum operator this insertion represents, and the classical field A_α that occurs in the action, depends on a choice of a regularization prescription, for instance, specified by a particular normal ordering. We will use the same symbol A_α both for the classical field in the action, and for the corresponding quantum operator. If there is danger of confusion, the latter will be denoted by $f_q(A_\alpha)$. The following identity holds

$$\frac{\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}(S(\phi)+\frac{1}{\pi}\int d^2z\, \mu_\alpha A_\alpha)}}{\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S(\phi)}} = \left\langle e^{-\frac{1}{\pi\hbar}\int d^2z\, \mu_\alpha f_q(A_\alpha)}\right\rangle_{OPE} \tag{3.1.16}$$

Using the representation (3.1.15) we derive a Ward identity for Γ†

$$\begin{aligned}
\pi\bar\partial\frac{\delta\Gamma}{\delta\mu_\alpha(z)} &= \bar\partial\frac{\left\langle A_\alpha(z)e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}\\
&= \bar\partial\frac{\left\langle \frac{-1}{\pi\hbar}\underbrace{A_\alpha(z)\int d^2w\sum_\beta \mu_\beta(w)A_\beta(w)}e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}
\end{aligned}$$

---

*For this reason these central terms are often identified with anomalies. We choose not to use this terminology here, because these terms can already be present at the classical level and are not pure quantum objects.

†In this derivation we use the fact that the correlation function of a product of A_α's can be evaluated using only the singular part of their OPE's. There are at least two different ways to see this. The first one is to express the fields in terms of their (Fourier) modes. The correlation function of a product of modes can be evaluated using only their commutators, and the commutators are completely fixed by the singular part of the OPE's. The second one is to put $\oint_C \frac{dz_1}{2\pi i}\,\epsilon(z_1)$ in front of $\langle A_1(z_1)\dots A_n(z_n)\rangle$, where the contour encloses all the points $z_2,\dots,z_n$. The contour can written as a sum of contours around each z_i separately, and to evaluate these contour integrals one only needs the singular part of the OPE of A_1 with A_i. Since the resulting identity holds for all ϵ, we find $\langle A_1(z_1)\dots A_n(z_n)\rangle = \sum_{i=2}^n\langle A_2(z_2)\dots A_{i-1}(z_{i-1})\underbrace{A_1(z_1)A_i}(z_i)A_{i+1}(z_{i+1})\dots A_n(z_n)\rangle$.

$$
\begin{aligned}
&= \bar\partial \frac{\left\langle \frac{-1}{\pi\hbar}\int d^2w \sum_{\beta,r>0}\mu_\beta(w)\frac{\{A_\alpha A_\beta\}_r(w)}{(z-w)^r} e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle} \\
&= \frac{\left\langle \frac{-1}{\hbar}\int d^2w \sum_{\beta,r>0}\mu_\beta(w)\{A_\alpha A_\beta\}_r(w)\frac{1}{(r-1)!}\partial_w^{r-1}\delta^2(z-w) e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle} \\
&= \frac{\left\langle \frac{1}{\hbar}\sum_{\beta,r>0}\frac{(-1)^r}{(r-1)!}\partial^{r-1}(\mu_\beta\{A_\alpha A_\beta\}_r) e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle} \\
&= \frac{1}{\hbar}\sum_{\beta,r>0}\frac{(-1)^r}{(r-1)!}\partial^{r-1}(\mu_\beta\{A_\alpha A_\beta\}_r)|_{A_\gamma\to\pi\frac{\delta\Gamma}{\delta\mu_\gamma},\ldots} \qquad (3.1.17)
\end{aligned}
$$

Here we used (2.1.28). In the last line the dots stand for the replacements of normal ordered products of fields by expressions in terms of $\Gamma[\mu_\alpha]$: in the one but last line the OPE of A_α with A_β can contain, besides the generators of $\mathcal{A}_{\rm sub}$, also normal ordered products of these generators. We want to replace these by expressions in terms of $\Gamma[\mu_\alpha]$, since our goal is to derive a Ward identity for $\Gamma[\mu_\alpha]$. If $\{A_\alpha A_\beta\}_r$ is linear in the generators of $\mathcal{A}_{\rm sub}$, this is easy, we use

$$
\frac{\left\langle A_\gamma(z) e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle_{S(\phi)}}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle} = \pi\frac{\delta\Gamma}{\delta\mu_\gamma(z)}. \qquad (3.1.18)
$$

If $\{A_\alpha A_\beta\}_r$ is non-linear in the generators, the situation is more complicated. To find the answer we must go back to the definition of the normal ordered product. It was defined in section 2.1.3 by point-splitting regularization,

$$
(A_1A_2)(z) \equiv \lim_{z\to w}\{A_1(z)A_2(w) - \text{terms that are singular as } z\to w\} \qquad (3.1.19)
$$

Replacing normal ordered products by their point-splitting regularized definition enables us to find expressions in terms of $\Gamma[\mu_\alpha]$. If, for example, $\{A_\beta A_\gamma\}_r$ contains the normal ordered product $(A_\gamma A_\delta)$, this leads to the following replacement rule for $(A_\gamma A_\delta)$

$$
\begin{aligned}
(A_\gamma A_\delta)(z) &\equiv \frac{\left\langle (A_\gamma A_\delta)(z)\, e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle_{S(\phi)}}{\left\langle e^{-\frac{1}{\pi\hbar}\int d^2z'\,\mu_\alpha A_\alpha}\right\rangle} \\
&= \lim_{z\to w}\left(\pi^2\frac{\delta\Gamma}{\delta\mu_\gamma(z)}\frac{\delta\Gamma}{\delta\mu_\delta(w)} - \hbar\pi^2\frac{\delta^2\Gamma}{\delta\mu_\gamma(z)\delta\mu_\delta(w)} - \ldots\right), \qquad (3.1.20)
\end{aligned}
$$

where the dots stand for the terms that are singular when $z\to w$.

Since the classical gauged algebra had no central terms, Γ is of order $\hbar$, and if the algebra is nonlinear we see from (3.1.20) that the Ward identity contains higher-order functional derivatives of Γ with respect to μ_α. Even if we expand $\Gamma = \sum_{i\geq 1} \hbar^i \Gamma^{(i)}$ and focus on the leading part $\Gamma^{(1)}$, the higher-order functional derivatives still contribute to the Ward identity. The presence of such terms makes it much more difficult to solve the Ward identities. If they are absent, the Ward identities are a set of nonlinear inhomogeneous partial differential equations for $\delta\Gamma/\delta\mu_\alpha$, which one could attempt to solve using standard techniques. So it seems that we are in a difficult situation to extract any information about Γ if the gauge algebra is nonlinear. However, it turns out that Γ can in many cases be expanded in another parameter, such that the Ward identities for the leading part of Γ in this expansion do not contain higher-order functional dervatives. This is the case if the gauge algebra $\mathcal{A}_{\rm sub}$ is the special case $\mathcal{A}_{\rm sub}(c_0;\hbar)$ of a one-parameter family of algebras $\mathcal{A}_{\rm sub}(c;\hbar)$, where the parameter c is equal to the central charge. Let $\Gamma[c,\hbar]$ denote a solution of the Ward identities for the gauge algebra $\mathcal{A}_{\rm sub}(c;\hbar)$. Suppose furthermore that the coefficients for the normal ordered products of k fields that occur in the operator product expansions of $\mathcal{A}_{\rm sub}(c;\hbar)$ behave for large c as $\mathcal{O}(c^{1-k})$ ‡. Then $\Gamma[c,\hbar]$ admits an expansion

$$\Gamma[c,\hbar] = \sum_{i\geq 0} c^{1-i}\Gamma^{(i)}[\hbar] \tag{3.1.21}$$

as can be seen from the Ward identities. It is clear from (3.1.20) that the higher-order functional derivatives of Γ do not contribute to the Ward identity for $\Gamma^{(0)}$. Also, the coefficients in the operator product expansions can be truncated to their leading c behavior if we want to extract the Ward identity for $\Gamma^{(0)}$ from (3.1.17). This truncated version of $\mathcal{A}_{\rm sub}(c;\hbar)$ is actually a classical operator product algebra. To see this, let $\mathcal{A}_{\rm sub}(c;\hbar;\hbar_2)$ denote the algebra obtained from $\mathcal{A}_{\rm sub}(c;\hbar)$ by rescaling $c \to c/\hbar_2$ and $A_\alpha \to A_\alpha/\hbar_2$. Then the classical limit of $\mathcal{A}_{\rm sub}(c;\hbar;\hbar_2)$ with respect to $\hbar_2$ is precisely the classical operator algebra that we need to compute $\Gamma^{(0)}$§.

As an example, suppose that we gauge the Virasoro algebra starting with a free scalar field. Then $\mathcal{A}_{\rm sub}(c;\hbar;\hbar_2)$ is the algebra

$$\underbrace{T(z)T(w)} = \frac{c\hbar\hbar_2/2}{(z-w)^4} + \frac{2\hbar\hbar_2 T(w)}{(z-w)^2} + \frac{\hbar\hbar_2\partial T(w)}{z-w}, \tag{3.1.22}$$

and $\mathcal{A}_{\rm sub}$ is the special case $c = c_0 = \hbar, \hbar_2 = 1$. The classical limit with respect to $\hbar_2$ is the Virasoro algebra with nonvanishing central charge, whereas the classical limit of $\mathcal{A}_{\rm sub}$ is a Virasoro algebra with $c = 0$. It is an amazing fact that the classical centerless

‡Almost all known operator algebras satisfy this condition. A counterexample is the $W_{4,6}$ algebra in [196].

§Recently, the same result for quadratic non-linear algebras was obtained in [290].

gauge algebra is promoted to a classical algebra with center when considering the Ward identity of $\Gamma^{(0)}$.

If we would have started with a gauge algebra with central terms, and would not have added the identity gauge field, we can still define the induced action as in (3.1.15), although we now expect that the induced action is of order $\hbar^0$ rather than $\hbar^1$, because the gauged action is not gauge invariant to start with. In any case, the whole derivation of the Ward identities given above still holds, and the structure is precisely the same. For example, for the free scalar field with background charge, the classical gauge algebra is a Virasoro algebra with central charge $-12\alpha_0^2$, and this gets promoted to a classical Virasoro algebra with central charge $c = 1 - 12\alpha_0^2$ when considering the $\Gamma^{(0)}$ Ward identity. Because the structure is the same whether we start with a gauge algebra with or without center, we will from now on treat both cases at the same time, assuming we did not include the identity gauge field in the gauging. The subtleties associated to the identity gauge field will be completely ignored in the what follows.

3.2. Solution of the Ward Identity

In this section we compute the lowest-order part $\Gamma^{(0)}$ of the induced action in case the classical algebra $\mathcal{A}^{\rm cl}_{\rm ind}$ that describes the Ward identity for $\Gamma^{(0)}$ is a W algebra related to an sl_2 embedding.

3.2.1. The Virasoro Algebra

In case $\mathcal{A}^{\rm cl}_{\rm ind}$ is a Virasoro algebra with central charge c, the Ward identity for $\Gamma^{(0)}[\mu]$ reads

$$\bar\partial \frac{\delta\Gamma}{\delta\mu} = \frac{c}{12\pi}\partial^3\mu + 2\frac{\delta\Gamma}{\delta\mu}\partial\mu + \mu\partial\frac{\delta\Gamma}{\delta\mu}. \tag{3.2.1}$$

Since the Virasoro algebra is linear, $\Gamma^{(0)}$ is equal to the all-order induced action $\Gamma[\mu]$. The equation is solved by rewriting it as

$$\frac{\delta\Gamma}{\delta\mu} = \frac{c}{12\pi}(\bar\partial - 2\partial\mu - \mu\partial)^{-1}\partial^3\mu \tag{3.2.2}$$

which can be integrated to [265]

$$\Gamma[\mu] = \frac{c}{24\pi}\int d^2z\, \mu\, \partial^2(1 - \frac{1}{\bar\partial}\,\mu\,\partial)^{-1}\frac{\partial}{\bar\partial}\,\mu \tag{3.2.3}$$

The exact solvability in due to the linearity of the Virasoro algebra. The same trick does not work for non-linear W algebras. However, the fact that W algebras can be seen as constrained current algebras will still enable us to write down a solution for Γ for all W algebras. The connection between $\Gamma[\mu]$ and sl_2 current algebra was first discussed in [265, 207, 36]. The induced action for the case when $\mathcal{A}^{\rm cl}_{\rm ind}$ is the W_3 algebra was constructed, using its connection with sl_3 current algebra, in [256]. Before we treat the general case, let us briefly illustrate the method by showing the relation between the Virasoro Ward identity (3.2.1) and constrained currents. If T denotes $\pi\delta\Gamma/\delta\mu$, then (3.2.1) can be written in the form of a zero-curvature equation

$$\left[\partial + \begin{pmatrix} 0 & \frac{\xi}{k} \\ \frac{T}{\xi} & 0 \end{pmatrix}, \bar{\partial} + \begin{pmatrix} \frac{1}{2}\partial\mu & \frac{\xi}{k}\mu \\ -\frac{k}{2\xi}\partial^2\mu + \frac{1}{\xi}\mu T & -\frac{1}{2}\partial\mu \end{pmatrix}\right] = 0. \tag{3.2.4}$$

Quite remarkably, the z-part of the connection is precisely of the form (2.2.10), whereas the $\bar{z}$-part is of the form (2.2.12)! This relation between the Ward identities and the machinery developed in chapter 2 holds for arbitrary W algebras. The next step in the computation of Γ is to notice that the WZNW action also satisfies a zero-curvature equation. Combining these facts enables us to express Γ in terms of a WZNW action, as we now demonstrate for the general case.

3.2.2. The General Case

To treat the general case and make contact with current algebra, we rewrite the Ward identity (with arbitrary value of the central charge, but $\hbar = 1$) using (2.1.17)

$$\begin{aligned} \pi\bar{\partial}\frac{\delta\Gamma^{(0)}}{\delta\mu_\alpha(z)} &= \sum_{\beta,r>0}\frac{(-1)^r\partial^{r-1}}{(r-1)!}(\mu_\beta\{A_\alpha A_\beta\}_r)\Big|_{A_\gamma\to\pi\frac{\delta\Gamma^{(0)}}{\delta\mu_\gamma}} \\ &= \sum_{\beta,r>0}\frac{(\partial^{r-1}\mu_\beta)}{(r-1)!}\{A_\beta A_\alpha\}_r\Big|_{A_\gamma\to\pi\frac{\delta\Gamma^{(0)}}{\delta\mu_\gamma}} \\ &= \sum_\beta\oint_0\frac{dz'}{2\pi i}\,\mu_\beta \underbrace{A_\beta(z')A_\alpha(z)}\Big|_{A_\gamma\to\pi\frac{\delta\Gamma^{(0)}}{\delta\mu_\gamma}} \\ &= \{\int dz'\sum_\beta\mu_\beta A_\beta(z'), A_\alpha(z)\}\Big|_{A_\gamma\to\pi\frac{\delta\Gamma^{(0)}}{\delta\mu_\gamma}}. \end{aligned} \tag{3.2.5}$$

The right hand side can be explicitly computed using (2.2.27). Since the algebra of the A_α's is isomorphic to that of the W^a's by assumption, we can choose a basis of A_α's in such a way that they are proportional the components of W, say

$$A_\alpha c_\alpha = W^{a(\alpha)}. \tag{3.2.6}$$

If we furthermore fix $F(\mu_\alpha) \in \ker \mathrm{ad}(\Lambda^+)$ by the requirement

$$\mathrm{Tr}(F(\mu_\alpha)W) = \sum_\alpha \mu_\alpha A_\alpha, \tag{3.2.7}$$

the Ward identity (3.2.5) can be rewritten as

$$[\partial + \frac{1}{k}(\xi\Lambda^+ + \mathrm{ad}(W)), \bar{\partial} + X(\mu_\alpha)]|_{W^{a(\alpha)} \to c_\alpha \pi \frac{\delta\Gamma^{(0)}}{\delta\mu_\alpha}} = 0, \tag{3.2.8}$$

where $X(\mu_\alpha)$ is expressed in terms of $F(\mu_\alpha)$ through (2.2.26). This is the generalization of (3.2.4) to arbitrary W algebras. We see that in the general case the Ward identity can also be written as a zero-curvature equation, with the z-part of the connection in the form of a constrained current, and with the $\bar{z}$-part in the form of a parameter of a constraint-preserving gauge transformation. This is a crucial observation, because it enables us to connect the Ward identity with the Ward identity for a WZNW theory. The action of the WZNW theory satisfies the following zero-curvature equation

$$[\partial + \frac{1}{k}\mathcal{J}, \bar{\partial} - k\pi\frac{\delta S^-_{wznw}}{\delta\mathcal{J}}] = 0. \tag{3.2.9}$$

It has almost the same form as (3.2.8). To bring (3.2.8) in the proper form we introduce the 'effective' action¶

$$e^{-\frac{1}{\hbar}\Gamma[A_\alpha]} = \int \prod_\alpha \mathcal{D}\mu_\alpha e^{-\frac{1}{\hbar}\Gamma[\mu_\alpha] + \frac{1}{\pi\hbar}\int d^2z\,\mu_\alpha A_\alpha} \tag{3.2.10}$$

which is essentially the Fourier transform of $\Gamma[\mu_\alpha]$. The effective action also admits an expansion $\Gamma[A_\alpha] = \sum_{i\geq 0} c^{1-i}\Gamma^{(i)}[A_\alpha]$, and

$$e^{-\frac{1}{\hbar}\Gamma^{(0)}[A_\alpha]} = \int_{\text{saddlepoint}} \prod_\alpha \mathcal{D}\mu_\alpha e^{-\frac{1}{\hbar}\Gamma^{(0)}[\mu_\alpha] + \frac{1}{\pi\hbar}\int d^2z\,\mu_\alpha A_\alpha}, \tag{3.2.11}$$

where the integral should be evaluated in the saddle point approximation around $c \to \infty$, which is not necessarily the same as $\hbar \to 0$. Equation (3.2.11) follows from (3.2.10), as the $\Gamma^{(i)}[\mu_\alpha]$ with $i > 0$ do not contribute to $\Gamma^{(0)}[A_\alpha]$, and neither do the one and higher

¶This terminology is a bit sloppy. Strictly speaking, $\Gamma[A_\alpha]$ is neither the effective action for the A_α, because that is defined as the Legendre transform of $\Gamma[\mu_\alpha]$, nor is it the effective action for the μ_α, which is the Legendre transform of $\Gamma[A_\alpha]$ itself. Nevertheless we will call $\Gamma[A_\alpha]$, which really is the generating functional for connected correlation functions of the μ_α, the effective action.

loop contributions of the path integral in (3.2.10). The Ward identity for $\Gamma^{(0)}[A_\alpha]$ is identical to that for $\Gamma^{(0)}[\mu_\alpha]$, with the replacements

$$\pi\frac{\delta\Gamma^{(0)}[\mu_\alpha]}{\delta\mu_\alpha} \leftrightarrow A_\alpha, \qquad \mu_\alpha \leftrightarrow -\pi\frac{\delta\Gamma^{(0)}[A_\alpha]}{\delta A_\alpha}. \tag{3.2.12}$$

Comparison of the Ward identity for $\Gamma^{(0)}[A_\alpha]$ with (3.2.9) now yields the important result that

$$\Gamma^{(0)}[A_\alpha] = kS^-_{wznw}(g)|_{kg^{-1}\partial g=\xi\Lambda^+ + W} \tag{3.2.13}$$

where A_α and W are connected through (3.2.6), and $c = -12\mathrm{Tr}(\Lambda^+\Lambda^-)k$ (see below (2.2.30)). The induced action $\Gamma[\mu_\alpha]$ is computed from (3.2.13) via

$$e^{-\Gamma^{(0)}[\mu_\alpha]} = \int_{\text{saddlepoint}} \prod_\alpha \mathcal{D}A_\alpha e^{-\Gamma^{(0)}[A_\alpha]-\frac{1}{\pi}\int d^2z\mu_\alpha A_\alpha}, \tag{3.2.14}$$

yielding

$$\Gamma^{(0)}[\mu_\alpha] = -kS^+_{wznw}(g) - \frac{\xi}{\pi}\int d^2z\mathrm{Tr}(\Lambda^+ g^{-1}\bar\partial g), \tag{3.2.15}$$

where g is expressed in terms of μ_α by the following two conditions

$$\Pi_{\mathrm{im\,ad}(\Lambda^+)}(kg^{-1}\partial g) = \xi\Lambda^+, \qquad \Pi_{\mathrm{ker\,ad}(\Lambda^+)}(g^{-1}\bar\partial g) = F(\mu_\alpha). \tag{3.2.16}$$

This concludes our discussion of the lowest-order contribution to the chiral induced action. We now turn to the quantization of the chiral induced action.

3.3. Quantization

The chiral induced action can provide a kinetic term for the gauge fields, as the example (3.2.3) demonstrates for the gauge field μ. Recalling that μ has the interpretation as a component of the metric, we see that $\Gamma[\mu]$ is a 'gravitational' action for the metric in a particular gauge, the chiral gauge. By analogy, we will call $\Gamma[\mu_\alpha]$ the chiral induced action for W gravity. This terminology will be further discussed in the next chapter. The quantization of the induced action is performed by computing the effective action (3.2.10), which is the generating functional for the correlation functions of the gauge fields. For gravity, we explicitly know $\Gamma[\mu] = \Gamma^{(0)}[\mu]$ to all orders (3.2.6), and we could in principle compute $\Gamma[T]$ order by order. This has been done up to one loop [241, 335, 207, 161], and the one loop result is $\Gamma^{(1)} = -25\Gamma^{(0)} + 13T\frac{\delta\Gamma^{(0)}}{\delta T}$, so that the total result up to one loop can be written as $(c-25)\Gamma^{(0)}[(1+\frac{13}{c})T]$. This suggests [335] that the all-order result for $\Gamma[T]$ is given by $Z_k c\Gamma^{(0)}[Z_T T]$, for certain constants Z_k and Z_T that are power series in $1/c$. However, it is clear that it will become more and more cumbersome to go to higher orders, so we will consider a different strategy for computing $\Gamma[T]$.

Going back to the original definition (3.1.15) we find that, in terms of the original conformal field theory $S(\phi)$, the effective action (3.2.10) is expressed as

$$e^{-\Gamma[T]} = \int \mathcal{D}\phi e^{-S(\phi)}\delta(T - T(\phi)). \tag{3.3.1}$$

In general, it is difficult to perform the path integral over the ϕ fields in the presence of this delta function, but it turns out that if we start with a constrained $Sl(2,\mathbb{R})$ WZNW theory as an action, it is possible to perform this path integral and thus to compute the effective action for gravity to all orders. This construction is closely related to the construction of W algebras from constrained current algebra discussed in chapter 2, and to the construction of the induced action in the previous section. The constrained WZNW model is given by the original WZNW model plus one extra term, which is a Lagrange multiplier times the constraint $J^+ - \xi$, that we used in sects. 2.2.4-6 to construct the Virasoro algebra. In section 2.2.6, we saw that this constraint generates a gauge invariance. The same gauge invariance is shared by the action for a constrained WZNW model, and can be used to put the current in the form (2.2.10)

$$\mathcal{J}_{\text{constr}} = \begin{pmatrix} 0 & \xi \\ J^- & 0 \end{pmatrix}. \tag{3.3.2}$$

The Poisson bracket for J^- is, in the presence of the constraints $J^+ - \xi = 0$ and $J^0 = 0$, the same as for an energy momentum tensor. This statement has counterparts both on the level of Ward identities and on the level of actions. On the level of Ward identities, the WZNW Ward identity becomes the Ward identity for $\Gamma[\mu]$. On the level of actions, J^- becomes the energy-momentum tensor of the theory. Therefore the delta function in (3.3.1) becomes a delta function for a current of the WZNW theory and can be integrated out. For this last step one has to perform a change of variables in the WZNW theory from the group variable g to the current $kg^{-1}\partial g$, and take the corresponding Jacobian into account. The result of all this is that the effective action $\Gamma[T]$ is indeed of the form $Z_k c\Gamma^{(0)}[Z_T T]$; Z_k and Z_T will be given later, see (3.3.49).

Here we present this calculation for W_3 gravity. Ordinary gravity can be treated in the same way, and we leave the corresponding but easier calculation for ordinary gravity to the reader. The induced action for W_3 gravity depends, besides on $\mu \equiv \mu_T$, on an extra field $\nu \equiv \mu_W$ that couples to the W_3 field. A difference with ordinary gravity is that the explicit form of $\Gamma[\mu,\nu]$ is not known; for W_3 gravity $\Gamma[\mu,\nu]$ is known up to $\Gamma^{(2)}$ [283]. Therefore we cannot simply try to quantize $\Gamma[\mu,\nu]$, and we have to rely on other methods, like the one presented here. For W_3 gravity we start with a constrained $SL(3,\mathbb{R})$ WZNW model. As we explained in chapter 2, imposing certain constraints on an sl_3 current algebra reduces the current algebra to a W_3 algebra. The fields $T(g)$ and $W(g)$ that couple to the W_3 gauge fields μ,ν are the generators of this W_3 algebra. Furthermore, $T(g)$ and $W(g)$ can be chosen such as to preserve the gauge invariance of the constrained WZNW model. This enables us to perform a BRST quantization of the model. Because the BRST operator is nilpotent only on-shell, we need the Batalin-Vilkovisky quantization procedure to compute the quantum action.

To complete the computation, we need the Jacobian for the change of variables from g to $g^{-1}\partial g$. This is a rather subtle point, of which our understanding is incomplete. This point is discussed in section 3.3.3, where it is shown that knowing this Jacobian is equivalent to knowing the effective action for ordinary WZNW theory. Using the ansatz that this effective action is proportional, up to an overall and field renormalization, to the WZNW action, we then complete the calculation of the effective action of W_3 gravity. The result agrees with one-loop calculations[161, 284] for W_3 gravity if the multiplicative renormalizations of the WZNW model agree with the one-loop calculations for the WZNW model given in [262, 280]. The resulting effective action for W_3 gravity is proportional to a constrained WZNW model, as conjectured in [284]. Thus, W_3 gravity can be seen as an example of completely integrable nonlocal field theory. The crucial ingredient in establishing this integrability is the requirement of BRST invariance at the quantum level. The result also shows that the level of the sl_3 current algebra in W_3 gravity is given by a KPZ-like formula as proposed in [236, 36]. We would like to stress that these conclusions only hold if the effective action of the WZNW model is proportional to a WZNW action itself.

Actually, Knizhnik, Polyakov and Zamolodchikov derived their result for the level of the sl_2 current algebra in gravity by an analysis of the gauge fixing of the covariant induced action for gravity [207]. This procedure is closely related to the one used here, and can be generalized to W_3 gravity by gauge fixing the covariant action, see the next chapter. The advantage of this approach is that it makes the sl_3 current algebra structure in W_3 gravity very clear. The disadvantage is that it is difficult to extract all-order results from it, because the covariant action is only known to lowest order in $1/c$.

3.3.1. The Induced Action of W_3 Gravity

We start with the action for a constrained $SL(3,\mathbb{R})$ WZNW model. To get the W_3 algebra we have to impose three first class constraints (cf section 2.2.11) that we add with corresponding Lagrange multipliers to the action. Thus, the action is given by [36]

$$S_1 = kS^-_{wznw}(g) - \tfrac{1}{\pi}\int d^2z \ (\bar{A}^1(J^1-\xi) + \bar{A}^2(J^2-\xi) + \bar{A}^3J^3), \qquad (3.3.3)$$

where the current $\mathcal{J} = kg^{-1}\partial g$ is parametrized as in (2.1.57). The action contains three gauge fields $\bar{A}^i$ $(i = 1\ldots 3)$, that play the role of Lagrange multipliers. The constraints $J^1 = J^2 = \xi, J^3 = 0$ reduce the sl_3 current algebra to a W_3 algebra, see section 2.2.11. The action (3.3.3) has an invariance under the gauge transformations generated by the subgroup $N^- = \exp(\mathbf{g}_-)$ of lower triangular matrices. Explicitly, the action (3.3.3) is invariant under $\delta_\epsilon \mathcal{J} = k\partial\epsilon + [\mathcal{J}, \epsilon]$ (or $\delta_\epsilon g = g\epsilon$) and $\delta_\epsilon \bar{A}^1 = -\bar{\partial}\epsilon_1$, $\delta_\epsilon \bar{A}^2 = -\bar{\partial}\epsilon_2$, and $\delta_\epsilon \bar{A}^3 = -\bar{\partial}\epsilon_3 + \bar{A}^2\epsilon_1 - \bar{A}^1\epsilon_2$, where

$$\epsilon = \begin{pmatrix} 0 & 0 & 0 \\ \epsilon_1 & 0 & 0 \\ \epsilon_3 & \epsilon_2 & 0 \end{pmatrix}. \qquad (3.3.4)$$

As explained in the previous section we intend to couple this theory to the W_3 gauge fields μ, ν by adding a term $\int \mu T(\mathcal{J}) + \int \nu W(\mathcal{J})$ to the action, while preserving the gauge invariance. To find $T(\mathcal{J})$ and $W(\mathcal{J})$ one uses the fact that there is a unique gauge transformation given by a lower triangular matrix n with ones on the diagonal (see section 2.2.6), such that

$$\begin{pmatrix} 0 & \xi & 0 \\ T(\mathcal{J})/2\xi & 0 & \xi \\ W(\mathcal{J})/\xi^2 & T(\mathcal{J})/2\xi & 0 \end{pmatrix} = n^{-1}\begin{pmatrix} H^0 & \xi & 0 \\ K^1 & H^1 - H^0 & \xi \\ K^3 & K^2 & -H^1 \end{pmatrix} n + kn^{-1}\partial n. \qquad (3.3.5)$$

The factors $1/2\xi$ and $1/\xi^2$ have been included for later convenience. The polynomials $T(\mathcal{J})$ and $W(\mathcal{J})$ are invariant under N^- gauge transformations of the constrained current $\mathcal{J}_{constr} = \mathcal{J}|_{J^1=\xi, J^2=\xi, J^3=0}$ that appears in (3.3.5). Under a gauge transformation of the full current $\mathcal{J}$, $T(\mathcal{J})$ and $W(\mathcal{J})$ are only invariant up to terms proportional to $\mathcal{J} - \mathcal{J}_{constr}$. Therefore, if we add $\int \mu T(\mathcal{J}) + \int \nu W(\mathcal{J})$ to the action (3.3.3), the action is N^- invariant up to terms proportional to the constraints. It is possible, by modifying the transformation rules for $\bar{A}^i$, to make the action exactly N^- invariant.

If we compute $T(\mathcal{J})$ and $W(\mathcal{J})$ from (3.3.5) and add these to the action (3.3.3), the resulting action $S_2(\bar{A}, g, \mu, \nu)$ reads

$$\begin{aligned} S_2 &= kS^-_{wznw}(g) - \tfrac{1}{\pi}\int d^2z \ (\bar{A}^1(J^1-\xi) + \bar{A}^2(J^2-\xi) + \bar{A}^3J^3) \\ &\quad + \tfrac{N_T}{\pi}\int d^2z\, \mu((H^0)2 - H^0H^1 + (H^1)^2 + \xi(K^1+K^2) - k\partial(H^0+H^1)) \end{aligned}$$

$$+\frac{N_W}{\pi}\int d^2z\,\nu((H^0)^2H^1 - H^0(H^1)^2 + \xi(H^1K^1 - H^0K^2) + \xi^2K^3 + \tfrac{1}{2}k\xi\partial(K^2 - K^1)$$
$$+\tfrac{1}{2}k^2\partial^2(H^0 - H^1) + k(-H^0\partial H^0 + H^1\partial H^1 + \tfrac{1}{2}H^0\partial H^1 - \tfrac{1}{2}H^1\partial H^0)), \qquad (3.3.6)$$

where we have introduced two normalization factors N_T and N_W. As explained above, the N^- transformations that leave this action invariant are still given by $\delta_\epsilon\mathcal{J} = k\partial\epsilon + [\mathcal{J},\epsilon]$ for the current, while for $\bar{A}^i$ they are extended to

$$\begin{aligned}
\delta_\epsilon\bar{A}^3 &= -\bar{\partial}\epsilon_3 + \bar{A}^2\epsilon_1 - \bar{A}^1\epsilon_2 - \epsilon_3(-N_T(\mu(H^0 + H^1) + 2k\partial\mu) \\
&\quad +N_W(\nu(-(H^0)^2 + (H^1)^2 + \xi(K^2 - K^1) + k\partial(H^0 - H^1)) + \tfrac{k}{2}\partial\nu(H^1 - H^0))), \\
\delta_\epsilon\bar{A}^2 &= -\bar{\partial}\epsilon_2 - N_T\mu(\epsilon_2(H^0 - 2H^1) - \xi\epsilon_3) + kN_T\epsilon_2\partial\mu \\
&\quad -N_W\nu(-\epsilon_2H^0(H^0 - 2H^1) - \xi\epsilon_3H^1 + \xi\epsilon_2K^1 + k\epsilon_2\partial H^0) \\
&\quad -\tfrac{k}{2}N_W(\epsilon_2\partial\nu(H^0 + 2H^1) - \xi\epsilon_3\partial\nu) - \tfrac{k^2}{2}N_W\epsilon_2\partial^2\nu, \\
\delta_\epsilon\bar{A}^1 &= -\bar{\partial}\epsilon_1 - N_T\mu(\epsilon_1(2H^0 - H^1) + \xi\epsilon_3) + kN_T\epsilon_1\partial\mu \\
&\quad +N_W\nu(-\epsilon_1H^1(H^1 - 2H^0) + \xi\epsilon_3H^0 - \xi\epsilon_1K^2 + k\epsilon_1\partial H^1) \\
&\quad +\tfrac{k}{2}N_W(\epsilon_1\partial\nu(2H^0 + H^1) + \xi\epsilon_3\partial\nu) + \tfrac{k^2}{2}N_W\epsilon_1\partial^2\nu.
\end{aligned} \qquad (3.3.7)$$

As a special case of (3.1.15) we consider the chiral induced action $\Gamma[\mu,\nu]$ defined by

$$e^{-\Gamma[\mu,\nu]} = \int \frac{\mathcal{D}\bar{A}\mathcal{D}g}{\text{gauge volume}} e^{-S_2(\bar{A},g,\mu,\nu)}. \qquad (3.3.8)$$

The quantization of $S_2(g,\bar{A},\mu,\nu)$ is most easily performed using BRST quantization (cf. [36]). The BRST transformation rules for g and $\bar{A}$ are defined by replacing the parameters ϵ_i of the gauge transformations $\delta_\epsilon g = g\epsilon$ and (3.3.7) by anti-commuting ghosts c_i. We denote these transformation rules by $\delta_B g$ and $\delta_B\bar{A}$. The BRST transformation rules for the ghosts read $\delta_B c_1 = \delta_B c_2 = 0$ and $\delta_B c_3 = c_1c_2$. However, due to the extra terms we added to the $\bar{A}$ transformation rules in (3.3.7), the BRST operator δ_B no longer satisfies $\delta_B^2 = 0$. It only satisfies $\delta_B^2 = 0$ when we use the $\bar{A}$ equations of motion. In such a case a proper quantization and BRST gauge fixing of the theory requires that we use the Batalin-Vilkovisky formalism [27].

For all fields in the theory we introduce antifields ($\bar{A}_i^*$, g^* and c^{i*}) with opposite statistics. Because $\delta_B^2 = 0$ only on-shell, we typically need to include terms that are quadratic in the ghosts c_α and in the anti-fields to find a solution to the master equation. Because only $\delta_B^2\bar{A}_i \neq 0$, the only terms quadratic in the antighosts that are needed are terms quadratic in $\bar{A}_i^*$. Furthermore, if we compute $\delta_B^2\bar{A}_i$, we find that each term in the answer contains at most one derivative, and that the answer is proportional to the $\bar{A}_i$ equations of motion. This leads us to write down the following ansatz for the minimal solution to the master equation

$$\begin{aligned} S_{min} &= S_2 + \int \bar{A}^*_i \delta_B \bar{A}^i + \int g^* \delta_B g - \int c^{3*} c_1 c_2 + \pi \int \bar{A}^*_i \bar{A}^*_j E^{ij,\alpha\beta} c_\alpha c_\beta \\ &+ \pi \int \bar{A}^*_i \bar{A}^*_j F^{ij,\alpha\beta} c_\alpha \partial c_\beta + \pi \int \bar{A}^*_i \partial \bar{A}^*_j G^{ij,\alpha\beta} c_\alpha c_\beta. \end{aligned} \tag{3.3.9}$$

If we denote by ϕ^I the set of fields $(\bar{A}^i, g, c_i)$ and by ϕ^*_I the corresponding set of anti-fields $(\bar{A}^*_i, g^*, c^{i*})$, then the master equation reads $(S_{min}, S_{min}) = 0$, where

$$(P, Q) = \frac{\overleftarrow{\partial} P}{\partial \phi^I} \frac{\overrightarrow{\partial} Q}{\partial \phi^*_I} - \frac{\overleftarrow{\partial} P}{\partial \phi^*_I} \frac{\overrightarrow{\partial} Q}{\partial \phi^I}. \tag{3.3.10}$$

Here, $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ correspond to right and left derivatives respectively. Working out the master equation for (3.3.9) yields, among others, the equation

$$\begin{aligned} \delta^2_B(\bar{A}^k) &= \frac{\delta \pi S_2}{\delta \bar{A}^j} \Big((2E^{jk,\alpha\beta} - \partial G^{jk,\alpha\beta}) c_\alpha c_\beta + (2F^{jk,\alpha\beta} - G^{jk,\alpha\beta} + G^{jk,\beta\alpha}) c_\alpha \partial c_\beta \Big) \\ &- \partial \left(\frac{\delta \pi S_2}{\delta \bar{A}^j} \right) (G^{jk,\alpha\beta} + G^{kj,\alpha\beta}) c_\alpha c_\beta. \end{aligned} \tag{3.3.11}$$

From this one can compute the tensors E, F and G. The components of these tensors either vanish, or can be determined from the following relations

$$\begin{aligned} E^{jk,\alpha\beta} &= = -E^{kj,\alpha\beta} = -E^{jk,\beta\alpha}, \\ E^{12,12} &= \tfrac{1}{4}(N_T \mu - 2N_W H^0 \nu + 2N_W H^1 \nu), \\ E^{13,13} &= \tfrac{1}{4}(-N_T \mu - 2N_W H^0 \nu - k N_W \partial \nu), \\ E^{23,23} &= \tfrac{1}{4}(-N_T \mu + 2N_W H^1 \nu + k N_W \partial \nu), \\ G^{jk,\alpha\beta} &= G^{kj,\alpha\beta} = -G^{jk,\beta\alpha}, \\ G^{12,12} &= G^{13,13} = -G^{23,23} = -\tfrac{k}{4} N_W \nu, \\ F^{jk,\alpha\beta} &= -F^{kj,\alpha\beta} = F^{jk,\beta\alpha}, \\ F^{12,12} &= F^{13,13} = -F^{23,23} = -\tfrac{k}{4} N_W \nu. \end{aligned} \tag{3.3.12}$$

If we substitute this back into (3.3.9), we find that the master equation is satisfied. The full quantum action is given by

$$S_q = S_{min} - \int d^2 z \, (b^*_1 B_1 + b^*_2 B_2 + b^*_3 B_3), \tag{3.3.13}$$

where b^*_i are the anti-fields for the anti-ghosts b_i, and the B_i are Lagrange multipliers, also known as the Nakanishi-Lautrup fields, that will impose the gauge condition. The gauge fixing is done by replacing the antifields ϕ^* by $\partial\Psi/\partial\phi$ in the full quantum action (3.3.13), where Ψ, the gauge fermion, represents a particular gauge choice. We will choose

$$\Psi = \frac{-1}{\pi}\int d^2z\,(b_1\bar{A}^1 + b_2\bar{A}^2 + b_3\bar{A}^3), \tag{3.3.14}$$

so that we put $c^{i*} = g^* = 0$, $\bar{A}^*_i = \frac{-1}{\pi}b_i$ and $b^*_i = \frac{-1}{\pi}\bar{A}^i$ in (3.3.13). The resulting gauge fixed action is off-shell BRST invariant under the BRST transformations

$$\delta'_B\phi^I = -\left.\frac{\overleftarrow{\partial} S_q}{\partial\phi^*_I}\right|_{\phi^*_I=\partial\Psi/\partial\phi^I}. \tag{3.3.15}$$

Note that the transformation rules for $\bar{A}$ with respect to δ'_B are different from those with respect to δ_B, but we are going to integrate out the $\bar{A}$, we do not give those (lengthy) transformation rules here. The gauge fixed action we have obtained can be written in a form that is remarkably similar to (3.3.6),

$$\begin{aligned}
S_{gf} \;=\;& kS^-_{wznw}(g) + \tfrac{1}{\pi}\int d^2z\;(b_1\bar{\partial}c_1 + b_2\bar{\partial}c_2 + b_3\bar{\partial}c_3)\\
&-\tfrac{1}{\pi}\int d^2z\;(\bar{A}^1(\hat{J}^1-\xi-B_1) + \bar{A}^2(\hat{J}^2-\xi-B_2) + \bar{A}^3(\hat{J}^3-B_3))\\
&+\tfrac{N_T}{\pi}\int d^2z\,\mu((\hat{H}^0)^2 - \hat{H}^0\hat{H}^1 + (\hat{H}^1)^2 + \xi(\hat{K}^1+\hat{K}^2) - k\partial(\hat{H}^0+\hat{H}^1))\\
&+\tfrac{N_W}{\pi}\int d^2z\,\nu((\hat{H}^0)^2\hat{H}^1 - \hat{H}^0(\hat{H}^1)^2 + \xi(\hat{H}^1\hat{K}^1 - \hat{H}^0\hat{K}^2) + \xi^2\hat{K}^3 + \tfrac{1}{2}k\xi\partial(\hat{K}^2-\hat{K}^1)\\
&+\tfrac{1}{2}k^2\partial^2(\hat{H}^0-\hat{H}^1) + k(-\hat{H}^0\partial\hat{H}^0 + \hat{H}^1\partial\hat{H}^1 + \tfrac{1}{2}\hat{H}^0\partial\hat{H}^1 - \tfrac{1}{2}\hat{H}^1\partial\hat{H}^0)),
\end{aligned} \tag{3.3.16}$$

where the hatted currents are the components of an $SL(3,\mathbb{R})$ valued object $\hat{\mathcal{J}}$ and are defined by

$$\begin{aligned}
&\hat{J}^1 = J^1 + c_2b_3, \qquad \hat{J}^2 = J^2 - c_1b_3, \qquad \hat{J}^3 = J^3,\\
&\hat{H}^0 = H^0 + c_1b_1 + c_3b_3, \qquad \hat{H}^1 = H^1 + c_2b_2 + c_3b_3,\\
&\hat{K}^1 = K^1 + c_3b_2, \qquad \hat{K}^2 = K^2 - c_3b_1, \qquad \hat{K}^3 = K^3.
\end{aligned} \tag{3.3.17}$$

A simple way to define these hatted quantities is by means of the following expression

$$\hat{\mathcal{J}} = \mathcal{J} - \left[\begin{pmatrix} 0 & 0 & 0\\ c_1 & 0 & 0\\ c_3 & c_2 & 0\end{pmatrix}, \begin{pmatrix} 0 & b_1 & b_3\\ 0 & 0 & b_2\\ 0 & 0 & 0\end{pmatrix}\right]_+, \tag{3.3.18}$$

where $[,]_+$ denotes an anticommutator. The meaning of these hatted currents becomes clear once we integrate out B_i from the gauge fixed action S_{gf}, giving

$$\begin{aligned} S_{gf2} &= kS^-_{wznw}(g) - \tfrac{1}{\pi}\int d^2z\ (b_1\bar{\partial}c_1 + b_2\bar{\partial}c_2 + b_3\bar{\partial}c_3) \\ &\quad + \tfrac{N_T}{\pi}\int d^2z\,\mu T(\hat{\mathcal{J}}) + \tfrac{N_W}{\pi}\int d^2z\,\nu W(\hat{\mathcal{J}}). \end{aligned} \tag{3.3.19}$$

The BRST transformation rules for the anti-ghosts b_i now read

$$\begin{aligned} \delta_B b_1 &= J^1 - \xi + c_2 b_3, \\ \delta_B b_2 &= J^2 - \xi - c_1 b_3, \\ \delta_B b_3 &= J^3. \end{aligned} \tag{3.3.20}$$

If we compare the BRST transformation rules of H^i and K^i with those for $\hat{H}^i$ and $\hat{K}^i$, we see that the transformation rules for $\hat{H}^i$ and $\hat{K}^i$ can be obtained from those for H^i and K^i by replacing J^1 and J^2 by ξ and J^3 by 0, and H^i and K^i by their hatted counterparts. The BRST transformation rules for $\hat{H}^i$ and $\hat{K}^i$ are therefore determined by the way the constrained current behaves under N^- gauge transformations, whereas the transformation rules for H^i and K^i were determined by the way in which the unconstrained current transformed under gauge transformations. Because $T(\mathcal{J})$ and $W(\mathcal{J})$ were constructed in such a way as to be exactly invariant under N^- gauge transformations of the constrained current, this automatically implies that $T(\hat{\mathcal{J}})$ and $W(\hat{\mathcal{J}})$ must be BRST invariant. As the reader may have observed, the hatted currents appeared previously, namely in section 2.2.8, where we quantized the W algebras using BRST quantization. Of course, this should not come as a surprise, as the calculation here is just a path integral counterpart of the algebraic procedure in section 2.2.8. It is amusing that the hatted currents appear quite naturally in the path integral framework, whereas they were introduced in section 2.2.8 only as an algebraic simplification. The classical BRST operator that generates the BRST transformations of (3.3.19) is identical to (2.2.33),

$$Q = \int dz(c_1(J^1 - \xi) + c_2(J^2 - \xi) + c_3 J^3 + b_3 c_1 c_2), \tag{3.3.21}$$

and $T(\hat{\mathcal{J}})$ and $W(\hat{\mathcal{J}})$ are the generators of the classical BRST cohomology of Q.

Now we come to a difficult question, namely what are the quantum operators that correspond to $T(\hat{\mathcal{J}})$ and $W(\hat{\mathcal{J}})$? Or, in the language of equation (3.1.16), what are $T_q \equiv f_q(T(\hat{\mathcal{J}}))$ and $W_q \equiv f_q(W(\hat{\mathcal{J}}))$? It is not clear how we should replace $T(\hat{\mathcal{J}})$ and $W(\hat{\mathcal{J}})$ by normal ordered expressions involving the currents and the ghosts. For instance, on the quantum level it is not true that $(H^0K^2) = (K^2H^0)$, so that we a priori do not know by what we must replace the classical product H^0K^2. Two normal orderings of a product of a certain number of currents always differ by terms that contain fewer currents than the original product. This indicates that the coefficient in front of the term with the largest number of currents is the same, both for the classical expressions $T(\hat{\mathcal{J}}), W(\hat{\mathcal{J}})$ and their normal ordered versions T_q, W_q[‖]. To obtain the full expressions for T_q and W_q, we need some extra regularization principle that tells us how to do this. The extra regularization principle we choose is that of BRST invariance. As $T(\hat{\mathcal{J}}), W(\hat{\mathcal{J}})$ were classically BRST invariant, we require that T_q, W_q are quantum BRST invariant. Together with the requirement that the coefficients for the terms with the largest number of currents do not change, this will completely fix the form of T_q and W_q. The quantum BRST operator is given by the same expression as (3.3.21), with products of fields replaced by normal ordered products. Notice that there is no normal ordering ambiguity in the definition of Q. The OPE's of the ghosts and the currents are given by

$$\begin{aligned} \underbrace{c_i(z)\, b_j(w)} &= \frac{\delta_{ij}}{(z-w)}, \\ \underbrace{J^a(z)\, J^b(w)} &= \frac{k\eta^{ab}}{(z-w)^2} + \frac{f^{ab}_{\ \ c} J^c(w)}{(z-w)}, \end{aligned} \tag{3.3.22}$$

where we decomposed the current $\mathcal{J} = J^a T_a$. It is now a straightforward to obtain T_q and W_q from (2.2.53) via a basis transformation

$$\begin{aligned} T_q &= (\hat{H}^0\hat{H}^0) - (\hat{H}^0\hat{H}^1) + (\hat{H}^1\hat{H}^1) + \xi(\hat{K}^1 + \hat{K}^2) - (k+2)\partial(\hat{H}^0 + \hat{H}^1), \\ W_q &= (\hat{H}^0(\hat{H}^0\hat{H}^1)) - (\hat{H}^0(\hat{H}^1\hat{H}^1)) + \xi((\hat{H}^1\hat{K}^1) - (\hat{H}^0\hat{K}^2)) + \xi^2\hat{K}^3 \\ &\quad + \tfrac{1}{2}(k+2)\xi\partial(\hat{K}^2 - \hat{K}^1) + \tfrac{1}{2}(k+2)^2\partial^2(\hat{H}^0 - \hat{H}^1) \\ &\quad + (k+2)(-(\hat{H}^0\partial\hat{H}^0) + (\hat{H}^1\partial\hat{H}^1) + \tfrac{1}{2}(\hat{H}^0\partial\hat{H}^1) - \tfrac{1}{2}(\hat{H}^1\partial\hat{H}^0)), \end{aligned} \tag{3.3.23}$$

form a quantum W_3 algebra, with central charge

$$c = 50 - 24\left((k+3) + \frac{1}{(k+3)}\right). \tag{3.3.24}$$

If the normalization constants N_T and N_W are chosen to be equal to

$$\begin{aligned} N_T &= \frac{1}{k+3}, \\ N_W &= \left(\frac{-6}{15k^4 + 146k^3 + 519k^2 + 792k + 432}\right)^{\frac{1}{2}} = \left(\frac{48}{(k+3)^3(5c+22)}\right)^{\frac{1}{2}} \end{aligned} \tag{3.3.25}$$

[‖]The same is true in ordinary quantum mechanics, if we want to replace the classical operator $p^a q^b$ by a quantum operator via the replacement $p \to \frac{\hbar}{i}\frac{\partial}{\partial q}$.

the gauge fields μ and ν couple to the generators of the W_3 algebra with their standard normalization.

This shows that $\Gamma[\mu,\nu]$ is indeed the all-order induced action for W_3 gravity, where c is related to k via (3.3.24). The constant ξ can be chosen arbitrarily.

To summarize, we have shown that the constrained WZNW model can be coupled to the W_3 gauge fields in such a way that the resulting induced action for the W_3 gauge fields is precisely the all-order (chiral) induced action for W_3 gravity.

3.3.2. The Effective Action of W_3 Gravity

The effective action for W_3 gravity is obtained by quantizing the induced action, and is defined by the following path integral (cf. (3.2.10))

$$e^{-\Gamma[T,W]} = \int \frac{\mathcal{D}g\mathcal{D}\bar{A}}{\text{gauge volume}} \mathcal{D}\mu\mathcal{D}\nu e^{\frac{1}{\pi}\int d^2z\,(\mu T+\nu W)-S_2(g,\bar{A},\mu,\nu)}, \tag{3.3.26}$$

where S_2 is the action (3.3.6). In the previous section we performed a BRST quantization of $S_2(g,\bar{A},\mu,\nu)$, by gauge fixing $\bar{A}^i = 0$. This is a convenient gauge condition for proving that the induced action for μ and ν is the same as the induced action for W_3 gravity, but not for the computation of the effective action. Therefore, we will use a different gauge here, namely $H^0 = H^1 = K^1 - K^2 = 0$. Because the BRST operator δ_B satisfies $\delta_B^2 H^0 = \delta_B^2 H^1 = \delta_B^2(K^1-K^2) = 0$, there is no need to use Batalin-Vilkovisky quantization here. Under gauge transformations H^0, H^1 and K^1-K^2 transform as

$$\begin{aligned}
\delta_\epsilon H^0 &= J^1\epsilon_1 + J^3\epsilon_3, \\
\delta_\epsilon H^1 &= J^2\epsilon_2 + J^3\epsilon_3, \\
\delta_\epsilon(K^1-K^2) &= (H^1-2H^0)\epsilon_1 + (H^0-2H^1)\epsilon_2 + (J^2-J^1)\epsilon_3 \\
&\quad + k\partial(\epsilon_1-\epsilon_2).
\end{aligned} \tag{3.3.27}$$

This shows that gauge fixing $H^0 = H^1 = K^1 - K^2 = 0$ produces a Faddeev-Popov contribution to the path integral which is equal to

$$\int \mathcal{D}\beta_1\mathcal{D}\gamma_1\mathcal{D}\beta_2\mathcal{D}\gamma_2\mathcal{D}\beta_3\mathcal{D}\gamma_3 \exp\left(-\frac{1}{\pi}\int d^2z\,(\xi\beta_1\gamma_1 + \xi\beta_2\gamma_2 + 2\xi\beta_3\gamma_3 + k\beta_3\partial(\gamma_1-\gamma_2))\right), \tag{3.3.28}$$

where we put $J^1 = J^2 = \xi$ and $J^3 = 0$, which can be done safely after performing the $\bar{A}$ integration. It is clear that (3.3.28) is just some numerical factor, and we will ignore this factor. Then we can remove the volume of the gauge group in (3.3.26) by inserting the combination $\delta(H^0)\delta(H^1)\delta(K^1-K^2)$ into the path integral. The $\bar{A}$ and μ,ν integrations yield five more delta function insertions in the path integral. Altogether this shows that

$$e^{-\Gamma[T,W]} = \int \mathcal{D}g\delta(J^1-\xi)\delta(J^2-\xi)\delta(J^3)\delta(H^0)\delta(H^1)\delta(K^1-K^2) \delta(T-N_T\xi(K^1+K^2))\delta(W-N_W\xi^2K^3)e^{-kS^-_{wznw}(g)}. \quad (3.3.29)$$

It seems that we are already done, as the delta functions absorb all the degrees of freedom, and that we are left with a constrained WZNW model. However, before we can integrate out the delta functions, we must first change variables from g to $g^{-1}\partial g$, and compute the corresponding Jacobian. This change of variables is a rather tricky point, which we now discuss in some detail.

3.3.3. The Effective Action of the WZNW Model

It is generally believed [261, 262], that the Jacobian corresponding to the change of variables from $A_z = g^{-1}\partial g$ to g leads to**

$$\mathcal{D}A_z = \exp(2h_G S^-_{wznw}(g))\,\mathcal{D}g, \quad (3.3.30)$$

where h_G is the dual Coxeter number of the group under consideration. The computation of this Jacobian proceeds by noticing that $\delta A_z = \partial_{A_z}(g^{-1}\delta g)$, so that the Jacobian is equal to $\det(\partial_{A_z})$, and then by writing this determinant as the path integral $\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp(-\int\bar{\psi}\partial_{A_z}\psi)$, where $\psi,\bar{\psi}$ are fermions transforming in the adjoint representation of the group. Finally, one can derive a Ward identity for this fermionic path integral and show that the solution to this Ward identity is indeed given by (3.3.30)††.

Actually, (3.3.30) is in disagreement with one-loop calculations for the WZNW model [262, 280]. If (3.3.30) were true, then one could easily compute the effective action for the WZNW model to all orders: first, we compute the generating functional of connected diagrams $G[J_{\bar{z}}]$, given by

$$\exp -G[J_{\bar{z}}] = \int \mathcal{D}A_z \exp(-kS^-_{wznw}(A_z) + \tfrac{1}{\pi}\int d^2z\,\mathrm{Tr}(A_zJ_{\bar{z}})). \quad (3.3.31)$$

In the spirit of the beginning of this chapter $G[J_{\bar{z}}]$ is the induced action for the WZNW model with its full chiral algebra gauged. If we change variables from A_z to g with

**The symbols A_z and $J_{\bar{z}}$ used in this section should not be confused with $\bar{A}^i$ and J_i used in the previous sections.

††See also chapter 5. In general, if r is a representation of $\mathbf{g}$, then $\log\det(\partial + r(A_z)) = S^-_{wznw}(A_z)$, where in the WZNW action the trace should be taken in the representation r.

$A_z = g^{-1}\partial g$, and parametrize $J_{\bar{z}}$ by $J_{\bar{z}} = (k - 2h_G)\bar{\partial} h h^{-1}$, we can use the Polyakov-Wiegmann identity (2.1.51) to write the right-hand side of (3.3.31) as

$$\int \mathcal{D}g \exp(-(k-2h_G)S^-_{wznw}(gh) + (k-2h_G)S^-_{wznw}(h)). \tag{3.3.32}$$

We can safely replace the variable g by $g' = gh^{-1}$, because this does not change the measure $\mathcal{D}g$, and we see that if we ignore an infinite factor, the generating functional $G[J_{\bar{z}}] = -(k-2h_G)S^-_{wznw}(h)$. The effective action $S_{eff}(A_z)$ is the Legendre transform of $G[J_{\bar{z}}]$,

$$\begin{aligned} S_{eff}(A_z) &= \min_{J_{\bar{z}}} \left(-G[J_{\bar{z}}] - \tfrac{1}{\pi} \int d^2z \, \mathrm{Tr}(A_z J_{\bar{z}}) \right) \\ &= \min_h \left((k-2h_G)S^-_{wznw}(h) - \tfrac{(k-2h_G)}{\pi} \int d^2z \, \mathrm{Tr}(A_z \bar{\partial} h h^{-1}) \right) \\ &= \min_h \left((k-2h_G)S^-_{wznw}(h^{-1}) + \tfrac{(k-2h_G)}{\pi} \int d^2z \, \mathrm{Tr}(A_z h^{-1} \bar{\partial} h) \right). \end{aligned} \tag{3.3.33}$$

The extremum is attained for $A_z = h^{-1}\partial h$, and we find that the effective action is simply

$$S_{eff}(A_z) = -(k-2h_G)S^-_{wznw}(A_z). \tag{3.3.34}$$

On the other hand, one can also perform a one-loop computation of the effective action [262, 280], and check the above result. In (3.3.31), the saddle point of the action $-kS^-_{wznw}(A_z) + \frac{1}{\pi}\int d^2z \, \mathrm{Tr}(A_z J_{\bar{z}})$ is at $A_z^{(0)}(J_{\bar{z}})$, where $A_z^{(0)}$ is defined by the equation $F(A_z^{(0)}, \frac{-1}{k} J_{\bar{z}}) = 0$[‡‡]. If we write $A_z = A_z^{(0)} + \tilde{A}_z$, and $J_{\bar{z}} = k\bar{\partial} h h^{-1}$, so that $A_z^{(0)} = -\partial h h^{-1}$, then we can expand (3.3.31)

$$\exp -G[J_{\bar{z}}] = \int \mathcal{D}\tilde{A}_z \exp(kS^-_{wznw}(h) + \frac{k}{2\pi} \int d^2z \, \mathrm{Tr}(\tilde{A}_z \partial^{-1}_{A_z^{(0)}} \bar{\partial}_{A_{\bar{z}}^{(0)}} \tilde{A}_z) + \ldots), \tag{3.3.35}$$

where $\partial_{A_z^{(0)}} = \partial + \mathrm{ad}(A_z^{(0)})$ and $\bar{\partial}_{A_{\bar{z}}^{(0)}} = \bar{\partial} + \mathrm{ad}(A_{\bar{z}}^{(0)})$, with $A_{\bar{z}}^{(0)} = -\frac{1}{k} J_{\bar{z}}$. This shows that the one-loop contribution to $G[J_{\bar{z}}]$ is given by $\frac{1}{2} \log\det(\partial^{-1}_{A_z^{(0)}} \bar{\partial}_{A_{\bar{z}}^{(0)}})$. If we *assume* that this determinant is equal to $\frac{1}{2}\log\det(\bar{\partial}_{A_{\bar{z}}^{(0)}}) - \frac{1}{2}\log\det(\partial_{A_z^{(0)}})$, then we can compute these determinants as explained below (3.3.30), to obtain

$$\begin{aligned} G_{one-loop}[J_{\bar{z}}] &= h_G S^+_{wznw}(A_{\bar{z}}^{(0)}) - h_G S^-_{wznw}(A_z^{(0)}) \\ &= h_G S^-_{wznw}(h) - h_G S^+_{wznw}(h) \\ &= 2h_G S^-_{wznw}(h) + \tfrac{h_G}{\pi} \int d^2z \, \mathrm{Tr}(h^{-1}\partial h h^{-1} \bar{\partial} h). \end{aligned} \tag{3.3.36}$$

[‡‡] $F(A_z, A_{\bar{z}})$ denotes the curvature of the connection $\partial + A_z + \bar{\partial} + A_{\bar{z}}$, and is given by $F = \partial A_{\bar{z}} - \bar{\partial} A_z + [A_z, A_{\bar{z}}]$.

The effective action up to one loop can be computed in the same way as in (3.3.33)

$$\begin{aligned} S_{eff}(A_z) &= \min_h \Big((k-2h_G) S^-_{wznw}(h^{-1}) - \tfrac{h_G}{\pi} \int d^2z\, \mathrm{Tr}(h^{-1}\partial h h^{-1}\bar\partial h) \\ &\qquad + \tfrac{k}{\pi} \int d^2z\, \mathrm{Tr}(A_z h^{-1}\bar\partial h) \Big) \\ &= \min_{h'} \Big((k-2h_G) S^+_{wznw}(h') + \tfrac{k-h_G}{\pi} \int d^2z\, \mathrm{Tr}(A_z h'^{-1}\bar\partial h') \Big), \end{aligned} \quad (3.3.37)$$

where in the last line we changed variables from h to h', with $h'^{-1}\bar\partial h' = (1+\frac{h_G}{k})h^{-1}\bar\partial h$. The extremum is at $A_z = (1-\frac{h_G}{k})h'^{-1}\partial h'$, and we find that up to one loop the effective action is given by

$$S_{eff}(A_z) = -(k-2h_G) S^-_{wznw}((1+\tfrac{h_G}{k})A_z). \quad (3.3.38)$$

The disagreement between (3.3.34) and (3.3.38) is due to the fact that the action $S^-_{wznw}(A_z)$ is non-renormalizable, and we can get any result for the effective action we want. At least, as long as we do not impose any additional constraint by hand. Consider, for example, the factorization of the determinants we used above. The same principle applied to $\frac{1}{2}\log\det(\bar\partial_{A^{(0)}_{\bar z}}\partial_{A^{(0)}_z})$ would yield $\frac{1}{2}\log\det(\bar\partial_{A^{(0)}_{\bar z}}) + \frac{1}{2}\log\det(\partial_{A^{(0)}_z})$, but $\frac{1}{2}\log\det(\bar\partial_{A^{(0)}_{\bar z}}\partial_{A^{(0)}_z})$ can also be computed with a regularization prescription that preserves the vector gauge invariance, and then the result is zero. Several proposals have appeared in the literature [262, 280, 291, 293], see also [302] and the related papers [99, 7, 214, 85]. All have one thing in common: the effective action for the WZNW model is given by

$$S_{eff}(A_z) = -kZ_k S^-_{wznw}(Z_A A_z), \quad (3.3.39)$$

where $Z_k = 1 - 2h_G/k$, and $Z_A = 1 + \mathcal{O}(h_G/k)$. The rest of our calculations are based upon this form of the effective action. We keep the explicit Z_A dependence in our final answers. If the effective action is of the form (3.3.39), one can deduce the following path integral identity

$$\int \mathcal{D}A_z\ f(A_z)\exp(-kS^-_{wznw}(A_z)) = \int \mathcal{D}g\ f(Z_A^{-1}g^{-1}\partial g)\exp(-kZ_k S^-_{wznw}(g)). \quad (3.3.40)$$

For the proof of this identity one first decomposes the function f into Fourier modes, and then parametrizes an arbitrary mode with a group valued variable h via

$$f_h(A_z) = \exp(\tfrac{1}{\pi}\int d^2z\, \mathrm{Tr}(Z_A Z_k k \bar\partial h h^{-1} A_z)). \quad (3.3.41)$$

Some manipulations, using the Polyakov-Wiegmann identity and the definition of the effective action, are then sufficient to derive (3.3.40) for an arbitrary Fourier mode, and thus for arbitrary functions f. If we take $f(A_z) = g(A_z)\exp(-lS^-_{wznw}(A_z))$ with an arbitrary functional g, we can evaluate the left hand side of (3.3.40) in two different ways. The two answers agree for generic g only if $Z_A = 1$ and $Z_k = 1 + \frac{a}{k}$ for some constant a. Thus, this suggests that self-consistency requires

$$Z_A(k) = 1. \tag{3.3.42}$$

This is precisely the value for Z_A obtained in (3.3.34), the value that follows from the 'KPZ' approach [291], and the value that follows from a calculation with a Pauli-Villars regularization method [293].

A different, but presumably related, question, is what the effective action of the WZNW action is, not as a function of A_z, but as a function of the group variable g. Since the WZNW action is renormalizable, this is a well-defined question. Some naive path integral manipulations show that the corresponding Z-factors are $Z_k^{wznw} = Z_A^{wznw} = 1$, which is confirmed by the analysis in [219]. On the other hand, in [302] it is argued that $Z_k^{wznw} = 1 - \frac{h_G}{k}$. The shift is half of that for the effective action of $S^-_{wznw}(A_z)$. The reason is that the latter is a chiral action, in contrast to $S^-_{wznw}(g)$, which is non-chiral. Recent two-loop calculations [314], in which the effective action is defined as that of the non-linear sigma model corresponding to the WZNW theory, yield $Z_k^{\text{sigma model}} = 1 - \frac{h_G}{k}$ and $Z_A^{\text{sigma model}} = 1$. It would be nice to have a better understanding of the relations between these different Z factors.

3.3.4. The Effective Action of W_3 Gravity, Continued

Using (3.3.40) and $Z_k = (1 - \frac{2h_G}{k})$ it is straightforward to work out the effective action for W_3 gravity. Starting with (3.3.29), and using that $h_G = 3$ for $SL(3,\mathbb{R})$, one finds:

$$\begin{aligned} e^{-\Gamma[T,W]} &= \int \mathcal{D}g\,\delta(J_1-\xi)\delta(J_2-\xi)\delta(J_3)\delta(H_0)\delta(H_1)\delta(K_1-K_2) \\ &\quad \delta(T - N_T\xi(K_1+K_2))\delta(W - N_W\xi^2K_3)e^{-kS^-_{wznw}(g)} \\ &= \int \mathcal{D}A_z\,\delta(J_1'-\xi)\delta(J_2'-\xi)\delta(J_3')\delta(H_0')\delta(H_1')\delta(K_1'-K_2') \\ &\quad \delta(T - N_T\xi(K_1'+K_2'))\delta(W - N_W\xi^2K_3')e^{-(k+6)S^-_{wznw}(A_z)}, \end{aligned} \tag{3.3.43}$$

where $J' = kZ_A(k+6)A_z$. We can substitute the delta functions into the WZNW action, and obtain the effective action for W_3 gravity to all orders. The final result reads, in terms of the renormalized level $k_c = k + 6$:

$$\Gamma[T, W] = k_c S^-_{wznw} \begin{pmatrix} 0 & \frac{\xi}{(k_c-6)Z_A(k_c)} & 0 \\ \frac{T}{2N_T\xi(k_c-6)Z_A(k_c)} & 0 & \frac{\xi}{(k_c-6)Z_A(k_c)} \\ \frac{W}{N_W\xi^2(k_c-6)Z_A(k_c)} & \frac{T}{2N_T(k_c-6)Z_A(k_c)} & 0 \end{pmatrix}. \tag{3.3.44}$$

The induced action for W_3 gravity to lowest order, $\Gamma^{(0)}[T, W]$, can be computed as was explained in section 3.2. If T and W are normalized to be the standard generators of the W_3 algebra, the result for $\Gamma^{(0)}[T, W]$ reads [256]

$$\Gamma^{(0)}[T, W] = k S^-_{wznw} \begin{pmatrix} 0 & \alpha & 0 \\ \beta T & 0 & \alpha \\ \gamma W & \beta T & 0 \end{pmatrix}, \tag{3.3.45}$$

where $c = -24k$, $2\alpha\beta k = 1$, and $\gamma^2 = -10\beta^2/\alpha^2$. Both (3.3.44) and (3.3.45) contain one free parameter, and we can choose $\xi/(k_c - 6)Z_A(k_c) = \alpha = 1$. This proves that

$$\Gamma[T, W] = Z_k \Gamma^{(0)}[Z_T T, Z_W W], \tag{3.3.46}$$

and using (3.3.24) and (3.3.25) we find that k_c and the central charge c are related through

$$c = 50 - 24\left((k_c - 3) + \frac{1}{(k_c - 3)}\right) \tag{3.3.47}$$

and that the renormalizations Z_k, Z_T and Z_W are given by

$$\begin{aligned} Z_k &= \frac{-24}{c} k_c = 1 - \frac{122}{c} + \ldots, \\ Z_T &= \frac{-c(k_c - 3)}{24(k_c - 6)^2 Z_A(k_c)^2}, \\ Z_W &= \frac{ic\sqrt{(5c + 22)}(k_c - 3)^{3/2}}{48\sqrt{30}(k_c - 6)^3 Z_A(k_c)^3}. \end{aligned} \tag{3.3.48}$$

These results are in agreement with the one-loop results obtained in [161, 284], if $Z_A(k_c) = 1 + \frac{3}{k_c} + \mathcal{O}(1/k_c)^2$, as predicted by (3.3.38). However, in [161] one has to deal with momentum routing ambiguities, whereas the calculations in [284] use the same factorization of determinants as the one that led to (3.3.38). Probably, the calculation of the effective action of W_3 gravity suffers from the same ambiguities as the calculation for the WZNW model. Note that the 'KPZ' relation between the level k_c and c given in (3.3.47) is independent of Z_A, and always comes out of this analysis as long as $Z_k = 1 - \frac{2h_G}{k}$. Clearly, the techniques used here can be applied to W_N gravity for arbitrary N, and in particular to 2-d quantum gravity, yielding

$$\begin{aligned}\Gamma[T] &= Z_k \Gamma^{(0)}[Z_T T], \\ c &= 13 - 6\left((k_c - 2) + \frac{1}{(k_c-2)}\right), \\ Z_k &= \frac{-6}{c} k_c, \\ Z_T &= \frac{-c(k_c-2)}{6(k_c-4)^2 Z_A(k_c)^2}.\end{aligned} \tag{3.3.49}$$

These results agree with those obtained in [207, 335, 241, 161], if $Z_A(k_c) = 1 + \frac{2}{k_c} + \mathcal{O}(1/k_c)^2$.

3.3.5. Remarks

The relation between the constrained WZNW model presented here and Toda theory becomes clear if one picks in (3.3.26) the gauge choice $K^1 = K^2 = K^3 = 0$. Ignoring the non-trivial contribution of the Faddeev-Popov ghosts in this case, the action (3.3.26) reduces to a Toda action, and T and W can be identified with the conserved currents of the Toda theory.

For a general W algebra associated to an sl_2 embedding, the effective action reads as follows*. As was explained in section 2.2.8, the BRST invariant expressions for the generators of the quantum W algebra are always of the form $A_\alpha = N_\alpha[k]\hat{J}^{\bar{\alpha}} + \ldots$, where $T_{\bar{\alpha}} \in \mathbf{g}_{lw}$ and k is the level of the current algebra. For large k, $N_\alpha[k]$ has an expansion $N_\alpha^{(0)} k^{d_\alpha} + N_\alpha^{(1)} k^{d_\alpha - 1} + \ldots$. Define k_c via the equation (cf (2.2.50))

$$c = d_0 - \frac{1}{2}\dim(\mathbf{g}_{\frac{1}{2}}) - 12\mathrm{Tr}\left(\frac{\rho}{\sqrt{k_c - h}} - t_0\sqrt{k_c - h}\right)^2, \tag{3.3.50}$$

and let h_α denote the conformal weight of A_α. Then

$$\Gamma[A_\alpha] = Z_k \Gamma^{(0)}[Z_\alpha A_\alpha], \tag{3.3.51}$$

where

$$Z_k = \frac{-12\mathrm{Tr}(t_0)^2}{c} k_c,$$

*Recently, lots of new examples were worked out, including supersymmetric ones; see [291, 292].

$$Z_\alpha \quad = \quad \frac{N_\alpha^{(0)}}{N_\alpha[k_c - 2h](k_c - 2h)^{h_\alpha} Z_A^{\mathbf{g}}(k_c)^{h_\alpha}} \left(\frac{-c}{12\mathrm{Tr}(t_0)^2} \right)^{d_\alpha + h_\alpha} . \tag{3.3.52}$$

4

Covariant W Gravity

4.1. Gauging Non-Chiral Algebras

In the previous chapter we explained how to gauge a subalgebra of the chiral algebra. The answer (3.1.4) was particularly simple, it contained only linear couplings to the gauge fields. In this section we take a look at the more general and more difficult problem of gauging a subalgebra of the full symmetry algebra $\mathcal{A} \times \bar{\mathcal{A}}$. In fact, the problem is that $\mathcal{A} \times \bar{\mathcal{A}}$ does not constitute a direct product in the sense of algebras, so that $\mathcal{A}_{\text{sub}} \times \bar{\mathcal{A}}_{\text{sub}}$ is not necessarily a closed subalgebra of $\mathcal{A} \times \bar{\mathcal{A}}$

4.1.1. Example: Free Scalar Field

Suppose we want to gauge both the holomorphic and anti-holomorphic Virasoro algebra for the free scalar field. Based on the construction of the previous chapter, we write down as a first attempt

$$S = \frac{1}{2\pi} \int d^2 z \, \partial X \bar{\partial} X + \frac{1}{\pi} \int d^2 z \, (\mu T + \bar{\mu} \bar{T}). \tag{4.1.1}$$

However, this action is not invariant. Under a transformation generated by

$$\oint_0 \frac{dz}{2\pi i} \, \epsilon(z, \bar{z}) T(z), \tag{4.1.2}$$

the anti-holomorphic stress-energy tensor transforms non-trivially, $\delta_\epsilon \bar{T} = -\bar{\partial} X \bar{\partial}(\epsilon \partial X)$, which implies

$$\delta_\epsilon S = \frac{1}{\pi} \int d^2 z \, \bar{\mu} \, (-\bar{\partial}\epsilon \, \partial X \, \bar{\partial} X + \epsilon \, \partial \bar{T}). \tag{4.1.3}$$

The first term is proportional to neither T nor $\bar{T}$, so that the the action cannot be made gauge invariant by a suitable modification of the μ and $\bar{\mu}$ transformation rules.

To restore the gauge invariance, we need to add terms of higher order in μ and $\bar{\mu}$. Rather than doing this by hand order by order, we can in this special case guess the final answer, owing to the observation made in section 3.1.1. There we noticed that the chiral gauged action was nothing but the free scalar field in the non-trivial background metric $ds^2 \sim (dz + \mu d\bar{z})d\bar{z}$. In the same way the anti-chiral gauged action corresponds to a background metric $ds^2 \sim dz(d\bar{z} + \bar{\mu}dz)$. This suggests that the fully gauged theory corresponds to the metric $ds^2 \sim (dz + \mu d\bar{z})(d\bar{z} + \bar{\mu}dz) \sim |dz + \mu d\bar{z}|^2$, yielding an action

$$S = \frac{1}{2\pi} \int d^2z \, \frac{1}{1 - \mu\bar{\mu}} \, (\partial - \bar{\mu}\bar{\partial})X \, (\bar{\partial} - \mu\partial)X. \tag{4.1.4}$$

To first order in μ and $\bar{\mu}$, this action coincides with (4.1.1), and the transformation rules for the fields agree with those in (4.1.1). Furthermore, (4.1.4) is invariant under general co-ordinate transformations (as is clear from (2.1.25)), and is therefore the correct gauged action. It contains terms of arbitrary order in $\mu, \bar{\mu}$, so that adding terms order by order to (4.1.1) would not have been a particularly efficient procedure to obtain the complete result.

What is important, is that by gauging both Virasoro algebras, we have re-obtained the covariant formulation of the free scalar field coupled to a metric, provided that we identify the gauge fields with components of the metric. This is our main motivation for studying the gauging of more general non-chiral algebras. We expect that the gauged actions are covariant (or even better, W-covariant) versions of the original action, and that the gauge fields can be identified as components of some W analogue of the metric. The ultimate goal of this program is to give some kind of W tensor calculus, so that one can write down actions that are manifestly W-invariant, in the same way as this can be done for co-ordinate transformations. Unfortunately, this goal is still far ahead. The problem of gauging an arbitrary algebra is still only partially solved, as we discuss below.

4.1.2. General Case

The reason why in the previous example the naive linear gauging did not work, is that there is a nonvanishing OPE between T and $\bar{T}$. There is an alternative way to think about this. In the linear gauged action (3.1.4) the gauge fields can be interpreted as Lagrange multipliers that impose the constraints $A_\alpha = 0$. The condition that the A_α generate a closed algebra is equivalent to the condition that the constraints $A_\alpha = 0$ are first-class. Thus, in terms of constraints, the problem is that $T = 0$ and $\bar{T} = 0$ do not form a set of first class constraints. Only a subalgebra that corresponds to a set of first class constraints can be linearly gauged. Therefore, the gauging is straightforward, once we manage to replace $T = 0$ and $\bar{T} = 0$ by two equivalent, first-class constraints.

This can be done by means of the following trick: we introduce a set of auxiliary fields, both for the generators of the chiral and of the anti-chiral algebra. We include them in the action in such a way that on-shell they are identical to the generators of the chiral and anti-chiral algebra. Subsequently, we define Poisson brackets for the auxiliary fields. With respect to these Poisson brackets, the auxiliary fields associated with the generators of the chiral algebra form an algebra that is isomorphic to the chiral algebra, but one that *does* commute with the copy of the anti-chiral algebra generated by the remaining auxiliary fields. If we replace in this set-up the subalgebra we want to gauge by the corresponding expressions in terms of auxiliary fields, we have a set of first class constraints that can be gauged in a linear fashion. Then the complete gauged action is obtained by integrating out the auxiliary fields. This method was first employed in [285], where the auxiliary fields were introduced by hand after an order-by-order construction of the gauged action for $W_3 \times \bar{W}_3$ symmetry. There, the auxiliary fields arise as components of a 'nested covariant derivative', see also [282]. The same action, but in a different form, was obtained in [32]. Yet another point of view was given in [242], where the same type of gauged actions were obtained from a Hamiltonian formulation. The problem with higher spin algebras like the W_3 algebra is that the gauged actions contain terms of order larger than two in the auxiliary fields, and integrating out the auxiliary fields yields a non-polynomial action. In general, it is thus more convenient to keep the auxiliary fields in the action.

We briefly sketch the Hamiltonian approach here, because it enables us to connect the different approaches. Let (q^i, p_i) be the canonical variables of the theory, and H_0 its Hamiltonian. The index i can be discrete, continuous or both. In the presence of a set of first-class constraints G_α, the action reads

$$S \sim \int dt\, (p_i \dot{q}^i - H_0 - \mu_\alpha G_\alpha), \tag{4.1.5}$$

where the gauge fields μ_α play the role of Lagrange multipliers. The charge $Q = \epsilon^\alpha G_\alpha$ generates a symmetry of the action,

$$\begin{aligned} \delta p_i &= \{Q, p_i\} \\ \delta q^i &= \{Q, q^i\} \\ \delta \mu_\alpha &= \dot{\epsilon}^\alpha - \mu_\beta \frac{D\{Q, G_\beta\}}{DG_\alpha} - \frac{D\{Q, H_0\}}{DG_\alpha} \end{aligned} \tag{4.1.6}$$

where the derivation 'D' is defined as in (3.1.6).

Now suppose that the Poisson algebra generated by q^i and p_i contains a copy of $\mathcal{A}$ and a copy of $\bar{\mathcal{A}}$ that mutually commute. In that case we can gauge an arbitrary subalgebra of $\mathcal{A} \times \bar{\mathcal{A}}$ without any problems, assuming that the Hamiltonian does preserve the algebra. If we gauge an algebra of symmetries, this is the case, and the scheme given here immediately yields the gauged action, and integrating out the momenta then gives the Lagrangian form of the gauged action. Integrating out the momenta is precisely what can generate the non-linear terms in the gauge fields.

This formulation is the one presented in [242]. To make contact with [285, 32], we introduce auxiliary fields F_α for each generator $A_\alpha(q,p)$ of the chiral algebra, and similarly $\bar{F}_\alpha$ for each generator of the anti-chiral algebra. The auxiliary fields are included in the action via

$$S \sim \int dt\,(p_i\dot{q}^i - H_0 - \mu_\alpha G_\alpha(F,\bar{F}) + \lambda_\alpha(F_\alpha - A_\alpha(q,p))^2 + \bar{\lambda}_\alpha(\bar{F}_\alpha - \bar{A}_\alpha(q,p))^2). \quad (4.1.7)$$

The first class constraints are now expressed in terms of F instead of A_α. The Poisson brackets of F_α are identical to those of $A_\alpha(q,p)$, which guarantees that (4.1.6) is still a symmetry of the action including the auxiliary fields. We can integrate out the momenta p_i from (4.1.7), which gives an action still containing the auxiliary fields $F_\alpha, \bar{F}_\alpha$. For a suitable choice of constants $\lambda_\alpha, \bar{\lambda}_\alpha$ this action is precisely the one obtained in [285, 32].

For the free scalar field this procedure is easily implemented. In terms of the coordinates τ,σ related to $z,\bar{z}$ via $\tau = z+\bar{z}, \sigma = z-\bar{z}$, the canonical variables for the free scalar field are X and $P = \partial_\tau X$. The chiral algebra is generated by ∂X, the anti-chiral algebra by $\bar{\partial}X$. The mutually commuting copies of these algebras are generated by the functions $P + \partial_\sigma X$ and $P - \partial_\sigma X$. With $\lambda = \bar{\lambda} = 1/2$ the action (4.1.7) becomes, when gauging the holomorphic and anti-holomorphic Virasoro algebra,

$$\begin{aligned} S \;\sim\; & \int d^2z\,\Big(P\partial_\tau X - (\frac{1}{2}P^2 + \frac{1}{2}(\partial_\sigma X)^2) + \frac{1}{2}(F - P - \partial_\sigma X)^2 + \frac{1}{2}(\bar{F} - P + \partial_\sigma X)^2 \\ & -\frac{\mu}{2}F^2 - \frac{\bar{\mu}}{2}\bar{F}^2\Big) \end{aligned} \quad (4.1.8)$$

Integrating over the momentum P gives

$$S \sim \int d^2z\left(\frac{1}{2}\partial X\bar{\partial}X - (F - \partial X)(\bar{F} - \bar{\partial}X) - \frac{\mu}{2}F^2 - \frac{\bar{\mu}}{2}\bar{F}^2\right), \quad (4.1.9)$$

and a subsequent elimination of the auxiliary fields leads to (4.1.4). The precise connection between this formulation and the Hamiltonian one without auxiliary fields can be found by first integrating out the auxiliary fields from (4.1.8), which yields

$$S \sim \int d^2z\left(P\partial_\tau X - (\frac{1}{2}P^2 + \frac{1}{2}(\partial_\sigma X)^2) - \frac{\mu}{2(1+\mu)}(P + \partial_\sigma X)^2 - \frac{\bar{\mu}}{2(1+\bar{\mu})}(P - \partial_\sigma X)^2\right). \quad (4.1.10)$$

Thus, the standard Hamiltonian formulation is equivalent to the one with auxiliary fields up to a field redefinition for μ and $\bar{\mu}$.

Given a realization of a W algebra in terms of an arbitrary number of free scalar fields with background charges, together with an arbitrary number of first-order systems, this procedure can directly be applied to gauge $W \times \bar{W}$. For first-order systems the holomorphic and anti-holomorphic chiral algebra are completely decoupled, and there is no need to go to the Hamiltonian formulation for that sector of the theory.

4.1.3. Gauging the WZNW Model

Besides free scalar fields and first-order systems we treated one more fundamental conformal field theory in section 2.1, the WZNW model. The phase space of the WZNW model is, from the Hamiltonian point of view, the cotangent bundle of the loop group of G. For a discussion of the phase space of the WZNW model, see [257]. The phase space variables are conveniently parametrized in terms of a Lie algebra valued current $\mathcal{J}$ and a group valued variable g. We also introduce what is going to be the anti-holomorphic current

$$\bar{\mathcal{J}} \equiv g\mathcal{J}g^{-1} - 2k\frac{\partial g}{\partial\sigma}g^{-1}. \tag{4.1.11}$$

The Hamiltonian is given by

$$H_0 = -\frac{1}{4\pi k}\int d^2z\,\mathrm{Tr}(\mathcal{J}^2 + \bar{\mathcal{J}}^2). \tag{4.1.12}$$

The variables $(g, \mathcal{J})$ are not canonical variables, and therefore the symplectic form [16] is not the usual one. This means that the expression for the action has to be modified to

$$S = \frac{1}{4i}\left(\int d^{-1}\omega - \int d\tau H_0\right) \tag{4.1.13}$$

where ω is the symplectic form. The factor of $1/4i$ has been included to make contact with our Euclidean formulation, $d\sigma d\tau = 4id^2z$. The symplectic form for the WZNW model is not closed, which leads to the non-local term in the WZNW action. We have [304]

$$\int d^{-1}\omega = \frac{-1}{\pi}\int d\sigma d\tau\mathrm{Tr}\left(\left(\mathcal{J} - kg^{-1}\frac{\partial g}{\partial\sigma}\right)g^{-1}\frac{\partial g}{\partial\tau}\right) - \frac{k}{3\pi}\int_{B_3}\mathrm{Tr}(g^{-1}dg)^3 \tag{4.1.14}$$

and integrating out the current $\mathcal{J}$ from (4.1.13) gives the usual WZNW action S^-_{wznw}. The nice thing is that $\mathcal{J}$ and $\bar{\mathcal{J}}$ commute, and the gauging of an arbitrary subalgebra can proceed along the same lines as in the previous section. In particular this enables us to write down non-chiral gauged actions for W algebras for which a Casimir construction is known [14]. The two different types of gauged actions that we described previously can be obtained from one action, by introducing a pair of auxiliary currents $\mathcal{F}, \bar{\mathcal{F}}$ and by adding the quadratic terms

$$-\frac{1}{2\pi k}\int d\sigma d\tau\, (\mathrm{Tr}(\mathcal{F}-\mathcal{J})^2 + \mathrm{Tr}(\bar{\mathcal{F}}-\bar{\mathcal{J}})^2) \tag{4.1.15}$$

to the action. The subalgebra to be gauged can be expressed in terms of either $\mathcal{J}$ and $\bar{\mathcal{J}}$, or in terms of $\mathcal{F}$ and $\bar{\mathcal{F}}$. In the latter formulation the current $\mathcal{J}$ can be integrated out, which gives for the ungauged action the expression

$$S = kS^-_{wznw}(g) + \frac{1}{\pi k}\int d^2z\, \mathrm{Tr}((\mathcal{F}-kg^{-1}\partial g)g^{-1}(\bar{\mathcal{F}}-k\bar{\partial}gg^{-1})g). \tag{4.1.16}$$

To illustrate the method, suppose we want to gauge $\bar{\mathbf{g}}_1\times\bar{\mathbf{g}}_2 \subset \bar{\mathbf{g}}\times\bar{\mathbf{g}}$, where $\bar{\mathbf{g}}$ denotes the affine Lie algebra based on $\mathbf{g}$. For this purpose we introduce a $\mathbf{g}_1$ valued gauge field A_z and a $\mathbf{g}_2$ valued gauge field $A_{\bar{z}}$, and add to the action (4.1.16) the term

$$\frac{1}{\pi}\int d^2z\, (\mu_{\mathbf{1}} - \mathrm{Tr}(A_z\bar{\mathcal{F}}) + \mathrm{Tr}(A_{\bar{z}}\mathcal{F})), \tag{4.1.17}$$

where we included a gauge field for the identity operator which is needed if $\bar{\mathbf{g}}_1\times\bar{\mathbf{g}}_2$ has a center. If we integrate out the auxiliary fields $\mathcal{F}, \bar{\mathcal{F}}$ we get

$$S = kS^-_{wznw}(g) + \frac{k}{\pi}\int d^2z\, (-\mathrm{Tr}(A_z\bar{\partial}gg^{-1}) + \mathrm{Tr}(g^{-1}\partial gA_{\bar{z}}) + \mathrm{Tr}(A_zgA_{\bar{z}}g^{-1})) + \frac{1}{\pi}\int d^2z\, \mu_{\mathbf{1}}. \tag{4.1.18}$$

The identity gauge field can be expressed in a non-local way in terms of A_z and $A_{\bar{z}}$. Write $A_z = g_1^{-1}\partial g_1$ and $A_{\bar{z}} = g_2^{-1}\bar{\partial}g_2$, then

$$\frac{1}{\pi}\int d^2z\, \mu_{\mathbf{1}} = kS^-_{wznw}(g_1) + kS^+_{wznw}(g_2). \tag{4.1.19}$$

and with the same parametrization the whole action is actually equal to $kS^-_{wznw}(g_1gg_2^{-1})$, which makes the $\mathbf{g}_1\times\mathbf{g}_2$ invariance completely obvious. However, a drawback of this action is that it is non-local in the gauge fields, unless $S^-_{wznw}(g_1) = S^+_{wznw}(g_2) = 0$, as is for instance the case if $\mathbf{g}_1$ and $\mathbf{g}_2$ are nilpotent.

The subgroups for which it is possible to write down an expression for the identity operator that is local in the gauge fields, are usually called anomaly-free subgroups (for a classification see [319]). The main anomaly-free subgroups are vector subgroups, and axial abelian subgroups. In the case of a vector subgroup ($\delta g = \epsilon g - g\epsilon$), we can replace

$$\frac{1}{\pi}\int d^2z\, \mu_{\mathbf{1}} \to -\frac{k}{\pi}\int d^2z\, \mathrm{Tr}(A_zA_{\bar{z}}), \tag{4.1.20}$$

with (for unitary groups) the additional requirement $A_{\bar{z}} = -A_z^\dagger$. This is the usual formulation of the gauged WZNW model [328, 145] that describes coset conformal field theories [23, 159]. In terms of the group variables $g_1, g, g_2 \equiv g_1$ the action reads

$$S = kS^-_{wznw}(g_1 g g_1^\dagger) - kS^-_{wznw}(g_1 g_1^\dagger), \tag{4.1.21}$$

but on a surface of non-trivial topology one cannot always globally write $A_z = g_1^{-1}\partial g_1$, and we have to go back to the formulation in terms of the gauge fields.

For axial abelian subgroups ($\delta g = \epsilon g + g\epsilon$) we can replace

$$\frac{1}{\pi}\int d^2z\,\mu_{\mathbf{1}} \to +\frac{k}{\pi}\int d^2z\,\mathrm{Tr}(A_z A_{\bar{z}}), \tag{4.1.22}$$

so that in terms of the group variables $g_1, g, g_2 \equiv g_1^\dagger$ the action equals

$$S = kS^-_{wznw}(g_1 g g_1) - kS^-_{wznw}(g_1 g_1^\dagger). \tag{4.1.23}$$

In this case the condition for A is $A_{\bar{z}} = A_z^\dagger$.

4.2. *W* Gravity

4.2.1. What is W Gravity?

In four dimensions, the field equations for the metric in Einstein's theory of general relativity can be recovered from the Einstein-Hilbert action

$$S = \frac{-1}{16\pi G}\int d^4x\,\sqrt{g}R, \tag{4.2.1}$$

and in the presence of matter fields by adding the action of a generally covariant field theory to this one. It is natural to start with the same definition of gravity in two dimensions. In two dimensions the Einstein-Hilbert action is, however, trivial:

$$\int_{\Sigma_h} d^2z\sqrt{g}R = -4\pi\chi(\Sigma), \tag{4.2.2}$$

where $\chi(\Sigma_h) = 2 - 2h$ is the Euler characteristic of Σ_h, with h is the number of handles of the Riemann surface Σ_h. Thus the definition of gravity in two dimensions is just 'any covariant field theory'. Now if this field theory happens to be a conformal field theory, the theory has besides the invariance under the group of general co-ordinate transformations also an invariance under Weyl rescalings of the metric. Since

$$\frac{\text{metrics on } \Sigma_h}{\text{co-ordinate + Weyl transformations}} = \mathcal{M}_h, \tag{4.2.3}$$

where $\mathcal{M}_h$ is the $3h-3$ complex dimensional moduli space of Riemann surfaces, one would naively expect that all the gravitational dynamics are reduced to some dynamics on the moduli space. However, we saw in the previous chapter that the chiral induced action $\Gamma[\mu]$ was non-vanishing and not invariant under co-ordinate transformations, because the classical centerless Virasoro algebra got replaced by a quantum Virasoro algebra with central charge $c \neq 0$ upon quantizing the matter part of the theory. This shows that the matter theory may induce some non-trivial dynamics for the metric. This is what we will call the induced action for gravity. Thus,

$$e^{-\Gamma_{ind}[g_{\alpha\beta}]} = \int \mathcal{D}\,\phi e^{-S(\phi, g_{\alpha\beta})}. \tag{4.2.4}$$

A priori the structure of $\Gamma[g_{\alpha\beta}]$ could depend on the precise details of the field theory defined by S. The fact that the structure of the induced action $\Gamma[\mu]$ only depended on the value of the central charge of the quantum Virasoro algebra already suggests that $\Gamma_{ind}[g_{\alpha\beta}]$ should not depend on the precise details of the field theory, but only on the value of central charge of the theory, which should play the role of a coupling constant for the induced action. Unfortunately, the induced action arises purely from an anomaly that is classically absent, which allows one the freedom to add any local counterterm to the induced action, corresponding to the different regularization schemes that one can use to compute the induced action (4.2.4). To fix its form, we will in addition require the induced action to be itself generally covariant, which is an essential property of the Einstein-Hilbert action.

Let us now compute the induced action for a simple case, namely for the free scalar field

$$e^{-\Gamma_{ind}[g_{\alpha\beta}]} = \int \mathcal{D}X e^{-\frac{1}{8\pi}\int d^2x \sqrt{g} g^{\alpha\beta}\partial_\alpha X \partial_\beta X}. \tag{4.2.5}$$

Following [173] we decompose $X = X_0 + X'$ where X_0 is the zero mode of the Laplacian $\Box = -\frac{1}{\sqrt{g}}\partial_\alpha \sqrt{g} g^{\alpha\beta}\partial_\beta$, and X' parametrizes the remaining modes of X. Then

$$e^{-\Gamma_{ind}[g_{\alpha\beta}]} = (\det \Box')^{-\frac{1}{2}} \int \mathcal{D}X_0 \int \mathcal{D}X' e^{-\frac{1}{8\pi}\int d^2x\, \sqrt{g}(X')^2} \tag{4.2.6}$$

The zero mode integral gives a factor proportional to the volume of space-time. The ultralocality principle [260] tells us that

$$\int \mathcal{D}X e^{-\frac{1}{8\pi}\int d^2x\,\sqrt{g}(X)^2} = e^{\mu_1^2\int d^2x\,\sqrt{g}}, \tag{4.2.7}$$

for some constant μ_1. Separating $\mathcal{D}X = \mathcal{D}X_0\mathcal{D}X'$ in this expression gives

$$\int \mathcal{D}X' e^{-\frac{1}{8\pi}\int d^2x\,\sqrt{g}(X')^2} = \left(\frac{8\pi^2}{\int d^2x\,\sqrt{g}}\right)^{-\frac{1}{2}} e^{\mu_1^2\int d^2x\,\sqrt{g}}. \tag{4.2.8}$$

If λ_i are the eigenvalues of $\Box$, then we use the manifestly covariant heat kernel regularization principle to derive

$$\begin{aligned} \delta\log\det\Box' &= \delta\sum_{\lambda_i>0}\log\lambda_i \\ &= \sum_{\lambda_i>0}\frac{\delta\lambda_i}{\lambda_i} \\ &= \lim_{\epsilon\to 0}\sum_{\lambda_i>0}\delta\lambda_i\int_\epsilon^\infty dt\, e^{-t\lambda_i} \\ &= \lim_{\epsilon\to 0}\int_\epsilon^\infty dt\,\mathrm{Tr}(\delta\Box e^{-t\Box}). \end{aligned} \tag{4.2.9}$$

Under a Weyl transformation $\delta g_{\alpha\beta} = \lambda g_{\alpha\beta}$ of the metric this gives

$$\begin{aligned} \delta\log\det\Box' &= \lim_{\epsilon\to 0}\int_\epsilon^\infty dt\,\mathrm{Tr}(-\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\Box e^{-t\Box}) \\ &= \lim_{\epsilon\to 0}\mathrm{Tr}(-\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}e^{-\epsilon\Box}) + \frac{\frac{1}{2}\int d^2x\,\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta}}{\int d^2x\,\sqrt{g}} \\ &= \lim_{\epsilon\to 0}\mathrm{Tr}(-\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}e^{-\epsilon\Box}) + \delta\log\int d^2x\,\sqrt{g}. \end{aligned} \tag{4.2.10}$$

The extra term in the two last lines originates from the $t\to\infty$ contribution to the integral in the first line. Altogether this shows that

$$\delta(\Gamma_{ind}[g_{\alpha\beta}] + \mu_1^2\int d^2x\,\sqrt{g}) = \frac{1}{2}\lim_{\epsilon\to 0}\mathrm{Tr}(-\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}e^{-\epsilon\Box}). \tag{4.2.11}$$

The heat kernel expansion of the Laplacian is well known [155]. It is

$$\mathrm{Tr}(Ye^{-\epsilon\Box}) = \frac{1}{4\pi\epsilon}\int d^2x\,\sqrt{g}Y - \frac{1}{24\pi}\int d^2x\,\sqrt{g}RY + \mathcal{O}(\epsilon). \tag{4.2.12}$$

The role of ϵ is the same as that of a cutoff. The eigenvalues of the Laplacian larger than $1/\epsilon$ are suppressed, corresponding to a cutoff $\Lambda \sim \epsilon^{-1/2}$. We find the following equation for the variation of the induced action under a Weyl transformation of g

$$\delta(\Gamma_{ind}[g_{\alpha\beta}] + (\mu_1^2 + \frac{\Lambda^2}{8\pi}) \int d^2x \sqrt{g}) = \frac{1}{96\pi} \int d^2x \sqrt{g} R g^{\alpha\beta} \delta g_{\alpha\beta}. \tag{4.2.13}$$

From the identity

$$R_{e^\rho g} = e^{-\rho}(R_g - \Box_g \rho) \tag{4.2.14}$$

we deduce that under a Weyl transformation

$$\delta(\sqrt{g}R) = -\frac{1}{2}\sqrt{g}\Box_g(g^{\alpha\beta}\delta g_{\alpha\beta}), \tag{4.2.15}$$

which implies that the right-hand side of (4.2.13) can be rewritten as

$$\delta\left(\frac{-1}{96\pi}\int d^2x\,(\sqrt{g}R)\frac{1}{\sqrt{g}\Box}(\sqrt{g}R)\right). \tag{4.2.16}$$

Since there are no Weyl invariant covariant functionals of the metric except the constant one, this shows that

$$\Gamma_{ind}[g_{\alpha\beta}] = \frac{-1}{96\pi}\int d^2x\,(\sqrt{g}R)\frac{1}{\sqrt{g}\Box}(\sqrt{g}R) - (\mu_1^2 + \frac{\Lambda^2}{8\pi})\int d^2x\,\sqrt{g} \tag{4.2.17}$$

This is the original result of Polyakov [267]. The second term is the cosmological constant term, and we will ignore it for the time being. It is easy to verify that if we would have started with an arbitrary conformal field theory with central charge c, the coefficient in front of (4.2.17) gets simply replaced by $-c/96\pi$. As a check, we work out the covariant action for the metric $ds^2 = (dz + \bar{\mu}d\bar{z})d\bar{z}$. Then $R = -4\partial^2\mu$, $\Box = 4(\partial\mu\partial - \partial\bar{\partial})$, and (4.2.17) reduces to (3.2.3). The structure of the induced action becomes somewhat clearer if we decompose the metric as $g = e^{-2\varphi}\hat{g}$, where $\hat{g}$ is the metric $ds^2 = |dz + \mu d\bar{z}|^2$. We find

$$\Gamma_{ind}[g_{\alpha\beta}] = S_L(\varphi, \hat{g}) + K[\mu, \bar{\mu}] + \Gamma[\mu] + \Gamma[\bar{\mu}] \tag{4.2.18}$$

where S_L is the Liouville action

$$S_L(\varphi, \hat{g}) = \frac{-c}{24\pi}\int d^2z\,(\sqrt{\hat{g}}\varphi\Box_{\hat{g}}\varphi + \sqrt{\hat{g}}R_{\hat{g}}\varphi), \tag{4.2.19}$$

and $K[\mu, \bar{\mu}]$ is a local expression involving μ and $\bar{\mu}$. Altogether we see the following picture arise: the covariant action for gravity consists of a local counterterm plus two non-local chiral induced actions $\Gamma[\mu]$ and $\Gamma[\bar{\mu}]$. The local counterterm shifts the diffeomorphism anomalies of $\Gamma[\mu]$ and $\Gamma[\bar{\mu}]$ to the Weyl anomaly of $\Gamma_{ind}[g_{\alpha\beta}]$ and restores the general covariance.

This leads to the following definition of covariant W gravity, where W can be any operator algebra that exists for generic values of the central charge. An action for covariant W gravity Γ_{cov} is an action that is (i) of the form $\Gamma[\mu_\alpha]+\Gamma[\bar{\mu}_\alpha]+$ a local term, (ii) is invariant under $W \times \bar{W}$ transformations and (iii) contains a free parameter that parametrizes the different values of the central charge. Such a covariant action will, like the induced action $\Gamma[\mu_\alpha]$, allow an expansion in $1/c$. The lowest-order term of the covariant action is the 'classical' covariant action for W gravity. The covariant action will in general contain extra fields in addition to $\mu_\alpha, \bar{\mu}_\alpha$. These fields can be auxiliary, or be analogues of the Liouville field in that they are components of a W analogue of the metric. However, as long as there is no canonical geometrical formulation of W gravity, there is no canonical choice of extra variables, and the covariant action is not uniquely defined by this definition. In fact, starting with any conformal field theory whose chiral algebra contains a W W algebra with center, we can define a covariant action, by gauging the $W \times \bar{W}$ algebra and replacing the identity gauge field by $-\Gamma^{(0)}[\mu_\alpha] - \Gamma^{(0)}[\bar{\mu}_\alpha]$. The resulting action is a candidate for $\Gamma^{(0)}_{cov}$. So, we need some kind of guiding principle that tells us how we should select the 'natural' covariant action for W gravity.

One guiding principle might be to look for the 'minimal' conformal field theory that contains the W algebra in its chiral algebra. Given any free field realization of the W algebra, a typical candidate would be the associated free field theory. The number of free fields is always equal to the number of generators of the W algebra. That number is the minimal number of extra fields that one needs in order to construct a covariant action for W gravity.

For those W algebras associated to sl_2 embeddings, a 'geometrical' interpretation was given in chapter 2, where we explained that W transformations can be seen as field dependent gauge transformations. In the next section we show that from this point of view there is a natural covariant action for these W algebras, that has the structure of a gauged Toda theory. Toda theories are the natural generalization of Liouville theory, and for this reason W gravity has sometimes been defined via Toda theory, although the full structure of the gauged Toda theory was unknown.

The main motivation to study W gravity lies in the fact that they present new examples of conformal field theories, and hence provide us with new string theories. If we start with a conformal field theory coupled to gravity, and quantize both the matter and gravitational degrees of freedom, everything will be tuned in such a way that the resulting theory has no conformal anomaly, *i.e.* $c = 0$. This can be understood from the path integral point of view: integrating over the metrics means that the resulting theory does not depend on the metric any more. This greatly simplifies the structure of these theories, and they essentially become topological theories, with only a few degrees of freedom of the original conformal field theory left. We will come back to this point in chapter 6. If the conformal field theory that couples to gravity has a central charge larger than one, gauging the gravitational degrees of freedom is not sufficient to produce a consistent theory. This is where W algebras come into play. If we do not only gauge the gravitational degrees of freedom, but the maximal symmetry algebra of the conformal field theory, it has a much better chance of becoming a simple consistent theory. In particular, if the conformal field theory is a rational conformal field theory, it will become a simple topological field theory. Thus, W algebras allow one to obtain string theories starting with conformal field theories with central charge larger than one*. Some of these string theories may have an interesting target space interpretation, like the black holes obtained from $c = 1$ matter coupled to gravity [325]. For example, $c = 2$ matter coupled to W_3 gravity seems to describe a string propagating in a black hole like geometry in a four-dimensional target space with signature $(2, 2)$.

For more on W gravity, see the review papers [286, 268, 183].

4.2.2. The Classical Covariant Action for W Gravity

To construct the covariant action we need to find the local counterterm $\Delta\Gamma[\mu_\alpha, \bar{\mu}_\alpha, \ldots]$ that should be added to $\Gamma[\mu_\alpha] + \Gamma[\bar{\mu}_\alpha]$ in order to make the action $W \times \bar{W}$ invariant. Using a Fourier transformation as in section 3.2.2, we find that $\Delta\Gamma[\mu_\alpha, \bar{\mu}_\alpha, \ldots]$ is related to a local counterterm $\Delta\Gamma[A_\alpha, \bar{A}_\alpha, \ldots]$ with the same properties by

$$e^{-\Delta\Gamma[\mu_\alpha,\bar{\mu}_\alpha,\ldots]} = \int \prod_\alpha \mathcal{D}A_\alpha \bar{A}_\alpha e^{-\Delta\Gamma[A_\alpha,\bar{A}_\alpha,\ldots]+\frac{1}{\pi}\int d^2z\,(\mu_\alpha A_\alpha + \bar{\mu}_\alpha \bar{A}_\alpha)}. \tag{4.2.20}$$

The lowest-order contribution to $\Gamma[\bar{A}_\alpha]$ can be found in the same way as in section 3.2.2. It is

$$\Gamma^{(0)}[\bar{A}_\alpha] = kS^+_{wznw}(g_2)|_{kg_2^{-1}\bar{\partial}g_2 = \bar{\xi}\bar{\Lambda}^+ + \bar{W}}, \tag{4.2.21}$$

where $\bar{\xi}$ is a new parameter not necessarily equal to ξ, $\bar{\Lambda}^+ \equiv (\Lambda^+)^\dagger$ and $\bar{W}$ contains the anti-holomorphic fields $\bar{A}_\alpha$. For convenience we write (3.2.13) in the form

$$\Gamma^{(0)}[A_\alpha] = kS^-_{wznw}(g_1)|_{kg_1^{-1}\partial g_1 = \xi\Lambda^+ + W}. \tag{4.2.22}$$

*For W algebras obtained from an sl_2 embedding, the $c = 1$ barrier is shifted to a barrier at $c = d_0 - \frac{1}{2}\dim(\mathbf{g}_{\frac{1}{2}}) - 24(|t_0||\rho| - \mathrm{Tr}(t_0\rho))$, see (2.2.50).

Now recall (section 2.2.7) that a W transformation acts on g_1 as a field dependent gauge transformation $\delta g_1 = g_1 X(\epsilon)$, where X is of the type (2.2.26). In a similar way $\bar{W}$ transformations act on g_2 via $\delta g_2 = g_2 \bar{X}(\bar{\epsilon})$. Such transformations can be compensated for by introducing an auxiliary group-valued field G with transformation rule $\delta G = -X(\epsilon)G + G\bar{X}(\bar{\epsilon})$, since then the combination $g_1 G g_2^{-1}$ is $W \times \bar{W}$ invariant. This leads to the following proposal

$$\Gamma^{(0)}[\bar{A}_\alpha] + \Gamma^{(0)}[A_\alpha] + \Delta\Gamma^{(0)}[A_\alpha, \bar{A}_\alpha, \ldots] = kS^-_{wznw}(g_1 G g_2^{-1}) \qquad (4.2.23)$$

which yields, using the Polyakov-Wiegmann identities (2.1.51), for $\Delta\Gamma$

$$\begin{aligned}\Delta\Gamma^{(0)}[A_\alpha, \bar{A}_\alpha, \ldots] &= kS^-_{wznw}(G) - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}((\xi\Lambda^+ + W)\bar{\partial}GG^{-1}) \\ &\quad + \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(G^{-1}\partial G(\bar{\xi}\bar{\Lambda}^+ + \bar{W})) \\ &\quad + \frac{1}{k\pi}\int d^2z\, \mathrm{Tr}((\xi\Lambda^+ + W)G(\bar{\xi}\bar{\Lambda}^+ + \bar{W})G^{-1}) \qquad (4.2.24)\end{aligned}$$

and which becomes local for a suitable parametrization of G. Thus this action satisfies all the criteria for a covariant action. It is quite similar to the action for a gauged WZNW model (4.1.18), with a $W \times \bar{W}$ algebra gauged instead of a Lie subalgebra. There is no simple reality condition for the pair of connections $(A_z, A_{\bar{z}}) = (\xi\Lambda^+ + W, \bar{\xi}\bar{\Lambda}^+ + \bar{W})$ unless $\bar{\xi} = -\xi$. Precisely in that case $A_{\bar{z}} = -A_z^\dagger$, the same condition as for unitary groups.

Next, we want to compute

$$e^{-\Delta\Gamma^{(0)}[\mu_\alpha, \bar{\mu}_\alpha, G]} = \int_{\text{saddlepoint}} \prod_\alpha \mathcal{D}A_\alpha \bar{A}_\alpha e^{-\Delta\Gamma^{(0)}[A_\alpha, \bar{A}_\alpha, \ldots] + \frac{1}{\pi}\int d^2z\, (\mu_\alpha A_\alpha + \bar{\mu}_\alpha \bar{A}_\alpha)}, \qquad (4.2.25)$$

by solving the equations of motion for W and $\bar{W}$. It is convenient to introduce $F(\mu_\alpha)$ and $\bar{F}(\bar{\mu}_\alpha)$ defined by

$$\mathrm{Tr}(F(\mu_\alpha)W) = \sum_\alpha \mu_\alpha A_\alpha, \qquad \mathrm{Tr}(\bar{F}(\bar{\mu}_\alpha)\bar{W}) = \sum_\alpha \bar{\mu}_\alpha \bar{A}_\alpha. \qquad (4.2.26)$$

Another ingredient that goes into the computation is a certain decomposition of the linear operator $\mathrm{Ad}_G : \mathbf{g} \to \mathbf{g}$ defined by $\mathrm{Ad}_G(X) = G^{-1}XG$ as a product of three other operators, $\mathrm{Ad}_G = L^G_+ L^G_0 L^G_-$. The Lie algebra $\mathbf{g}$ can be written as the direct sum $V_1 \oplus V_2$, where $V_2 = \ker \mathrm{ad}(\Lambda^-)$, and V_1 its orthocomplement. A second decomposition $\mathbf{g} = W_1 \oplus W_2$ is defined by taking $W_1 = \ker \mathrm{ad}(\Lambda^-)^\dagger$ and W_2 its orthocomplement. With these definitions, the three operators L_+^G, L_0^G, L_-^G are uniquely defined by requiring that with respect to $\mathbf{g} = V_1 \oplus V_2$ they have the form

$$L_+^G \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad L_0^G L_-^G \sim \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \tag{4.2.27}$$

and with respect to $\mathbf{g} = W_1 \oplus W_2$ the form

$$L_-^G \sim \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \qquad L_+^G L_0^G \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}. \tag{4.2.28}$$

With these bits of notation the saddle point equations for W and $\bar{W}$ can be solved to give

$$\begin{aligned} W &= (L_-^G)^{-1}(L_0^G)^{-1}(\Pi_{V_2}(-kG^{-1}\partial G - \xi G^{-1}\Lambda^+ G + k\bar{F})), \\ \bar{W} &= L_+^G L_0^G(\Pi_{W_1}(k\bar{\partial}GG^{-1} - \bar{\xi}G\bar{\Lambda}^+ G^{-1} + kF)), \end{aligned} \tag{4.2.29}$$

and

$$\begin{aligned} \Delta\Gamma^{(0)}[\mu_\alpha, \bar{\mu}_\alpha, G] &= kS_{wznw}^-(G) - \frac{1}{\pi}\int d^2z \,\mathrm{Tr}(\xi\Lambda^+\bar{\partial}GG^{-1}) \\ &\quad + \frac{1}{\pi}\int d^2z \,\mathrm{Tr}(\bar{\xi}G^{-1}\partial G\bar{\Lambda}^+) \\ &\quad + \frac{1}{k\pi}\int d^2z \,\mathrm{Tr}(\xi\bar{\xi}\Lambda^+ G\bar{\Lambda}^+ G^{-1}) \\ &\quad - \frac{1}{k\pi}\int d^2z \,\mathrm{Tr}(\Pi_{V_2}(-kG^{-1}\partial G - \xi G^{-1}\Lambda^+ G + k\bar{F}) \\ &\qquad\qquad L_0^G(\Pi_{W_1}(k\bar{\partial}GG^{-1} - \bar{\xi}G\bar{\Lambda}^+ G^{-1} + kF))). \end{aligned} \tag{4.2.30}$$

This is the final result for the covariant action. The transformation rules for the fields under W and $\bar{W}$ transformations can easily be obtained from those for the fields in the Fourier transformed picture.

As an example, we work out the action for gravity. Decomposing G as

$$G = \begin{pmatrix} 1 & 0 \\ \frac{k}{\xi}\omega & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \begin{pmatrix} 1 & -\frac{k}{\xi}\bar{\omega} \\ 0 & 1 \end{pmatrix} \tag{4.2.31}$$

yields

$$\Delta\Gamma = \frac{k}{\pi}\int d^2z \,\partial\varphi\bar{\partial}\varphi - \frac{2k}{\pi}\int d^2\,(\omega + \partial\varphi)(\bar{\omega} + \bar{\partial}\varphi)$$

$$+\frac{1}{k\pi}\int d^2z\,\xi\bar{\xi}(1-\mu\bar{\mu})e^{-2\varphi}$$
$$+\frac{k}{\pi}\int d^2z\,\mu(\partial\omega-\omega^2)+\frac{k}{\pi}\int d^2z\,\bar{\mu}(\bar{\partial}\bar{\omega}-\bar{\omega}^2). \tag{4.2.32}$$

The first and the third line of this expression are precisely like the expressions for a free scalar field with a gauged holomorphic and anti-holomorphic Virasoro algebra that we considered in the beginning of this chapter. The form of the stress-energy tensor, $\partial\omega-\omega^2$, shows that the free scalar field has a background charge, and that the Virasoro algebra has a center, in agreement with the discussion in the previous section. If we integrate out the auxiliary fields $\omega, \bar{\omega}$, the first and third line become exactly $S_L(\varphi,\hat{g})+K[\mu,\bar{\mu}]$ as in (4.2.18). The second line of (4.2.32) is a cosmological constant term

$$\frac{\xi\bar{\xi}}{k\pi}\int d^2z\sqrt{g}. \tag{4.2.33}$$

The sign of the cosmological constant term depends on the way in which we choose the sl_2 embeddings. Only the sign is important, the magnitude of the cosmological constant can be changed by shifting φ with a constant. The cosmological constant term forms, together with the action for the free scalar field, a Toda theory, and the covariant action is essentially a gauged Toda theory. The vertex operator is the one which reduces the chiral algebra generated by $\partial\varphi$ to the Virasoro algebra. It is possible to get rid of the vertex operator by taking $\xi=\bar{\xi}\to 0$ in the action (4.2.32) and in the transformation rules. Then the covariant action reduces exactly to $R\Box^{-1}R$.

In conclusion, the group theoretically motivated construction for the covariant action is, for ordinary gravity, identical to the covariant action calculated using a covariant regularization procedure.

4.2.3. The General Structure of the Covariant Action

In general the classical covariant action is a gauged generalized Toda theory [274, 20]. By varying the sl_2 embeddings, one can tune the coefficients in front of the vertex operators that are part of the Toda action. It is possible to scale them away altogether, resulting in the 'minimal' covariant action.

The original derivation of the covariant action for W_N gravity [50, 49] used the connection between the induced action $\Gamma[\mu_\alpha]$ and Chern-Simons theory, that was discovered previously in the context of gravity [309]. In this setup, extra fields are introduced in addition to μ_α, and a local term is added to $\Gamma[\mu_\alpha]$ in such a way that the resulting action is a wave function in Chern-Simons theory. The inner product of these Chern-Simons theory wave functions is then expressed as a path integral of an action that is identical to the covariant action for W gravity.

The construction of the covariant action given here is based on the geometrical interpretation of W transformations as field dependent gauge transformations, and this is reflected in the fact that the auxiliary fields we needed to introduce were group valued. Alternative approaches, not using the group theoretical structure, have been attempted for W_3 gravity in [78, 180]. In [78] a part of the covariant action was constructed by hand working with the formalism of [282], and in [180] it was shown that to lowest order in the gauge fields the covariant action has an $R\frac{1}{\Box}R$ like structure for some generalized curvature. Whether there exist another geometrical formulation of W_3 gravity, besides the group-theoretical one, in which a simple nice formula for the covariant action can be written down is unknown. The most obvious generalization of the metric to W gravity is to introduce symmetric tensor fields $g_{\mu_1\ldots\mu_s}$ for each pair of generators $W, \bar{W}$ of conformal weight $(s, 0)$ and $(0, s)$ respectively. The components of such a tensor correspond to fields of spins† $s, s-2, \ldots, -s$. The spectrum of spins of the fields $\mu_\alpha, \bar{\mu}_\alpha, G$ of the covariant action consists of a sequence $s, s-1, \ldots, -s$ for each generator of spin s of the W algebra. This indicates that the covariant action might allow for a formulation where the fields have been organized in symmetric tensor fields, and for each spin s generator of the W algebra there are two tensor fields, one with s indices and one with $s-1$ indices. It would be interesting to investigate this possibility in some more detail.

It is often claimed that gravity has a hidden $SL(2, \mathbb{R})$ symmetry. It is at this stage not (yet) clear where the reality condition should come from. The equation $kg^{-1}\partial g = \xi\Lambda^+ + W$ has only under special circumstances a solution with $g \in SL(2, \mathbb{R})$, and G in the covariant action is definitely not real, for $\bar{\xi} = -\xi$ it is Hermitian, which is quite something different. The covariant action is real. The Liouville part is bounded from below for $c < 0$ (recall $c \sim -6k$) and positive cosmological constant. To check whether the full spectrum is bounded from below is more difficult. This requires a detailed analysis of the quantization of the Liouville theory [288]. For general W gravity, analogous statements hold, with Liouville theory replaced by the appropriate Toda theory. We will come back to the connection with $sl(N, \mathbb{R})$ in the next chapter.

4.3. Quantum Aspects of the Covariant Action

4.3.1. All-Order Results for the Covariant Action

So far we only succeeded in computing the lowest-order part $\Delta\Gamma^{(0)}$ of the covariant action. Thanks to our analysis in section 3.3, we know what the all-order result for $\Gamma[A_\alpha]$ looks like (3.3.51). It is equal to $k_c S^-_{wznw}(g_1)$, where g_1 is expressed in terms of

†The spin of a field with conformal weights $(h, \bar{h})$ is $h - \bar{h}$.

the A_α involving quite complicated factors. In the same way, the all-order result for $\Gamma[\bar{A}_\alpha]$ is $k_c S^+_{wznw}(g_2)$ where g_2 is expressed in terms of $\bar{A}_\alpha$. Thus, the all-order covariant action that is invariant under the renormalized $W \times \bar{W}$ transformations is given by

$$\Gamma_{cov}[A_\alpha, \bar{A}_\alpha, G] = k_c S^-_{wznw}(g_1 G g_2^{-1}). \tag{4.3.1}$$

From this one can in principle compute $\Delta\Gamma$ in terms of μ and $\bar{\mu}$. It is not sufficient to simply solve the $A_\alpha, \bar{A}_\alpha$ equations of motion and to substitute these back into the action. There are extra contributions that are a function of $\det(M)$ if the quadratic terms in $\Delta\Gamma$ are of the form $A_\alpha M_{\alpha\beta} \bar{A}_\beta$. For a discussion of such terms, see [325, 70]. Here, we just want to show that

$$\Delta\Gamma[G] \equiv \Delta\Gamma[\mu_\alpha, \bar{\mu}_\alpha, G]|_{\mu_\alpha=0, \bar{\mu}_\alpha=0} \tag{4.3.2}$$

is identical to a gauged WZNW model that describes a Toda theory. Actually, the equivalence can be established for

$$e^{-\Delta\Gamma[G_0]} \equiv \int \mathcal{D}G_+ \mathcal{D}G_- e^{-\Delta\Gamma[G]} \tag{4.3.3}$$

rather than for G. Here, the decomposition $G = G_- G_0 G_+$ is a Gauss-like decomposition that corresponds on the Lie algebra level to the decomposition of $\mathbf{g}$ discussed in section 2.2.7. We introduce $B = g_1^{-1}\partial g_1$ and $\bar{B} = g_2^{-1}\bar{\partial} g_2$, and call the renormalized constraints, that are imposed on g_1 and g_2 in order to obtain the all-order result for $\Gamma[A_\alpha]$, respectively $k_c B = \xi_r \Lambda^+ + W_r$ and $k_c \bar{B} = \bar{\xi}_r \bar{\Lambda}^+ + \bar{W}_r$. Then

$$\begin{aligned}
e^{-\Delta\Gamma[G_0]} &= \int \prod_\alpha \mathcal{D}A_\alpha \mathcal{D}\bar{A}_\alpha \mathcal{D}G_- \mathcal{D}G_+ \\
&\quad \times \exp\left(-k_c S^-_{wznw}(g_1 G g_2^{-1}) + k_c S^-_{wznw}(g_1) + k_c S^+_{wznw}(g_2)\right) \\
&= \int \prod_\alpha \mathcal{D}A_\alpha \mathcal{D}\bar{A}_\alpha \mathcal{D}G_- \mathcal{D}G_+ \mathcal{D}B \mathcal{D}\bar{B} \\
&\quad \times \delta(k_c B - \xi_r \Lambda^+ + W_r)\delta(k_c \bar{B} - \bar{\xi}_r \bar{\Lambda}^+ + \bar{W}_r) \\
&\quad \times \exp\left(-k_c S^-_{wznw}(g_1 G g_2^{-1}) + k_c S^-_{wznw}(g_1) + k_c S^+_{wznw}(g_2)\right) \\
&= \int \mathcal{D}G_- \mathcal{D}G_+ \mathcal{D}B \mathcal{D}\bar{B} \delta(\Pi_{V_1}(k_c B - \xi_r \Lambda^+))\delta(\Pi_{W_2}(k_c \bar{B} - \bar{\xi}_r \bar{\Lambda}^+)) \\
&\quad \times \exp\left(-k_c S^-_{wznw}(g_1 G g_2^{-1}) + k_c S^-_{wznw}(g_1) + k_c S^+_{wznw}(g_2)\right),
\end{aligned} \tag{4.3.4}$$

where V_1 and W_2 were defined above (4.2.27). We can now replace (cf. the analysis in section 3.3; we are following a reverse route compared with that one) half of the delta functions by integrals over Lagrange multipliers A and $\bar{A}$ that are $\mathbf{g}_+$ and $\mathbf{g}_-$ valued gauge fields. The remaining delta functions constrain the components of $B, \bar{B}$ in $V_1^{0-} \equiv V_1 \cap (\mathbf{g}_0 \oplus \mathbf{g}_-)$ and $W_2^{0+} \equiv W_2 \cap (\mathbf{g}_0 \oplus \mathbf{g}_+)$. We get

$$\begin{aligned} e^{-\Delta\Gamma[G_0]} &= \int \mathcal{D}G_-\mathcal{D}G_+\mathcal{D}B\mathcal{D}\bar{B}\mathcal{D}A\mathcal{D}\bar{A}\delta(\Pi_{V_1^{0-}}(k_cB))\delta(\Pi_{W_2^{0+}}(k_c\bar{B})) \\ &\times \exp\Big(-k_cS^-_{wznw}(g_1Gg_2^{-1}) + k_cS^-_{wznw}(g_1) + k_cS^+_{wznw}(g_2) \\ &+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(\bar{A}(k_cB - \xi_r\Lambda^+) + A(k_c\bar{B} - \bar{\xi}_r\bar{\Lambda}^+))\Big). \end{aligned} \tag{4.3.5}$$

In the same way as we gauge fixed the gauge symmetry in section 3.3.2, we can now restore a $G_- \times G_+$ symmetry and eliminate the delta functions. Again, the Faddeev-Popov contribution is just a numerical factor that can be ignored. Although the measures for B and $\bar{B}$ are not gauge invariant, one can replace them by a gauge invariant measure. This renormalizes the coefficients in front of $S^-_{wznw}(g_1)$ and $S^+_{wznw}(g_2)$. One can compensate for this change by rescaling A and $\bar{A}$ so that the exponential in (4.3.5) contains a constrained WZNW model at a shifted value of the level, which is manifestly gauge invariant. The remaining part of the action, $S^-_{wznw}(g_1G_-G_0G_+g_2^{-1})$ is also manifestly gauge invariant because we integrate over G_- and G_+. The result is

$$\begin{aligned} e^{-\Delta\Gamma[G_0]} &= \int \frac{\mathcal{D}G_-\mathcal{D}G_+\mathcal{D}B\mathcal{D}\bar{B}\mathcal{D}A\mathcal{D}\bar{A}}{\text{gauge volume}} \\ &\times \exp\Big(-k_cS^-_{wznw}(g_1Gg_2^{-1}) + k_cS^-_{wznw}(g_1) + k_cS^+_{wznw}(g_2) \\ &+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(\bar{A}(k_cB - \xi_r\Lambda^+) + A(k_c\bar{B} - \bar{\xi}_r\bar{\Lambda}^+))\Big). \end{aligned} \tag{4.3.6}$$

The $B, \bar{B}$ integration can be performed, because the action is simply quadratic in B and $\bar{B}$. The quadratic term is proportional to $BG\bar{B}G^{-1}$. In principal integration over $B, \bar{B}$ could give rise to extra terms proportional to the determinant of $\mathrm{ad}(G)$. However, since we integrate over all $\mathbf{g}$-valued gauge fields B, it is easy to see that this determinant is equal to one‡. In view of this observation we can simply replace $B, \bar{B}$ by the solutions of their respective equations of motion. If we in addition replace $G \to G^{-1}$ we get

$$\begin{aligned} e^{-\Delta\Gamma[G_0^{-1}]} &= \int \frac{\mathcal{D}G_-\mathcal{D}G_+\mathcal{D}A\mathcal{D}\bar{A}}{\text{gauge volume}} \\ &\times \exp\Bigg(k_cS^-_{wznw}(G) + \frac{k_c}{\pi}\int d^2z\,\mathrm{Tr}(AG\bar{A}G^{-1}) \\ &+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(\bar{A}(k_cG^{-1}\partial G - \xi_r\Lambda^+)) \\ &+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(A(-k_c\bar{\partial}GG^{-1} - \xi_r\Lambda^+))\Bigg). \end{aligned} \tag{4.3.7}$$

‡This need not be true if one integrates over connections $B, \bar{B}$ with values in a subalgebra.

This is the standard form of a gauged WZNW model (cf. (4.1.18) and section 3.3). It is the field theoretic counterpart of the construction of W algebras from the constrained current algebra point of view [21, 19]. If one extracts Liouville theory (4.2.19) from the covariant action, the measure for the Liouville field φ is not yet the one for a free scalar field. One can replace the the measure by the one of a free field; this renormalizes the background charge of Liouville theory. This renormalization has been computed in a somewhat heuristic way (the so-called DDK analysis) [82, 97], and later more rigorously [237, 175]. Alternatively, one can start with the gauged WZNW theory given here and derive correctly normalized Liouville theory from it [273, 95, 185].

The DDK analysis has been repeated for W_N gravity [232], starting from the ansatz that the effective action for W_N gravity in the conformal gauge is given by a Toda theory. The same results have been reproduced [189] using the sigma model interpretation of gauged WZNW models [302, 26]. The analysis in this section proves that the Toda actions obtained from gauged WZNW models indeed describe the all-order effective action for W gravity in the conformal gauge, and the exact values of the background charges of the Toda theories can be computed in this way.

The gauged WZNW model (4.3.7) is based on a WZNW model of level $-k_c$. Furthermore, if we gauge fix the $W, \bar{W}$ symmetries of the covariant action in order to put $\mu_\alpha = \bar{\mu}_\alpha = 0$, we have to introduce Faddeev-Popov ghost systems corresponding to each generator of the W algebra. The central charge of the gauged WZNW model (4.3.7) is given by (2.2.50), with k replaced by $-k_c$. The central charge of the ghosts is

$$\begin{aligned} c_{gh} &= -2\sum_i (6h_i(h_i-1)+1) \\ &= -2\sum_{h_i\in\mathbb{Z}}(6h_i(h_i-1)+1) - 2\sum_{h_i\in\mathbb{Z}+\frac{1}{2}}(6h_i(h_i-1)+1) \\ &= -12\sum_{h_i\in\mathbb{Z}} h_i(h_i-1) - 12\sum_{h_i\in\mathbb{Z}+\frac{1}{2}}(h_i-\frac{1}{2})^2 - 2d_0 + \dim(\mathbf{g}_{\frac{1}{2}}) \\ &= -24\sum_{h_i\in\mathbb{Z}}\sum_{k=1}^{h_i-1} k - 24\sum_{h_i\in\mathbb{Z}+\frac{1}{2}}\sum_{k=0}^{h_i-\frac{3}{2}}(k+\frac{1}{2}) - 2d_0 + \dim(\mathbf{g}_{\frac{1}{2}}) \\ &= -24\sum_{\alpha\in\Delta^+}\mathrm{Tr}(t_0 h_\alpha) - 2d_0 + \dim(\mathbf{g}_{\frac{1}{2}}) \\ &= -48\mathrm{Tr}(t_0\rho) - 2d_0 + \dim(\mathbf{g}_{\frac{1}{2}}) \end{aligned} \qquad (4.3.8)$$

where the sum is over the generators of the W algebra and h_i is the conformal weight of each of the generators. Finally, the central charge of the original matter system was related to k_c via (3.3.50). Thus the total central charge of the gauge fixed system is

$$c_{tot} = c_{mat} + c_{toda} + c_{ghost}$$

$$
\begin{aligned}
&= d_0 - \frac{1}{2}\dim(\mathbf{g}_{\frac{1}{2}}) - 12\mathrm{Tr}\left(\frac{\rho}{\sqrt{k_c - h}} - t_0\sqrt{k_c - h}\right)^2 \\
&\quad + d_0 - \frac{1}{2}\dim(\mathbf{g}_{\frac{1}{2}}) - 12\mathrm{Tr}\left(\frac{\rho}{\sqrt{-k_c + h}} - t_0\sqrt{-k_c + h}\right)^2 \\
&\quad - 48\mathrm{Tr}(t_0\rho) - 2d_0 + \dim(\mathbf{g}_{\frac{1}{2}}) \\
&= 0.
\end{aligned}
\tag{4.3.9}
$$

This is in perfect agreement with our discussion at the end of section 4.2.1. Starting with any matter system, coupling to W gravity will produce an exactly conformally invariant theory, that is a candidate to be a consistent string theory.

4.3.2. KPZ Approach

We saw that the gauge fixed covariant action has total central charge zero. A different way to obtain this result is to start with the classical covariant action $\Gamma^{(0)}_{cov}[\mu_\alpha, \bar{\mu}_\alpha, G] = \Gamma^{(0)}[\mu_\alpha] + \Delta\Gamma^{(0)}[\mu_\alpha, \bar{\mu}_\alpha, G] + \Gamma^{(0)}[\bar{\mu}_\alpha]$. This action is invariant under $W \times \bar{W}$ transformations, whose transformation rules can easily be obtained from the framework presented in this chapter. One can then perform a BRST quantization of this action. Decomposing $G = G_- G_0 G_+$, and using the $W \times \bar{W}$ symmetries to impose the gauge fixing conditions $\bar{\mu}_\alpha = 0$ and $G_0 = 0$ leads to the generalization of the KPZ analysis of gravity [207]. The resulting gauge fixed action has a BRST symmetry. This BRST operator is, not surprisingly, closely related to the BRST operator that arises in quantum Hamiltonian reduction, and makes the $\mathbf{g}$ structure of W gravity very clear. Demanding that this BRST operator satisfies $Q^2 = 0$ on the quantum level then gives the same kind of renormalizations for the level k as we have obtained in the last two chapters. The physical states can now be found by computing the cohomology of Q.

An alternative gauge fixing of the $W \times \bar{W}$ symmetry is to fix $\mu_\alpha = \bar{\mu}_\alpha = 0$, which leads to covariant W gravity in the conformal gauge. Again, the (classical) BRST operator follows directly from the transformation rules for the fields in the covariant action, and demanding closure $Q^2 = 0$ on the quantum level gives the quantum BRST operator of W gravity in the conformal gauge. For W_3 this procedure is worked out in detail in [48], and gives the same BRST operator as was found previously for non-critical W_3 gravity in [40, 31].

5

The Moduli Space of W Gravity

5.1. W Algebras on Riemann Surfaces

So far we restricted our attention to W algebras and actions that were defined on the complex plane. In particular we assumed that it is possible to choose the conformal gauge $\mu_\alpha = \bar{\mu}_\alpha = 0$. This is certainly true on the complex plane, but on higher-genus Riemann surfaces there may be global obstructions to choosing this gauge. In general, the space of W gauge fields $\mu_\alpha, \bar{\mu}_\alpha$ modulo W transformations will not be a point, but a finite dimensional moduli space. Such a moduli space can for instance parametrize winding numbers of the gauge fields along non-contractable loops in the Riemann surface. If we want to answer questions like 'what is the moduli-space of W algebras', we have to discuss W algebras on higher-genus Riemann surfaces, as on the complex plane one can choose globally well-defined co-ordinates, and it is sufficient to express everything in terms of these co-ordinates. One need not bother about the transformation properties of the different objects one encounters under a change of co-ordinates, and generally there are no moduli. In this section we study W algebras on general Riemann surfaces. In previous chapters we have seen how one can use the formulation of W algebras in terms of zero-curvature equations and field dependent gauge-transformations to obtain the chiral and covariant actions for W gravity, and we will use the same formulation to go to arbitrary Riemann surfaces.

To illustrate the problem, consider the constraints (2.2.10). Naively all the components of a current $\mathcal{J}$ are primary fields of dimension $(1,0)$, because $\mathcal{J} = kg^{-1}\partial g$ and g is simply a map $\Sigma \to G$. If we view $A_z \equiv \frac{1}{k}\mathcal{J}$ as the $(1,0)$ part of a connection, then the fact that g is simply a map $\Sigma \to G$ is translated in the fact that A_z is part of a connection on a trivial bundle. If V is the fundamental representation of SL_2, then A_z is also a connection for this associated (trivial) vector bundle. Now consider the equation $(\partial + A_z)\psi = 0$, with ψ a two-component vector in the fundamental representation of sl_2. The second component of ψ can be expressed in terms of the first one, $\psi_2 = -\frac{k}{\xi}\partial\psi_1$. Under the gauge transformation (2.2.12) ψ transforms as

$$\begin{pmatrix} \delta\psi_1 \\ -\frac{k}{\xi}\partial\delta\psi_1 \end{pmatrix} = -\begin{pmatrix} \frac{k}{2\xi}\partial\epsilon & \epsilon \\ -\frac{k^2}{2\xi^2}\partial^2\epsilon + \frac{1}{\xi}J^-\epsilon & -\frac{k}{2\xi}\partial\epsilon \end{pmatrix}\begin{pmatrix} \psi_1 \\ -\frac{k}{\xi}\partial\psi_1 \end{pmatrix}, \tag{5.1.1}$$

from which we find

$$\delta\psi_1 = -\frac{1}{2}\partial(\frac{k}{\xi}\epsilon)\psi_1 + (\frac{k}{\xi}\epsilon)\partial\psi_1. \tag{5.1.2}$$

Since we want to identify these gauge transformations according to the 'soldering' philosophy with co-ordinate transformations, this equation tells us that ψ_1 is an object with weight $(-\frac{1}{2},0)$. Or, if we denote by K the holomorphic cotangent line bundle $T^*_{\mathbb{C}}\Sigma$, ψ_1 must transform as a section of $K^{-\frac{1}{2}}$. On an arbitrary Riemann surface, this is generically a non-trivial line bundle, and we conclude that V is no longer a trivial vector bundle, and $\partial + A_z$ is no longer a connection on a trivial bundle. In the case at hand, $V \simeq K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$. This is the geometrical counterpart of the soldering procedure. The constraints force us to twist the trivial bundle V into a nontrivial one. Correspondingly, the components of A_z transform no longer as objects of dimension $(1,0)$, but as

$$\begin{array}{ccccc} V & : & \begin{pmatrix} K^0 \\ K^0 \end{pmatrix} & \rightarrow & \begin{pmatrix} K^{-\frac{1}{2}} \\ K^{\frac{1}{2}} \end{pmatrix} \\ A_z & : & \begin{pmatrix} K^1 & K^1 \\ K^1 & K^1 \end{pmatrix} & \rightarrow & \begin{pmatrix} K^1 & K^0 \\ K^2 & K^1 \end{pmatrix}. \end{array} \tag{5.1.3}$$

On the algebraic level, the same twisting of the spin assignments of the components of the current is realized by adding the improvement term $-\mathrm{Tr}(t_0\partial\mathcal{J})$ to the standard Sugawara energy-momentum tensor (2.1.55). This improvement is necessary to let the top right component of A_z transform as a scalar, so that it can be put equal to a constant globally on a Riemann surface.

However, apart from the necessity to change the global structure of the bundles involved, there is a second problem, related to the fact that A_z does not transform as a one-form, but as a connection. This means that as we go from one co-ordinate patch to another, A_z does not transform into $g^{-1}A_z g$ but picks up an extra term $g^{-1}\partial g$. This extra term makes it impossible, even if we twist the bundle for which A_z is a connection, to put a component of A_z globally equal to constant. The resolution of this problem is to introduce a fixed background connection B_z^0, and to impose the constraints on $A_z - B_z^0$ rather than A_z. The different choices of B_z^0 correspond to different regularization prescriptions, which will become clear when we discuss WZNW action for twisted bundles. With these observations in mind we now describe the structure of W algebras related to sl_2 embeddings on arbitrary Riemann surfaces.

5.1.1. General Case

Consider a Lie algebra $\mathbf{g}$ with sl_2 embedding t_-, t_0, t_+, and an n-dimensional representation $R : G \rightarrow V$ of G, the Lie group corresponding to $\mathbf{g}$. Using, if necessary, an inner automorphism of $\mathbf{g}$, one can put t_0 in the Cartan subalgebra of $\mathbf{g}$, and $R(t_0)$ is diagonal, say $\mathrm{diag}(d_1, \ldots, d_n)$. The twisted vector bundle relevant for this sl_2 embedding is

$$V = K^{-d_1} \oplus \cdots \oplus K^{-d_n}. \tag{5.1.4}$$

The vector bundle V determines a non-trivial principal fiber bundle P over Σ with structure group G. This bundle is constructed from an auxiliary $n \times n$ matrix vector bundle W, whose columns are isomorphic to V. To go from $GL(V)$ to $R(G)$ we need to impose extra restrictions f_i on $M \in GL(V)$. The same restrictions, when imposed on $M \in W$, determine locally a copy $R(G) \subset W$. To find out whether these patch together properly, consider the group element $g = \exp(rt_0)$. It acts on V by multiplying the j^{th} component of V by e^{rd_j}. Since $f_i(M) = 0 \Leftrightarrow f_i(gM) = 0$, the conditions that characterize $R(G)$ as subset of W are co-ordinate invariant, and $R(G) \subset W$ patch together to form a non-trivial principal fiber bundle P over Σ with structure group G. However, due to the special structure of V and P the structure group can be further reduced to $GL(1,\mathbb{C})$.

A connection on P is locally a one-form with values in $\mathrm{ad}(P)$, the adjoint bundle of P. It is isomorphic to the restriction to $R(\mathbf{g})$ of $\mathrm{End}(V)$, the bundle of endomorphisms of V. The structure of both P and $\mathrm{ad}(P)$ is independent of the representation R one chose in the beginning. Locally, a section of $\mathrm{ad}(P)$ can be written as $S^a T_a$, and it is easily verified that S^a transforms as a section of $K^{1-\delta_a}$, where $[t_0, T_a] = \delta_a T_a$. This is the same twisting as is accomplished on the algebraic level by adding the improvement term $-\mathrm{Tr}(t_0 \partial \mathcal{J})$ to the stress-energy tensor.

The description of W algebras via field dependent gauge transformations is now relatively straightforward. We introduce once and for all a fixed background connection B_z^0 on the vector bundle V. The connection B_z^0 acts on V via the representation R, but we will from here on no longer display representation R. The constraints on A_z are

$$A_z - B_z^0 = \frac{\xi}{k}\Lambda^+ + \frac{1}{k}W \tag{5.1.5}$$

and the W transformations are by definition those gauge transformations that preserve the constraints. To convert these to operator product expansions is more complicated than on the complex plane, because we need the exact Green function $G(z,w)$ for the operator $\bar{\partial}$, that was for the complex plane simply given by $1/(z-w)$, but is more difficult on non-trivial Riemann surfaces. Nevertheless, explicit expressions for the W transformations can be given. They are particularly simple if we work in isothermal co-ordinates $ds^2 = \rho dz d\bar{z}$. Then a globally well-defined background connection B_z^0 is

$$\nabla_z \equiv \partial + B_z^0 = \partial + \mathrm{diag}(d_1 \partial \log \rho, \ldots, d_n \partial \log \rho) \tag{5.1.6}$$

and with this choice the W transformations act via the following generalization of (2.2.27)

$$\begin{aligned}&\int dz\,\{\mathrm{Tr}(F(z)W(z)),W^a(w)\}_{\mathrm{dirac}}T_a\\&\quad=(k\partial+k\mathrm{ad}(B_z^0)+\xi\mathrm{ad}(\Lambda^+)+\mathrm{ad}(W))\frac{1}{1+\xi^{-1}L(k\partial+k\mathrm{ad}(B_z^0)+\mathrm{ad}(W))}F.\end{aligned}\tag{5.1.7}$$

Equation (2.2.30) is accordingly modified to

$$\begin{aligned}&\int dz\,\{\epsilon(z)T(z),W^a(w)\}_{\mathrm{dirac}}T_a\\&\quad=-\frac{k^2}{\xi}\Lambda^-(\partial-\partial\log\rho)\partial(\partial+\partial\log\rho)\epsilon+(1-\mathrm{ad}\Lambda^0)(W)(\partial+\partial\log\rho)\epsilon\\&\qquad+(\partial-(1-\mathrm{ad}\Lambda^0)\partial\log\rho)W\epsilon\\&\quad=-\frac{k^2}{\xi}\Lambda^-(\partial-\partial\log\rho)\partial(\partial+\partial\log\rho)\epsilon+(1-\mathrm{ad}\Lambda^0)(W)\partial\epsilon+\partial W\epsilon.\end{aligned}\tag{5.1.8}$$

These results can all easily be found by replacing ∂ by covariant derivatives. The metric $ds^2=\rho dzd\bar{z}$ induces a natural metric on K^j and the associated covariant derivative on sections of K^j is just $\partial-j\partial\log\rho$. A different but related construction of W algebras on a Riemann surface has been given in [130]. For a discussion of operator product expansions on arbitrary Riemann surfaces, see [107].

5.2. The Moduli Space Associated to a W Algebra

5.2.1. The Moduli Space Associated to a Chiral Algebra

In the previous chapter we tacitly assumed that it is possible to use the $W\times\bar{W}$ symmetries to gauge away the gauge fields μ_α and $\bar{\mu}_\alpha$ completely. This is true on the complex plane, but on an arbitrary Riemann surface there may be obstructions to doing this. The simplest example is gravity. The gauge fields $\mu,\bar{\mu}$, also known as Beltrami differentials, cannot be gauged away. The quotient

$$\frac{\text{Beltrami differentials}}{\mathrm{Vir}\times\bar{\mathrm{Vir}}}\tag{5.2.1}$$

is the Teichmüller space of the Riemann surface. The moduli space is the quotient of this space by the modular group, which consists of the diffeomorphisms of the Riemann

surface that cannot continuously be deformed into the identity map. The space of Beltrami differentials is canonically dual to the space of quadratic differentials via the pairing $\int d^2z\mu T$, and Teichmüller space admits an equivalent description as the space of quadratic differentials modulo Vir $\times$ $\bar{\text{V}}$ir. If we generalize this to an arbitrary chiral algebra (cf. [326]), this gives the following definition of $\mathcal{A}$-Teichmüller space:
$\mathcal{A}$-Teichmüller space is the quotient of the space of smooth fields $A_\alpha(z,\bar{z})$ modulo the space of $\mathcal{A}$-transformations. $\mathcal{A}$-moduli space is the quotient of $\mathcal{A}$ Teichmüller space by the action of the modular group.

As long as there is no geometrical interpretation of $\mathcal{A}$-transformations, there is no definition of global $\mathcal{A}$ transformations, apart from the modular group which can be seen as the group of global Virasoro transformations. Furthermore, since we started by gauging a set of local symmetries, this is the natural definition at this point. We now specialize to the case where the chiral algebra is a W algebra corresponding to an sl_2 embedding.

5.2.2. A Finite Dimensional Model of the Moduli Space for W Algebras

The space of smooth W fields is the same as the set of connections $M = \{\nabla_z + \Lambda^+ + W\}$. For simplicity we take $\xi = k = 1$ in the rest of this section. Each operator $D' \in M$ defines an anti-holomorphic structure on the bundle V (5.1.4). Such an anti-holomorphic structure is determined by defining what the local anti-holomorphic sections of the vector bundle are. In the anti-holomorphic structure corresponding to D' these are just the local sections s that satisfy $D's = 0$. The space of anti-holomorphic structures M must be divided by the set of W transformations. To do this, introduce the following equivalence relation on M: two operators $D'_1, D'_2 \in M$ are equivalent, $D'_1 \sim D'_2$, if there is a gauge transformation, $g \in \pi_0(\mathcal{G}_c)$ relating the two, $D'_1 = (D'_2)^g$. The moduli space we are looking for is the space $\mathcal{M}_W = M/\sim$. The transformations that relate two different D' are what one might call global W transformations. The infinitesimal transformations of this type are precisely the W transformations considered previously. Note that the equivalence relation $D'_1 \sim D'_2$ is not generated by the action of a group on M, as the precise form of the gauge transformation relating two different D' depends explicitly on the precise form of these D'. Thus, we cannot view $\mathcal{M}_W$ as the quotient of some space by a group action, and this makes the study of $\mathcal{M}_W$ somewhat more difficult. One of the things we would in particular like to compute is the dimension of the $\mathcal{M}_W$, or equivalently, of its tangent space. If one were to consider the full set of anti-holomorphic structures on V modulo gauge transformations, *i.e.* the space $\mathcal{M} = \{\nabla_z + A_z\}/\pi_0(\mathcal{G}_c)$, the tangent space $T_{D'}\mathcal{M}$ at $D' \in \mathcal{M}$ is given by the $(1,0)$-cohomology of the short complex

$$0 \xrightarrow{D'} \Omega^0(\Sigma; \text{ad}(P)) \xrightarrow{D'} \Omega^{1,0}(\Sigma; \text{ad}(P)) \xrightarrow{D'} 0. \qquad (5.2.2)$$

Here, $\Omega^{p,q}(\Sigma; \mathrm{ad}(P))$ denotes the space of (p,q)-forms with values in $\mathrm{ad}(P)$. To compute the tangent space $T_{D'}\mathcal{M}_W$ for $\mathcal{M}_W$, we should replace this complex by some kind of W complex containing the W transformations. There is an interesting connection between the two, which we will now explain. This connection relies heavily on the existence of the operator L that was defined as the inverse of $\mathrm{ad}(\Lambda^+)$ above equation (2.2.24). Because L is an operator of degree -1 with respect to the gradation defined by t_0, it provides us with an 'integration' operator

$$\Omega^{1,0}(\Sigma; \mathrm{ad}(P)) \xrightarrow{L} \Omega^0(\Sigma; \mathrm{ad}(P)). \tag{5.2.3}$$

As an analogy one might think of the operation of integrating over the n^{th} co-ordinate in $\mathbb{R}^n$, which maps p-forms on $\mathbb{R}^n$ to $(p-1)$-forms on $\mathbb{R}^{n-1}$. This latter operator can be used to show that the cohomology of $\mathbb{R}^n$ is the same as the cohomology of $\mathbb{R}^{n-1}$, by constructing a so-called homotopy-equivalence between the de Rham complexes for $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ [56]. Here we can perform a similar construction using the 'integration' operator L. Defining the two operators

$$f_0 = 1 - L \circ D', \quad f_1 = 1 - D' \circ L, \tag{5.2.4}$$

we can construct the following commutative diagram

$$\begin{array}{ccccccc} 0 & \xrightarrow{D'} & \Omega^0(\Sigma; \mathrm{ad}(P)) & \xrightarrow{D'} & \Omega^{1,0}(\Sigma; \mathrm{ad}(P)) & \xrightarrow{D'} & 0 \\ & & \Big\downarrow f_0 & & \Big\downarrow f_1 & & \\ 0 & \xrightarrow{D'} & f_0(\Omega^0(\Sigma; \mathrm{ad}(P))) & \xrightarrow{D'} & f_1(\Omega^{1,0}(\Sigma; \mathrm{ad}(P))) & \xrightarrow{D'} & 0 \end{array} \tag{5.2.5}$$

which gives actually a homotopy equivalence of complexes*, implying that the cohomology of both complexes in (5.2.5) is the same. The next step is to iterate this construction a number of times, until the complex does not change anymore. Let us denote the corresponding limit complex, if it exists, by

$$0 \xrightarrow{D'} f_0^\infty(\Omega^0(\Sigma; \mathrm{ad}(P))) \xrightarrow{D'} f_1^\infty(\Omega^{1,0}(\Sigma; \mathrm{ad}(P))) \xrightarrow{D'} 0. \tag{5.2.6}$$

Using the properties of L one can show that a sufficient condition for the limit complex to exist is that the operator $L \circ (D' - \mathrm{ad}(\Lambda^+))$ is nilpotent, and that in that case

$$\begin{aligned} f_0^\infty = (1 - L \circ D')^\infty &= \frac{1}{1 + L(D' - \mathrm{ad}(\Lambda^+))} \circ \Pi_{\ker\, \mathrm{ad}(\Lambda^+)}, \\ f_1^\infty = (1 - D' \circ L)^\infty &= \Pi_{\ker\, \mathrm{ad}(\Lambda^-)} \circ \frac{1}{1 + (D' - \mathrm{ad}(\Lambda^+))L}. \end{aligned} \tag{5.2.7}$$

*Indeed, denoting by f both maps f_0 and f_1, we have $1 - f = L \circ D' + D' \circ L$, so that L is precisely an homotopy operator as defined in [56].

Specializing to the case of W algebras, we take $D' = \nabla_z + \Lambda^+ + W$ and find, upon comparing the limit complex with (5.1.7), that the limit complex precisely contains the W transformations, and is the W complex we were looking for. To illustrate how this works in practice, we take the simplest example $G = Sl(2)$ as in section 2.2. The limit complex is reached by applying f_0 and f_1 two times. The operator L is given by

$$L : \begin{pmatrix} p^0 & p^+ \\ p^- & -p^0 \end{pmatrix} = \begin{pmatrix} -p^+/2 & 0 \\ p^0 & p^+/2 \end{pmatrix}, \qquad (5.2.8)$$

and if we represent an arbitrary element of $\Omega^0(\Sigma; \mathrm{ad}(P))$ by $\begin{pmatrix} \epsilon^0 & \epsilon^+ \\ \epsilon^- & -\epsilon^0 \end{pmatrix}$, and an element of $\Omega^{1,0}(\Sigma; \mathrm{ad}(P))$ by $\begin{pmatrix} a^0 & a^+ \\ a^- & -a^0 \end{pmatrix}$, we find the following diagram, where $\gamma = \partial \log \rho$:

$$\begin{array}{ccc}
\begin{pmatrix} \epsilon^0 & \epsilon^+ \\ \epsilon^- & -\epsilon^0 \end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix} a^0 & a^+ \\ a^- & -a^0 \end{pmatrix} \\
\big\downarrow f_0 & & \big\downarrow f_1 \\
\begin{pmatrix} \frac{1}{2}\partial\epsilon^+ + \frac{1}{2}\epsilon^+\gamma & \epsilon^+ \\ \epsilon^+ T - \partial\epsilon^0 & -\frac{1}{2}\partial\epsilon^+ - \frac{1}{2}\epsilon^+\gamma \end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix} \frac{1}{2}\partial a^+ & 0 \\ a^- - (\partial - \gamma)a^0 + Ta^+ & -\frac{1}{2}\partial a^+ \end{pmatrix} \\
\big\downarrow f_0 & & \big\downarrow f_1 \\
\begin{pmatrix} \frac{1}{2}\partial\epsilon^+ + \frac{1}{2}\epsilon^+\gamma & \epsilon^+ \\ \epsilon^+ T - \frac{1}{2}\partial^2\epsilon^+ - \frac{1}{2}\partial(\epsilon^+\gamma) & -\frac{1}{2}\partial\epsilon^+ - \frac{1}{2}\epsilon^+\gamma \end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix} 0 & 0 \\ a^- - (\partial - \gamma)(a^0 + (\frac{1}{2}\partial - T)a^+) & 0 \end{pmatrix}
\end{array}$$

Working out the action of D' in the last line we find

$$D' \begin{pmatrix} \frac{1}{2}\partial\epsilon^+ + \frac{1}{2}\epsilon^+\gamma & \epsilon^+ \\ \epsilon^+ T - \frac{1}{2}\partial^2\epsilon^+ - \frac{1}{2}\partial(\epsilon\gamma) & -\frac{1}{2}\partial\epsilon^+ - \frac{1}{2}\epsilon^+\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \delta_{\epsilon^+} T & 0 \end{pmatrix}, \qquad (5.2.9)$$

where $\delta_{\epsilon^+} T = -\frac{1}{2}(\partial - \gamma)\partial(\partial + \gamma)\epsilon^+ + 2\partial\epsilon^+ T + \epsilon^+ \partial T$ which indeed describes the transformation of T under a co-ordinate transformation.

Altogether we reach the remarkable conclusion that W transformations are nothing but a homotopic contraction of ordinary gauge transformations. Under a homotopy equivalence the cohomology does not change, and therefore the Riemann-Roch theorem can be applied to the W complex (5.2.6) to give

$$\dim H^{1,0} - \dim H^{0,0} = (g-1) \dim G. \qquad (5.2.10)$$

This is a useful formula which we need to prove the following: for genus $g > 1$, $\mathcal{M}_W = M/\sim = M^{hol}/\sim$, where $M^{hol} = \{\nabla_z + \Lambda^+ + W \mid \bar{\partial} W = 0\}^\dagger$. In other words, the fields

†Notice that $\bar{\partial}$ is a globally well-defined connection on the holomorphic bundle V (5.1.4).

W can always be made holomorphic using a global W transformation. To prove this, it is sufficient to show that if we write down an even further reduced complex containing $D' \in M^{hol}$ and only those W transformations that preserve the condition $D' \in M^{hol}$, this complex still has the same cohomology as (5.2.6). It might happen that in this way one misses certain connected components of $\mathcal{M}_W$, but that is not a problem here: $\mathcal{M} = \mathcal{M}_W$ is connected, because we are working with bundles of a fixed topological type.

The infinitesimal gauge transformations that preserve the condition $\bar{\partial}W = 0$, are given by the ϵ satisfying

$$\bar{\partial}(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))\epsilon = 0. \tag{5.2.11}$$

If we choose a metric of constant curvature $R_{z\bar{z}} = [\nabla_z, \bar{\partial}]$, then $L(R_{z\bar{z}})$ is proportional to the Lie-algebra element Λ^-. This shows that $[L(R_{z\bar{z}}), W] = 0$ and

$$[\nabla_z + \Lambda^+ + W, \bar{\partial} - L(R_{z\bar{z}})] = 0 \tag{5.2.12}$$

for the W that satisfy $\bar{\partial}W$=0. Note that (5.2.12) gives a solution to the zero-curvature equations for these W. The ϵ that satisfy (5.2.11) must also be of the form $\epsilon = (1 + L(\nabla_z + \text{ad}(W)))^{-1}F$ with $F \in \Pi_{\text{ker ad}(\Lambda^+)}\mathbf{g}$ in order to preserve the form of W. If we substitute this in (5.2.11) and use the fact that $[L(R_{z\bar{z}}), \delta W] = 0$, (5.2.11) can be rewritten as

$$\begin{aligned}
(\bar{\partial} - L(R_{z\bar{z}}))(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))\frac{1}{1 + L(\nabla_z + \text{ad}(W))}F &= 0 \Leftrightarrow \\
(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))(\bar{\partial} - L(R_{z\bar{z}}))\frac{1}{1 + L(\nabla_z + \text{ad}(W))}F &= 0 \Leftrightarrow \\
(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))\frac{1}{1 + L(\nabla_z + \text{ad}(W))}\bar{\partial}F &= 0.
\end{aligned} \tag{5.2.13}$$

Locally, $\bar{\partial}F$ can be written as $\sum_\alpha f_\alpha(\bar{z})G_\alpha(z)$, where the f_α are linearly independent antiholomorphic functions, and the G_α are holomorphic sections with respect to $\bar{\partial}$ of $(\Pi_{\text{ker ad}(\Lambda^+)}\text{ad}(P)) \otimes \bar{K}$. Substituting this in (5.2.13) yields

$$\sum_\alpha f_\alpha(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))\frac{1}{1 + L(\nabla_z + \text{ad}(W))}G_\alpha = 0. \tag{5.2.14}$$

Because the f_α are linearly independent, each G_α must satisfy $(\nabla_z + \text{ad}(\Lambda^+) + \text{ad}(W))(1 + L(\nabla_z + \text{ad}(W)))^{-1}G_\alpha = 0$. Locally, there are a finite number of solutions G_α to this equation. Globally, such G_α do not exist, as $(\Pi_{\mathrm{ker\ ad}(\Lambda^+)}\mathrm{ad}(P)) \otimes \bar{K}$ is a direct sum of line bundles K^r with $r < 0$ (upon identifying $\bar{K}$ with K^{-1}), and these do not have any global holomorphic sections. Therefore (5.2.13) implies that $\bar{\partial} F = 0$. F is a section of a direct sum of line bundles K^r with $r \leq 0$. These do, for genus $g > 1$, not have global holomorphic sections unless $r = 0$, in which case the only holomorphic sections are the constant ones. The piece of F which transforms as a section of K^0 is precisely the piece that has degree zero with respect to the gradation of the Lie algebra. The subalgebra in which this piece of F lives is $\Pi_0 \mathbf{g}$[‡], the centralizer of the sl_2 embedding (see (2.2.29)). For a constant $F \in \Pi_0 \mathbf{g}$ the parameter ϵ of the gauge transformation is given by $\epsilon = (1 + L(\nabla_z + \mathrm{ad}(W)))^{-1} F = F$, and the gauge transformation reads $\delta W = [W, F]$. The reduced complex we were looking for is

$$0 \xrightarrow{D'} \Pi_0 \mathbf{g} \xrightarrow{D'} T_{D'} M^{hol} \xrightarrow{D'} 0. \tag{5.2.15}$$

To show that the cohomology of this complex agrees with that of (5.2.6) we need only compute the difference $\dim H^{1,0} - \dim H^{0,0}$ of (5.2.15). In (5.2.15) only finite dimensional spaces occur, and therefore the index $\dim H^{1,0} - \dim H^{0,0}$ equals $\dim T_{D'} M^{hol} - \dim \Pi_0 \mathbf{g}$. The dimension of M^{hol} equals $\sum_i H^0_{\bar{\partial}}(\Sigma; K^{s_i})$, where s_i are the spins of the different components of W. The dimension of $H^0_{\bar{\partial}}(\Sigma; K^r)$ equals $(2r-1)(g-1)$ for $r > 1$, and g for $r = 1$. Thus we find

$$\begin{aligned} \dim H^{1,0} - \dim H^{0,0} &= \sum_{i, s_i > 1} (g-1)(2s_i - 1) + \sum_{i, s_i = 1} g - \dim \Pi_0 \mathbf{g} \\ &= \sum_i (g-1)(2s_i - 1) = (g-1) \dim G, \end{aligned} \tag{5.2.16}$$

which indeed agrees with (5.2.10). Altogether this proves that $\mathcal{M}_W = M^{hol}/\sim$, and so we have a simple finite dimensional model of W moduli space at our disposal.

5.2.3. The Connection with Higgs Bundles

The dimension (5.2.16) is equal to the dimension of the moduli space of flat $G_{\mathbb{R}}$ connections, where $G_{\mathbb{R}}$ is the maximal non-compact real subgroup of G. Furthermore, topological field theory and the matrix model approach to two-dimensional gravity seem to suggest that the moduli space for W_N gravity is somehow related to the moduli space of flat $SL(N, \mathbb{R})$ bundles [89, 225, 94]. However, it is at this stage not clear what the

[‡]That $\Pi_0 \mathbf{g}$ is actually a subalgebra is related to the fact that $\Pi_0 \mathbf{g}$ contains the Kac-Moody symmetries that survive the reduction to the W algebra. One can in principle impose further constraints on the W algebra so as to get rid of these residual Kac-Moody symmetries [117], but we will not do that here.

precise relation between our moduli space and the moduli space of flat $G_{\mathbb{R}}$ bundles should be. The zero-curvature equations associate a flat connection to all operators $D' = \nabla_z + \mathrm{ad}(\Lambda^+) + \mathrm{ad}(W)$, but a priori these flat connections are flat $G_{\mathbb{C}}$ connections, and it is not easy to see whether they can be written as flat $G_{\mathbb{R}}$ connections using an appropriate gauge transformation. The main difficulty is that $G_{\mathbb{R}}$ is a non-compact group, and therefore one cannot simply use the Narasimhan-Seshadri theorem [252] (see also [11]), which essentially states that for compact groups the space of anti-holomorphic structures on an associated vector bundle modulo complexified gauge transformations is the same as the space of flat connections modulo ordinary gauge transformations. In this theorem, the anti-holomorphic structure is required to satisfy a certain condition called stability, and this condition is not valid for the special bundles under consideration.

There exists an extension of the work of Narasimhan-Seshadri where the compact group is replaced by the general complex Lie group $G_{\mathbb{C}}$. This is the theory of Higgs bundles [170, 294], and this seems to be the natural setting for W moduli space. A Higgs bundle is a pair consisting of a holomorphic vector bundle V and a holomorphic section $\theta \in H^0(\Sigma; \mathrm{End}(V) \otimes K)$. In our case we are interested in the situation where V is given by (5.1.4), the holomorphic structure is given by the operator $\bar{\partial}$, the group $Gl(n,\mathbb{C})$ is reduced to $G_{\mathbb{C}}$ and $\theta = \Lambda^+ + W$, where W is holomorphic. The group $G_{\mathbb{C}}$ acts in a natural way on Higgs bundles, and one can define a moduli space for Higgs bundles by identifying two that are equivalent under a $G_{\mathbb{C}}$-transformation. To obtain a good moduli space one has to impose a condition on the Higgs bundle that is also called stability. A Higgs-bundle is called stable if for every holomorphic subbundle $V' \subset V$ that satisfies $\theta(V') \subset V' \otimes K$, the slope $\mu(V')$ of V' is smaller than the slope $\mu(V)$ of V. The slope is defined as the first Chern class divided by the rank of the bundle.

Let us see whether the Higgs bundle with $\theta = \Lambda^+ + W$ is stable. The slope of V vanishes, and therefore every subbundle V' with $\theta(V') \subset V' \otimes K$ must have a negative slope for stability. The sl_2 algebra $\{t_-, t_0, t_+\}$ acts via left multiplication on the vector bundle V. Under this action the n-dimensional representation furnished by V decomposes in a direct sum of irreducible sl_2-representations, $V = \bigoplus_{l=1}^{n_l} V_l$, of spin j_l, and $V_l \simeq K^{-j_l} \bigoplus K^{1-j_l} \bigoplus \ldots \bigoplus K^{j_l}$. The slope of each of the V_l is zero, as they have vanishing Chern class. Λ^+ preserves V_l and all subbundles of V_l of the type $K^{-j_l} \bigoplus K^{1-j_l} \bigoplus \ldots \bigoplus K^{j_l - t}$ for some $t > 0$. These all have strictly negative slope, and therefore the only problematic subbundles of V are direct sums of the V_l, as these are the only holomorphic subbundles preserved by Λ^+ that have a nonnegative slope. The same bundles are also the bundles that might threaten the stability of (V, θ) with $\theta = \Lambda^+ + W$. A component W corresponding to an irreducible subrepresentation of $V_l \otimes V_{l'}$ mixes between the bundles V_l and $V_{l'}$. Therefore if sufficiently many of these components are nonzero, no direct sum of V_l's will be invariant under $\Lambda^+ + W$ anymore, because V was obtained from an irreducible representation of G, and all proper holomorphic subbundles, if they exist, will have negative slope.

Another, equivalent way to express this condition is to demand that $\ker \mathrm{ad}(W)(\Pi_0 \mathbf{g}) = 0$, so that $\Pi_0\mathbf{g}$ acts faithfully on M^{hol}. If we therefore define $M^{hol}_{red} = \{\nabla_z + \Lambda^+ + W \mid \bar\partial W = 0 \wedge \ker \mathrm{ad}(W)(\Pi_0\mathbf{g}) = 0\}$, then the quotient space $M^{hol}_{red}/\Pi_0\mathbf{g}$ has no singularities, and it is naturally a subspace of the moduli space of stable Higgs bundles, of complex dimension $(g-1)\dim(G)$.

The dimension of the moduli space of Higgs bundles is $2(g-1)\dim(G)$, which is twice as large as the dimension of the W moduli space. These correspond to flat irreducible $G_{\mathbb{C}}$ bundles over the Riemann surface Σ [170, 295]. One might wonder which property characterizes the flat $G_{\mathbb{C}}$ connections that correspond to points in the W moduli space. For general W algebras we do not know the answer to this question, but for the 'standard' W_N algebras the answer is, that only those flat $Sl(N,\mathbb{C})$ connections which are reducible to a flat $Sl(N,\mathbb{R})$ connections can correspond to points in the W moduli space. To prove this, we use lemma 3.20 in [294]. This lemma states that a Higgs bundle (V,θ) corresponds to a flat real connection if and only if there exists a bilinear symmetric form $S(u,v)$ on $V\otimes V_C$, where V_C is the Higgs bundle $(V,-\theta)$, such that

$$\bar\partial S(u,v) = S((\bar\partial+\theta)u,v) + S(u,(\bar\partial-\theta)v). \tag{5.2.17}$$

For the standard W algebras such a symmetric form S exists. The vector bundle V is in this case

$$V = \bigoplus_{l=1}^{N} V_l, \qquad V_l \simeq K^{\frac{l-N-1}{2}} \tag{5.2.18}$$

and the symmetric form S is given by

$$S(u,v) = \sum_{l=1}^{N} u_l \;\, v_{N+1-l}. \tag{5.2.19}$$

It is an easy exercise to show that (5.2.17) indeed holds for (5.2.19). Putting everything together we conclude that, for standard W algebras, W moduli space is a component of the moduli space of flat irreducible $Sl(N,\mathbb{R})$ connections. This parametrization of a component of the moduli space of flat $Sl(N,\mathbb{R})$ connections in terms of certain Higgs bundles is similar to the one studied by Hitchin [169, 172]. The relevant component is specified by the topological type of the real vector bundle on which the flat $Sl(N,\mathbb{R})$ connection lives. To really construct this flat connection explicitly, one needs to know the so-called Hermitian-Yang-Mills metric on the Higgs bundle, see [295]. This metric, and the associated flat connection are very easy to describe if $W=0$. In that case one picks a constant curvature metric on the Riemann surface, and uses the metric this induces on K to construct a metric on V. This is already the Hermitian-Yang-Mills metric and the corresponding flat connection is

$$D = \nabla_z + \bar\partial + \Lambda^+ - L(R_{z\bar z}). \tag{5.2.20}$$

The real vector bundle for which this defines an $Sl(N,\mathbb{R})$ connection is given by the bundle left invariant by an involution of V that commutes with D. The involution $\sigma : V \to V$ is given by sending $u \in K^r \to \bar{u} \in \bar{K}^r \simeq K^{-r}$, where the metric is used to identify $\bar{K}$ with K^{-1}.

As an example, consider ordinary gravity. In that case $G = Sl(2,\mathbb{C})$, and $V = K^{-\frac{1}{2}} \bigoplus K^{\frac{1}{2}}$. The involution σ is in local co-ordinates with metric $ds^2 = \rho dz d\bar{z}$ given by

$$\sigma : \left(\begin{array}{c} u_1 \\ u_2 \end{array} \right) \to \left(\begin{array}{c} \bar{u}_2/\sqrt{\rho} \\ \bar{u}_1\sqrt{\rho} \end{array} \right). \tag{5.2.21}$$

The corresponding real bundle is one of Euler class $2(g-1)$, and the W moduli space as defined here is the Teichmüller space of Σ, which is indeed closely related to the moduli space of Riemann surfaces, and has occurred before in studies of 2D quantum gravity [309]. For details, see [170].

It is a very interesting problem to characterize the flat $Sl(N,\mathbb{C})$-bundles that are related to the moduli space of the nonstandard W algebras obtainable from sl_N. We have checked for a few cases that it is impossible to construct a symmetric bilinear form satisfying (5.2.17) for those W algebras, and therefore they do not correspond to flat $Sl(N,\mathbb{R})$ connections. If we replace V by $Sl(V)$, and consider the Higgs bundle $(Sl(V),\theta)$ with $\theta : Sl(V) \to Sl(V) \otimes K$ given by $\theta(X) = [\Lambda^+ + W, X]$, then the existence of symmetric bilinear form satisfying (5.2.17) for this Higgs bundle would show that the flat $Sl(N,\mathbb{C})$ connections for these W algebras are always reducible to a flat $\mathbf{g}$ connection, where $\mathbf{g}$ is some Lie algebra whose complexification is $sl(N,\mathbb{C})$. However, for the cases we checked it was impossible to construct a symmetric form S for $(Sl(V),\theta)$ either, and therefore it is still unclear what precisely characterizes the nonstandard W moduli spaces.

Instead of looking at $M^{hol}_{red}/\Pi_0\mathbf{g}$, one could also look at different 'strata' of M^{hol}, by defining $M^{hol}_k = \{\nabla_z + \Lambda^+ + W \mid \bar{\partial}W = 0 \wedge \dim\ker \mathrm{ad}(W)(\Pi_0\mathbf{g}) = k\}$. The space $\mathcal{M}_{W,k} = M^{hol}_k/\Pi_0\mathbf{g}$ is presumably related to 'singular' configurations of W fields, and deserves some further study as well.

The simplest nonstandard W algebra is $W_3^{(2)}$ (section 2.2.12). For this W algebra, Λ^+ and W are given by

$$\Lambda^+ = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad W = \left(\begin{array}{ccc} H/2 & 0 & 0 \\ G^+ & -H & 0 \\ T & -G^- & H/2 \end{array} \right), \tag{5.2.22}$$

where J has spin 1, G^+, G^- have spin 3/2 and T has spin 2. The space M^{hol} has dimension $8g-7$, and $\Pi_0\mathbf{g}$ consists of the constant matrices

$$X = \begin{pmatrix} \epsilon/2 & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & \epsilon/2 \end{pmatrix}, \quad \text{and} \quad \delta_\epsilon W = [W, X] = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{2}\epsilon G^+ & 0 & 0 \\ 0 & \frac{3}{2}\epsilon G^- & 0 \end{pmatrix}. \qquad (5.2.23)$$

From this we see that $M^{hol}_{red} = \{J, G^+, G^-, T | G^+ \neq 0 \text{ or } G^- \neq 0\}$, and that $\mathcal{M}_{W,0}$ is topologically the product of $\mathbb{C}^{4g-3}$ and a weighted projective space of dimension $4g-5$. Clearly, $\mathcal{M}_{W,1}$ is topologically a vector space of dimension $4g-3$. We see that for this W algebra the moduli space is non trivial.

The discussion of W moduli space in this section has so far been limited to genus $g>1$. Most of the analysis can also be carried through for genus $g=0,1$. The main difference with $g>1$ is that for the latter case, W moduli space is in a natural way a subspace of the moduli space of stable Higgs bundles. For $g=0,1$ this is no longer the case, because the Higgs bundles one obtains for $g=0,1$ are not stable any more, the reason for this being the fact that the first Chern class of K is given by $c_1(K)=2(g-1)$, and changes sign at $g=1$. Let us briefly indicate what the moduli spaces for $g=0,1$ look like.

For $g=0$, the line bundle K^r with $r>0$ has no global holomorphic sections. Therefore M^{hol} is contains only $D' = \nabla + \Lambda^+$ and has dimension 0. The gauge transformations that act trivially on $\nabla + \Lambda^+$ are given by $\delta\Lambda^+ = 0 = [\Lambda^+, F]$, where F is an arbitrary holomorphic section of $\Pi_{\text{ker ad}(\Lambda^+)}\text{ad}(P) \simeq \oplus_i K^{1-s_i}$. For genus $g=0$ the line bundle K^{1-s_i} has $(2s_i-1)$ holomorphic sections, and the dimension of the space of gauge transformations that act on M^{hol} equals $\sum_i(2s_i-1) = \dim G$. This shows that $\dim H^{1,0} - \dim H^{0,0} = -\dim G$, in agreement with the Riemann-Roch theorem (5.2.6), which is valid for arbitrary genus.

For $g=1$, the line bundles K^r are all trivial and have precisely one holomorphic section. If d_W denotes the number of generators of the W algebras, then $\dim M^{hol} = d_W$, and the space of gauge transformations that acts on M^{hol} also has dimension d_W. These gauge transformations act on M^{hol} via $\delta W = [\Lambda^+ + W, (1 + L\text{ad}_W)^{-1}F]$, where F is an arbitrary holomorphic section of $\Pi_{\text{ker ad}(\Lambda^+)}\text{ad}(P)$. Again, (5.2.6) is satisfied. A natural candidate for the moduli space is in this case

$$\mathcal{M}_W = \{\nabla + \Lambda^+ + W | \dim C_{\mathbf{g}}(\Lambda^+ + W) = \text{rank } \mathbf{g}\}/\sim \ , \qquad (5.2.24)$$

where $C_{\mathbf{g}}(X)$ is the centralizer of X, *i.e.* the set of elements of $\mathbf{g}$ that commute with X, and $\mathbf{g}$ should be identified with the holomorphic sections of $\text{ad}(P) \otimes K$. The dimension of the genus 1 moduli space equals the rank of $\mathbf{g}$.

5.2.4. Reconstruction of the Metric

For gravity, it is possible to construct the flat SL_2 connection explicitly for every point in the moduli space, *i.e.* for every quadratic differential $T \in H^0_{\bar\partial}(\Sigma; K^2)$. It is instructive to see how this works. We start with a Higgs bundle (V, θ). A hermitian metric for the bundle V is locally given by a hermitian matrix Ω that determines the inner product of two sections $s_1, s_2 \in \Gamma(V)$ via $(s_1, s_2) = \int_\Sigma d^2z\, \rho\, s_1^\dagger \Omega s_2$. If the holomorphic structure on V is given by the operator $\bar\partial + \bar A$, then the metric determines a connection D_Ω on V,

$$D_\Omega = \partial + \Omega^{-1}\partial\Omega - \Omega^{-1}\bar A^\dagger\Omega + \theta + \bar\partial + \bar A + \Omega^{-1}\theta^\dagger\Omega. \tag{5.2.25}$$

The condition that this connection is flat is equivalent to the statement that the metric Ω is a Hermitian-Yang-Mills metric. (this equivalence holds because the first Chern class of V vanishes [294]). The zero-curvature equation for D_Ω is

$$[\Omega^{-1}(\partial - \bar A^\dagger)\Omega, \bar\partial + \bar A] + [\theta, \Omega^{-1}\theta^\dagger\Omega] = 0. \tag{5.2.26}$$

For the holomorphic bundle V with $\bar A = 0$, this equation reduces to

$$-\bar\partial(\Omega^{-1}\partial\Omega) + [\theta, \Omega^{-1}\theta^\dagger\Omega] = 0. \tag{5.2.27}$$

For gravity, with

$$\theta = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \tag{5.2.28}$$

the zero curvature equation is satisfied with

$$\Omega = \begin{pmatrix} \rho^{\frac{1}{2}} & 0 \\ 0 & \rho^{-\frac{1}{2}} \end{pmatrix} \tag{5.2.29}$$

if the background metric ρ satisfies

$$\frac{1}{2\rho}\partial\bar\partial\rho = \left(1 - \left|\frac{T}{\rho}\right|^2\right). \tag{5.2.30}$$

For $T = 0$ this equations says that $ds^2 = \rho\, dz\, d\bar z$ is a constant curvature metric. For $T \neq 0$ the equation is a kind of vortex equation [190]. It is equivalent to the constant curvature equation for the metric

$$ds^2 = \rho \left| dz + \frac{\bar T}{\rho} d\bar z \right|^2. \tag{5.2.31}$$

This metric is a representative of the point in Teichmüller space corresponding to T [170]. The equation (5.2.30) is at the same time the equation of motion for Liouville theory coupled to the background metric (5.2.31) with an appropriate choice of cosmological constant term. This is nothing new; the relation between Liouville theory and the geometry of Riemann surfaces is well known [299]. The merit of the construction given here is that it is easy to generalize it to arbitrary W algebras, and seems a fruitful direction to search for the geometrical meaning of W algebras. However, for W algebras the analysis is much more difficult. Already for W_3 it is not sufficient to take a diagonal Ω to solve (5.2.27), which could well be related to the fact that the covariant action contains auxiliary fields besides the Toda fields that cannot be integrated out explicitly. Whether the solution has any other interpretation apart from being a flat $SL(3,\mathbb{R})$ connection, is unclear.

For genus $h \leq 1$, curious things can happen. Consider for example gravity on the torus, which we identify with the square $[0,1]\times[0,1]$ with metric $ds^2 = dzd\bar{z}$. The relation between the modular parameter τ of the torus and the constant T is

$$\tau = \frac{1-\bar{T}}{1+\bar{T}} i, \tag{5.2.32}$$

so that the modular transformation $\tau \to -1/\tau$ simply corresponds to $T \to -T$. On the torus, gauge transformations can exist that are not homotopic to the identity gauge transformation, and relate different values of T with each other. The equation

$$G^{-1}\begin{pmatrix} 0 & 1 \\ T_1 & 0 \end{pmatrix} G + G^{-1}\partial G = \begin{pmatrix} 0 & 1 \\ T_2 & 0 \end{pmatrix} \tag{5.2.33}$$

can be solved for G, resulting in

$$G = \begin{pmatrix} \sqrt[4]{\frac{T_2}{T_1}}\cos(2\alpha) & \frac{-i}{\sqrt[4]{T_1 T_2}}\sin(2\alpha) \\ -i\sqrt[4]{T_1 T_2}\sin(2\alpha) & \sqrt[4]{\frac{T_1}{T_2}}\cos(2\alpha) \end{pmatrix} \tag{5.2.34}$$

where

$$\alpha = 2\mathrm{Im}(z(\sqrt{T_1} - \sqrt{T_2})). \tag{5.2.35}$$

The gauge transformation is well-defined globally on the torus if

$$\sqrt{T_1} - \sqrt{T_2} \in \pi i \mathbb{Z} + \pi \mathbb{Z}. \tag{5.2.36}$$

The interpretation of this condition and the corresponding gauge transformations is completely unclear to us. They certainly do not correspond to modular transformations.

5.2.5. Relation to Self-Duality Equations

The name Higgs bundle suggests that these geometrical objects have something to do with a Higgs field. In this section we explain this connection [169].

We start with the self-duality equations on $\mathbb{R}^4$ for some principal bundle P with connection d_A and associated curvature two-form F. The self-duality equations are

$$F = *F \tag{5.2.37}$$

where $*$ is the Hodge star operator[§]. The self-duality equations arise naturally when considering the extrema of the Yang-Mills action

$$\int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge *F). \tag{5.2.38}$$

Here, Tr is a positive definite bilinear form on $\mathbf{g}$. In terms of standard co-ordinates (x^1, x^2, x^3, x^4) on $\mathbb{R}^4$ the connection can be written as $A_1 dx^1 + A_2 dx^2 + A_3 dx^3 + A_4 dx^4$.

The idea is now to dimensionally reduce the self-duality equations. Therefore we assume that A does not depend on one co-ordinate, say x_4. Calling $\phi = A_4$, the self-duality equations can be rewritten as

$$\begin{aligned} F_{12} = F_{34} &= (d_A)_3\phi, \\ F_{13} = F_{42} &= -(d_A)_2\phi, \\ F_{23} = F_{14} &= (d_A)_1\phi \end{aligned} \tag{5.2.39}$$

which is equivalent to

$$F = *d_A\phi. \tag{5.2.40}$$

These equations are known as the Bogomolny equations. They arise when considering the minima of the dimensionally reduced Yang-Mills action, which reads

$$\int_{\mathbb{R}^3} (\mathrm{Tr}(F \wedge *F) + \mathrm{Tr}(d_A\phi \wedge *d_A\phi)). \tag{5.2.41}$$

This action is the $\lambda \to 0$ limit of the Yang-Mills-Higgs functional

$$\int_{\mathbb{R}^3} (\mathrm{Tr}(F \wedge *F) + \mathrm{Tr}(d_A\phi \wedge *d_A\phi) + \lambda(1 - \mathrm{Tr}(\phi^2))^2). \tag{5.2.42}$$

[§]On a Riemannian manifold of dimension n the Hodge star operator is the operator that maps p forms to $n-p$ forms defined by the condition that for all p-forms α the equation $\alpha \wedge *\beta = (\alpha \mid \beta)\tau$ holds. Here, τ is the volume form $\sqrt{g} dx^1 \wedge \ldots \wedge dx^n$, and the inner product $(\alpha|\beta)$ of two p-forms $\frac{1}{p!} A_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ and $\frac{1}{p!} B_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ is $\frac{1}{p!} A_{i_1 \ldots i_p} B^{i_1 \ldots i_p}$.

Taking the limit $\lambda \to 0$ but keeping the asymptotic condition $\mathrm{Tr}(\phi^2) \to 1$ as $x \to \infty$ is known as the Bogomolny-Prasad-Sommerfield limit, and in that limit the solutions of the Bogomolny equations are called magnetic monopoles.

Higgs bundles appear by a further dimensional reduction of the self-duality equations to $\mathbb{R}^2$. Thus, assume that the connection A does not depend on both x_3 and x_4, and call $A_3 = \phi_1$ and $A_4 = \phi_2$. The self-duality equations become

$$\begin{aligned} F_{12} = F_{34} &= [\phi_1, \phi_2], \\ (d_A)_1\phi_1 = F_{13} = F_{42} &= -(d_A)_2\phi_2, \\ (d_A)_2\phi_1 = F_{23} = F_{14} &= (d_A)_1\phi_2. \end{aligned} \tag{5.2.43}$$

In terms of the complex co-ordinate $z = x^1 + ix^2$ and $\Phi = (\phi_1 - i\phi_2)/2$ they become

$$\begin{aligned} F_{z\bar{z}} &= -[\Phi, \Phi^*], \\ (d_A)_{\bar{z}}\Phi &= 0. \end{aligned} \tag{5.2.44}$$

The second equation expresses the fact that Φ is a holomorphic section, so that with $\theta = \Phi$ this is precisely our previous definition of a Higgs bundle. To go back from a Higgs bundle (V, θ) to the two equations given here, we need the Hermitian-Yang-Mills metric (5.2.26). For a stable Higgs bundle such a metric always exists, and given a solution Ω one defines $\Phi^* = \Omega^{-1}\Phi^\dagger\Omega$ and $A_z = \Omega^{-1}\partial\Omega - \Omega^{-1}\bar{A}_{\bar{z}}{}^\dagger\Omega$.

Although equations (5.2.44) are defined on $\mathbb{R}^2$, they are conformally invariant and can therefore also be defined on a compact Riemann surface. Alternatively, one could have started with the self-duality equations on $\Sigma \times \mathbb{R}^2$ and reduce these.

The Higgs bundles that are relevant for the W moduli space were of a special type with $\theta = \Lambda^+ + W$. For $W = 0$, these special Higgs bundles have an interpretation in terms of the self-duality equations, namely they are related to spherically symmetric solutions of the self-duality equations [306, 222]. These solutions are constructed as follows: we start with the self-duality equations on $\mathbb{R}^4$, and replace x^2, x^3, x^4 by spherical co-ordinates r, θ, φ. Subsequently one imposes spherical symmetry under the rotations generated by the angular momentum operator $J_i = L_i + T_i, i = 2, 3, 4$, where L generates rotations in x^2, x^3, x^4 space and T generates an $SU(2)$ subgroup of the group G which was the fiber of the principal bundle P. The self-duality equations, subject to the condition of spherical symmetry, take a simple form in the complex co-ordinates y, z defined by

$$dy = dr + idt, \qquad dz = r(d\theta - i\sin(\theta)d\varphi). \tag{5.2.45}$$

Expressed in terms of the gauge fields

$$B_y = A_y + iT_y/r, \qquad B_{\bar{y}} = A_{\bar{y}} - iT_{\bar{y}}/r \tag{5.2.46}$$

and

$$\Phi = A_z + iT_z/r \tag{5.2.47}$$

the self-duality equations are equivalent to the equations (5.2.44) for a Higgs bundle. This is not very surprising, as spherical symmetry can be used to eliminate the θ, φ dependence, and causes also a dimensional reduction from four to two dimensions. A difference with the previous derivation of (5.2.44) is that the rotation group has three generators instead of two. This gives an extra constraint on Φ, and in an appropriate gauge the constraint is

$$\Phi = f(z)\Lambda^+. \tag{5.2.48}$$

Since $z, \bar{z}$ are co-ordinates on the right half-plane, there is a freedom to multiply Φ by an holomorphic function. If we replace the right half-plane by a compact Riemann surface, on which no global holomorphic functions exist, this freedom disappears and what remains is $\Phi \sim \Lambda^+$, precisely what we found in the case of W algebras.

The reason that the same Φ occurs both for spherical symmetric solutions of the self-duality equations and for W algebras, is related to the fact that both have to do with a kind of soldering procedure. The conditions for spherical symmetry are with respect to $J = L + T$, a combination of space and gauge transformations. The same combination of co-ordinate transformations and gauge transformations led to the W algebra in section 2.2.5. This raises the interesting question to what kind of solutions to the self-duality equations the Higgs bundles with $\theta = \Lambda^+ + W$ for generic W correspond. The equation (5.2.27) with $\theta = \Lambda^+$ and Ω restricted to $\exp(\mathbf{g}_0)$ is the equation of motion of a generalized Toda theory. This suggest (as is true for gravity) that for $W \neq 0$ they are related to Toda equations coupled to a non-trivial W background. Going back to the self duality equations, this would mean that the corresponding solutions of the self-duality equations are spherical symmetric with respect to some kind of W-metric for the co-ordinates x^1 and r. This would be an interesting observation, as it would enable us to construct spherically symmetric solutions of the self-duality equations on a manifold with non-trivial metric, and might give rise to new monopole solutions. These issues are left to future investigation.

5.3. Generalized Actions

Having formulated W algebras on arbitrary Riemann surfaces, a next natural step is to reconsider the induced and covariant action for W gravity on an arbitrary Riemann surface. As the WZNW action played an essential role in these constructions, our first goal will be to generalize the WZNW action to non-trivial principal fiber bundles over a Riemann surface.

As mentioned in the beginning of section 3.3.3, the WZNW action can be obtained from a fermionic path integral, $S^-_{wznw}(A_z) \sim \log\det(\partial + A_z)$. This definition is easy to generalize to non-trivial bundles. Let P be a principal fiber bundle with group G over Σ, and A a connection on P. Furthermore, let V be the vector bundle associated to a representation r of G, and consider the action

$$S(\psi, A_z) = \frac{1}{\pi}\int d^2z\, \mathrm{Tr}\left(\tilde{\psi}(\partial + A_z)\psi\right) \tag{5.3.1}$$

and its induced action

$$e^{S_{ind}(A_z)} = \int \mathcal{D}\tilde{\psi}\mathcal{D}\psi e^{-S}, \tag{5.3.2}$$

where ψ transforms as a section of $V \otimes (\bar{K})^{\frac{1}{2}}$, and $\tilde{\psi}$ as a section of $\tilde{V} \otimes (\bar{K})^{\frac{1}{2}}$. The bundle $\tilde{V}$ is associated to the representation r^{-1} of G, acting from the right on the same vector space as the one on which r acts. The action (5.3.1) is gauge invariant under $\tilde{\psi} \to \tilde{\psi}h$, $\psi \to h^{-1}\psi$, and $A \to h^{-1}Ah + h^{-1}\partial h$. The two-point function $G_{pk}(A_z; z, w) = \langle \psi_p(z)\tilde{\psi}_k(w)\rangle$ satisfies $(\partial\delta_{lp} + (A_z)_{lp}(z))G_{pk}(A_z; z, w) = \pi\delta_{lk}\delta(z-w)$. From this one finds the following rule for the change of G under a gauge transformation h: $G_{p'k'}(A^h_z; z, w) = h(z)^{-1}_{p'p}G_{pk}(A_z; z, w)h_{kk'}(w)$. For $z \to w$ G behaves as

$$G \sim \frac{1}{\bar{z} - \bar{w}} + \chi_{\bar{z}}(z) - A_z(z)\frac{z - w}{\bar{z} - \bar{w}} + \text{terms that vanish as } z \to w. \tag{5.3.3}$$

Under the gauge transformation $A_z \to A^h_z = h^{-1}\partial h + h^{-1}A_z h$ we find, by expanding $h(w) = h(z) + (\bar{z} - \bar{w})\bar{\partial}h(z) + (z - w)\partial h(z) + \ldots$, that $\chi_{\bar{z}} \to h^{-1}\chi_{\bar{z}}h + h^{-1}\bar{\partial}h$, *i.e.* $\chi_{\bar{z}}$ transforms as a connection. Furthermore, if locally $A_z = 0$ we know that G is exactly given by $1/(\bar{z} - \bar{w})$. Combining these facts we deduce that the curvature of the connection one-form $A_z dz + \chi_{\bar{z}} d\bar{z}$ must vanish. To express this fact in terms of the current

$$(J_{\bar{z}})_{pk} \equiv -\pi\frac{\delta S_{ind}(A_z)}{\delta(A_z)_{pk}}, \tag{5.3.4}$$

we must first define what we mean by $J_{\bar{z}}$. The naive definition $J_{\bar{z}} = \lim_{z\to w} G(A_z; z, w)$ does not work, due to the singularity of G as $z \to w$. This means we have to regularize $J_{\bar{z}}$, and a standard way of doing this is by using point splitting regularization: one defines $J_{\bar{z}} = \lim_{z\to w}(G(A_z; z, w) - G_0(A_z; z, w))$ where G_0 is some function that has the same singular behavior as G. This means that

$$G_0 \sim \frac{1}{\bar{z} - \bar{w}} + B^0_{\bar{z}}(z) - A_z(z)\frac{z - w}{\bar{z} - \bar{w}} + \text{terms that vanish as } z \to w. \tag{5.3.5}$$

Here, $B^0_{\bar{z}}(z)$ is some fixed field that transforms as the $d\bar{z}$-part of a connection. Unfortunately, one cannot just take $B^0_{\bar{z}}$ equal to zero, because zero is not a globally well-defined connection. The current $J_{\bar{z}}$ is

$$J_{\bar{z}} = \chi_{\bar{z}} - B^0_{\bar{z}}, \tag{5.3.6}$$

and the Ward-identity (expressing the fact that the current $J_{\bar{z}}$ is not conserved) reads

$$D_{A_z} J_{\bar{z}} = \bar{\partial} A_z + D_{A_z} B^0_{\bar{z}}. \tag{5.3.7}$$

This identity should come from some kind of generalized WZNW action. The WZNW action $S^-_{wznw}(g)$ is really a functional on the space of gauge transformations, and on the complex plane this space is isomorphic to maps from the plane to G. The terms $g^{-1}\partial g$ in the WZNW action are just $\mathbf{0}^g$, where $\mathbf{0}$ is the trivial connection on the complex plane, and the superscript g refers as usual to a gauge transformation. Even on the complex plane one sees that the WZNW action is not invariant if one chooses a different trivialization of the (trivial) bundle P. If one chooses a different trivialization, related to the first one via a gauge transformation h, then $\mathbf{0}^g \to (\mathbf{0}^h)^{h^{-1}gh}$, because g transforms as a section of the adjoint bundle $\mathrm{Ad}(P)$¶. However, $(\mathbf{0}^h)^{h^{-1}gh} = (gh)^{-1}\partial(gh)$, and as is well known, $S_{wznw}(gh) \neq S_{wznw}(g)$. As the WZNW action should not depend on the choice of trivialization of P, there is something wrong with the identification of $g^{-1}\partial g$ with $\mathbf{0}^g$. Actually, there is another possibility, which turns out to be the right one, namely to identify $g^{-1}\partial g$ with $\mathbf{0}^g - \mathbf{0}$. Under a change of trivialization

$$\mathbf{0}^g - \mathbf{0} \to (\mathbf{0}^h)^{h^{-1}gh} - (\mathbf{0}^h) = h^{-1}(\mathbf{0}^g - \mathbf{0})h, \tag{5.3.8}$$

and the WZNW action is invariant, because the h and h^{-1} cancel each other inside the traces in the WZNW action.

If P is a non-trivial bundle, one cannot take $\mathbf{0}$ as a well-defined connection, and it must be replaced by some other, fixed connection $B = B^0_z dz + B^0_{\bar{z}} d\bar{z}$. It turns out that we must require B to be flat. As A_z is identified with $g^{-1}\partial g$, we must replace $g^{-1}\bar{\partial} g$ by a function $A_{\bar{z}}(A_z)$ determined by requiring the curvature of the connection one-form $A(A_z) = A_z dz + A_{\bar{z}} d\bar{z}$ to vanish. This leads to the following definition of the WZNW action

$$kS^{\pm}_{wznw}(A;B) = -\tfrac{k}{8\pi}\int_\Sigma \mathrm{Tr}((A - B^0)\wedge *(A - B^0)) \mp \tfrac{ik}{12\pi}\int_M \mathrm{Tr}(\tilde{A} - \tilde{B}^0)^3, \tag{5.3.9}$$

where $\partial M = \Sigma$, and $\tilde{A}, \tilde{B}^0$ denote flat extensions of A, B^0 on a bundle $\tilde{P}$ on B that restricts to P on Σ. The $*$ is the Hodge star on the Riemann surface Σ, where we

¶$\mathrm{Ad}(P)$ is defined as $(P \times G)/G$, where one the G-action on $P \times G$ is given by the standard left action on P and by the adjoint action on G.

assume that some metric compatible with the complex structure on Σ is given. That an extension $\tilde{P}$ of P exists can be seen as follows [56]: it is sufficient to construct a complex vector bundle $\tilde{V}$ over B, that extends V (5.1.4). The vector bundle V is the pull-back of a universal vector bundle over a certain Grassmannian Gr. The map we use to pull back this universal vector bundle maps Σ into an element of the second homology of the Grassmannian, and it is precisely this element of $H_2(Gr)$ that gives the obstruction to construct an extension $\tilde{V}$. Because this element is essentially the first Chern class of V, and the first Chern class of V vanishes, we know Σ maps to zero in $H_2(Gr)$ and an extension $\tilde{V}$ indeed exists. It may seem surprising that one needs an additional connection B to write down the WZNW action, but this is necessary if one wants to write down an action for chiral fermions only. It also appears when considering determinant bundles associated to operators such as D_{A_z} [272].

Let us demonstrate that (5.3.9) indeed satisfies the Ward-identity (5.3.7), if we identify the two $B^0_{\bar{z}}$'s with each other. Consider a small variation $A \to A + \delta A$. From the zero-curvature equation $dA + A \wedge A = 0$ we find $d\delta A + A \wedge \delta A + \delta A \wedge A = 0$, with similar equations for $\tilde{A}$. Using this we compute

$$\begin{aligned} \delta \mathrm{Tr}(\tilde{A} - \tilde{B}^0)^3 &= 3\mathrm{Tr}(\delta\tilde{A} \wedge (\tilde{A} - \tilde{B}^0) \wedge (\tilde{A} - \tilde{B}^0)) \\ &= 3\mathrm{Tr}(\tilde{A} \wedge \delta\tilde{A} \wedge (\tilde{A} - \tilde{B}^0) + \delta\tilde{A} \wedge \tilde{A} \wedge (\tilde{A} - \tilde{B}^0) - \\ &\quad \delta\tilde{A} \wedge \tilde{A} \wedge \tilde{A} + \delta\tilde{B}^0 \wedge \tilde{B}^0 \wedge \tilde{B}^0) \\ &= 3\mathrm{Tr}(-d(\delta\tilde{A}) \wedge (\tilde{A} - \tilde{B}^0) + \delta\tilde{A} \wedge (d\tilde{A} - d\tilde{B}^0)) \\ &= -3\mathrm{Tr}(d(\delta\tilde{A} \wedge (\tilde{A} - \tilde{B}^0))). \end{aligned} \tag{5.3.10}$$

This gives for the total variation of the WZNW action

$$k\delta S^-_{wznw}(A_z; B^0) = \tfrac{-k}{4\pi} \int_\Sigma \mathrm{Tr}(\delta A \wedge (* - i)(A - B^0)) = \tfrac{-k}{\pi} \int_\Sigma d^2z \; \mathrm{Tr}(\delta A_z (A - B^0)_{\bar{z}}), \tag{5.3.11}$$

because $(* \mp i)/2$ are precisely the operators that define the complex structure. This shows that this action indeed solves the Ward-identity (5.3.7) with $J_{\bar{z}} = -\pi \frac{\delta S^-_{wznw}(A_z)}{\delta A_z}$, if we identify the $\bar{B}^0$'s with each other. Therefore the induced action $S_{ind}(A_z)$ is precisely $S^-_{wznw}(A_z)$ with $k = 1$ and the trace taken in the same representation as the fermions live in.

Having defined a generalized WZNW action, it is interesting to see whether this action shares some of the properties of the ordinary WZNW action. Using a calculation similar as (5.3.10), one can verify the following version of the Polyakov-Wiegmann formula

$$kS^-_{wznw}(A; B) = kS^-_{wznw}(A; C) + kS^-_{wznw}(C; B) - \tfrac{k}{\pi} \int_\Sigma d^2z \, \mathrm{Tr}((A-C)_z (C-B)_{\bar{z}}), \tag{5.3.12}$$

from which the usual Polyakov-Wiegmann formula follows by putting $A = (gh)^{-1}d(gh)$, $C = h^{-1}dh$ and $B = 0$ for a trivial bundle P. Another issue is whether $S_{wznw}(A;B)$ depends on the choice of extension $\tilde{A}$ and $\tilde{B}$. Choosing a different extension will change the action by a term $\frac{k}{12\pi}\int_M \mathrm{Tr}(\tilde{A}-\tilde{B})^3$, where now $\Sigma \subset B$ and $\partial B = \emptyset$. Let $\mathcal{U}$ denote the space of flat connections $\tilde{A}$ on such a B such that $\tilde{A}\mid_\Sigma = A$, and consider the function $r_{\tilde{B}} : \mathcal{U} \to \mathbb{C}$ given by $r_{\tilde{B}}(\tilde{A}) = \frac{k}{12\pi}\int \mathrm{Tr}(\tilde{A}-\tilde{B})^3$. From the identity

$$\mathrm{Tr}((\tilde{A}-\tilde{B})^3 + (\tilde{B}-\tilde{C})^3 + (\tilde{C}-\tilde{A})^3) = 3\mathrm{Tr}(d((\tilde{A}-\tilde{B}) \wedge (\tilde{B}-\tilde{C}))) \qquad (5.3.13)$$

it follows that $r_{\tilde{B}}(\tilde{A}) + r_{\tilde{C}}(\tilde{B}) = r_{\tilde{C}}(\tilde{A})$. This implies that $r_{\tilde{B}}(\tilde{A}+\delta\tilde{A}) - r_{\tilde{B}}(\tilde{A})$ is of third order in $\delta\tilde{A}$, and therefore that $r_{\tilde{B}}$ is locally constant on $\mathcal{U}$, *i.e.* $r_{\tilde{B}}$ descends to a map $\pi_0(\mathcal{U}) \to \mathbb{C}$. We see that the WZNW action is invariant under a continuous change of the choice of extension. To find out whether or not k is quantized is not very easy as it requires knowledge of $\pi_0(\mathcal{U})$. However, in the case that $\mathcal{U}/\mathcal{G}$ is connected, where $\mathcal{G}$ is the space of gauge transformations acting on $\mathcal{U}$, one can say a little bit more, using the fact that is this case all connected components of $\mathcal{U}$ can be reached from a fixed one using gauge transformations. To do this, one has to take a slightly different look at the function $r_{\tilde{B}}$. For any group G one can write down an element of $H^3(G)$ by extending the three-form $\omega(X,Y,Z) = \frac{k}{12\pi}\mathrm{Tr}(X[Y,Z])$ on the Lie algebra of G all over the group G. One can choose k such that ω defines actually an (possibly trivial) element of $H^3(G,\mathbb{Z})$ (see section 2.1.6). This three-form is invariant under the adjoint action of G, and therefore defines an element of $\tilde{\omega} \in H^3(\mathrm{Ad}(\tilde{P}),\mathbb{Z})$ which restricts to ω on each fiber. A simple computation now shows that $r_{\tilde{B}}(\tilde{B}^g) = \int_M g^*\tilde{\omega}$, which is an integer, because $g^*\tilde{\omega}$ is an element of integral cohomology and evaluating such an element on a three manifold without boundary always gives an integer. We conclude that k must sometimes be restricted to those values for which $\tilde{\omega}$ is an element of integral cohomology (so that upon quantizing the model everything is independent of the choice of extension), but if for instance $\tilde{\omega} = 0$ in $H^3(\mathrm{Ad}(\tilde{P}),\mathbb{Z})$ for a $k \neq 0$, k can be taken arbitrarily.

5.3.1. Classical Actions for Chiral and Covariant Gravity on Higher Genus

With the generalized WZNW action, it is straightforward to write down the form of the classical chiral and covariant actions. The chiral actions depend on a choice of basepoint, but in the covariant action the basepoint dependence drops out. This is a natural thing to happen (cf. [272]), since the covariant action cannot be factorized in a holomorphic and anti-holomorphic part while keeping the $W, \bar{W}$ invariance.

We choose once and for all a fixed flat background connection B^0, representing a choice of regularization procedure. The chiral induced action (3.2.13) on an arbitrary

Riemann surface reads

$$\Gamma^{(0)}[A_\alpha; B^0] = kS^-_{wznw}(A; B^0)|_{kA_z=\xi\Lambda^+ + W + kB^0_z}. \tag{5.3.14}$$

The Ward identity (3.2.8) is replaced by

$$[\partial + B^0_z + \frac{1}{k}(\xi\Lambda^+ + \mathrm{ad}(W)), \bar\partial + X(\mu_\alpha; B_0)]|_{W^{a(\alpha)} \to c_\alpha \pi \frac{\delta\Gamma^{(0)}}{\delta\mu_\alpha}} = 0, \tag{5.3.15}$$

where $X(\mu_\alpha; B^0)$ is a modified version of (2.2.26). If all nonzero components of B^0_z have degree less than or equal to zero, then an explicit expression for X can be given,

$$X = \frac{1}{1 + \xi^{-1}L(k\partial + k\mathrm{ad}(B^0_z) + \mathrm{ad}(W))}(F + \frac{k}{\xi}L(\bar\partial B^0_z)). \tag{5.3.16}$$

The anti-chiral induced action is

$$\Gamma^{(0)}[\bar A_\alpha; B] = kS^+(A; B)|_{kA_{\bar z}=\bar\xi\bar\Lambda^+ + \bar W + kB^0_{\bar z}}, \tag{5.3.17}$$

and the full covariant action is given by the following extremely simple expression, in which the basepoint no longer appears

$$\Gamma^{(0)}[A_\alpha; B] + \Gamma^{(0)}[\bar A_\alpha; B] + \Delta\Gamma^{(0)}[A_\alpha; \bar A_\alpha; G; B] = kS^-_{wznw}((A_1)^G, A_2) \tag{5.3.18}$$

where $(A_1)^G$ is the gauge transform of A_1 and

$$\begin{aligned} k(A_1)_z &= \xi\Lambda^+ + W + kB^0_z, \\ k(A_2)_{\bar z} &= \bar\xi\bar\Lambda^+ + \bar W + kB^0_{\bar z}. \end{aligned} \tag{5.3.19}$$

The same framework can also be used to write down the correct actions for constrained and gauged WZNW models on an arbitrary Riemann surface, if the principal G bundle is non-trivial.

5.3.2. Example: Gravity

Let us demonstrate what this implies for gravity. We take a basepoint

$$B^0_z = \begin{pmatrix} b_0 & 0 \\ b_- & -b_0 \end{pmatrix} \tag{5.3.20}$$

so that

$$B_z^0 + \frac{1}{k}(\xi\Lambda^+ + \mathrm{ad}(W)) = \begin{pmatrix} b_0 & \frac{\xi}{k} \\ b_- + \frac{T}{\xi} & -b_0 \end{pmatrix} \tag{5.3.21}$$

and

$$X(\mu; B) = \begin{pmatrix} b_0\mu + \frac{1}{2}\partial\mu & \frac{\xi}{k}\mu \\ b_-\mu + \frac{1}{\xi}\mu T + \frac{k}{\xi}\bar{\partial} b_0 - \frac{k}{\xi}\mu\partial b_0 - \frac{k}{\xi} b_0 \partial\mu - \frac{k}{2\xi}\partial^2\mu & -b_0\mu - \frac{1}{2}\partial\mu \end{pmatrix}. \tag{5.3.22}$$

The Ward identity that follows from the zero-curvature equation is

$$(\bar{\partial} - \mu\partial - 2\partial\mu)(T + \xi b_- + k b_0^2 - k\partial b_0) = -\frac{k}{2}\partial^3\mu. \tag{5.3.23}$$

This form is equivalent to the Ward identity derived in [216], if one takes $b_0 = \frac{1}{2}\partial \log \rho$, and b_- a quadratic differential such that the sum of the projective connection $kb_0^2 - k\partial b_0$ and ξb_- is a holomorphic projective connection.

The explicit form of the covariant action on an arbitrary Riemann surface is easy to obtain in this framework [47]. The result agrees with the expressions obtained in [206, 330, 336] (for the generalization to supergravity on super Riemann surfaces, see [2]). The modifications with respect to the actions on the complex plane are quite modest. The basically involve the introduction of certain background projective connections, and the covariantization of the differential operators that occur in the action. Covariantization of differential operators can also serve as a starting point to study W algebras, see [119, 134, 154]. A construction of Toda field theories on Riemann surfaces that is in spirit closely related to the constructions in this chapter is given in [5].

5.4. Remarks

5.4.1. The Geometrical Description of W Algebras

In the literature, several attempts at a geometrical description of W algebras have appeared, and we want to indicate briefly what the relation is with the approach in this chapter. The advantage of our approach is that it makes maximal use of the underlying group theoretical structure of W algebras, and that the moduli space is relatively easy to obtain. However, an interpretation of W transformations as the natural co-ordinate transformations in some W-superspace has not yet been found.

In [153, 297] W geometry is connected with the geometry of surfaces embedded in certain target spaces. The connection with constrained connections is most easily understood by looking at the differential operators $\partial + \Lambda^+ + W$. The vector bundle V has a natural filtration

$$V \supset \Lambda^+(V) \supset (\Lambda^+)^2 V \supset \cdots \supset (\Lambda^+)^r V \supset 0, \tag{5.4.1}$$

where r is the largest integer such that $(\Lambda^+)^r V \neq 0$. We denote $(\Lambda^+)^r V$ by V_{red}. Two sections f, g of V_{red} are equivalent at $z \in \Sigma$ if $(1 + L\partial)^{-1} f = (1 + L\partial)^{-1} g$ in z. The equivalence classes of the sections of V_{red} over z give a generalized jet, which patch together to the jet bundle Γ^Λ. An admissible local frame of V is a collection of n independent sections ψ_i, where n is the rank of V, such that the matrix which has the ψ_i as columns is an element of G, and such that each section can be written as $(1+L(\partial+b))^{-1}\psi_i^{red}$, with $\psi_i^{red} \in V_{red}$ and $b \in B_- \equiv \exp(\mathbf{g}_- \oplus \mathbf{g}_0)$. Two local admissible frames $\{\phi_i\}, \{\psi_i'\}$ are equivalent if there is an $n \in \exp(\mathbf{g}_-)$ such that for each i, $\psi_i' = n\psi_i$. Then the equivalence classes of admissible frames are in one-to-one correspondence with a set of W fields, and deformations of these frames correspond to W transformations. Alternatively, this can be described in terms of deformations of flags of type (5.4.1) for the jet bundle Γ^Λ. From this point of view, W geometry has been discussed in [235].

The nice feature of the standard W_N algebras is that there is a natural gauge choice for the space of admissible frames modulo equivalence. Take an arbitrary section of $\mathbb{P}(V_{red}^{\oplus n})$, the projectivization of the direct sum of n copies of V_{red} (which is one-dimensional), say $(\psi_1, \ldots, \psi_n)$ in homogeneous co-ordinates, and define the frame by $((1 + L\partial)^{-1}\psi_1, \ldots, (1 + L\partial)^{-1}\psi_n)$. This frame defines a set of W fields. The fact that V_{red} is globally non-trivial on a Riemann surface of genus $h \neq 1$ makes it in general impossible to view $\psi_1, \ldots \psi_n$ as co-ordinates on $\mathbb{CP}^n$, so that the interpretation of W 'surfaces' as Riemann surfaces immersed in $\mathbb{CP}^n$ is not obvious. If any such interpretation exists, it is probably closely related to the branched covers of the Riemann surface that arise as the solution space of the equation $\det(x - \Lambda^+ - W) = 0$, and are subspaces of the cotangent bundle of the Σ [168].

A related parametrization of W fields by sections ψ_i has been used to compute the symplectic leaves of the W algebra on the circle [201].

A different approach to W geometry has been proposed in [184]. The idea is to take a W algebra that is realized in terms of one free scalar field ϕ (such realizations will usually be reducible), and to gauge the $W \times \bar{W}$ symmetry as in section 4.1. The resulting action is written as

$$\int d^2x\, F(x^\alpha, \partial_\alpha \phi), \tag{5.4.2}$$

with

$$F(x^\alpha, y_\alpha) = \sum_{n\geq 2} \frac{1}{n} A^{\alpha_1\ldots\alpha_n}(x^\alpha) y_{\alpha_1}\ldots y_{\alpha_n}. \tag{5.4.3}$$

The symmetric tensor fields $A^{\alpha_1\ldots\alpha_n}$ contain the W gauge fields. The function F satisfies a complicated set of constraints, that can be obtained from the much simpler Hamiltonian formulation of the gauged action via a Legendre transformation. The constraints on F are translated in constraints on a family of Kähler potentials K_x defined by

$$K_x(\zeta, \bar\zeta) = F(x^\alpha, \zeta_\alpha + \bar\zeta_\alpha) \tag{5.4.4}$$

where $\zeta_\alpha, \bar\zeta_\alpha$ are complex co-ordinates on $\mathbb{R}^4$. The metrics corresponding to these Kähler potentials describe self-dual geometries on $\mathbb{R}^4$ with two Killing vectors. This looks a lot like our discussion in section 5.2.5, where we examined the relation between Higgs bundles and self-duality equations. It would be interesting if we could make this connection more precise. The relation between the self-duality equations in four dimensions and two-dimensional theories has been considered from different points of view in [164, 10].

5.4.2. The Moduli Space for a Surface with Punctures

In this section we compute the dimension of the moduli space in the presence of marked points. For gravity, the answer is well-known, $\dim \mathcal{M}_{g,n} = 3g-3+n$. The n extra moduli correspond to the locations of the marked points. The vector field $\frac{\partial}{\partial z}|_{z=z_i}$ moves the marked point located at $z=z_i$ around, and can be seen as a tangent vector to the moduli space. Thus, the moduli space in the presence of marked points is not the quotient of the space of complex structures modulo diffeomorphisms, but modulo a subgroup of the group of diffeomorphisms, obtained by modding out the vector fields $\frac{\partial}{\partial z}|_{z=z_i}$ from the group of all diffeomorphisms. The vector field $\frac{\partial}{\partial z}$ is locally a section of K^{-1}. In the SL_2 picture, the parameters of gauge transformations transform as sections of $\mathrm{ad}(P)$, and K^{-1} corresponds to the top-right component. The remaining two components span a parabolic subalgebra b of sl_2, and modding out certain vector fields at $z=z_i$ can in the bundle language be rephrased by saying that at $z=z_i$ the bundle is reduced to B, the parabolic subgroup corresponding to b.

To generalize this to W gravity, we need to know what the generalization of the vector field $\frac{\partial}{\partial z}|_{z=z_i}$ is to 'W punctures'. Since all negative powers of K have an interpretation as a bundle of differential operators, we take as the generalization of the vector fields for gravity those gauge transformations whose parameter at $z=z_i$ is an element of the subalgebra n_+ of $\mathbf{g}$ corresponding to negative powers of K, or to a positive eigenvalue of $\mathrm{ad}(t_0)$. The corresponding parabolic algebra is the subalgebra b of $\mathbf{g}$ consisting of eigenvalues less than or equal to zero of t_0. The moduli space for W gravity with marked points is the same as for the moduli space without marked points, but in addition we require that the corresponding bundle P has a reduction to B at all marked points z_i (cf. [171, 239]).

This definition is expressed in terms of gauge transformations. To find out what this means for W-transformations, we have to push this definition through the homotopy contraction that related the gauge and W transformations. The parameters ϵ_α of the W transformations transform as a section of $\Pi_{\ker \mathrm{ad}\Lambda^+}(\mathrm{ad}(P)) \simeq \oplus_\alpha K^{1-h_\alpha}$. The W transformations that should be seen as tangent vectors to the moduli space have parameters that are in the neighborhood of z_i given by

$$\epsilon_\alpha = \sum_{l=0}^{[h_\alpha - 3/2]} \epsilon_\alpha^{(l)} (z - z_i)^l. \tag{5.4.5}$$

The dimension of the moduli space is

$$\begin{aligned} \dim \mathcal{M}_{g,n} &= (g-1)\dim(G) + n\dim(n_+) \\ &= (g-1)\dim(G) + \frac{n}{2}(\dim(G) - d_0) \end{aligned} \tag{5.4.6}$$

5.4.3. From Teichmüller space to the Moduli Space

The final step in finding the moduli space for W algebras is to take the quotient of the Teichmüller spaces we computed so far (although we called them moduli spaces) by the action of the modular group. Unfortunately, this is not so straightforward as it seems, because for this one needs to know the action of the modular group on $H^0(\Sigma; K^{h_\alpha})$. Since we worked in a fixed complex structure all the time, the action cannot simply be induced from its action on Σ. We have to specify the embedding of the ordinary Teichmüller space in the W Teichmüller space. The most obvious way to do this is via the sl_2 embedding. In that case, W moduli space is simply given by a bundle over ordinary moduli space, whose fiber is the W moduli space restricted to $T = 0$. However, the details of this procedure are not yet entirely clear to us.

For arbitrary chiral algebras $\mathcal{A}$, the analysis for W algebras suggests the following result for the Teichmüller space: it is given by the vector space of holomorphic $\mathcal{A}$ fields, modulo the transformations generated by the spin one currents in $\mathcal{A}$ with constant parameter. The moduli space is a bundle over ordinary moduli space with fiber given by $\mathcal{A}$ Teichmüller space restricted to $T = 0$. If $\{h_\alpha\}$ denotes the set of weights of the generators of $\mathcal{A}$, the dimension of the moduli space is

$$\dim \mathcal{M}^{\mathcal{A}}_{g,n} = \sum_\alpha (g-1)(2h_\alpha - 1) + n \sum_\alpha [h_\alpha - \frac{1}{2}]. \tag{5.4.7}$$

6

Towards an Exact Solution?

In the previous chapters we have focused our attention on the construction of the action that describes (induced) W gravity coupled to W matter. One of the main results is given in section 4.3.1, where we showed that in the conformal gauge the action consists of three pieces: a matter part, which can be represented by a constrained WZNW model at level $k_c - 2h$, a gravitational part, represented by a constrained WZNW model at level $-k_c$, and a free ghost part. The total central charge is equal to zero. The obvious next step is investigate the properties of this action, in particular one may investigate whether it describes a sensible theory, what is the spectrum and what are the correlation functions.

For gravity, there has been tremendous progress in answering these questions, especially when the matter theory is a minimal model or a $c = 1$ theory. There are three different ways in which one can study gravity coupled to matter, each having its own merits. These are the continuum approach, the topological approach and the discrete approach. In this chapter we take a closer look at each of these, and comment on their possible extensions to W gravity.

6.1. Continuum Approach

6.1.1. Gravity

Consider the action of the constrained SL_2 WZNW theory at level k,

$$S = kS^{-}_{wznw}(g) - \frac{1}{\pi}\int d^2z\, \bar{A}(J^+ - \xi). \tag{6.1.1}$$

It can be argued that this action, when integrating out the gauge field, becomes equivalent to the action of a free scalar field [273, 95, 185, 189]

$$S = \frac{1}{8\pi}\int d^2x\,\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{i\alpha_0}{4\pi}\int d^2x\,\sqrt{\hat{g}}R_{\hat{g}}\phi \tag{6.1.2}$$

with background charge

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{k+2}} - \sqrt{k+2}\right). \tag{6.1.3}$$

In (6.1.2) we restored the dependence on an arbitrary background metric $\hat{g}$, to make clear that the free field has a background charge. We ignore the vertex operator that should be added to (6.1.2) to reduce its chiral algebra to the Virasoro algebra (cf. section 2.1.4). More precisely, the chiral algebra of (6.1.1), which is the Virasoro algebra, coincides with the centralizer of this vertex operator acting on the chiral algebra of (6.1.2) under the equivalence (6.1.1)↔(6.1.2). Using this equivalence and the results of section 4.3.1. we can express both the matter part and the gravity part of the covariant action (in the conformal gauge) in terms of one scalar field each. Denoting these by ϕ_L (the Liouville field) and ϕ_M (the matter field), the covariant action reads (with $\hat{g} \sim dz\, d\bar{z}$)

$$S_{cov} = \frac{1}{2\pi}\int d^2z\,(\partial\phi_M\bar{\partial}\phi_M + \partial\phi_L\bar{\partial}\phi_L) + \frac{1}{4\pi}\int d^2z\,\sqrt{\hat{g}}R_{\hat{g}}(i\alpha_M\phi_M + \alpha_L\phi_L) + \frac{1}{\pi}\int d^2z\, b\bar{\partial}c. \tag{6.1.4}$$

The values of the background charges are $\alpha_M = \sqrt{(1-c)/12}$ and $\alpha_L = \sqrt{(25-c)/12}$. The action has problems for $c > 1$, because tachyonic divergencies appear in the partition function for for $c > 1$ [287, 213]. See also the remark below (6.1.9). Alternatively, the breakdown of this approach for $c > 1$ can be explained by the fact that in this regime the branched-polymer configurations dominate the functional integral over 2-d metrics [105]. To avoid all these problems, we assume henceforth that $c \leq 1$.

The action (6.1.4) has been extensively studied, especially the Liouville part of it. We refer the reader to [79, 152, 151, 259] and the review papers [234, 288, 287, 259, 6, 156, 84, 82, 174] for a more detailed discussion.

In section 4.3.2 we explained that (6.1.4) is a gauge-fixed version of the covariant action, and that there is a BRST operator associated to this gauge fixing. It is given by

$$Q = \oint \frac{dz}{2\pi i}\, c(T_M + T_L + \frac{1}{2}T_{gh}) \tag{6.1.5}$$

where $T_{L,M,gh}$ denote the energy-momentum tensors of the respective sectors of (6.1.4). The BRST operator is nilpotent because the sum of the central charges $c_M + c_L + c_{gh} = 0$. The physical states of (6.1.4) are given by the BRST cohomology of Q. This BRST cohomology has been computed in [223, 61], see also [316, 323, 266, 204, 246]. For $c = 1$ there are only BRST non-trivial physical states at a finite set of ghost numbers. This is different for $c < 1$. For the (p, q) minimal model (cf. (2.1.14)) with

$$c = c(p, q) = 1 - \frac{6(p-q)^2}{pq} \tag{6.1.6}$$

the cohomology is spanned by operators, labeled by an index n, of the form

$$\mathcal{O}_n e^{\alpha_n \phi_L}, \qquad \frac{\alpha_n}{\gamma} = \frac{p+q-n}{2q}, \qquad n \geq 1,\, n \neq 0 \bmod p,\, n \neq 0 \bmod q, \tag{6.1.7}$$

where * $\gamma = \frac{1}{\sqrt{12}}(\sqrt{25-c} - \sqrt{1-c})$, and $\mathcal{O}_n$ is an operator made of matter, ghosts and derivatives of ϕ_L. The ghost number of $\mathcal{O}_n$ depends linearly on n.

The simplest explicit example of operators that are in the BRST cohomology are operators of the type (the 'Distler-Kawai' states [97])

$$V = c\, e^{i\lambda_M \phi_M}\, e^{\lambda_L \phi_L}. \tag{6.1.8}$$

A straightforward calculation shows that $[Q, V] = 0$ if V has vanishing conformal weight,

$$\frac{\lambda_M^2}{2} + \alpha_M \lambda_M - \frac{\lambda_L^2}{2} + \alpha_L \lambda_L - 1 = 0. \tag{6.1.9}$$

We rewrite this equation as

$$\Delta_M - \frac{1}{2}(\lambda_L - \alpha_L)^2 + \frac{1-c}{24} = 0, \tag{6.1.10}$$

which can be read as an Euclidean on-shell condition in target space [156]

$$\frac{1}{2}E^2 + \frac{1}{2}p^2 + \frac{1}{2}m^2 = 0. \tag{6.1.11}$$

Here, $E = i(\lambda_L - \alpha_L)$. We see that the condition for no tachyonic divergencies, *i.e.* $p^2 + m^2 > 0$, reads

$$\min_M(\Delta_M + \frac{1-c}{24}) \geq 0. \tag{6.1.12}$$

For unitary theories, $\min_M \Delta_M = 0$, and beyond the $c = 1$ barrier the theory suffers from tachyonic divergencies.

The expressions for the gravitational dressed scaling dimensions derived in [207] can be found from (6.1.9). They are defined by

$$\Delta = 1 - \frac{\lambda_L(h)}{\gamma} \tag{6.1.13}$$

*The parameter γ is defined such that the operator $e^{\gamma\phi_L}$ has weight one; this operator plays the role of the cosmological constant operator.

where γ was defined under (6.1.7), and $\lambda_L(h)$ is the solution of (6.1.9) with $\lambda_M^2/2+\alpha_M\lambda_M$ replaced by the conformal weight h of the matter part of the operator V.

The operators (6.1.8) have conformal weight zero, and it makes sense to consider the correlation function of products of these operators located at certain fixed points on the surface. So far we tacitly assumed that it is possible to completely gauge away the Beltrami differential μ. This is true on the complex plane, but not on a general surface. On a general surface one can only impose the gauge-fixing condition $\mu = \hat{\mu}$, with $\hat{\mu}$ representing a point in the moduli space $\mathcal{M}_{g,n}$. The path integral includes an integration over the moduli space. Since the coupling of the covariant action (4.2.30) to the moduli $\hat{\mu}$ is quite complicated, it is difficult to compute correlation functions in this way. For this reason, progress has been limited to the computation of correlation functions for genus 0 and 1 surfaces [163, 278, 266]. In genus zero, one can replace the integration over the moduli space $\mathcal{M}_{0,n}$ by an integration over the locations of the vertex operators. To do this, we first have to change the operator V into a BRST invariant operator $V^{(1)}$ of conformal weight one, in order to be able to integrate it over the surface. This 'change of picture' is accomplished via the descent equation (cf. [321])

$$[Q, V^{(1)}] = [\oint \frac{dz}{2\pi i}\,[Q,b], V]. \tag{6.1.14}$$

Notice that $[Q,b]$ is nothing but the total stress-energy tensor of the system, and its contour integral is the L_{-1} Fourier mode that acts as $\partial/\partial z$. The operator $V^{(1)}$ is given by V without the c-ghost,

$$V^{(1)} = e^{i\lambda_M\phi_M}e^{\lambda_L\phi_L}. \tag{6.1.15}$$

On the sphere, the computation of correlation functions of products of vertex operators is now reduced to the computation of correlators of operators $V^{(1)}$, together with an integration over the location of these operators. In the latter picture, no explicit knowledge of the moduli is needed, which is the reason for the solvability of the genus-zero correlation functions.

Let us now discuss what extra complications occur if we try to generalize this discussion to W gravity.

6.1.2. Generalization to W Gravity: BRST Operator

The first problem we have to deal with is the construction of the BRST operator for W gravity coupled to W matter (see also section 4.3.2). For critical ($c_M = 100$) matter, the gravitational W algebra decouples, and the construction of the BRST operator can proceed as in [300, 281]. However, for non-critical W gravity coupled to W matter, it is a priori not at all clear that such an operator should exists. The problem is the non-linearity of the W algebra. For linear algebras with central terms, one can construct a new algebra from two commuting copies of the original algebra by taking the diagonal subalgebra. The central charges simply add up. For non-linear algebras, this statement is definitely not true. For W gravity coupled to W matter, the algebra we want gauge and which gives us the BRST operator is generated by the sum of the matter and gravity W algebra, and does not have the structure of a W algebra. On the other hand, the fact that we were able to construct a covariant action for W gravity guarantees that a BRST operator must exist, obtained by gauge fixing the covariant action. So what is going on here?

Actually, it is not necessary to have a closed non-linear algebra to be able to write down a BRST operator. According to [133, 167], it is sufficient to have a closed algebra of soft type, *i.e.* with an closed algebra with field-dependent structure constants. We claim that this is precisely what is going on here, at least classically: the sum of the W generators of the gravity and matter sector form a closed algebra with field-dependent structure constants.

To demonstrate this, we employ the general expression (2.2.27). Consider two copies of the same W algebra, obtained from the WZNW model at level k_i with constraints $\xi_i\Lambda^+$, $i = 1, 2$. We compute

$$\begin{aligned}
&\{\textstyle\int \mathrm{Tr}(F(W_1+W_2)), W_1+W_2\}_{\rm dirac} = \\
&\left\{(k_1\partial+\mathrm{ad}W_1)\tfrac{1}{1+\xi_1^{-1}L(k_1\partial+\mathrm{ad}W_1)} + (k_2\partial+\mathrm{ad}W_2)\tfrac{1}{1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2)}\right\}F = \\
&\left\{\tfrac{1}{1+(k_1\partial+\mathrm{ad}W_1)\xi_1^{-1}L}(k_1\partial+\mathrm{ad}W_1) + (k_2\partial+\mathrm{ad}W_2)\tfrac{1}{1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2)}\right\}F = \\
&\tfrac{1}{1+(k_1\partial+\mathrm{ad}W_1)\xi_1^{-1}L}\left\{(k_1\partial+\mathrm{ad}W_1)(1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2))\right. \\
&\qquad\qquad \left.+(1+(k_1\partial+\mathrm{ad}W_1)\xi_1^{-1}L)(k_2\partial+\mathrm{ad}W_2)\right\}\tfrac{1}{1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2)}F = \\
&\tfrac{1}{1+(k_1\partial+\mathrm{ad}W_1)\xi_1^{-1}L}\left\{(k_1+k_2)\partial+\mathrm{ad}(W_1+W_2)\right. \\
&\qquad\qquad \left.+(k_1\partial+\mathrm{ad}W_1)L(k_2\partial+\mathrm{ad}W_2)(\xi_1^{-1}+\xi_2^{-1})\right\}\tfrac{1}{1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2)}F.
\end{aligned} \tag{6.1.16}$$

Taking $k_1 + k_2 = \xi_1^{-1} + \xi_2^{-1} = 0$, this expression simplifies to

$$\frac{1}{1+(k_1\partial+\mathrm{ad}W_1)\xi_1^{-1}L}\mathrm{ad}(W_1+W_2)\frac{1}{1+\xi_2^{-1}L(k_2\partial+\mathrm{ad}W_2)}F. \tag{6.1.17}$$

We therefore reach the important conclusion that for $k_1 = -k_2$ and $\xi_1 = -\xi_2$ the sum of the two W algebras forms a closed algebra with field dependent structure constants. Remakably, (4.3.7) shows that (classically) the gravitational constrained WZNW model has k and ξ with the opposite sign compared with those of the matter constrained WZNW model.

In conclusion, the classical BRST operator can be obtained in the standard fashion from

$$\begin{aligned} S &= kS^-_{wznw}(g_1) - \frac{1}{\pi}\int d^2z\,\bar{A}_1(J_1 - \xi\Lambda^+) - kS^-_{wznw}(g_2) - \frac{1}{\pi}\int d^2z\,\bar{A}_2(J_2 + \xi\Lambda^+) \\ &\quad + \frac{1}{\pi}\int d^2z\,\mathrm{Tr}(F(\mu_\alpha)(W_1 + W_2)), \end{aligned} \tag{6.1.18}$$

by gauge fixing the symmetries generated by $W_1 + W_2$. In (6.1.18), W_1 and W_2 contain the gauge invariant polynomials of the respective constrained WZNW models.

To find the quantum BRST operator, one can use the classical one as inspiration. Often, it is sufficient to replace the coefficients in front of each term in the classical BRST operator by an arbitrary one, followed by demanding that Q^2 on the quantum level. An explicit expression for the W_3 BRST operator is given in [48, 40, 31]. Further properties of the BRST operator are studied in [30].

6.1.3. The Spectrum

The spectrum of W_3 gravity coupled to W_3 matter and of the critical ($c_{\text{matter}} = 100$) W_3 string has been analyzed in [40, 39, 48, 276, 80, 270, 269, 136, 230, 63]. Due to the complicated structure of the W_3 BRST operator, the determination of the spectrum is is a difficult problem, and has not yet completely been solved.

For the W algebras that are not obtained from the principal sl_2 embedding, we do not even know what the irreducible representations look like, and nothing is known about the spectrum of these theories.

A class of states that can be written down for all W gravity theories are the generalizations of the states (6.1.8). Any weight Λ of $\mathbf{g}$ gives a state in a reduced WZNW theory based on $\mathbf{g}$. Combining two such states for the W matter and the W gravity part gives

$$|\psi\rangle = |\Lambda^M\rangle \otimes |\Lambda^L\rangle \otimes |0\rangle_{gh} \tag{6.1.19}$$

where $|0\rangle_{gh}$ is a state in the ghost Hilbert space that will be specified in a moment. The conformal weight of the state $|\psi >$ can be computed using the expressions for the energy-momentum tensors of the two constrained WZNW theories,

$$h_\psi = \frac{(\Lambda_M|\Lambda_M + 2\rho)}{2(k_c - h)} - (t_0|\Lambda_M) + \frac{(\Lambda_L|\Lambda_L + 2\rho)}{2(h - k_c)} - (t_0|\Lambda_L) + h_{gh} \tag{6.1.20}$$

Introducing the 'background charges' $\alpha_M = \rho - (k_c - h)t_0$ and $\alpha_L = \rho + (k_c - h)t_0$, this equation can be rewritten as

$$h_\psi = \frac{(\Lambda_M + \alpha_M|\Lambda_M + \alpha_M)}{2(k_c - h)} + \frac{(\Lambda_L + \alpha_L|\Lambda_L + \alpha_L)}{2(h - k_c)} + 2(\rho|t_0) + h_{gh}. \qquad (6.1.21)$$

If some Weyl group element $w \in W$ exists such that

$$w(\Lambda_M + \alpha_M) = \Lambda_L + \alpha_L \qquad (6.1.22)$$

then the first two terms in (6.1.21) cancel against each other, and

$$h_\psi = 2(\rho|t_0) + h_{gh}. \qquad (6.1.23)$$

Now there is a ghost c_α of weight $1 - h_\alpha$ for every generator of the W_α of the W algebra. If $|0\rangle_{sl_2}$ denotes the standard sl_2 invariant ghost vacuum, then the ghost Hilbert space contains the state

$$|0\rangle_{gh} \equiv \prod_\alpha \prod_{i=0}^{[h_\alpha - 3/2]} \partial^i c_\alpha |0\rangle_{sl_2} \qquad (6.1.24)$$

and no states with lower conformal weight, since the ghosts anti-commute. The conformal weight of $|0\rangle_{gh}$ is easily computed, it is

$$h_{gh} = -\sum_{\alpha\in\Delta^+} (t_0|H_\alpha) = -2(t_0|\rho). \qquad (6.1.25)$$

Combining everything, we have obtained a state of conformal weight zero for every weight Λ_M together with a Weyl group element w that specifies the W gravitational dressing of the matter field. For W_N gravity these states are indeed in the BRST cohomology for generic values of Λ_M [63]. For gravity, the Weyl group contains two elements, plus one and minus one. If Λ_M is in the fundamental Weyl chamber, there are physical reasons to keep only the state with $w = +1$, known as the Seiberg bound [288, 287]. See also section 3.6 in [156]. One expects a similar criterion to hold for W gravity, but its precise form is unknown [39]. Another open problem is to prove that the states (6.1.19) are in the BRST cohomology for nonprincipal embeddings†. In addition the BRST cohomology can have additional states at other ghost numbers or with an other structure. The states with ghost number zero and conformal weight zero constitute the so-called ground ring [323, 316]. The structure of the ground ring for

†Before doing this it would be nice to have a good description of the representation theory of these W algebras, and to know which minimal models (or RCFT's) can be made from them

W_3 gravity is conjectured in [63]. To prove and extend these results to arbitrary W algebras is yet another open problem. Finally, let us mention two different approaches to the computation of the spectrum of W gravity plus W matter. The first one is to use the relation with G/G models, which we discuss in section 6.2.3. The second one is to compute the spectrum in a different gauge than the conformal gauge, for instance in the light-cone gauge [207]. In the light-cone gauge, the BRST operator differs significantly from the one in the conformal gauge. For gravity, the spectrum in the light-cone gauge has been studied in [187] and more recently it has been shown [233] that the spectrum agrees with the spectrum obtained in the conformal gauge. The BRST operator for W_3 gravity in the light-cone gauge, together with a preliminary investigation of its cohomology, is given in [48, 160].

6.1.4. Correlators

Correlators of states of the type (6.1.19) can in principle be computed by going to a free field realization of the Toda theories that are part of the covariant action, and by using free field techniques similar to those employed for gravity. These correlation functions include an integral over the moduli space of a punctured W surface. This can also be seen from the fact that the ghost number of the operators (6.1.19) is equal to the number of extra moduli that are introduced by a puncture in a W surface (section 5.4.2). The structure of the ghost part of the operators (6.1.19) closely resembles the structure of the vector fields we modded out in section 5.4.2 to obtain the punctured moduli space. This correspondence does not hold for, for instance, ground ring operators, or more general for operators whose ghost structure is not given by (6.1.24). If such operators have conformal weight zero, we can consider their correlation functions by attaching them to some marked points on the Riemann surface. The computation of the correlation function now involves an integral over a modified moduli space, whose structure resembles the ghost structure of the operators. These moduli spaces correspond to the case where, in the terminology of section 5.4.2, the bundle P has reductions to different parabolic subalgebras B_i at the marked points z_i [239]. For ground ring generators, this parabolic subalgebra is $\mathbf{g}$ itself, so that the punctures do not introduce any extra moduli.

For gravity, we could trade of the integral over the genus-zero moduli space for an integral over the locations of the punctures. For W gravity, a similar procedure may be conceivable once we have a better understanding of the structure of the moduli space of a surface with punctures. If an explicit description of this moduli space were available, one could write down generalized descent equations, and replace the integral over the moduli space by an integral over the location of the operator plus some extra internal space, representing part of the W moduli space. This would at the same time answer the question what W geometry is, because the extra internal space is precisely the W 'superspace', in which all the W transformations have a clear geometrical meaning.

Unfortunately, this final goal is still far ahead.

For critical W_3 strings, correlation functions in genus zero were computed in [136, 230]. These computations are, however, not generic. The rely on a special realization of the $c_M = 100$ W_3 algebra, in terms of a scalar field φ with background charge and a matter system with $c_M = \frac{51}{2}$. For this special realization of the W_3 algebra the correlation functions can be expressed in terms of correlation functions of the Ising model and a $c = \frac{51}{2}$ Virasoro string, and no special new W techniques are needed to compute them.

6.2. Topological Approach

The second approach to the solution of gravity we discuss is the topological approach. In this approach, the independence of the metric is built in from the start, and does not arise from some 'anomalous quantization' procedure like the one we used to compute the covariant action. A whole zoo of topological field theories exists, each of which seems to be equivalent to theories of ordinary ($c < 1$) matter coupled to ordinary gravity. Most of the proofs of the equivalences of these theories consist of a computation of the spectrum and the correlation functions. This is a very elaborate method, and it would be nice to understand the equivalence of all these theories on some higher level. Let us now turn to a discussion of some of these models. (see also [217])

6.2.1. Topological W Conformal Field Theory

A topological quantum field theory is a quantum field theory in which all the correlation functions are independent of the metric [317, 318]. In the topological field theories of cohomological type [44], this is guaranteed by the presence of a nilpotent BRST-like operator, such that the physical states are in one-to-one correspondence with states in the BRST cohomology, and such that the stress-energy tensor is BRST-exact,

$$T_{\alpha\beta} = [Q, G_{\alpha,\beta}]. \tag{6.2.1}$$

The latter condition guarantees that all correlation functions are metric independent.

A subset of the set of topological field theories are the topological conformal field theories [93, 91]. These are characterized, in addition to being a topological field theory, by the fact that the stress-energy tensor is traceless. This means in particular (section 2.1.2) that the theory can be decomposed in a holomorphic and an anti-holomorphic sector. Usually, we will only focus on the holomorphic part of the theory, and denote the holomorphic part of the BRST operator, the stress-energy tensor and the fermionic spin 2 supercurrent $G_{\alpha\beta}$ by $Q, T(z)$ and $G^-(z)$ respectively. Furthermore, the BRST current will be denoted by $G^+(z)$, so that $Q = \oint \frac{dz}{2\pi i} G^+(z)$, and since Q is BRST exact, we can express it in terms of a $U(1)$ current, $G^+(z) = [Q, J(z)]$. The four fields $T, G^\pm$ and J form a twisted $N = 2$ superconformal algebra, also known as the topological conformal algebra [321, 317, 318, 108]. The OPE's of this algebra are given by

$$\begin{aligned}
\underbrace{T(z)\,T}(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
\underbrace{T(z)\,G^\pm}(w) &= \frac{(\frac{3}{2} \mp \frac{1}{2})G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}, \\
\underbrace{T(z)\,J}(w) &= \frac{-\hat{c}/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \\
\underbrace{J(z)\,J}(w) &= \frac{\hat{c}/3}{(z-w)^2}, \\
\underbrace{J(z)\,G^\pm}(w) &= \pm\frac{G^\pm(w)}{z-w}, \\
\underbrace{G^+(z)\,G^-}(w) &= \frac{\hat{c}/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{z-w}, \\
\underbrace{G^\pm(z)\,G^\pm}(w) &= 0.
\end{aligned} \tag{6.2.2}$$

The parameter $\hat{c}$ is the central charge of the untwisted $N = 2$ superconformal algebra that follows from (6.2.2) by taking $\hat{T} = T - \frac{1}{2}\partial J$.

The natural requirement for W gravity is that the correlators are invariant not only under a variation of the metric, but under an arbitrary variation of the 'W' metric. In other words, the holomorphic W fields must be BRST exact. For those W algebras obtained from an sl_2 embedding in sl_n, a natural $N = 2$ extension was given in section 2.2.15, and inspired by the definition of a topological conformal field theory, we define a topological W conformal field theory by the presence of a twisted version of these $N = 2$ W algebras. For other W algebras, a similar definition should exists, but we do not have an explicit description of the relevant $N = 2$ extensions.

A natural question arises, whether the theories of W matter coupled to W gravity are W topological. Naively, one would expect this to be the case, since the theory includes an integral over the W 'metrics'. In [91] it was observed that for ordinary gravity (see previous section) there is almost an $N = 2$ algebra, upon identifying the anti-ghost b with G^-. Later, it was realized that one can always add a total derivative to the BRST current and that this can be done in such a way that one recovers exactly the $N = 2$ algebra [143]. This observation was generalized to W_N gravity in [39, 40].

For gravity, it is easy to give the explicit form of the realization of the twisted $N = 2$

algebra. We introduce two parameters y_L, y_M, that satisfy (everything is in the notation of section 6.1.1)

$$(y_L - \alpha_L)^2 + (y_M - i\alpha_M)^2 = 0. \tag{6.2.3}$$

The four generators of the topological conformal algebra are

$$\begin{aligned} J &= (cb) - y_l \partial\phi_L - y_M \partial\phi_M, \\ G^+ &= c(T_L + T_M + T_{gh}) + \frac{1}{2}(1 - y_L^2 - y_M^2)\partial^2 c + y_L \partial(c\partial\phi_L) + y_M \partial(c\partial\phi_M), \\ G^- &= b, \\ T &= T_L + T_M + T_{gh}, \end{aligned} \tag{6.2.4}$$

and

$$\hat{c} = 3(1 - y_L^2 - y_M^2). \tag{6.2.5}$$

Similar explicit but extremely cumbersome expressions exist for the generators of the $N = 2$ super W_3 algebra in the case of W_3 gravity. In the next section we investigate whether the presence of the $N = 2$ structure can be understood more directly.

6.2.2. $N = 2$ Structure in W Gravity

We start with the classical action (6.1.18) that describes the holomorphic part of the (ungauged) classical covariant action for W matter coupled to W gravity. The action has three gauge invariances; two are generated by the gauge groups $G_- \equiv \exp \mathbf{g}_-$ for respectively g_1 and g_2, and the third one consists of the transformations generated by $W_1 + W_2$ that we will loosely speaking simply call W transformations, although they do not form a W algebra. Using a gauge fixing procedure as in section 3.3.2, one sees that the gauge invariant polynomials $W_{1,2}$ can replaced by the currents in $\ker \mathrm{ad}(\Lambda^-)$, if one modifies the $g_{1,2}$ and $\bar{A}_{1,2}$ transformation rules accordingly. This brings the $\int F(\mu_\alpha)(W_1 + W_2)$ term on the same footing as the two terms containing $\bar{A}_{1,2}$, in that the μ_α are now just some Lagrange multipliers that set the currents $W_1 + W_2$ equal to zero. The next step is to perform a BRST gauge-fixing of the action as in section 3.3.1, but only for the symmetries generated by $W_1 + W_2$. This dresses some of the currents with ghost contributions, and kills, if we denote the total current by $\mathcal{J}_+ = \mathcal{J}_1 + \mathcal{J}_2$, the highest and lowest weight components of all the sl_2 representations in which $\mathcal{J}_+$ branches under the sl_2 embedding. For sl_2 embeddings in sl_n under which the fundamental representation branches as $\underline{n} \to \oplus \underline{n_i}$, the surviving components of $\mathcal{J}_+$ are almost equal to the the tensor product of $\oplus \underline{n_i - 1}$. Finally, we have to consider the ghosts that arise from the BRST gauge fixing of the W symmetry, and the surviving currents altogether and map them into some superalgebra. For embeddings in sl_n, the above reasoning suggest that the proper superalgebra is $sl(n|n')$, the one we considered in section 2.2.15. The ghosts should be identified with some of the supercurrents. The remaining constrains imposed by $\bar{A}_{1,2}$ are to be imposed on the superalgebra. For these constraints, standard Hamiltonian reduction can be applied. If the constraints take the form as in section 2.2.15, we know that the resulting algebra will be an $N = 2$ extended W algebra. For W_N, this construction works and the resulting Hamiltonian reduction was given in [39]. It provides the explicit realization of the $N = 2$ algebra in terms of the Toda fields and the ghosts.

One word of caution, however. Since neither $\partial\phi_L$ nor $\partial\phi_M$ survives to the chiral algebra of the matter coupled to gravity system, the $U(1)$ current in (6.2.4) is strictly speaking not part of the symmetry algebra of the covariant action. An exception is when the cosmological constant vanishes, in that case both $\partial\phi_L$ and $\partial\phi_M$ are present in the chiral algebra. In the reasoning of the previous paragraph this problem occurs at the end, when the constraints that arise from the integration over $\bar{A}_{1,2}$ have to be imposed. These will not be exactly identical to those that give the $N = 2$ algebra. This implies that the action for W gravity coupled to W matter for generic value of the cosmological constant is a conformal perturbation of the underlying $N = 2$ theory, but not equivalent to it.

Whether or not this correspondence with $N = 2$ theories is useful for computations in general W matter systems coupled to W gravity is not clear. For special matter systems, namely those for which the Kac-Moody level of the matter sector $k_c = 2h + k$ with k an integer, it certainly is. From the gravitational point of view, those values correspond to the minimal models $(1, k+2)$; from the $N = 2$ viewpoint they correspond to the unitary $N = 2$ minimal models. While the structure of unitary $N = 2$ theories is relatively well-understood [248, 218], the situation for non-unitary representations is more difficult [275, 277]. The cohomology of the BRST operator (in that context, usually identified with $G^+_{-\frac{1}{2}}$) for unitary $N = 2$ theories gives the so-called chiral ring of these theories. This ring is isomorphic to $\mathbb{C}[x]/(x^{k+1})$. It does not contain all the states of the matter plus gravity theory, the gravitational descendants are missing. This is due the subtleties mentioned in the previous paragraph. The correspondence can be restored by a modification of the BRST operator of the $N = 2$ theory, which corrects for the change in sl_2 constraints that had to be made when passing from the standard formulation to the $N = 2$ formulation.

In summary, matter plus gravity is equivalent to $N = 2$ theories with a modified BRST operator, at least for some special matter systems. Next, we consider the relation between matter plus gravity and topological G/G models.

6.2.3. Relation with G/G models

The action of the G/H model is given by (cf. section 4.1.3)

$$S = kS^-_{wznw}(g) + \frac{k}{\pi}\int d^2z\,(-\mathrm{Tr}(A_z\bar{\partial}gg^{-1}) + \mathrm{Tr}(g^{-1}\partial gA_{bz}) + \mathrm{Tr}(A_zgA_{\bar{z}}g^{-1}) - \mathrm{Tr}(A_zA_{\bar{z}})), \tag{6.2.6}$$

where A is a H valued gauge field. The action of the supersymmetric G/H model is obtained by adding to (6.2.6) the term

$$\frac{1}{2\pi}\int d^2z\,\mathrm{Tr}(\psi(\bar{\partial} + \mathrm{ad}(A_{\bar{z}}))\psi + \bar{\psi}(\partial + \mathrm{ad}(A_z))\bar{\psi}) \tag{6.2.7}$$

where $\psi, \bar{\psi}$ are Weyl fermions with values in $\mathbf{g}/\mathbf{h}$, where $\mathbf{h}$ is the Lie algebra of H and we made an orthogonal decomposition $\mathbf{g} = \mathbf{h} \oplus (\mathbf{g}/\mathbf{h})$. If G/H is Kähler, the theory has an $N = 2$ supersymmetry, and describes the Kazama-Suzuki models [200]. In this case the Lie algebra $\mathbf{g}/\mathbf{h}$ can be decomposed in two closed subalgebras $\mathbf{g}/\mathbf{h}^+$ and $\mathbf{g}/\mathbf{h}^-$ such that Tr is zero when restricted to $\mathbf{g}/\mathbf{h}^\pm$. Denote by $\rho, \bar{\rho}$ the projections of ψ on $\mathbf{g}/\mathbf{h}^-$ and $\mathbf{g}/\mathbf{h}^+$ respectively, and by $\chi, \bar{\chi}$ the projections of $\bar{\psi}$ on $\mathbf{g}/\mathbf{h}^+$ and $\mathbf{g}/\mathbf{h}^-$. The extra term (6.2.7) can be rewritten as

$$S = \frac{1}{\pi}\int d^2z\,\mathrm{Tr}(\rho(\bar{\partial} + \mathrm{ad}(A_{\bar{z}}))\chi + \bar{\rho}(\partial + \mathrm{ad}(A_z))\bar{\chi}). \tag{6.2.8}$$

Since ψ, χ transforms as a section of $K^{1/2}$, so does ρ. The twisted G/H model is defined by a change of the spins of ρ, χ to one and zero respectively. This turns the theory into a topological field theory. From the algebraic point of view these topological field theories have been studied in [111]. The twist of the G/H model is the same twist that turns the $N = 2$ superconformal algebra in the topological conformal algebra (section 6.1.1).

Now consider the same model, but for the non-trivial principal G bundle defined in chapter 5. This involves a replacement of the WZNW actions by generalized WZNW actions. Furthermore, ρ transforms as a section of the restriction to $\mathbf{g}/\mathbf{h}^-$ of $\mathrm{ad}(P) \otimes K$, and χ as the appropriate restriction of $\mathrm{ad}(P)$. Following the arguments of [3], one can derive a quantum action for the twisted G/H model, if one ignores the possible moduli in the H valued gauge field (these can easily be included in the action, see [249]). Introduce group valued fields h^i for every simple component H^i of H, a vector valued scalar field $\boldsymbol{\varphi}$ for the abelian part of H, and $\mathbf{h}$ valued ghosts b_H, c_H. The quantum action is

$$\begin{aligned} S &= kS^-_{wznw}(g) + \sum_i(-k - h_G - h_{H^i})S^-_{wznw}(h^i) \\ &\quad + \frac{1}{2\pi}\int d^2z\,\mathrm{Tr}(\partial\boldsymbol{\varphi}\bar{\partial}\boldsymbol{\varphi} + \frac{1}{2}R_{\hat{g}}(\frac{\rho_G}{\sqrt{k + h_G}} + t_0\sqrt{k + h_G})\boldsymbol{\varphi}) \end{aligned}$$

$$+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(b_H\bar{\partial}c_H+\bar{b}_H\partial\bar{c}_H)+\frac{1}{\pi}\int d^2z\,\mathrm{Tr}(\rho\bar{\partial}\chi+\bar{\rho}\partial\bar{\chi}). \qquad (6.2.9)$$

In section 4.3.1 we showed that the full covariant action for W matter plus W gravity in the conformal gauge is given by a constrained WZNW model at level $k_c - 2h$, one at level $-k_c$ and a free ghost system corresponding to the generators of the W algebra. A BRST gauge fixing of the two constrained WZNW models as in section 3.3.1 leads to a sum of a WZNW action at level $k_c - 2h$, one at level $-k_c$, and the sum of three ghost actions. The three ghost actions can be combined into one ghost action with a G valued ghost field, with spin determined by the bundle $\mathrm{ad}(P)$. The resulting action is *exactly* identical to (6.2.9) with $H = G$, if we put $k = k_c - 2h$. Thus, on the level of actions, the gauge fixed action for W matter coupled to W gravity is identical to a twisted G/G model based upon the bundle P.

To establish the equivalence of the two theories, we need to compare the spectra of the two theories. Although the action is the same, the two BRST operators are completely different. Nevertheless, computations of the BRST cohomology in G/G models [177, 4, 298, 276] and in G/H models [3, 296, 250, 251, 249, 186, 322] show that starting with the proper modules for the G and H affine Lie algebras, the spectra of both theories coincide for the case $G = SL(2)$, and irrespective of whether $H = SL(2)$ or $H = U(1)$. Thus, gravity plus matter is equivalent to topological G/H models.

There is also a direct relation between G/G theories and $N = 2$ theories, they are related by a perturbation [127] by what in the gravity language is the cosmological constant operator. The same operator reduces the chiral algebra of the covariant action for gravity. In the presence of a cosmological constant, $\partial\phi_L$ is no longer part of the chiral algebra.

The ground ring of gravity plus the $(1, k + 2)$ minimal model is, upon omitting the gravitational descendants, equal to the chiral ring of the $N = 2$ theory. In the presence of a cosmological constant, the chiral ring is deformed, and for the right value of the cosmological constant it is identical to the fusion ring of $SU(2)$ at level k [149], which is at the same time identical to the fusion ring of the topological $SU(2)_k/SU(2)_k$ models [298, 319]. Thus, the equivalences between all these models allow us to obtain quite easily part of the ground ring of the matter plus gravity system.

The generalization of this structure to W_N gravity has not yet been accomplished. The cohomology for the G/H models has been computed [3, 296, 276], but the cohomology of the BRST operator for W gravity coupled to W matter has only been conjectured [63]. It is easy to guess what the generalizations of the statements to the case of W_N gravity should be. We expect that the ground ring is an extension by ordinary and W gravitational descendants of the fusion ring of G at level k, if the W matter is a minimal W model with $k_c = 2h + k$, etc. Rather than comparing the spectra, we will now give some additional evidence for the equivalence of G/G theories with W matter plus W

gravity, by demonstrating that (classically) both can be obtained by gauge fixing one action. This action is (6.1.18),

$$\begin{aligned} S &= kS^-_{wznw}(g_1) - \frac{1}{\pi}\int d^2z\, \bar{A}_1(J_1 - \xi\Lambda^+) - kS^-_{wznw}(g_2) - \frac{1}{\pi}\int d^2z\, \bar{A}_2(J_2 + \xi\Lambda^+) \\ &\quad + \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(F(\mu_\alpha)(W_1 + W_2)). \end{aligned} \tag{6.2.10}$$

The standard formulation of W matter plus W gravity in the conformal gauge follows from (6.1.18) by first gauge-fixing $\bar{A}_{1,2} = 0$, followed by gauge-fixing $\mu_\alpha = 0$. The resulting BRST operator is the sum of two quantum Drinfeld-Sokolov operators $Q^{DS}_{1,2}$ and the non-critical W BRST operator Q^W. The role of $Q^{DS}_{1,2}$ is to reduce modules of the affine Lie algebras based on **g** to representations of the matter and gravity W algebra respectively; they commute with Q^W.

The second gauge fixing of constrained WZNW models given in section 3.3.2 is also convenient here. This gauge fixing can be accomplished by adding to the action (6.1.18) the gauge fixing term

$$\frac{-1}{\pi}\int d^2z\, \mathrm{Tr}(B_1 J_1 + B_2 J_2) \tag{6.2.11}$$

where, in the notation of (4.3.5), the gauge fields are V_1^{0-} valued. The path integral over these extra gauge fields generates the insertion of the gauge-fixing delta functions. In the presence of these delta functions it is allowed to replace the gauge invariant polynomials that form W_1 and W_2 by the corresponding $\ker \mathrm{ad}(\Lambda^-)$ valued currents J_1 and J_2. The reasoning is similar as in section 6.2.2. Introduce new gauge fields $A^\pm = \frac{1}{2}(\bar{A}_1 \pm \bar{A}_2)$, and $B^\pm = \frac{1}{2}(B_1 \pm B_2)$. The gauge fields A^+, B^+ and μ_α span the entire Lie algebra **g**, and can be combined in one **g** valued gauge field $\mathcal{B}$. The action reads, in terms of these new variables,

$$\begin{aligned} S &= kS^-_{wznw}(g_1) - kS^-_{wznw}(g_2) - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(\mathcal{B}(J_1 + J_2)) \\ &\quad - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(A^-(J_1 - J_2 - 2\xi\Lambda^+) + B^-(J_1 - J_2)). \end{aligned} \tag{6.2.12}$$

The last two terms in this action are gauge fixing terms for the G symmetry of the action

$$S = kS^-_{wznw}(g_1) - kS^-_{wznw}(g_2) - \frac{1}{\pi}\int d^2z\, \mathrm{Tr}(\mathcal{B}(J_1 + J_2)) \tag{6.2.13}$$

and by reversing the procedure in section 3.3.2 we discard them and restore the G symmetry. It is maybe not clear right away that (6.2.13) is a G invariant action. It is crucial that the two WZNW actions have opposite values for k, so that the sum of the currents generates a centerless affine Lie algebra. In the terminology of chapter 3, this implies that there is no need to introduce a gauge field for the identity operator. Gauge fixing (6.2.13) along the lines of section 3.3.1 gives the same action as for W matter plus W gravity in the conformal gauge, but the BRST operator is now equal to the BRST operator of the G/G model. This demonstrates, at least at the classical level, that the two theories are equivalent. The equivalence of G/G and other G/H models can also be understood by comparing different gauge fixings of one action. The G/H action is a partially gauge fixed version of the G/G action [251].

Three remarks are in order. The first one is that there is a curious one-to-one correspondence between the choices of H such that G/H is Kähler, and the choices $\mathbf{g}_0 \subset \mathbf{g}$ related to different sl_2 embeddings as in section 2.2.7. Furthermore, the shift of the level in equation (2.2.45) agrees with the shifts in the level of the h sector in the quantum action for the G/H model (6.2.9). Although the details are unclear, this suggests that there is a close relation between the G/H model and the W gravity based upon the sl_2 embedding with $H = \mathbf{g}_0$. They may well both correspond to the same perturbation of the G/G theory.

A second remark concerns the fact that we neglected moduli throughout. In the G/H model we ignored the fact that the H gauge field A can not be gauged away completely, but only up to some moduli. In [145] these moduli are given by the moduli space of flat connections, but for our twisted bundle this correspondence is not valid, since it is not a stable bundle. It is more natural to parametrize them with the help of Higgs bundles, so as to establish a direct link with the W moduli. These moduli are not a problem for the genus-zero correlation functions, as long as we restrict attention to the states that are in the ground ring, *i.e.* that have conformal weight zero and ghost number zero. The ghost number of a state counts the number of extra moduli it introduces, so ground ring generators simply do not give any moduli, and since the sphere has no moduli itself, there are no moduli to integrate over. Once we go to higher genus, we have to take the moduli of the surface into account, and the situation gets more difficult.

Finally, a third remark is that a different relation between gravity coupled to matter, the twisted $N = 2$ superconformal algebra and the $SL(2)/SL(2)$ theory has recently been found purely on the algebraic level [277]. In this paper a kind of algebraic version of the path integral manipulations in this section is proposed. More precisely, a decomposition of the BRST operator of the $SL(2)/SL(2)$ model is given in terms of two commuting parts. The second term in the spectral sequence associated with this decomposition contains the twisted $N = 2$ superconformal algebra, and using a resolution of representations of the superconformal algebra in terms of '$2-d$ gravity modules' one can then show that the second term in the spectral sequence has the same cohomology as the BRST complex for the gravity plus matter theory. It would be interesting to understand this kind of Hamiltonian reduction of the $SL(2)/SL(2)$ theory from the path integral point of view, in order to be able to generalize it to arbitrary W algebras coupled to W matter.

6.2.4. Topological Gravity

A different connection between gravity and matter on the one hand and certain topological field theories on the other hand was known before one discussed in the previous sections was found. Namely, it was known that the theory of topological gravity [215, 307, 98] is equivalent to ordinary gravity [321, 307, 209], and that topological gravity coupled to topological conformal field theories [92, 224, 108, 89] describes ordinary gravity coupled to some matter theory. It is quite puzzling that topological matter, either without or with topological gravity, can describe ordinary gravity plus matter. Recently, this strange fact was clarified in [109, 227]; these authors showed that all the states in the topological gravity plus matter theory have BRST representatives that live purely in the matter sector. A particular nice argument was given in [109]: the total BRST operator of topological matter plus gravity is equivalent, via a similarity transformation, to the BRST operator of the topological matter theory. The realization of gravitational descendants purely in terms of matter fields enables one to compute their correlation functions using the topological Landau-Ginzburg methods of [92]. This forms the link with the discrete approach to gravity, and we continue with the discussion of this method in the section 6.3.

6.2.5. Topological W Gravity

The extension of topological gravity to topological W_N gravity has been considered in [225, 326, 229, 212, 176]. The extension to arbitrary W algebras is straightforward. Since we know the moduli space for W algebras, we can use the general method [317, 28] for obtaining the topological field theory associated with some moduli space. The action is given by

$$S = \int \delta_S \mathrm{Tr}(\chi F) = \int \mathrm{Tr}(\pi F - \chi D\Psi) \tag{6.2.14}$$

where F is the curvature two-form for a $\mathbf{g}$ valued connection A, δ_S is the 'shift' symmetry, ψ is a fermionic one-form and π and χ are Lagrange multipliers. The action is invariant under the BRST symmetry

$$\begin{array}{llllll} \delta_B A_\mu = \Psi_\mu - D_\mu c & \delta_B \chi = \pi + [c,\chi] & \delta_B c = \frac{1}{2}[c,c] - \gamma \\ \delta_B \Psi_\mu = [c,\Psi_\mu] - D_\mu \gamma & \delta_B \pi = [c,\pi] + [\gamma,\chi] & \delta_B \gamma = [c,\gamma] \end{array} \tag{6.2.15}$$

The BRST symmetry can be used to put A_μ equal to a flat connection parametrized by the moduli space of flat connections. The idea of [307] was to perform a different (partial) gauge fixing, ignoring the moduli but allowing instead for delta-function singularities in the curvature. Performing the gauge fixing procedure as in [307, 212, 176] results in the action

$$S = \int \mathrm{Tr}(\pi_0 F - \chi_0 D\Psi) + \text{ghosts} \tag{6.2.16}$$

where π_0 and χ_0 are $\mathbf{g}_0$ valued. The connection A and fermionic one-form ψ are expressed in terms of a $G_0 \equiv \exp \mathbf{g}_0$ valued field Ω and a $\mathbf{g}_0$ valued fermionic field Ω_Ψ according to

$$\begin{aligned} A_z &= \Omega^{-1}\partial\Omega + \Lambda^+, \\ A_{\bar{z}} &= \Omega^{-1}(\Lambda^+)^\dagger\Omega, \\ \Psi_z &= \Omega^{-1}(\partial\Omega_\psi)\Omega + [\Lambda^+, \Omega_\psi], \\ \Psi_{\bar{z}} &= \bar{\partial}\Omega_\Psi + \Omega^{-1}[(\Lambda^+)^\dagger, \Omega_\Psi]\Omega \end{aligned} \tag{6.2.17}$$

These parametrizations are such that both F and $D\Psi$ are $\mathbf{g}_0$ valued. The equation $F = 0$ is precisely the same as (5.2.27) with $\theta = \Lambda$, and there is no explicit moduli dependence. Thus, the equation for the Hermitian-Yang-Mills metric is imposed by the Lagrange multiplier π_0.

There are several interesting open problems for topological W gravity:

- Can on show as in [109] that the BRST-cohomology of topological W matter coupled to topological W gravity can be represented entirely within the matter sector?
- Can the action for topological W gravity be obtained as the induced action for a theory with an $N = 2$ extended W algebra as its chiral algebra? This would mean that (6.2.16) is somehow equivalent to an $N = 2$ Toda action. For gravity, (6.2.16) indeed closely resembles the action for $N = 2$ Liouville theory [131].
- The holomorphic formulation of Higgs bundles in terms of a holomorphic vector bundle and an holomorphic section of $\mathrm{End}(V) \otimes K$ is closely related to constrained WZNW models. On the other hand, the covariant formulation of Higgs bundles in terms of the Hermitian-Yang-Mills metric is closely related to actions of the type (6.2.14). Is it possible to relate these two approaches directly on the level of actions? In other words, is there an analogue of the Narasimhan-Seshadri theorem for Higgs bundles [252, 294] on the level of actions?

6.2.6. Intersection Theory

One of the nice features of topological gravity is that the correlation functions can be expressed purely in geometric terms, which reduces the solution of the theory to a problem in algebraic geometry [321]. Briefly, the relation is as follows. Let $\bar{\mathcal{M}}_{g,s}$ be the stable or Deligne-Mumford compactification of the moduli space of genus g Riemann surfaces with s punctures. The cotangent space at the i^{th} puncture of the Riemann surface gives a line bundle $\mathcal{L}_i$ over $\bar{\mathcal{M}}_{g,s}$. The physical observables of topological gravity are labeled by a positive integer n, corresponding to the n^{th} gravitational descendant of the identity operator. Since there is no matter in pure topological gravity, this is the complete Hilbert space. The n^{th} observable is usually denoted by σ_n. Geometrically, σ_n corresponds to the n^{th} power of the first Chern-class of the line bundle $\mathcal{L}_i$. With this correspondence, the correlation functions of topological gravity are defined by

$$\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle = \int_{\bar{\mathcal{M}}_{g,s}} c_1(\mathcal{L}_1)^{n_1} \wedge \dots \wedge c_1(\mathcal{L})^{n_s}. \tag{6.2.18}$$

For a more detailed discussion, see [321, 226, 96, 327]. Using a topological G/H model with $G = SU(2)$ and $H = U(1)$, Witten [322, 315] has generalized this geometrical description to the minimal models of type $(1, k+2)$ coupled to gravity. Take integers $p_1, \dots, p_s \in \{0, \dots, k\}$ and an integer g such that $2g - 2 - \sum p_i$ is divisible by $k+2$. Let $\Sigma \in \bar{\mathcal{M}}_{g,s}$, and S be the line bundle $K \otimes_i \mathcal{O}(z_i)^{-p_i}$. It has $(k+2)^{2g}$ different $(k+2)$th roots, and let $\mathcal{T}$ denote one of them. Generically, $T \equiv H^0(\Sigma; \mathcal{T}^* \otimes K)$ will be a vector space of dimension

$$d = \frac{k}{k+2}(g-1) + \sum \frac{p_i}{k+2}. \tag{6.2.19}$$

To give an expression for the correlation functions of operators σ_{n_i,p_i}, which is the n_i^{th} gravitational descendant of the p_i^{th} primary field of the $(1, k+2)$ minimal model, we need an extended moduli space $\bar{\mathcal{M}}(g, s, \mathcal{T})$ of Riemann surfaces with s punctures and a choice of $(k+2)$th root $\mathcal{T}$. It is a branched cover of degree $(k+2)^{2g}$ of $\bar{\mathcal{M}}_{g,s}$ The vector spaces T form a vector bundle of rank d over $\bar{\mathcal{M}}(g, s, \mathcal{T})$. Let $e(T)$ denote its Euler class. Then

$$\langle \sigma_{n_1,p_1} \dots \sigma_{n_s,p_s} \rangle = (k+2)^{-g} \int_{\bar{\mathcal{M}}(g,s,\mathcal{T})} c_1(\mathcal{L}_1)^{n_1} \wedge \dots \wedge c_1(\mathcal{L})^{n_s} \wedge e(T). \tag{6.2.20}$$

What about the extension to W gravity? For simplicity we restrict ourselves to W_N gravity. It is natural to expect that the moduli space of punctured Riemann surfaces should be replaced by the moduli space of punctured W surfaces. So let us first think about the appropriate generalization of the vector bundle S. In the terminology of section 6.1.3., the states in the matter sector are labeled by a weight Λ_M. Purely from the WZNW model, a product of operators with matter weights $\Lambda_M^{(i)}$ is only non-vanishing if

$$\sum_i \Lambda_M^{(i)} = -\left[\rho - (k_c - h)t_0\right](2 - 2g) \tag{6.2.21}$$

For W_N gravity, $t_0 = \rho$; recall that t_0 prescribes the geometry of the vector bundle V (5.1.4) that played such a prominent role in chapter 5. Now the upshot of [322] seems to be that to get the proper geometrical prescription, one should allow t_0 to be varied, and change the geometry of (5.1.4) correspondingly. To be more precise, $t_0(2-2g)$ represents the direct sum of the first Chern classes of the line bundles that the Cartan subbundle of the principle fiber bundle P constitutes. Thus, the geometry of P gets 'renormalized' on the quantum level. We expect that all of this can be made more rigorous be a detailed investigation of the topological twisted $SU(N)/U(N-1)$ model. This leads for the following proposal for the bundle S, in symbolic notation

$$S \simeq K^{\rho} \otimes_i \mathcal{O}(z_i)^{\Lambda_M^{(i)}} \tag{6.2.22}$$

which should be read as an equation in the weight space of $\mathbf{g}$, and the tensor product should be seen as addition in the weight space. The bundle $\mathcal{T}$ is a (k_c-h)th root of this one. Both S and $\mathcal{T}$ have rank $N-1$. It is now straightforward to write down the expression for a correlation function involving only gravitational descendants. Our lack of understanding the nature of W descendants prevents us from giving the correlation functions for these operators as well. It seems quite remarkable that we have to modify the geometry of the bundle V, specified by the solution t_0 of (6.2.21), depending on the choice of operators for which we compute the correlation functions. Recent work on G/H models with k integral [249] shows that the correlation functions with different values for t_0 are related by spectral flow isomorphisms, and that the proper definition of the correlation functions involves a sum over the possible vector of first Chern classes of V. A generalization of this work to fractional level G/H models could explain why the theory automatically selects the proper geometry for the vector bundle V.

6.3. The Discrete Approach

By far the most powerful approach to the calculation of correlation functions in gravity is the discrete approach or matrix model approach. The basic idea is to replace the path integral over the metrics that enters in the path-integral description of gravity by a sum over discretized surfaces. This sum can be represented in terms of a matrix integral that can be computed exactly. The limit where the discretized surface becomes smooth corresponds in the matrix model to the so-called double scaling limit. It was shown in [66] that in this limit the model is described by an integrable hierarchy of differential equations, the Korteweg-deVries hierarchy. This can be used to compute all the correlation functions explicitly. The equivalence of this approach to (6.2.18) has been proven by Kontsevich [209]. The equivalence matrix models to topological matter coupled to topological gravity has been shown [109]. A different way to see this relation is given in [143, 144].

The reader who is interested in more background material for matrix models can consult any of the review papers [96, 327, 82, 234, 156, 84, 198, 158, 43, 67, 231, 87, 199, 205, 244] and references therein.

The $N=2$ twisted minimal models that describe gravity coupled to a minimal model of type $(1, k+2)$ can be described in terms of a topological Landau-Ginzburg model [305]. This fact has been used in [92, 227, 109, 110] to compute the correlation functions in these theories. It is also the most direct way to make contact between matrix models and topological $N=2$ models. The relation between topological Landau-Ginzburg models and more general topological conformal field theories on the one hand and integrable hierarchies on the other hand has been further analyzed in [211, 103, 104]. In [104] it is shown that if the primary chiral algebra of the topological conformal field theory is decomposable, the 'Witten-Dijkgraaf-Verlinde-Verlinde' equations ([89, 321, 93]) of topological field theory are gauge equivalent to the integrable system

$$
\begin{aligned}
\partial_k \gamma_{ij}(u) &= \gamma_{ik}(u)\gamma_{kj}(u), \qquad i,j,k \quad \text{distinct}, \\
\sum_{k=1}^{N} \partial_k \gamma_{ij}(u) &= 0, \\
\gamma_{ji}(u) &= \gamma_{ij}(u).
\end{aligned} \tag{6.3.1}
$$

Here, N is the number of primary field in the operator algebra, u refers to new coordinates on the coupling constant space and $\partial_i = \partial/\partial u^i$. The system describes a so-called N wave interaction system. The decomposability assumption on the primary chiral algebra is certainly satisfied for the Landau Ginzburg topological models that describe the G/G fusion rules, since the fusion rules can be diagonalized by a modular transformation. Thus the equations (6.3.1) include the equations that describe topological W matter theories in genus zero. Now it was shown in [191] that the same equations can be obtained from the n-component KP hierarchy. This hierarchy can be expressed in the same form as the usual KP hierarchy, but the differential operators have to be replaced by $n \times n$ matrix differential operators. If I_n denotes the $n \times n$ identity matrix, and E_{ij} the $n \times n$ matrix with a one in its i,j entry and zeroes everywhere else, the n-component KP hierarchy is described by the following set of equations: L and $C^{(i)}$, $i = 1 \dots n$ are formal pseudo-differential operators of the form

$$
\begin{aligned}
L &= I_n \partial + \sum_{j=1}^{\infty} U^{(j)}(x) \partial^{-j}, \\
C^{(i)} &= E_{ii} + \sum_{j=1}^{\infty} C^{(i,j)}(x) \partial^{-j},
\end{aligned} \tag{6.3.2}
$$

subject to the conditions

$$
\sum_{i=1}^{n} C^{(i)} = I_n, \quad C^{(i)} L = L C^{(i)}, \quad C^{(i)} C^{(j)} = \delta_{ij} C^{(i)}. \tag{6.3.3}
$$

They satisfy the following set of equations for some pseudo-differential operator $P = I_n + P^{(1)}\partial^{-1} + \ldots$

$$\begin{aligned} LP &= P\partial, \\ C^{(i)}P &= PE_{ii}, \\ \frac{\partial P}{\partial x_k^{(i)}} &= -(L^{(i)k})_- P, \quad \text{where } L^{(i)} = C^{(i)}L. \end{aligned} \tag{6.3.4}$$

The last equation describes the generalized KP flows; as usual, the subscript $-$ means restricting to negative powers of ∂ only. The differential ∂ is the formal object

$$\partial = \frac{\partial}{\partial x_1^{(1)}} + \ldots + \frac{\partial}{\partial x_1^{(n)}}. \tag{6.3.5}$$

The rest of this section is devoted to showing how the genus-zero version of the n-component KP hierarchy can be obtained from the standard matrix model by taking a different continuum limit, as proposed in [55]. We propose that this is the matrix model that describes W gravity.

6.3.1. Review of the p-Matrix Model

The partition function of a general multimatrix model is given by [101]

$$Z = \int \prod_{i=1}^{p} dM_i \exp \beta\, tr \left(-\sum_{i=1}^{p} V_i(M_i) + \sum_{i=1}^{p-1} c_i M_i M_{i+1} \right) \tag{6.3.6}$$

where the M_i are Hermitian $N \times N$ matrices. The integral over the angular parts of the M_i can be done and we are left with the following integral over the eigenvalues $\lambda_{i,n}$ of the M_i

$$Z = \text{const} \int \prod_{i=1}^{p} \prod_{n=1}^{N} d\lambda_{i,n} \triangle(\lambda_{\alpha,1}) \triangle(\lambda_{\alpha,p}) \exp \beta \left(-\sum_{i=1}^{p} \sum_{n=1}^{N} V_i(\lambda_{i,n}) + \sum_{i=1}^{p-1} \sum_{n=1}^{N} c_i \lambda_{i,n} \lambda_{i+1,n} \right) \tag{6.3.7}$$

where $\triangle(\lambda_{\alpha,r}) = \prod_{a<b}(\lambda_{a,r} - \lambda_{b,r})$ is a Vandermonde determinant.

Next introduce (following [240]) orthogonal polynomials of order n $A_n(x) = x^n + \ldots$ and $B_n(x) = x^n + \ldots$ satisfying

$$h_n \delta_{n,m} = \int \prod_{i=1}^{p} d\lambda_i A_n(\lambda_1) \exp \beta \left(-\sum_{i=1}^{p} V_i(\lambda_i) + \sum_{i=1}^{p-1} c_i \lambda_i \lambda_{i+1} \right) B_m(\lambda_p) \tag{6.3.8}$$

We can write $\triangle(\lambda_{\alpha,1})$ as $\det_{\alpha\beta}(A_\alpha(\lambda_{\beta,1}))$ and $\triangle(\lambda_{\alpha,p})$ as $\det_{\alpha\beta}(B_\alpha(\lambda_{\beta,p}))$. Substituting this into the partition function (6.3.7) and expanding the determinant yields the following well-known expression for the partition function

$$Z = \text{const} \times N! \prod_{i=0}^{N-1} h_i \tag{6.3.9}$$

From now on we will use orthonormal polynomials, i.e. we make redefinitions $A_n \to A_n\sqrt{h_n}$ and $B_n \to B_n\sqrt{h_n}$. For the sake of brevity, write $\exp(-\beta\mu)$ for the exponential occurring in (6.3.8). As usual, we define certain infinite matrices by their matrix elements with respect to the orthonormal polynomials A_n and B_m

$$\begin{aligned}
Q(j)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i \, \lambda_j A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \quad 1 \le j \le p \\
P(1)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i \, A'_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \\
P(p)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i \, A_n(\lambda_1) e^{-\beta\mu} B'_m(\lambda_p)
\end{aligned} \tag{6.3.10}$$

In these equations, the prime denotes differentiation with respect to λ_1 and λ_p respectively. The use of indices may look a bit strange, but guarantees e.g. that the matrix corresponding to an insertion of λ_1^2 is just $Q(1)^2_{nm} \equiv \sum_r Q(1)_{nr} Q(1)_{rm}$. It is straightforward to verify the following properties of the matrices $P(i)$ and $Q(i)$

$$\begin{aligned}
&P(1)_{nm} = 0 \quad m \le n, \quad P(1)_{m,m+1} = (m+1)\sqrt{h_m/h_{m+1}} \\
&P(p)_{nm} = 0 \quad m \ge n, \quad P(p)_{m+1,m} = (m+1)\sqrt{h_m/h_{m+1}} \\
&Q(1)_{nm} = 0 \quad m < n-1, \quad Q(1)_{m+1,m} = \sqrt{h_{m+1}/h_m} \\
&Q(p)_{nm} = 0 \quad n < m-1, \quad Q(p)_{m,m+1} = \sqrt{h_{m+1}/h_m}
\end{aligned} \tag{6.3.11}$$

Another set of important identities can be obtained by considering

$$\int \prod_{i=1}^{p} d\lambda_i \frac{d}{d\lambda_r} \Big(A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \Big) = 0 \tag{6.3.12}$$

for $r = 1, \ldots, p$. This gives a set of relations expressing all matrices in terms of $P(1)$ and $Q(1)$:

$$\begin{aligned}
\beta^{-1} P(1) - V'_1(Q(1)) + c_1 Q(2) &= 0 \\
c_{r-1} Q(r-1) - V'_r(Q(r)) + c_r Q(r+1) &= 0 \quad 2 \le r \le p-1 \\
\beta^{-1} P(p) - V'_p(Q(p)) + c_{p-1} Q(p-1) &= 0
\end{aligned} \tag{6.3.13}$$

Finally, the multimatrix model has a set of discrete 'string equations'. Two of them, $[P(1), Q(1)] = [Q(p), P(p)] = 1$, can be directly obtained from the definitions of P and Q (6.3.10), the others then follow from (6.3.13) and simply read

$$\beta c_r [Q(r), Q(r+1)] = 1 \tag{6.3.14}$$

These equations together are sufficient to determine the h_i and therefore to evaluate the partition function (6.3.9).

If all potentials are of finite degree, one can check that $P(1)$, $P(p)$ and $Q(r)$ are Jacobi matrices. This means that the (a, b)-matrix element is only nonvanishing if $|a - b| \leq K$ for some integer K. For instance, using (6.3.11) and (6.3.13) one finds that for $Q(1)$ we can take $K = \prod_{r=2}^{p}(deg(V_r) - 1)$. In the continuum limit, Jacobi matrices are expected to become finite-order differential operators.

To proceed, we define

$$f_i = \frac{h_i}{h_{i-1}} \tag{6.3.15}$$

and

$$Q(1)_{n-l,n} = \sqrt{\frac{h_{n-l}}{h_n}} R_n^{(l)} \tag{6.3.16}$$

for $l \geq 0$. Similar expansions can be defined for the other matrices $Q(j)$. Inserting these into (6.3.13) and restricting the first and third equation to the matrix elements where $P(1)$ and $P(p)$ vanish, yield the usual recursion relations, which in general are very complicated.

If we take the $(m-1, m)$ matrix element of the first equation in (6.3.13) we find an equation, that later will turn out to be equivalent to the string equation. It reads

$$\frac{m}{\beta} = (V_1'(Q(1))_{m-1,m} - c_1 Q(2)_{m-1,m}) \sqrt{\frac{h_m}{h_{m-1}}} \tag{6.3.17}$$

Using the recursion relations mentioned above, we can in general eliminate all the variables like the $R_n^{(l)}$ occurring in (6.3.16), so that the only variables left will be the f_i defined in (6.3.15). Then (6.3.17) takes the form

$$\frac{m}{\beta} = W(f_i) \tag{6.3.18}$$

We now take the scaling limit in the standard way [66]. We let $N \to \infty$, $\beta/N \to 1$, and replace discrete by continuous variables: $x = m/\beta$, $\epsilon = 1/N$, $f_i \to f(x)$, $R_n^{(l)} \to R^{(l)}(x)$, etc. If in the planar limit, which will be described in just a moment, the function W occurring in (6.3.18) behaves as

$$W(f) \simeq W_c - (f - f_c)^k \tag{6.3.19}$$

we can take the double scaling limit which essentially amounts to amplifying the region around $f = f_c$. Let $\gamma = -1/k$ denote the string susceptibility, and define the lattice spacing a and the renormalized cosmological constant μ_R by

$$\lambda a^{2-\gamma} = \epsilon \tag{6.3.20}$$

$$W_c - a^2 \mu_R = x \tag{6.3.21}$$

Here λ is the parameter that controls the genus expansion of the partition function, $Z = \sum \lambda^{2g-2} Z_g$. We will assume $\lambda = 1$, which can be accomplished by a redefinition of a and μ_R. To obtain the string equation in the usual form, make expansions for f and the R's in terms of $a^{-\gamma}$,

$$f(\mu_R) = f_c + a^{-2\gamma} f^{(1)}(\mu_R) + a^{-3\gamma} f^{(2)}(\mu_R) + \ldots \tag{6.3.22}$$

and similar for the R's. Substituting these expansions back into the recursions relations obtained from (6.3.13) and letting $a \to 0$ turns equation (6.3.18) into the string equation.

6.3.2. Genus-Zero Formulation

To find the critical points, i.e. the potentials that yield a behavior as in (6.3.19), we will now restrict ourselves to the planar limit. This means that we will neglect the dependence of f_i and $R_n^{(l)}$ on i and n, because this dependence is only relevant for higher genus as can be seen from (6.3.20). The matrices can now be represented as power series in the 'shift' operator

$$z = \sum_r \delta_{r-1,r} \tag{6.3.23}$$

As can be seen from (6.3.11), the expansion for $Q(1)$ reads

$$Q(1) = \frac{\sqrt{f}}{z} + \sum_{l \geq 0} R^{(l)} \left(\frac{z}{\sqrt{f}} \right)^l \tag{6.3.24}$$

and from (6.3.11) and (6.3.13) we see that

$$\frac{P^{(1)}(z)}{\beta} = V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z) = \frac{x}{\sqrt{f}} z + \mathcal{O}(z^2) \tag{6.3.25}$$

Given the V_i, we can in principal determine the $P^{(i)}(z)$ and $Q^{(i)}(z)$ as functions of f. These equations are however highly nonlinear and difficult to solve. We will, therefore, follow a reverse route, and will assume that the $Q^{(i)}$ are given. One may then try to construct the potentials by using (6.3.13). The equations for the coefficients occurring in the potentials are linear, but do not always admit a solution. For the time being, we will restrict our attention to (6.3.13). From this equation it is easy to see that, given $Q^{(1)}$ and $Q^{(2)}$, V_1 is completely and uniquely determined by requiring $V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z)$ to be of order $\mathcal{O}(z)$. Clearly, if $Q^{(2)}(z) = az^{-n}$+higher order, V_1 will be of order $n+1$.

The requirement that $V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z)$ is of order $\mathcal{O}(z)$ is met for every value of f. Taking for instance $u = f$, we see that there exist $Q^{(1)}(z,u)$ and $Q^{(2)}(z,u)$ labeled by one extra parameter u, such that $V_1'(Q^{(1)}(z,u)) - c_1 Q^{(2)}(z,u)$ is still of order $\mathcal{O}(z)$. The reason that we bother to introduce a new variable u here, is that we will assume that everything depends analytically on u, which need not necessarily be the case for $u = f$. To find the exact u dependence would require a knowledge of V_1. We can, however, find an equation which does not explicitly depend upon V_1, by differentiating (6.3.13) with respect to both z and u, which gives two equations from which V_1'' can be eliminated. The result of this is the following equation

$$\beta^{-1}\{P^{(1)}, Q^{(1)}\}_{z,u} = c_1\{Q^{(1)}, Q^{(2)}\}_{z,u} \tag{6.3.26}$$

where the 'Poisson' bracket $\{\}_{z,u}$ is defined by

$$\{A(z,u), B(z,u)\}_{z,u} = z\frac{\partial A}{\partial z}\frac{\partial B}{\partial u} - \frac{\partial A}{\partial u}z\frac{\partial B}{\partial z} \tag{6.3.27}$$

The extra z has been introduced for later convenience. (6.3.26) looks like a 'classical' analog of the equation $\beta^{-1}[P^{(1)}, Q^{(1)}] = c_1[Q^{(1)}, Q^{(2)}]$ which is valid in the original matrix model. As we will now show, the planar approximation is nothing but the replacement of 'quantum' commutators by 'classical' Poisson brackets. Consider $A = X(u)z^l$ and $B = Y(u)z^k$. On the level of matrices, this means $A = \sum_\alpha X_\alpha(u)\delta_{\alpha-l,\alpha}$ and $B = \sum_\beta Y_\beta(u)\delta_{\beta-k,\beta}$. In the planar approximation the commutator of A and B can be calculated as follows

$$\begin{aligned}
[A,B] &= \sum_\alpha (X_{\alpha-k}Y_\alpha - X_\alpha Y_{\alpha-l})\delta_{\alpha-k-l,\alpha} \\
&= \sum_\alpha \left[(X_\alpha - k\frac{\partial X_\alpha}{\partial i})Y_\alpha - X_\alpha(Y_\alpha - l\frac{\partial Y_\alpha}{\partial i})\right]\delta_{\alpha-k-l,\alpha} \\
&= \sum_\alpha \left(lX_\alpha\frac{\partial Y_\alpha}{\partial i} - k\frac{\partial X_\alpha}{\partial i}Y_\alpha\right)\delta_{\alpha-k-l,\alpha} \\
&\rightarrow \left(lX\frac{\partial Y}{\partial i} - k\frac{\partial X}{\partial i}Y\right)z^{k+l}
\end{aligned}$$

$$
\begin{aligned}
&= z\frac{\partial A}{\partial z}\frac{\partial B}{\partial i} - \frac{\partial A}{\partial i} z\frac{\partial B}{\partial z} \\
&= \frac{\partial u}{\partial i}\{A, B\}_{z,u} \\
&= \beta^{-1}\frac{\partial u}{\partial x}\{A, B\}_{z,u}
\end{aligned}
$$

In particular, this gives the following version of the planar string equation

$$\{P^{(1)}(z,u), Q^{(1)}(z,u)\}_{z,u} = \beta\left(\frac{\partial u}{\partial x}\right)^{-1} \tag{6.3.28}$$

Indeed, taking $P^{(1)}(z,u) \equiv P(z,u) = \beta z W(f)/\sqrt{f} + \mathcal{O}(z^2)$ and $Q^{(1)}(z,u) \equiv Q(z,u) = \sqrt{f}/z + \mathcal{O}(1)$, a short calculation shows

$$\{P, Q\}_{z,u} = \beta\frac{\partial f}{\partial u}W'(f) + \mathcal{O}(z) \tag{6.3.29}$$

which combined with (6.3.28) gives $\partial W(f)/\partial x = 1$, in agreement with (6.3.18).

6.3.3. The Continuum Limit in Genus Zero

Let us now consider the differential operators that P and Q will become in the continuum limit. In that limit we have

$$z = e^{-\epsilon\partial/\partial x} = e^{a^{-\gamma}\partial/\partial\mu_R} \tag{6.3.30}$$

In the planar approximation $\partial/\partial\mu_R$ commutes with everything, and can be replaced by a commuting object which we will denote by ξ. Instead of z and u we can also think of P and Q as functions depending on ξ and u. Because $z\partial/\partial z = a^\gamma\partial/\partial\xi$ we find that

$$\{A(z,u), B(z,u)\}_{z,u} = a^\gamma\{A(\xi,u), B(\xi,u)\}_{\xi,u} \tag{6.3.31}$$

where $\{\}_{\xi,u}$ denotes the usual Poisson bracket $\{A,B\}_{\xi,u} = \partial_\xi A\partial_u B - \partial_u A\partial_\xi B$. As $\beta \sim a^{\gamma-2}$, and by using (6.3.21), we find the string equation in terms of ξ

$$-\frac{\partial u}{\partial\mu_R}\{P(\xi,u), Q(\xi,u)\}_{\xi,u} = 1 \tag{6.3.32}$$

From now on we will assume that $u = a^{2\gamma}(f - f_c)$; u will be finite if we let $a \to 0$ at fixed μ_R. Obviously, we can write $\beta^{-1}P$ as $P_0 + \sum_{n\geq p} P_n a^{-n\gamma}$ and $Q = Q_0 + \sum_{n\geq q} Q_n a^{-n\gamma}$, where P_n and Q_n are polynomials in ξ and u of degree n (ξ has degree 1 and u has degree 2). P_0 and Q_0 will not contribute to the string equation and we will ignore these constants. Naively, one would think that p and q are the orders of the differential operators that P and Q become in the double scaling limit. This is, however, not always true. If, for instance, $p = q = 2$ and $P_2 = Q_2$, it is clear that $\{P_2, Q_2\} = 0$ and that we cannot consider P_2 and Q_2 as the scaling limit of P and Q while preserving the string equation. What is essential is that (6.3.32) is invariant under $P \to P + \sum a_i Q^i$ and also under $Q \to Q + \sum b_i P^i$, and in some cases such redefinitions are necessary to find then true orders of P and Q. If e.g. $q \leq p$, we try to find the operator $P' = P + \sum a_i Q^i$ that is of highest order, as it is this what survives in the string equation, and not P[‡]. Actually, we only need to determine the true orders of P and Q for $u = 0$, because (6.3.32) can be used to find the u-dependence, without lowering the order of either P or Q. (6.3.32) (More precisely, we need (6.3.28)) together with (6.3.13) for $u = 0$ imply that (6.3.13) is also valid for $u \neq 0$.

If $Q = Q_0 + \sum_{n \geq q} Q_n a^{-n\gamma}$ and $\beta^{-1} P' = P'_0 + \sum_{n \geq p} P'_n a^{-n\gamma}$, we define

$$\hat{Q} = \lim_{a \to 0} a^{q\gamma}(Q - Q_0) \tag{6.3.33}$$

$$\hat{P} = \lim_{a \to 0} a^{p\gamma}(P'_0 - \beta^{-1} P') \tag{6.3.34}$$

The string equation turns into ($\mu = \mu_R$)

$$\frac{\partial u}{\partial \mu} \beta a^{-(p+q)\gamma} \{\hat{P}, \hat{Q}\}_{\xi,u} = 1 \tag{6.3.35}$$

To get something finite, we must have $\gamma - 2 - p\gamma - q\gamma = 0$, so that $\gamma = -2/(p + q - 1)$, coinciding with the KPZ result for a (p, q)-minimal model [207]. Furthermore, $\{\hat{P}, \hat{Q}\}_{\xi,u}$ must be independent of ξ and is therefore proportional to $u^{(p+q-3)/2}$. Then the string equation implies $u \sim \mu^{-\gamma}$.

Something interesting happens when $p + q$ =even. In this case, the above does not work, as $\{\hat{P}, \hat{Q}\}$ can never be independent of ξ (unless it vanishes). The reason that it does not work is that γ^{-1} is not an integer. Looking at (6.3.19) we see that the parameter upon which everything depends analytically is $\sqrt{f - f_c}$ rather that $(f - f_c)$. Indeed, we can repeat the above taking $u = a^\gamma \sqrt{f - f_c}$, and we find correctly that $\gamma = -2/(p + q - 1)$ and that $\sqrt{f - f_c} \sim u \sim \mu^{-\gamma/2}$.

One should bear this in mind when comparing the above with the spherical formalism of [86]. There, the commutator of $f\partial^a$ and $g\partial^b$ is computed in the spherical approximation by keeping only first derivatives and dropping higher ones. The result is therefore $(afg' - bgf')\partial^{a+b-1}$. If we replace ∂ by ξ and assume f and g depend on a

[‡]There is some ambiguity in these redefinitions, but these are irrelevant.

certain variable u, then this is precisely equal to $\partial u/\partial\mu\{f\xi^a, g\xi^b\}_{\xi,u}$. Thus, our formalism is equivalent to their formalism. However, we should be careful when using their formalism to expand everything in either u or $\sqrt{u}$ depending on whether $p+q$ is odd or even.

It is also instructive to compare this formalism with that of the topological Landau Ginzburg models in [93, 91]. What they call x is our ξ and their t_0 is our μ. Therefore any result in the differential operator formalism of matrix models can be simply translated into the Landau-Ginzburg formulation, by replacing commutators by a Poisson bracket with respect to x and t_0. For instance (for details see [93, 91])

$$[L_+^{i+1}, L] = \{L_+^{i+1}, L\}_{x,t_0} = \left(\frac{\partial L_+^{i+1}}{\partial x}\frac{\partial L}{\partial t_0} - \frac{\partial L_+^{i+1}}{\partial t_0}\frac{\partial L}{\partial x}\right) = \partial_x L_+^{i+1} \tag{6.3.36}$$

This establishes on the most direct level the equivalence between topological matter in genus zero and the matrix model in genus zero. Using the observations in [227, 109, 110] this equivalence can be extended to include the gravitational descendants.

6.3.4. The New Continuum Limit

The new continuum limit is one in which we end up with matrix valued differential operators rather than scalar ones. The idea is to treat the indices of the infinite matrices in (6.3.10) in a nonuniform way, depending on the value of the index modulo M, and to keep this distinction in the continuum limit. This will give rise to $M \times M$ matrix valued differential operators. In terms of discretized Riemann surfaces, this corresponds to a certain continuum limit of the set of colored triangulations of the Riemann surface with M different colors.

Let us now make this idea a bit more explicit. Introduce the following matrices

$$\begin{aligned}
Q(j)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i\, \lambda_j^M A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \quad 1 \le j \le p \\
P(1)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i\, \lambda_1 A_n'(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \\
P(p)_{mn} &= \int \prod_{i=1}^{p} d\lambda_i\, A_n(\lambda_1) e^{-\beta\mu} \lambda_p B_m'(\lambda_p) \\
C(1)^{(j)}_{mn} &= \int \prod_{i=1}^{p} d\lambda_i\, \Pi^{(j)} A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \quad 1 \le j \le M \\
C(p)^{(j)}_{mn} &= \int \prod_{i=1}^{p} d\lambda_i\, A_n(\lambda_1) e^{-\beta\mu} \Pi^{(j)} B_m(\lambda_p) \quad 1 \le j \le M
\end{aligned} \tag{6.3.37}$$

where $\Pi^{(j)}$ is a projection operator for polynomials in some variable λ; it satisfies $\Pi^{(j)}\lambda^k = \lambda^k$ if $j = k \bmod M$ and 0 otherwise. These matrices satisfy (6.3.3) with L replaced by either P or Q. Furthermore we have $[P(1), Q(1)] = MQ(1)$ and $[Q(p), P(p)] = MQ(p)$.

Instead of the shift operator (6.3.23) we now expand all the matrices in terms of

$$\mathcal{E}_{ij} = \sum_r \delta_{Mr+i,Mr+j}, \qquad 1 \le i \le M, \ \ 1 \le j \le M, \tag{6.3.38}$$

and

$$z = \sum_r \delta_{r,r+M}. \tag{6.3.39}$$

With these definitions it is relatively straightforward to repeat the analysis in the previous sections for this case, and to see that in the continuum limit we get differential operators, once we replace

$$z \to \exp(a^{-\gamma}\partial) \tag{6.3.40}$$

with

$$\partial = \frac{\partial}{\partial \mu_R^{(1)}} + \dots \frac{\partial}{\partial \mu_R^{(M)}}. \tag{6.3.41}$$

The different $\mu_R^{(j)}$ here refer to the indices that are equal to $j \bmod M$. The matrix z decreases all these indices by M, which explains why z contains the sum (6.3.41).

To get nontrivial, *i.e.* non-diagonal differential operators, we presumably have to allow for potentials that include projection operators such as $\Pi^{(j)}$. The continuum limit provides in any case several differential operators, P, Q and $C^{(j)}$, that satisfy (6.3.3) with L replaced by either P or Q, and in addition the 'string' equation $[P, Q] = MQ$. Since the $C^{(j)}$ are not represented by Jacobi matrices, the $C^{(j)}$ are expected to become pseudo-differential operators rather than differential operators.

The genus zero formulation can be worked out, and gives a Poisson bracket version of the M-component KP hierarchy.

To get the full M component KP hierarchy requires more work, although life might be not too hard due to the following observation: There seems to be a curious principle, that if we can obtain genus-zero correlation functions from integrable systems that are expressed in terms of Poisson brackets, we can obtain higher genus correlation functions by simply replacing the Poisson brackets by commutators, as in ordinary quantum mechanics. This is certainly true for the ordinary matrix models. If it also applies for these new scaling limits, it would mean that all correlation functions for W gravity can really be computed from the M component KP hierarchy, once we know precisely which critical points correspond to W gravity. We might not have to wait too long anymore to see the exact solution of W gravity appear.

Bibliography

[1] R. Abraham, J. E. Marsden, 'Foundations of Classical Mechanics', second edition, The Benjamin/Cummings Publishing Company inc.

[2] J.-P. Ader, F. Gieres and Y. Noirot, 'Relating Weyl and Diffeomorphism Anomalies on Super Riemann Surfaces', MPI-Ph-92/38, LPTB 92-6; F. Gieres, 'Maurer-Cartan Forms and Equations for Two-Dimensional Superdiffeomorphisms', MPI-Ph/92-43; J.-P. Ader and H. Kachkachi, Class. Qauntum Grav. **10** (1993) 417

[3] O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, 'On the Twisted G/H Topological Models', TAUP-1990-92

[4] O. Aharony, O. Ganor, N. Sochen, J. Sonnenschein and S. Yankielowicz, 'Physical States in the G/G Models and Two-Dimensional Gravity', TAUP-1947-92, to appear in Nucl. Phys. **B**; O. Aharony, J. Sonnenschein and S. Yankielowicz, Phys. Lett. **B289** (1992) 309

[5] E. Aldrovandi and L. Bonora, 'Liouville and Toda Field Theories on Riemann Surfaces', SISSA-ISAS 27/93/EP

[6] L. Alvarez-Gaumé and C. Gómez, 'Topics in Liouville Theory', to appear in the proceedings of the 1991 Trieste spring school, CERN-TH.6175/91; UGVA-PHY-07-737/91

[7] O. Alvarez, Nucl. Phys. **B238** (1984) 61

[8] G. Anderson and G. Moore, Commun. Math. Phys. **117** (1988) 441

[9] I. Antoniadis, J. Ellis, R. Lacaze and D. V. Nanopoulos, Phys. Lett. **B268** (1991) 188; S. Kalara, J. L. Lopez and D. V. Nanopoulos, Phys. Lett. **B269** (1991) 84; I. Antoniadis, J. Ellis, S. Kelley and D. V. Nanopoulos, Phys. Lett. **B272** (1991) 31; L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. **B355** (1991) 649; L. E. Ibáñez, D. Lüst and G. G. Ross, Phys. Lett. **B272** (1991) 251; D. Bailin and A. Love, Phys. Lett. **B278** (1992) 125; Phys. Lett. **B280** (1992) 26; Phys. Lett. **B292** (1992) 315; M. K. Gaillard and R. Xiu, Phys. Lett. **B296** (1992) 71 L. Ibáñez and D. Lüst, Nucl. Phys. **B382** (1992) 305; D. Lüst, 'Consistency and Phenomenology of Four-Dimensional Strings', CERN-TH.6819/93, talk at 26$^{\rm th}$ workshop 'From Superstrings to Supergravity', Erice, December 1992

[10] M. F. Atiyah, Commun. Math. Phys. **93** (1984) 437

[11] M. F. Atiyah and R. Bott, Phil. Trans. R. Soc. Lond. **A 308** (1982) 523.

[12] F. A. Bais, J. de Boer, J. Goeree, T. Tjin and P. van Driel, 'W Algebras, W Gravities and their Moduli Spaces', to appear in the proceedings of the Trieste Summer School 1992

[13] F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. **B304** (1988) 348

[14] F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. **B304** (1988) 348; Nucl. Phys. **B304** (1988) 371

[15] F. A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357** (1991) 632

[16] I. Bakas and D. McMullan, Phys. Lett. **B189** (1987) 141; K. Gawędzki, Commun. Math. Phys. **139** (1991) 201; J. Harnad and B. A. Kupershmidt, 'Symplectic Geometries on $T^*\widetilde{G}$, Hamiltonian Group Actions and Integrable Systems, Montreal preprint (1991); M F. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. **B266** (1991) 71

[17] I. Bakas, Nucl. Phys. **B302** (1988) 189

[18] I. Bakas, Phys. Lett. **B213** (1988) 313; Commun. Math. Phys. **123** (1989) 627

[19] J. Balog, L. Fehér, P. Forgács, L. O'Raifeartaigh and A. Wipf, Ann. Phys. **203** (1990) 76

[20] J. Balog, L. Fehér, P. Forgács, L. O'Raifeartaigh and A. Wipf, 'On the General Structure of Hamiltonian Reductions of WZW Theory', Phys. Rep. **222** (1992) 1

[21] J. Balog, L. Fehér, P. Forgács, L. O'Raifeartaigh and A. Wipf, Phys. Lett. **B227** (1989) 214; Phys. Lett. **B244** (1990) 435

[22] T. Banks and E. Martinec, Nucl. Phys. **B294** (1987) 733

[23] K. Bardacki and M. B. Halpern, Phys. Rev. **D3** (1971) 2493; M. B. Halpern, Phys. Rev. **D4** (1971) 2398

[24] G. T. Barkema and J. de Boer, 'A Statistical Model for Ice at Low Temperatures', THU-93/03, to appear in J. Chem. Phys.

[25] G. T. Barkema and J. de Boer, Phys. Rev. **A44** (1991) 8000

[26] I. Bars and K. Sfetsos, 'Exact Effective Action and Spacetime Geometry in Gauged WZW Models', USC-93/HEP-B1

[27] I.A. Batalin, G.A. Vilkovisky, Phys. Lett. **B102** (1981) 27; Phys. Rev. **D28** (1983) 2567.

[28] L. Baulieu and I. Singer, Nucl. Phys. **B** (Proc. Suppl.) **5B** (1988) 12; S. Ouvry, R. Stora and P. van Baal, Phys. Lett. **B220** (1989) 159; P. van Baal, Acta. Phys. Pol. **B21** (1990) 73

[29] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333

[30] E. Bergshoeff, H. J. Boonstra, M. de Roo, S. Panda and A. Sevrin, 'On the BRST Operator of W-Strings', UG-2/93, UCB-PTH-93/05, LBL-33737

[31] E. Bergshoeff, A. Sevrin and X. Shen, Phys. Lett. **B296** (1992) 95 Noncritical W Strings', UG-8/92, CERN-TH.6674/92, UCB-PTH-92/27, LBL-32806

[32] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, X. Shen and K. S. Stelle, Phys. Lett. **B243** (1990) 350

[33] Z. Bern and D. Kosower, Phys. Rev. Lett. **66** (1991) 1669; Nucl. Phys. **B379** (1992) 451; Z. Bern and D. C. Dunbar, Nucl. Phys. **B379** (1992) 562

[34] D. Bernard and A. LeClair, Commun. Math. Phys. **142** (1991) 99

[35] D. Bernard and G. Felder, Commun. Math. Phys. **127** (1990) 145

[36] M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126** (1989) 49

[37] M. Bershadsky and H. Ooguri, Phys. Lett. **B229** (1989) 374

[38] M. Bershadsky, Commun. Math. Phys. **139** (1991) 71

[39] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, 'Extended $N = 2$ Superconformal Structure of Gravity and W Gravity Coupled to Matter', CALT-68-1832, CERN-TH.6694/92, HUTP-A061/92 and USC-92/021, to appear in Nucl. Phys. B.

[40] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, Phys. Lett. **B292** (1992) 35

[41] A. Bilal and J.-L. Gervais, Nucl. Phys. **B326** (1989) 222

[42] A. Bilal and J.-L. Gervais, Phys. Lett. **B206** (1988) 412; Nucl. Phys. **B314** (1989) 646; Nucl. Phys. **B318** (1989) 579; in 'Infinite Dimensional Lie Algebras and Lie Groups', ed. V. Kac, Proc. CIRM-Luminy conference, World Scientific, Singapore, 1989; Adv. Ser. Math. Phys. **7** (1988) 483

[43] A. Bilal, '2d Gravity from Matrix Models', John Hopkins Lectures, CERN-TH.5867/90

[44] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, 'Topological Field Theory', Phys. Rep. **209** (1991) 129

[45] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, Nucl. Phys. **B361** (1991) 255

[46] J. de Boer and J. Goeree, Commun. Math. Phys. **139** (1991) 267

[47] J. de Boer and J. Goeree, 'Covariant W Gravity and Its Moduli Space from Gauge Theory' THU-92/14, to appear in Nucl. Phys. **B**

[48] J. de Boer and J. Goeree, 'KPZ Analysis for W_3 Gravity', THU-92/34, to appear in Nucl. Phys. **B**

[49] J. de Boer and J. Goeree, Nucl. Phys. **B381** (1992) 329

[50] J. de Boer and J. Goeree, Phys. Lett. **B274** (1992) 289

[51] J. de Boer and J. Goeree, 'The Effective Action of W_3 Gravity to All Orders', THU-92/33, to appear in Nucl. Phys. **B**

[52] J. de Boer and J. Goeree, 'W Gravity', to appear in proceedings of International Conference on Mathematical Physics, Quantum Gravity and String Theory, Rakhov, 1992

[53] J. de Boer and T. Tjin, 'Quantization and Representation Theory of Finite W Algebras', THU-92/32, ITFA-28-92, to appear in Commun. Math. Phys.

[54] J. de Boer and T. Tjin, 'The Relation between Quantum W Algebras and Lie Algebras', THU-93/05, ITFA-02-93

[55] J. de Boer Nucl. Phys. **B366** (1991) 602

[56] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology, Springer Graduate Texts in Mathematics 82, Springer-Verlag, New York, 1982

[57] P. Bouwknegt and K. Schoutens, '$\mathcal{W}$-Symmetry in Conformal Field Theory', Phys. Rep. **223** (1993) 183

[58] P. Bouwknegt, in 'Infinite Dimensional Lie Algebras and Lie Groups', ed. V. Kac, Proc. CIRM-Luminy conference, World Scientific, Singapore, 1989

[59] P. Bouwknegt, J. McCarthy and K. Pilch, Commun. Math. Phys. **131** (1990) 125

[60] P. Bouwknegt, J. McCarthy and K. Pilch, in proceedings 'Strings and Symmetries', eds. N. Berkovitz et al., World Scientific, 1991

[61] P. Bouwknegt, J. McCarthy and K. Pilch, Lett. Math. Phys. **23** (1991) 193; Commun. Math. Phys. **145** (1992) 541

[62] P. Bouwknegt, J. McCarthy and K. Pilch, Progr. Theor. Phys. Supp. **102** (1990) 67

[63] P. Bouwknegt, J. McCarthy and K. Pilch, 'Semi-Infinite Cohomology of W-Algebras', USC-93/11, ADP-93-200/M15; 'On the BRST Structure of W_3 Gravity Coupled to $c = 2$ Matter', USC-93/14, ADP-93-203/M17

[64] P. Bowcock and G. M. T. Watts, 'On the Classification of Quantum W-Algebras', EFI-91-63, DTP-91-63

[65] P. Bowcock, Nucl. Phys. **B356** (1991) 367

[66] E. Brézin and V. Kazakov, Phys. Lett. **B236** (1990) 144; M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635; D. J. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127; Nucl. Phys. **B340** (1990) 333

[67] E. Brézin, 'Large N Limit and Discretized Two-Dimensional Quantum Gravity', in 'Two Dimensional Quantum gravity and Random Surfaces', proceedings of Jerusalem Winter School 1990/91, Eds. D. Gross et al.; D. Gross, 'The $c = 1$ matrix models', in the same proceedings

[68] R. Brustein and K. Roland, Nucl. Phys. **B372** (1992) 201

[69] R. Brustein and S. de Alwis, Nucl. Phys. **B352** (1991) 451

[70] T. H. Buscher, Phys. Lett. **B201** (1988) 466; A. A. Tseytlin, Mod. Phys. Lett. **A6** (1991) 1721; A. S. Schwarz and A. A. Tseytlin, 'Dilaton Shift under Duality and Torsion of Elliptic Complex', Imperial/TP/92-93/01

[71] C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. **B262** (1985) 593

[72] C. G. Callan, I. R. Klebanov and M. J. Perry, Nucl. Phys. **B278** (1986) 78

[73] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. **D45** (1992) R1005

[74] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46

[75] J. L. Cardy, in 'Fields, Strings and Critical Phenomena', Les Houches session XLIX, eds. E. Brézin and J. Zinn-Justin, North-Holland, 1990

[76] J. L. Cardy, in 'Phase Transitions and Critical Phenomena', vol. 11, eds. C. Domb and J. Lebowitz, Academic Press, 1986'

[77] J. L. Cardy, Nucl. Phys. **B270** (1986) 186

[78] A. Ceresole, M. Frau, J. McCarthy and A. Lerda, Phys. Lett. **B265** (1991) 72

[79] T. L. Curtright and C. B. Thorn, Phys. Rev. Lett. **48** (1982) 1309; E. Braaten, T. L. Curtright and C. B. Thorn, Phys. Lett. **B118** (1982) 115; Ann. Phys. (N.Y.) **147** (1983) 365; E. Braaten, T. L. Curtright, G. Ghandour and C. B. Thorn, Phys. Rev. Lett. **153**

(1984) 147; Ann. Phys. (N.Y.) **153** (1984) 147 E. D'Hoker and R. Jackiw, Phys. Rev. Lett. **50** (1983) 1719; Phys. Rev. **D26** (1982) 3517; E. D'Hoker, D. Freedman and R. Jackiw, Phys. Rev. **D28** (1983) 2583

[80] S. R. Das, A. Dhar and S. K. Rama, Mod. Phys. Lett. **A6** (1991) 3055; Int. Journal of Mod. Phys. **A7** (1992) 2295; S. K. Rama, Mod. Phys. Lett. **A6** (1991) 3531

[81] F. David, Mod. Phys. Lett. **A3** (1988) 1651

[82] F. David, 'Simplicial Quantum Gravity and Random Lattices', lectures given at Les Houches 1992 summer school, Saclay preprint T93/20.

[83] A. Deckmyn, Phys. Lett. **B298** (1993) 318

[84] K. Demeterfi, Int. Journal of Mod. Phys. **A8** (1993) 1185

[85] C. Destri and H. J. de Vega, Phys. Lett. **B208** (1988) 255

[86] P. Di Francesco, D. Kutasov, Nucl. Phys. **B342** (1990) 589; P. Di Francesco, D. Kutasov, 'Integrable Models of two Dimensional Quantum gravity', PUPT-1206

[87] P. Di Fransesco, P. Ginsparg and J. Zinn-Justin, '2D Gravity and Random Matrices', Physics Reports, to appear

[88] R. Dijkgraaf and E. Verlinde, Nucl. Phys. **B** (Proc. Suppl.) **5B** (1988) 87

[89] R. Dijkgraaf and E. Witten, Nucl. Phys. **B342** (1990) 486

[90] R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. **115** (1988) 649

[91] R. Dijkgraaf, E. Verlinde and H. Verlinde, 'Notes on Topological String Theory and 2d Quantum Gravity', Proceedings of the Trieste Spring School 1990, Eds. M. Green et al., World Scientific, 1991

[92] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59

[93] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59

[94] R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. **B348** (1991) 435

[95] R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. **B371** (1992) 269

[96] R. Dijkgraaf, in 'New Symmetry Principles in Quantum Field Theory', Eds. J. Fröhlich et al., proceedings of NATO Advanced Study Institute, Cargèse 1991, Plenum Press, New York (1992)

[97] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 501

[98] J. Distler, Nucl. Phys. **B342** (1990) 523

[99] P. Di Vecchia and P. Rossi, Phys. Lett. **B140** (1984) 344; P. Di Vecchia, B. Durhuus and J. Petersen, Phys. Lett. **B144** (1984) 245

[100] V. Dotsenko, Nucl. Phys. **B235** (1984) 54

[101] M.R. Douglas, Phys. Lett. **B238** (1990) 176

[102] V. Drinfel'd and V. Sokolov, J. Sov. Math **30** (1984) 1975; Sov. Math. Doklady **62** (1981) 403

[103] B. A. Dubrovin, Commun. Math. Phys. **145** (1992) 195; 'Integrable Systems and Classification of 2-Dimensional Topological Field Theories', SISSA 162/92/FM; 'Differential

Geometry of the Space of Orbits of a Coxeter Group', SISSA 29/93/FM

[104] B. A. Dubrovin, Nucl. Phys. **B379** (1992) 627

[105] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. **B240** (1984) 453; F. David, Nucl. Phys. **B368** (1992) 671; D. Förster, Nucl. Phys. **B283** (1987) 669; M. Cates, Europhys. Lett. **8** (1988) 719; A. Krzywicki, Phys. Rev. **D41** (1990) 3086

[106] E. B. Dynkin, Amer. Math. Soc. Transl. **6** (1967) 111

[107] T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987) 308

[108] T. Eguchi and S. Yang, Mod. Phys. Lett. **A4** (1990) 1693

[109] T. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, 'Topological Strings, Flat Coordinates and Gravitational Descendants', Univestity of Tokyo preprint UT 630

[110] T. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, 'Topological Field Theories and the Period Integrals', Univestity of Tokyo preprint UT 641

[111] T. Eguchi, S. Hosono and S.-K. Yang, Commun. Math. Phys. **140** (1991) 159; W. Lerche, Phys. Lett. **B252** (1990) 349

[112] J. Ellis, S. Kelley and D. V. Nanopoulos, Phys. Lett. **B249** (1990) 441; Phys. Lett. **B260** (1991) 131; P. Langacker, Univ. of Pennsylvania preprint UPR-0435T (1990); U. Amaldi, W. de Boer and F. Fürstenau, Phys. Lett. **B260** (1991) 447; P. Langacker, M.-X. Luo, Phys. Rev. **D44** (1991) 817; F. Anselmo, L. Cifarelli, A. Peterman and A. Zichichi, Nuovo Cim. **104A** (1991) 817; Nuovo Cim. **105 A** (1992) 581; J. Hisano, H. Murayama and T. Yanagida, Phys. Rev. Lett. **69** (1992) 1014; R. Roberts and G. G. Ross, Nucl. Phys. **B377** (1992) 571

[113] A. E. Faraggi, 'Construction of Realistic Standard-Like Models', talk at the TEXAS/PACHOS conference, Berkeley, December 1992; WIS-92/103/DEC-PH

[114] V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. **B280** (1987) 644

[115] V. A. Fateev and A. B. Zamolodchikov, Sov. J. Nucl. Phys. **43** (4) (1986) 657; P. Christe and R. Flume, Nucl. Phys. **B282** (1987) 466; E. Date, M. Jimbo, A. Matsuo and T. Miwa, Int. J. Mod. Phys. **B4** (1990) 1049; A. Matsuo, Commun. Math. Phys. **134** (1990) 65; V. V. Schechtman and A. N. Varchenko, 'Integral Representations of n-Point Conformal Correlators in the WZW Model', MPI/89-51; Lett. Math. Phys. **20** (1990) 279; H. Awata, A. Tsuchiya and Y. Yamada, Nucl. Phys. **B365** (1991) 680; H. Awata, Progr. Theor. Phys. Supp. **110** (1992) 303

[116] V. A. Fateev and S. L. Lykyanov, Int. Journ. Mod. Phys. **A3** (1988) 507; Sov. Phys. JETP **67** (1988) 447; Sov. Sci. Rev. A. Phys. **15** (1990) 1

[117] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, 'Polynomial and Primary Field Character of W_n^l-Algebras, DIAS-STP-91-42; L. Fehér, L. O'Raifeartaigh, P. Ruelle and I. Tsutsui, Phys. Lett. **B283** (1992) 243

[118] L. Fehér, L. O'Raifeartaigh, P. Ruelle and I. Tsutsui, 'On the Completeness of the Set of Classical W Algebras Obtained from DS Reductions', BONN-HE-93-14, DIAS-STP-93-02

[119] B. L. Feigin and D. B. Fuchs, Funct. Anal. Appl. **16** (1982) 114

[120] B. L. Feigin and E. Frenkel, 'Affine Kac-Moody Algebras at the Critical Level and Gelfand-Dikii Algebras', MSRI-04029-91

[121] B. L. Feigin and E. Frenkel, Phys. Lett. **B246** (1990) 75

[122] B. L. Feigin and E. V. Frenkel, Usp. Mat. Nauk **43** (no. 5) (1988) 227

[123] G. Felder, K. Gawędzki and A. Kupiainen, Commun. Math. Phys. **117** (1988) 127

[124] G. Felder, Nucl. Phys. **B315** (1989) 217

[125] P. Fendley and K. Intrilligator, Nucl. Phys. **B372** (1992) 533; 'Scattering and Thermodynamics in Integrable $N = 2$ Theories', BUHEP-92-5/HUTP-91-A067 (1992)

[126] P. Fendley Nucl. Phys. **B374** (1992) 667

[127] P. Fendley, W. Lerche, S. Mathur and N. P. Warner, Nucl. Phys. **B348** (1991) 66; S. Cecotti and C. Vafa, Nucl. Phys. **B367** (1991) 359; D. Nemeschansky and N. P. Warner, Nucl. Phys. **B380** (1992) 241

[128] J. M. Figueroa-O'Farrill, Nucl. Phys. **B343** (1990) 450

[129] A. H. Díaz and J. M. Figueroa-O'Farrill, Nucl. Phys. **B349** (1991) 237

[130] V. Fock, talk at 'International Conference on Mathematical Physics, String Theory and Quantum Gravity', Rakhov, November 1992; 'Towards the Geometrical Sense of Operator Product Expansions for Chiral Currents and W Algebras', preprint.

[131] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B106** (1981) 63

[132] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B158** (1985) 316; Nucl. Phys. **B261** (1985) 1

[133] E. S. Fradkin and T. E. Fradkina, Phys. Lett. **B72** (1978) 343

[134] P. di Fransesco, C. Itzykson and J.-B. Zuber, Commun. Math. Phys. **140** (1991) 543

[135] D. S. Freed, J. Diff. Geom. **28** (1988) 223

[136] M. D. Freeman and P. West, Phys. Lett. **B299** (1993) 30; 'The Covariant Scattering and Cohomology of W_3 Strings', KCL-TH-93-2

[137] E. Frenkel, PhD thesis, Harvard University, 1991

[138] E. Frenkel, V. Kac and M. Wakimoto, Commun. Math. Phys. **147** (1992) 295

[139] E. Frenkel 'W-Algebras and Langlands-Drinfeld Correspondence', in 'New Symmetry Principles in Quantum Field Theory', Eds. J. Fröhlich et al., proceedings of NATO Advanced Study Institute, Cargèse 1991, Plenum Press, New York (1992)

[140] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93

[141] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575; in 'Vertex Operators and Mathematical Physics', ed. J. Lepowski et al., Springer Verlag, 1984; Commun. Math. Phys. **107** (1986) 535

[142] J. Fröhlich and U. M. Studer, 'Gauge Invariance and Current Algebra in Non-Relativistic Many-Body Theory', Zürich preprint; A. Capelli, G. V. Dunne, C. A. Trugenberger and G. R. Zemba, 'Conformal Symmetry and Universal Properties of Quantum Hall States' CERN-TH.6702/92

[143] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. **B288** (1992) 38

[144] B. Gato-Rivera and A. M. Semikhatov, 'Singular Vectors and Topological Theories from Virasoro Constraints via the Kontsevich-Miwa Transform', CERN-TH.6752/92

[145] K. Gawędzki and A. Kupiainen, Phys. Lett. **B215** (1988) 119

[146] K. Gawędzki, in 'New Symmetry Principles in Quantum Field Theory', Eds. J. Fröhlich et al., proceedings of NATO Advanced Study Institute, Cargèse 1991, Plenum Press, New York (1992)

[147] I. M. Gelfan'd and L. A. Dickey, Funct. Anal. Appl. **10:4** (1976) 13; **11:2** (1977) 93

[148] D. Gepner and E. Witten, Nucl. Phys. **B278** (1986) 493

[149] D. Gepner, Commun. Math. Phys. **141** (1991) 381; M. Spiegelglas, Phys. Lett. **B274** (1992) 21; D. Gepner and A. Schwimmer, 'Symplectic Fusion Rules and Their Metric', WIS-92/34/March-ph

[150] A. Gerasimov, A. Morozov, M. Olshanetsky, A. Marshakov and S. Shatashvili, Int. J. Mod. Phys. **A5** (1990) 2495

[151] J.-L. Gervais and A. Neveu, Nucl. Phys. **B199** (1982) 59; Nucl. Phys. **B224** (1983) 329; Nucl. Phys. **B238** (1984) 125; Nucl. Phys. **B238** (1984) 396; Phys. Lett. **B151** (1985) 271

[152] J.-L Gervais and A. Neveu, Nucl. Phys. **B209** (1982) 125; Phys. Lett. **B160** (1985) 277; Nucl. Phys. **B264** (1986) 557

[153] J.-L. Gervais and Y. Matsuo, Phys. Lett. **B274** (1992) 309; in 'Proceedings of the Workshop on New Aspects of Quantum Field Theories', Eds. S. Nojiri, Institute for Nuclear Study, University of Tokyo, INS-T-513, 1992; Commun. Math. Phys. **152** (1993) 317

[154] F. Gieres, Int. Journal of Mod. Phys. **A8** (1993) 1

[155] P. B. Gilkey, 'Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem', Mathematics Lecture Series 11, Publish or Perish, 1984

[156] P. Ginsparg and G. Moore, 'Lectures on 2D Gravity and 2D String Theory', lectures given at 1992 TASI summer school, YCTP-P23-92, LA-UR-92-3479

[157] P. Ginsparg, in 'Fields, Strings and Critical Phenomena', Les Houches session XLIX, eds. E. Brézin and J. Zinn-Justin, North-Holland, 1990

[158] P. Ginsparg, 'Matrix Models of 2d Gravity', lectures at the Trieste Spring School 1991, Los Alamos preprint LA-UR-91-4101

[159] P. Goddard, A. Kent and D. Olive, Phys. Lett. **B152** (1986) 88; Commun. Math. Phys. **103** (1986) 105

[160] J. Goeree, 'Higher Spin Extensions of Two-Dimensional Gravity', PhD thesis, Utrecht, 1993

[161] M. T. Grisaru and P. van Nieuwenhuizen, Int. Journal of Mod. Phys. **A7** (1992) 5891

[162] M. F. de Groot, T. J. Hollowood and J. L. Miramontes, Commun. Math. Phys. **145** (1992) 57; N. J. Burroughs, M. F. de Groot, T. J. Hollowood and J. L. Miramontes,

Commun. Math. Phys. **153** (1993) 187

[163] A. Gupta, S. Trivedi and M. Wise, Nucl. Phys. **B340** (1990) 475; M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051; Y. Kitazawa, Phys. Lett. **B265** (1991) 262; N. Sakai and Y. Tanii, Progr. Theor. Phys. **86** (1991) 547; Int. Journal of Mod. Phys. **A6** (1991) 2743; N. Sakai, '$c = 1$ Two Dimensional Quantum Gravity', TIT/HEP-205; L. Alvarez-Gaumé, J. L. F. Barbón and C. Gómez, Nucl. Phys. **B368** (1991) 57; V. Dotsenko, Mod. Phys. Lett. **A6** (1991) 3601; P. Di Fransesco and D. Kutasov, Phys. Lett. **B261** (1991) 385; Nucl. Phys. **B375** (1992) 119; M. Bershadsky and I. Klebanov, Phys. Rev. Lett. **65** (1990) 3088; Nucl. Phys. **B360** (1991) 559; K. Aoki and E. D'Hoker, 'On the Liouville Approach to Correlation Functions for 2d gravity', UCLA/91/TEP/32; D. Ghoshal and S. Mahapatra, Mod. Phys. Lett. **A8** (1993) 197; S. Yamaguchi, Mod. Phys. Lett. **A8** (1993) 327

[164] Q.-Han Park, Phys. Lett. **B236** (1990) 429; Phys. Lett. **B238** (1990) 287; Phys. Lett. **B257** (1991) 105

[165] J. A. Harvey and A. Strominger, 'Quantum Aspects of Black Holes', EFI-92-41

[166] N. Hayashi, Nucl. Phys. **B363** (1991) 681

[167] M. Henneaux, Phys. Rep. **126** (1985) 1

[168] N. Hitchin, Duke Math. J. **54** (1987) 91; M. F. Atiyah, 'The Geometry and Physics of Knots', Cambridge University Press, Cambridge, 1990

[169] N.J. Hitchin, In 'Global Geometry and Mathematical Physics', Eds. M. Francaviglia, F. Gherardelli, Lecture Notes in Mathematics 1451, Springer-Verlag, Berlin, 1990.

[170] N.J. Hitchin, Proc. London Math. Soc. (3) **55** (1987) 59.

[171] N. J. Hitchin, Progr. Theor. Phys. Supp. **102** (1990) 159

[172] N.J. Hitchin, Topology **31** (1992) 449.

[173] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60** (1988) 917

[174] E. D'Hoker, in 'Strings and Symmetries 1991', Eds. N. Berkovits et al., World Scientific, Singapore, 1992

[175] E. D'Hoker, Mod. Phys. Lett. **A6** (1991) 745

[176] S. Hosono, 'Algebraic Definition of Topological W-Gravity', UT-588

[177] H.-L. Hu and M. Yu, Phys. Lett. **B289** (1992) 302; Nucl. Phys. **B391** (1993) 389

[178] J. Hughes, J. Liu and J. Polchinski, Nucl. Phys. **B316** (1989) 15

[179] C. M. Hull and P. K. Townsend, Nucl. Phys. **B274** (1986) 349

[180] C. M. Hull, Nucl. Phys. **B367** (1991) 731

[181] C. M. Hull, Phys. Lett. **B240** (1990) 110

[182] C. M. Hull, Phys. Lett. **B259** (1991) 68

[183] C. M. Hull, 'Classical and Quantum W Gravity', in 'Strings and Symmetries 1991', Eds. N. Berkovits et al., World Scientific, Singapore, 1992; 'Lectures on W-Gravity, W-Geometry and W-Strings', QMW-93-2, to appear in proceedings of Trieste Summer School in High Energy Physics and Cosmology, 1992

[184] C. M. Hull, Phys. Lett. **B269** (1991) 257; 'W Geometry', QMW-92-6, to appear in Commun. Math. Phys.; 'The Geometric Structure of W_N-Gravity', QMW-93-6

[185] N. Ishibashi, Nucl. Phys. **B379** (1992) 220

[186] H. Ishikawa and M. Kato, Phys. Lett. **B302** (1993) 209; S. Mukhi and C. Vafa, 'Two Dimensional Black-hole as a Topological Coset Model of $c = 1$ String Theory', HUTP-93/A002, TIFR/TH/93-01; O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, '$c = 1$ String Theory as a Topological G/G Model', TAUP-2032-93

[187] K. Itoh, Nucl. Phys. **B342** (1990) 449; T. Kuramoto, Phys. Lett. **B233** (1989) 363; Z. Horváth, L. Palla and P. Vecsernyés, Int. Journal of Mod. Phys. **A4** (1989) 5261

[188] K. Ito, Phys. Lett. **B295** (1991) 73; Nucl. Phys. **B370** (1992) 123

[189] I. Jack and J. Panvel, 'WZW-Toda Reduction Using the Casimir Operator', LTH 304

[190] A. Jaffe and C. Taubes, 'Vortices and Monopoles', Birkhäuser, Boston, 1980

[191] V. G. Kac and J. W. van de Leur, 'The n-Component KP Hierarchy and Representation Theory', preprint

[192] V. Kac, 'Infinite Dimensional Lie Algebras', 3d ed., Cambridge University Press, 1990

[193] L. P. Kadanoff, Ann. Phys. **120** (1979) 39; L. P. Kadanoff and A. C. Brown, Ann. Phys. **121** (1979) 318

[194] V. Kaplunovsky, Nucl. Phys. **B307** (1988) 145; I. Antoniadis, J. Ellis, R. Lacaze and D. V. Nanopoulos, Phys. Lett. **B268** (1991) 188

[195] M. Karowski, Nucl. Phys. **B153** (1979) 244

[196] H. G. Kausch and G. M. T. Watts, Nucl. Phys. **B354** (1991) 740

[197] T. Kawai and T. Nakatsu, Mod. Phys. Lett. **A6** (1991) 3557; erratum, Mod. Phys. Lett. **A7** (1992) 267

[198] V. A. Kazakov and A. A. Migdal, Nucl. Phys. **B311** (1988) 171

[199] V. Kazakov, 'Bosonic Strings and String Field Theories in One-Dimensional Target Space', in proceedings of 1990 Cargése workshop, 'Random Surfaces and Quantum Gravity', Eds. O. Alvarez et al., Plenum Press, 1991

[200] Y. Kazama and H. Suzuki, Phys. Lett. **B216** (1989) 112; Nucl. Phys. **B321** (1989) 232

[201] B. A. Khesin, Usp. Mat. Nauk **45** (1) (1990); V. Yu. Ovsienko and B. A. Khesin, Funct. Anal. Appl. **24** (1990) 33; B. A. Khesin and B. Z. Shapiro, Commun. Math. Phys. **145** (1992) 357

[202] T. G. Khovanova, Funct. Anal. Appl. **21** (1987) 332

[203] T. R. Klassen and E. Melzer, Nucl. Phys. **B362** (1991) 329; 'Kinks in Finite Volume', CLNS-92-1190/ITP-SB-92-01; Nucl. Phys. **B338** (1990) 485; Nucl. Phys. **B350** (1991) 635; 'Sine-Gordon $\neq$ Massive Thirring, and Related Heresies', CLNS-92/1149,ITP-SB-92-36

[204] I. Klebanov and A. M. Polyakov, Mod. Phys. Lett. **A6** (1991) 3273

[205] I. Klebanov, 'String Theory in Two Dimensions', lectures at 1991 Trieste Spring School on 'String Theory and Quantum Gravity', PUPT-1271

[206] M. Knecht, S. Lazzarini and F. Thuillier, Phys. Lett. **B251** (1990) 279; M. Knecht, S. Lazzarini and R. Stora Phys. Lett. **B262** (1991) 25; 'On Holomorphic Factorization of Free Conformal Fields II', IPNO/TH 91-70, SISSA-140/91/FM, LAPP-TH-351/91, CERN-TH.6262/91

[207] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819

[208] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247** (1984) 83

[209] M. Kontsevich, Commun. Math. Phys. **147** (1992) 1

[210] B. Kostant and S. Sternberg, Ann. Phys. **176** (1987) 49

[211] I. Krichever, Commun. Math. Phys. **143** (1992) 415; 'The τ-function of the Universal Whitham Type Hierarchy, Matrix Models and Topological Field Theories', LPTENS-92/18; 'Whitham Theory for Integrable Systems and Topological Field Theories' in 'New Symmetry Principles in Quantum Field Theory', Eds. J. Fröhlich et al., proceedings of NATO Advanced Study Institute, Cargèse 1991, Plenum Press, New York (1992)

[212] H. Kunimoto, 'Topological $SL(3,R)$ Gravity and the Quantum W_3 Symmetry', YITP/K-919

[213] D. Kutasov and N. Seiberg, Nucl. Phys. **B358** (1991) 600

[214] D. Kutasov, Phys. Lett. **B233** (1989) 369

[215] J. M. F. Labastida, M. Pernici and E. Witten, Nucl. Phys. **B310** (1988) 611; D. Montano and J. Sonnenschein, Nucl. Phys. **B313** (1989) 258; R. C. Myers and V. Periwal, Nucl. Phys. **B333** (1990) 536

[216] S. Lazzarini and R. Stora, proceedings 13^{th} John Hopkins Workshop on Common Trends in Mathematics and Quantum Field Theories, 'Knots, Topology and Quantum Field Theories', ed. L. Lusanna, World Scientific, Singapore, 1989.

[217] W. Lerche, 'Chiral Rings in Topological (W-)Gravity', talk given at 26^{th} workshop 'From Superstrings to Supergravity', Erice, December 1992, CERN-TH.6812/93

[218] W. Lerche, C. Vafa and N. Warner, Nucl. Phys. **B324** (1989) 427

[219] H. Leutwyler and M. Shifman, Int. Journal of Mod. Phys. **A7** (1992) 795

[220] A. N. Leznov and M. V. Savaliev, 'Group-Theoretical Methods for Integration of Non-Linear Dynamical Systems', Birkhäuser, Basel, 1992

[221] A. N. Leznov and M. V. Savaliev, Phys. Lett. **B79** (1978) 294; Lett. Math. Phys. **3** (1979) 207; Lett. Math. Phys. **6** (1982) 505; Commun. Math. Phys. **89** (1983) 59

[222] A. N. Leznov and M. V. Saveliev, Commun. Math. Phys. **74** (1980) 111

[223] B. Lian and G. Zuckerman, Phys. Lett. **B254** (1991) 417; Phys. Lett. **B266** (1991) 21; Commun. Math. Phys. **135** (1991) 547; Commun. Math. Phys. **145** (1992) 561

[224] K. Li, Nucl. Phys. **B354** (1991) 711; Nucl. Phys. **B354** (1991) 725

[225] K. Li, Phys. Lett. **B251** (1990) 54; Nucl. Phys. **B346** (1990) 329

[226] E. Looijenga, 'Intersection Theory on Deligne-Mumford Compactifications', Séminaire Bourbaki, 45éme année, 1992-93, no 768

[227] A. Lossev, ‘Descendants Constructed from Matter Field and K. Saito Higher Residue Pairing in Landau-Ginsburg Theories Coupled to Topological Gravity’, TPI-MINN-xx-92/6-T; ‘Descendants Constructed from Matter Field in Topological Landau-Ginzburg Theories Coupled to Topological Gravity’, talk at the Kiev conference on String Theory, June 1992; talk at ‘International Conference on Mathematical Physics, String Theory and Quantum Gravity’, Rakhov, November 1992

[228] C. Lovelace, Nucl. Phys. **B273** (1986) 413

[229] H. Lu, C. N. Pope and X. Shen, Nucl. Phys. **B366** (1991) 95

[230] H. Lu, C. N. Pope, S. Schrans and X. J. Wang, ‘On the Spectrum and Scattering of W_3 Strings’, CTP TAMU-4/93; ‘The Interacting W_3 String’, CTP TAMU-86/92

[231] J. Mañes and Y. Lozano, ‘Introduction to Non-Perturbative 2d Quantum Gravity’, Fortschr. Phys. **41** (1993) 187

[232] P. Mansfield and B. Spence, Phys. Lett. **B258** (1991) 49

[233] N. Marcus and Y. Oz, Nucl. Phys. **B392** (1993) 281

[234] E. Martinec, ‘An Introduction to 2d Gravity and Solvable String Models’, lectures at 1991 Trieste spring school, Rutgers preprint RU-91-51

[235] Y. Matsuo, ‘Classical W_n-Symmetry and Grassmannian Manifold’, LPTENS-91/19; R. Zucchini, ‘Light Cone W_n Geometry and its Symmetries and Projective Field Theory’, DFUB-92; A. Bilal, V. V. Fock and I. I. Kogan, ‘On the Origin of W Algebras’, CERN-TH.5965/90; A. Gerasimov, A. Levin and A. Marshakov, Nucl. Phys. **B360** (1991) 537

[236] Y. Matsuo, Phys. Lett. **B227** (1989) 209.

[237] N. Mavromontes and J. Miramontes, Mod. Phys. Lett **A4** (1989) 1849; E. D’Hoker and P. S. Kurzepa, Mod. Phys. Lett **A5** (1990) 1411

[238] J. McCleary, ‘User’s Guide to Spectral Sequences’, Mathematics Lecture Series 12, Publish or Perish, Washington, 1982

[239] V. B. Mehta and C. S. Seshadri, Math. Ann. **248** (1980) 205

[240] M. Mehta, Commun. Math. Phys. **79** (1981) 327; S. Chadha, G. Mahoux, M. Mehta, J. Phys. **A14** (1981) 579

[241] K. A. Meissner and J. Pawelczyk, Mod. Phys. Lett. **A10** (1990) 763

[242] A. Miković, Phys. Lett. **B260** (1991) 75; Phys. Lett. **B278** (1991) 51

[243] G. Moore and N. Seiberg, Phys. Lett. **B212** (1988) 451; Nucl. Phys. **B313** (1989) 16; Commun. Math. Phys. **123** (1989) 177

[244] A. Morozov, ‘Integrability and Matrix Models’, ITEP-M2/93

[245] A. Morozov, JETP Lett. **49** (1989) 345; M. Kuwahara and H. Suzuki, Phys. Lett. **B235** (1990) 52; Phys. Lett. **B235** (1990) 57; K. Ito and Y. Kazama, Mod. Phys. Lett. **A5** (1990) 215; Mod. Phys. Lett. **A6** (1991) 581

[246] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. **B266** (1991) 337; H. Kanno and M Sarmadi, IC/92/150; K. Itoh and N. Ohta, Nucl. Phys. **B377** (1992) 113; N. Chair, V. Dobrev and H. Kanno, Phys. Lett. **B283** (1992) 194; S. Govindarajan, T. Jayamaran,

V. John and P. Majumdar, Mod. Phys. Lett. **A7** (1992) 1063; S. Govindarajan, T. Jayamaran and V. John, 'Chiral Rings and Physical States in $c < 1$ String Theory', IMSc-92/30

[247] M. Lässig, G. Mussardo and J. L. Cardy, Nucl. Phys. **B348** (1991) 591

[248] G. Mussardo, G. Sotkov and M. Stanichkov, Int. Journal of Mod. Phys. **A4** (1989) 1135; K. Ito, Nucl. Phys. **B332** (1990) 566

[249] T. Nakatsu and Y. Sugawara, 'Instanton and Spectral Flow in Topological Conformal Field Theories', UT-637

[250] T. Nakatsu and Y. Sugawara, Mod. Phys. Lett. **A7** (1992) 2215

[251] T. Nakatsu and Y. Sugawara, Nucl. Phys. **B385** (1992) 276

[252] M. S. Narasimhan and C. S. Seshadri, Ann. Math. **82** (1965) 540.

[253] D. Nemeschansky and S. Yankielowicz, '$N = 2$ W Algebra, Kazama-Suzuki Models and Drinfeld-Sokolov Reduction', USC-91/005

[254] P. van Nieuwenhuizen, Phys. Rep. **68** (1981) 189

[255] S. P. Novikov, Sov. Math. Doklady **24** (1981) 222; Usp. Math. Nauk. **37** (1982) 3

[256] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **145** (1992) 515

[257] G. Papadopoulos and B. Spence, 'The Canonical Structure of Wess-Zumino-Witten Models', Imperial 91-92/19, QMW 92/2; 'The Global Phase Space Structure of the Wess-Zumino-Witten Model', Imperial 91-92/28, QMW 92/9; 'The Phase Space of the Wess-Zumino-Witten Model', Imperial 91-92/32, QMW 92/15

[258] T. Kimura, Commun. Math. Phys. **151** (1993) 155; M. Dubois-Violette, Ann. Inst. Fourier **37** (1987) 45 ; D. Wilbour, J. M. Arms, 'Reduction procedures for Poisson manifolds', Washington Preprint.

[259] J. Polchinski, in 'Strings '90', Proceedings of Texas A& M Workshop, Eds. B. Arnowitt et al., World Scientific, Singapore, 1991

[260] J. Polschinski, Commun. Math. Phys. **104** (1986) 37

[261] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **B131** (1983) 121; Phys. Lett. **B141** (1984) 233

[262] A. M. Polyakov, in Les Houches 1988, Fields, Strings and Critical Phenomena, eds. E. Brézin and J. Zinn-Justin, North-Holland, 1990.

[263] A. M. Polyakov, in 'Physics and Mathematics of Strings', eds. L. Brink, D. Friedan and A. M. Polyakov, World Scientific, Singapore, 1990

[264] A. Polyakov, Int. J. Mod. Phys. **A5** (1990) 833

[265] A. M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893

[266] A. M. Polyakov, Mod. Phys. Lett. **A6** (1991) 635

[267] A. M. Polyakov, Phys. Lett. **B103** (1981) 207

[268] C. N. Pope, 'Lectures on W Algebras and W Gravity', CTP TAMU-103/91, to appear in proceedings of Trieste Summer School on High-Energy Physics, 1991

[269] C. N. Pope, E. Sezgin, K. S. Stelle and X. J. Wang, Phys. Lett. **B299** (1993) 247; P. West, 'On the Spectrum, No Ghost Theorem and Modular Invariance of W_3 Strings', KCL-TH-92-7; H. Lu, B. E. W. Nilsson, C. N. Pope, K. S. Stelle and P. C. West, 'The Low-Level Spectrum of the W_3 String', CTP TAMU-70/92; C. N. Pope and X. J. Wang, 'The Ground Ring of the W_3 String', CTP TAMU-3/93

[270] C. N. Pope, L. J. Romans and K. S. Stelle, Phys. Lett. **B268** (1991) 167; Phys. Lett. **B269** (1991) 287; C. N. Pope, L. J. Romans, E. Sezgin and K. S. Stelle, Phys. Lett. **B274** (1992) 298; H. Lu, C. N. Pope, S. Schrans and K. W. Xu, Nucl. Phys. **B385** (1992) 99; H. Lu, C. N. Pope, S. Schrans and X. J. Wang, Nucl. Phys. **B379** (1992) 47

[271] A. Pressley and G. Segal, 'Loop Groups', Oxford University Press, 1986

[272] D. Quillen, Funct. Anal. Appl. **19** (1985) 31.

[273] L. O'Raifeartaigh, P. Ruelle and I. Tsutsui, Phys. Lett. **B258** (1991) 359

[274] L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Commun. Math. Phys. **143** (1992) 333

[275] V. Sadov, 'Free Field Resolutions for Nonunitary Representations of $N = 2$ SuperVirasoro Algebra', HUTP-92/A070

[276] V. Sadov, 'On the Spectra of $\hat{sl}(N)_k/\hat{sl}(N)_k$-cosets and W_N Gravities I', HUTP-92/A055

[277] V. Sadov, 'The Hamiltonian Reduction of the BRST Complex and $N = 2$ SUSY', HUTP-93/A006

[278] N. Sakai and Y. Tanii, Phys. Lett. **B276** (1992) 41; Prog. Theor. Phys. Supp. **110** (1992) 117

[279] H. J. Schnitzer, Nucl. Phys. **B324** (1989) 412;

[280] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, 'Induced Gauge Theories and W gravity', in proceedings 'Strings and Symmetries', eds. N. Berkovitz et al., World Scientific, 1991

[281] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **124** (1989) 87

[282] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B349** (1991) 791; Int. Journal of Mod. Phys. **A6** (1991) 2891

[283] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B364** (1991) 584

[284] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B371** (1992) 315.

[285] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. **B243** (1990) 245

[286] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, 'Example of a Quantum Field Theory Based on a Non-Linear Lie Algebra', in 'Festschrift on the Occasion of M. Veltman's 60th birthday', Eds. R. Akhoury et al.

[287] N. Seiberg, in 'Random Surfaces and Quantum Gravity', Eds. O. Alvarez, E. Marinari and P. Windey, NATO ASI series B: vol **262**, Plenum Press, 1991

[288] N. Seiberg, Progr. Theor. Phys. **102** (1990) 319

[289] A. Sen, Phys. Rev. **D32** (1985) 2102

[290] A. Sevrin and P. van Nieuwenhuizen, 'Quantum Gauging from Classical Gauging of Nonlinear Algebras', ITP-SB-92-67, LBL-33339, UCB-PTH-92/43

[291] A. Sevrin, K. Thielemans and W. Troost, 'Induced and Effective Gravity Theories in $D = 2$', LBL-33738, UCB-PTH-93/06, KUL-TF-93/09

[292] A. Sevrin, K. Thielemans and W. Troost, 'The Relation between Linear and Non-Linear $N = 3, 4$ Supergravity Theories', LBL-33738, UCB-PTH-93/06, KUL-TF-93/09

[293] A. Sevrin, R. Siebelink and W. Troost, in preparation

[294] C.T. Simpson, 'Higgs Bundles and Local Systems', Princeton University Preprint.

[295] C.T. Simpson, Journal of the A.M.S. **1** (1988) 867.

[296] J. Sonnenschein, 'Physical States in Topological Coset Models', TAUP-1999-92

[297] G. Sotkov and M. Stanishkov, Nucl. Phys. **B356** (1991) 439; G. Sotkov, M. Stanishkov and C. J. Zhu, Nucl. Phys. **B356** (1991) 245

[298] M. Spiegelglas and S. Yankielowicz, Nucl. Phys. **B393** (1993) 30; 'Fusion Rules as Amplitudes in G/G Theories', Technion PH-35-90

[299] L. Takhtajan, in 'New Symmetry Principles in Quantum Field Theory', Eds. J. Fröhlich et al., Plenum Press, New york, 1992 (Proceedings of NATO Advanced Study Institute, Cargése, July 1992)

[300] J. Thierry-Mieg, Phys. Lett. **B197** (1987) 368

[301] T. Tjin, PhD thesis, Amsterdam, to appear

[302] A. A. Tseytlin, 'Effective Action of Gauged WZW Model and Exact String Solutions', Imperial/TP/92-93/10; 'Conformal Sigma Models Corresponding to Gauged Wess-Zumino-Witten Theories', CERN-TH.6804/93

[303] A. A. Tseytlin, Int. J. Mod. Phys. **A5** (1990) 589

[304] I. Tsutsui and L. Fehér, 'On the Langrangian Realization of the WNZW Reductions', DIAS-STP-92-09, UdeM-LPN-TH-92/92

[305] C. Vafa, Mod. Phys. Lett. **A6** (1991) 337

[306] W. P. G. van Velthoven and F. A. Bais, Physica **139A** (1986) 326

[307] E. Verlinde and H. Verlinde, Nucl. Phys. **B348** (1991) 457

[308] E. Verlinde, Nucl. Phys. **B300** (1988) 360

[309] H. Verlinde, Nucl. Phys. **B337** (1990) 652

[310] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240** (1984) 312; Nucl. Phys. **B251** (1985) 691

[311] M. Wakimoto, Commun. Math. Phys. **104** (1986) 605

[312] G. M. T. Watts, Phys. Lett. **B245** (1990) 65

[313] J. Wess and B. Zumino, Phys. Lett. **B37** (1971) 95

[314] B. de Wit, M. Grisaru, P. van Nieuwenhuizen, in preparation

[315] E. Witten, 'Algebraic Geometry Associated With Matrix Models of Two-Dimensional Gravity', IAS preprint IASSNS-HEP-91/74

[316] E. Witten and B. Zwiebach, Nucl. Phys. **B377** (1992) 55

[317] E. Witten, Commun. Math. Phys. **117** (1988) 353

[318] E. Witten, Commun. Math. Phys. **118** (1988) 411

[319] E. Witten, Commun. Math. Phys. **144** (1992) 189

[320] E. Witten, Commun. Math. Phys. **92** (1984) 455

[321] E. Witten, Nucl. Phys. **B340** (1990) 281

[322] E. Witten, Nucl. Phys. **B371** (1992) 191

[323] E. Witten, Nucl. Phys. **B373** (1992) 187

[324] E. Witten, 'On Background Independence in Open-String Field Theory', IASSNS-HEP-92/53

[325] E. Witten, Phys. Rev. **D44** (1991) 314

[326] E. Witten, 'Surprises with Topological Field Theories', Proc. of the Superstring Workshop, Texas A & M University (1990), eds. R. Arnowitt *et. al.*, World Scientific, Singapore (1991).

[327] E. Witten, Surveys in Diff. Geom. **1** (1991) 243

[328] Y. S. Wu, Phys. Lett. **B153** (1985) 70; W. Nahm, Duke Math. J. **54** (1987) 579; E. Guadagnini, M. Martellini and M. Minchev, Phys. Lett. **B191** (1987) 69; K. Bardacki, E. Rabinovici and B. Säring, Nucl. Phys. **B299** (1988) 157; A. Altschuler, K. Bardacki and E. Rabinovici Commun. Math. Phys. **118** (1988) 241; K. Gawędzki and A. Kupiainen, Nucl. Phys. **B320** (1989) 625; D. Karabali, Q.-H. Park, H. J. Schnitzer and Z. Yang, Phys. Lett. **B216** (1989) 307; H. J. Schnitzer, Nucl. Phys. **B324** (1989) 412; D. Karabali and H. J. Schnitzer, Nucl. Phys. **B329** (1990) 649

[329] K. Yamagishi, Phys. Lett. **B205** (1988) 466; P. Mathieu, Phys. Lett. **B208** (1988) 101; O. Babelon, Phys. Lett. **B215** (1988) 523

[330] K. Yoshida, Mod. Phys. Lett. **A4** (1989) 71; K. Yoshida, Int. J. Mod. Phys. **A5** (1990) 4559

[331] A. B. Zamolodchikov, Int. Journ. Mod. Phys. **A3** (1988) 743; Advanced Studies in Pure Mathematics **19** (1989) 1

[332] A. B. Zamolodchikov, JETP Lett. **43** (1986) 730

[333] A. B. Zamolodchikov, Theor. Math. Phys. **65** (1985) 1205

[334] Al. B. Zamolodchikov, Nucl. Phys. **B342** (1990) 695

[335] Al. B. Zamolodchikov, preprint ITEP 84-89 (1989)

[336] R. Zucchini, Phys. Lett. **B260** (1990) 296; Commun. Math. Phys. **152** (1993) 269

Samenvatting

De hedendaagse theoretische natuurkunde kent twee zeer succesvolle theorieën, die beide in goede overeenstemming met experimentele gegevens zijn, maar die (nog) niet op consistente wijze in een overkoepelende theorie ondergebracht kunnen worden. Aan de ene kant is dit de algemene relativiteitstheorie van Einstein, die de gravitationele wisselwerking tussen materie beschrijft, en nauwkeurige voorspellingen doet voor de omloopbanen van planeten, de rotatiesnelheid van binaire pulsars, enz. Aan de andere kant is er het standaardmodel van de elektromagnetische krachten en de zwakke en sterke kernkrachten, dat de niet gravitationele wisselwerkingen tussen de elementaire deeltjes beschrijft, en met grote precisie de uitkomsten van experimenten voorspelt die gedaan worden met deeltjesversnellers zoals die in het CERN. Het is tot nu toe niet mogelijk geweest een experiment uit te voeren, waarvan de uitkomst alleen voorspeld kan worden door gebruik te maken van zowel de algemene relativiteitstheorie als het standaard model. Zo'n experiment zou cruciale informatie omtrent de structuur van een overkoepelende theorie kunnen leveren, maar lijkt helaas ook in de naast toekomst niet haalbaar. Daarom moeten we ons bij het zoeken naar deze overkoepelende theorie laten leiden door andere criteria, namelijk is de theorie consistent en is het mogelijk om zowel de algemene relativiteitstheorie als het standaard model eruit af te leiden?

Een voorstel voor zo'n overkoepelende theorie is de zogenaamde stringtheorie. Het basisidee van stringtheorie is om elementaire deeltjes niet langer als puntdeeltjes te beschouwen, maar als verschillende manifestaties van een string (een 'touwtje'). De verschillende manieren waarop een string kan trillen corresponderen dan met de verschillende elementaire deeltjes. Er zijn vele verschillende stringtheorieën, en diverse ervan bevatten in bepaalde limieten zowel de algemene relativiteitstheorie als een uitbreiding van het standaard model. De eis dat stringtheorieën consistent moeten zijn geeft echter restricties. Zo is de eenvoudigste stringtheorie alleen consistent als de string in 26 in plaats van 4 dimensies leeft. Vandaar dat het van belang is om een classificatie te maken van alle mogelijke consistente stringtheorieën. In dit proefschrift wordt een grote klasse van consistente stringtheorieën geconstrueerd, en een begin gemaakt met het exact oplossen van deze modellen.

Een string kan het beste gezien worden als een rond gesloten rubber touwtje. De trillingen van het touwtje kunnen gezien worden als een superpositie van linksom en rechtsom bewegende golven. Meestal beperken we ons alleen tot de linksom draaiende golven, die horen bij de zogenaamde chirale sector van de stringtheorie. De rechtsom draaiende golven vormen de anti-chirale sector. Als we beide golven tegelijk bekijken, dan spreken we van een covariante formulering van de stringtheorie. Een stringtheorie heeft in het algemeen een enorm grote symmetriegroep. Als het rubber touwtje een bepaalde vorm heeft, kunnen we het touwtje op de ene plaats wat uit elkaar trekken en op een andere plaats wat in elkaar duwen zonder dat de vorm van het touwtje veranderd. Al deze operaties samen vormen een oneindige symmetriegroep, die iedere stringtheorie bezit: de theorie hangt alleen af van de vorm van het rubber touwtje, niet van de manier waarop we het rubber opgerekt en in elkaar geduwd hebben.

Wat we tot nu toe hebben beschreven is een 'kale' string theorie. Er zijn nog veel meer stringtheorieën, waarbij er van alles en nog wat op het rubber touwtje rondloopt, bijvoorbeeld (symbolisch) een aantal mannetjes met gekleurde vlaggetjes. Deze theorieën hebben vaak een nog grotere symmetriegroep, behalve van de manier waarop we de spanning over het rubber touwtje verdelen hangt de theorie er bijvoorbeeld ook niet van af of een mannetje rechtop staat of op zijn kop loopt. Deze symmetrieën hebben soms een merkwaardig karakter, wat het moeilijk maakt om er mee te werken. Ze kunnen namelijk niet lineair zijn. Om te illustreren wat dit betekent kunnen we bijvoorbeeld eens denken aan een muur met een aantal tennisballen wat ervoor op de grond ligt. We voeren twee operaties (zeg O_1 en O_2) uit op de ballen. De eerste (O_1) is om ze allemaal een meter verder van de muur te leggen, de tweede (O_2) is om ze allemaal tweemaal zover van de muur af te leggen als ze lagen. Nu maakt het wat uit in welke volgorde we de operaties uitvoeren. Als we eerst O_1 en dan O_2 uitvoeren, dan komen de ballen een meter verder van de muur te liggen als wanneer we eerst O_2 en dan O_1 uitvoeren. Met andere woorden, de commutator van O_1 en O_2 is gelijk aan O_1. Dit is een voorbeeld van een lineaire algebra. Voor niet-lineaire algebras wordt de commutator van twee operaties gegeven door het product van een aantal operaties, wat dat dan ook mag betekenen. De symmetrie groepen die we in dit proefschrift bekijken, de zogenaamde W algebras, zijn meestal van dit niet-lineaire soort.

In hoofdstuk twee geven we een korte inleiding in de twee-dimensionale veldentheorieën, de zogenaamde conforme veldentheorieën, die stringtheorieën beschrijven. Vervolgens bevat het hoofdstuk een analyse van W algebras, zowel van de klassieke als van de quantummechanische versies van W algebras.

In hoofdstuk drie leggen we uit hoe we een stringtheorie die een bepaalde symmetrie algebra in de chirale sector heeft, van nog veel meer symmetrieën kunnen voorzien, door ijkvelden in te voeren (extra mannetjes op het rubber touwtje). We berekenen de effectieve quantummechanische theorie voor deze ijkvelden, en quantiseren deze effectieve theorie tot op alle ordes in storingstheorie.

In hoofdstuk vier doen we hetzelfde, maar dan voor de covariante formulering van de theorie. Het blijkt dat zelfs als we met een niet consistente stringtheorie begonnen waren, de extra velden die we invoeren precies zodanig zijn dat de stringtheorie consistent wordt. Dit biedt de mogelijkheid om een consistente stringtheorie te verkrijgen, uitgaande van een niet-consistente.

De correlatiefuncties in deze stringtheorieën worden niet gegeven door een som over Feynman diagrammen, maar door een integraal over een eindig-dimensionale ruimte, de moduli ruimte, die de Feynman diagrammen voor de string parametriseert. Zo'n diagram is een twee-dimensionaal oppervlak met een aantal in- en uitgaande cylinders die de in- en uitgaande strings weergeven. In hoofdstuk vijf bepalen we de moduli ruimte voor stringtheorieën met een W algebra als symmetriegroep.

Tenslotte bekijken we in hoofdstuk zes het spectrum van deze theorieën en geven een aantal manieren waarlangs (een gedeelte van) de exacte oplossing van deze theorieën verkregen zouden kunnen worden.

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Curriculum Vitae

Ik ben op 29 juni 1967 geboren te Sint Nicolaasga. Van 1978 tot 1984 bezocht ik de Rijksscholengemeenschap in Sneek, waar ik in 1984 mijn gymnasium B diploma behaalde. Vervolgens ben ik wiskunde en natuurkunde gaan studeren aan de Rijksuniversiteit Groningen. In augustus 1989 heb ik de doctoraal examens (cum laude) voor deze studies behaald. Daarna ben ik, in dienst van de stichting FOM, als onderzoeker in opleiding met het hier afgeronde promotieonderzoek begonnen aan het instituut voor theoretische fysica te Utrecht. Na deze promotie ga ik in september voor twee jaar als postdoc naar de State University of New York at Stony Brook in de Verenigde Staten.