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Black Hole Bound States and Their Quantization

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We briefly review the construction of multi-centered black hole solutions in type IIA string theory. We then discuss a decoupling limit which embeds these solutions in M-theory on $\text{AdS}_3 \times \text{S}^2 \times \text{CY}$, and discuss some aspects of their dual CFT interpretation. Finally, we consider the quantization of these solutions and applications to the wall-crossing formula and the “fuzzball” proposal.

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1. Introduction

Since the initial success of string theory in accounting for the entropy of supersymmetric black holes by counting states in a field-theory¹ there has been an ongoing effort to understand exactly what the structure of these microstates is and how they manifest themselves in gravity. In addition, the entropy of a large variety of black objects has been successfully reproduced by counting states in suitable dual field theories. In many, but not all cases (the BMPV black hole being a notable exception), black objects can be embedded in a spacetime which is asymptotically AdS by taking a suitable decoupling limit. The relevant dual field theory is then simply the one that one obtains via AdS/CFT duality.

Here, we will consider a large class of black objects, namely bound states of four-dimensional black holes in type IIA string theory compactified on a Calabi-Yau manifold. These are of interest, since they are the largest known family of supersymmetric black objects, they include supersymmetric black holes with a large macroscopic horizon so that several fundamental issues in black hole physics can be addressed, and they also include a large family of smooth solutions which are candidate microstates of supersymmetric black holes, as explained later. Furthermore, these black hole bound states play an important role in trying to find a mathematically precise formulation of the notion of bound states of D-branes.

In what follows, we will describe in some detail the nature of the four-dimensional black hole bound states and a decoupling limit which embeds them in $\text{AdS}_3 \times \text{S}^2 \times \text{CY}$, showing that they should be amenable to a description in a dual

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two-dimensional conformal field theory. We will then describe some features of the space of solutions and its quantization, and finally conclude with some possible applications of our results. Most of what we describe is based on results presented in Refs 2 and 3.

2. Review of Four-Dimensional Solutions

In four dimensions Bates and Denef⁴ have constructed general multi-centered BPS solutions of generic $\mathcal{N} = 2$ supergravity theories, and in Ref. 5 Bena and Warner classify the full set of BPS solutions for the special case of the five-dimensional $\mathcal{N} = 2$ supergravity theory which is the truncation of the $\mathcal{N} = 8$ theory (i.e. the theory is invariant under 8 instead of 32 supersymmetries). The latter require specifying a four-dimensional base metric that is restricted to be hyperkähler. A particularly appealing class of hyperkähler manifolds are Gibbons-Hawking or multi-Taub-NUT geometries which are asymptotically $\mathbb{R}^3 \times S^1$ and for which we have explicit metrics. Moreover, it has been shown that the five dimensional solutions constructed using a Gibbons-Hawking base manifold⁶ correspond to the four dimensional ones via the 4d/5d connection^{7,8} making them an interesting class of solutions to study.

The five dimensional solutions, although relatively complicated, are determined entirely in terms of $2b_2 + 2$ harmonic functions where b_2 is the second betti number of the compactification Calabi-Yau, X ,

$$\begin{aligned} H^0 &= \sum_a \frac{p_a^0}{|\mathbf{x} - \mathbf{x}_a|} + h^0, & H^A &= \sum_a \frac{p_a^A}{|\mathbf{x} - \mathbf{x}_a|} + h^A, \\ H_A &= \sum_a \frac{q_A^a}{|\mathbf{x} - \mathbf{x}_a|} + h_A, & H_0 &= \sum_a \frac{q_0^a}{|\mathbf{x} - \mathbf{x}_a|} + h_0. \end{aligned} \quad (1)$$

Here the coordinate vector \mathbf{x}_a gives the position in the spatial \mathbb{R}^3 of the a 'th center with charge $\Gamma_a = (p_a^0, p_a^A, q_A^a, q_0^a)$ (note here A runs from $1, \dots, b_2$). The IIA interpretation of these charges is (D6,D4,D2,D0) wrapping cycles of X while in M-theory the charge vector is (KK,M5,M2,P). Note that the harmonics have $2b_2 + 2$ constants $h = (h^0, h^A, h_A, h_0)$ that together determine the asymptotic behaviour of the harmonics and hence the solutions. We will also have frequent occasion to use the notation $\Gamma = (p^0, p^A, q_A, q_0)$ to refer to the total charge $\Gamma = \sum_a \Gamma_a$.

The position vectors have to satisfy the integrability constraints

$$\sum_b \frac{\langle \Gamma_a, \Gamma_b \rangle}{|\mathbf{x}_a - \mathbf{x}_b|} = \langle h, \Gamma_a \rangle, \quad (2)$$

where we define the symplectic intersection product

$$\langle \Gamma_1, \Gamma_2 \rangle := -p_1^0 q_2^0 + p_1^A q_2^A - q_1^A p_2^A + q_1^0 p_2^0. \quad (3)$$

By summing (2) over a we find that the constants h have to obey $\langle h, \Gamma \rangle = 0$. Note that even once the charges of each center have been fixed there is a large space of solutions that may even have several disconnected components. In particular,

the constraint (2) implies that the positions of the centers are generally restricted, defining a complicated moduli space of (generically) bound solutions.

The metric, gauge field and Kähler scalars of the solution are now given in terms of the harmonics by

$$\begin{aligned}
 ds_{5d}^2 &= 2^{-2/3} Q^{-2} [-(H^0)^2(dt + \omega)^2 - 2L(dt + \omega)(d\psi + \omega_0) + \Sigma^2(d\psi + \omega_0)^2] \\
 &\quad + 2^{-2/3} Q dx^i dx^i, \\
 A_{5d}^A &= \frac{-H^0}{Q^{3/2}}(dt + \omega) + \frac{1}{H^0} \left(H^A - \frac{Ly^A}{Q^{3/2}} \right) (d\psi + \omega_0) + \mathcal{A}_d^A, \\
 Y^A &= \frac{2^{1/3} y^A}{\sqrt{Q}},
 \end{aligned} \tag{4}$$

where $x^i \in \mathbb{R}^3$ and ψ is an angular coordinate with period 4π , and the functions appearing satisfy the relations

$$\begin{aligned}
 d\omega_0 &= \star dH^0, \\
 d\mathcal{A}_d^A &= \star dH^A, \\
 \star d\omega &= \langle dH, H \rangle \\
 \Sigma &= \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \\
 L &= H_0(H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0, \\
 Q &= \left(\frac{1}{3} D_{ABC} y^A y^B y^C \right)^{2/3}, \\
 D_{ABC} y^A y^B &= -2H_C H^0 + D_{ABC} H^A H^B.
 \end{aligned} \tag{5}$$

Here the Hodge star is with respect to the flat \mathbb{R}^3 spanned by the coordinates x^i and D_{ABC} are the triple intersection numbers of the chosen basis of $H^2(X)$. Note that the only equation in (5) for which there is no general solution in closed form is the last one. In some cases, e.g. when $b_2 = 1$, it is even possible to obtain a solution, in closed form, to this equation.

From (4) and (5) it may seem that the solutions are singular if H^0 vanishes but this is not the case as various terms in Q and L cancel any possible divergences due to negative powers of H^0 (in fact, the BTZ black hole can, in the decoupling limit introduced in the next section, be mapped to such a solution with H^0 vanishing everywhere).

An additional complication is the fact that even solutions satisfying the constraint equation (2) may still suffer from various pathologies, most notably closed time-like curves (CTCs). For instance, the prefactor to the $d\psi^2$ term in the metric may become negative if Σ becomes imaginary^a. Unfortunately there is no simple

^aAs described in Ref. 8 this would also imply that the 4-dimensional metric associated with this 5-d solution (via the 4d/5d connection of Ref. 7) becomes imaginary as Σ appears directly in the former.

criterion which can be used to determine if a given solution is pathology free. To fill in this gap Refs. 9 and 10 conjectured the physically well-motivated but mathematically unproven *attractor flow conjecture*, a putative criterion for the existence of (well-behaved) solutions.

An essential feature of these solutions is that they are stationary but not static. In particular they carry quantized intrinsic angular momentum associated with the crossed electric and magnetic fields of the dyonic centers⁹

$$\vec{J} = \frac{1}{2} \sum_{a < b} \frac{\langle \Gamma_a, \Gamma_b \rangle \vec{x}_{ab}}{r_{ab}}. \quad (6)$$

This will be important when quantizing the solution space as it is a necessary (but not sufficient) condition for the latter to be a proper phase space with a non-degenerate symplectic form. A solution space with vanishing angular momentum does not enjoy this property and must be completed to a phase space by the addition of velocities (see e.g. Ref. 11).

3. Decoupling

To study the solutions described above using AdS/CFT we need to find a suitable decoupling limit. Such a limit indeed exists and it can be taken in any proper $\mathcal{N} = 2$ theory, yielding an $\text{AdS}_3 \times \text{S}^2$ near horizon geometry³. This decoupling limit only works for microstate solutions whose total charge does not contain any overall $D6/\text{KK}$ -monopole charge so the relevant CFT is essentially the MSW CFT^b. Although the latter is not under very good control it is nonetheless possible to determine, from the asymptotics of a given geometry, the CFT quantum numbers of the corresponding state. It is also possible to use general CFT properties to determine various quantities such as the number of states in a given charge sector. It would, of course, be desirable to make progress in understanding this CFT as this may yield significant insight into the microstate geometries.

In³ the decoupling limit of the solutions described above is defined by taking $\ell_5 \rightarrow 0$ (ℓ_5 is the 5-d plank length) while keeping fixed the mass of $M2$ branes stretched between the various centers and wrapping the M -theory circle. In doing so we also fix the volume of the Calabi-Yau and the length of the M -theory circle, R , as measured in 5-d plank units. Since the mass of such membranes is given by $m_{M2} \sim R \Delta x / \ell_5^3$, the coordinate distances between the centers, Δx , must be rescaled as ℓ_5^3 . Alternatively, we can see this limit as a rescaling of the 5-d metric under which the Einstein part of the action is invariant.

We now define new rescaled coordinates, x^i , and harmonic functions, H , as

$$x^i = \ell_5^{-3} \mathcal{X}^i \quad H = \ell_5^{3/2} \mathbf{H} \quad (7)$$

^bThe ‘‘MSW’’ CFT is the theory that arises as the low-energy description of $M5$ -branes wrapping an ample divisor in the Calabi-Yau. It is an $N = (0, 4)$ superconformal field theory and it owes its name to the three authors of Ref. 12.

By restricting to the region of finite x^i we are essentially keeping the mass of transverse, open membranes fixed while rescaling the original coordinates, x^i . One can see that, in these new variables, the structure of the solution (in terms of the harmonics) does not change in the decoupling limit except for an overall scaling of the metric by a factor of ℓ_5^{-2} . The rescaled harmonics do take a new form, however,

$$\begin{aligned} H^0 &= \sum_a \frac{p_a^0}{|x - x_a|}, & H^A &= \sum_a \frac{p_a^A}{|x - x_a|}, \\ H_A &= \sum_a \frac{q_A^a}{|x - x_a|}, & H_0 &= \sum_a \frac{q_0^a}{|x - x_a|} - \frac{R^{3/2}}{4}. \end{aligned} \quad (8)$$

In particular note that all the constants have disappeared except the $D0$ -brane constant which now takes a fixed value. Related to this is the fact that the asymptotic value of the moduli are forced to the attractor point, $Y^A \sim p^A$ (this corresponds to sending the 4-d Kähler moduli to $J^A = \infty p^A$).

Recall that the coordinate locations of the centers must satisfy the integrability constraint (2) and that this constraint depends on the value of the constants in the harmonic functions. As a consequence it is possible that, in taking the decoupling limit, some solutions cease to exist. Furthermore, even if a set of charges admits a solutions that satisfies the constraint equations in the decoupling limit the solution may develop other pathologies such as CTCs.

The decoupled solutions are asymptotically $\text{AdS}_3 \times \text{S}^2$ and by studying the sub-leading behavior of the metric we can read off that

$$\begin{aligned} L_0 &= \frac{(p^A q_A)^2}{2p^3} - q_0 + \frac{p^3}{24}, \\ \tilde{L}_0 &= \frac{(p^A q_A)^2}{2p^3} + \frac{p^3}{24}. \end{aligned} \quad (9)$$

and that the $SU(2)$ R-symmetry charge associated with a solution is equal to the angular momentum defined in (6). This charge plays a distinguished role in the quantization of the system as its presence is necessary in order to have a non-trivial symplectic form on the phase space.

4. Quantization

Next, we would like to quantize the phase space of supergravity solutions described above directly. The quantization we will perform will be quite general in that it will cover the original 4-d multi-center black hole configurations⁴, their 5-d uplift discussed in section 2, and the decoupled version of the latter (which can be related to the (0,4) CFT). The first step in the quantization procedure is to determine the symplectic form on the phase space of solutions. This can, in principle, be derived from the supergravity action as was done, for instance, in Ref. 13. In this case, however, it is far more tractable to take a different approach³. As discussed in Ref. 14, the four dimensional multi-centered solutions can also be analyzed in the

probe approximation by studying the quiver quantum mechanics of D-branes in a multicentered supergravity background. Moreover, a non-renormalization theorem¹⁴ implies that the terms in the quiver quantum mechanics Lagrangian linear in the velocities do not receive corrections, either perturbatively or non-perturbatively. We can use this fact to calculate the symplectic form in the probe regime and extend it to the fully back-reacted solution; this because, for time-independent solutions, the symplectic form depends only on the terms in the action linear in the velocity.

In Ref. 3 the symplectic form on the solution space is determined. The result is that

$$\tilde{\Omega} = \frac{1}{4} \sum_{p \neq q} \langle \Gamma_p, \Gamma_q \rangle \frac{\epsilon_{ijk} (\delta(x_p - x_q)^i \wedge \delta(x_p - x_q)^j) (x_p - x_q)^k}{|\mathbf{x}_p - \mathbf{x}_q|^3}. \quad (10)$$

This is a two form on the $(2N-2)$ -dimensional solution space which is a submanifold of \mathbb{R}^{3N-3} defined by (2). Moreover, one can show that, on this submanifold, this form is closed and, in the cases we will investigate below, non-degenerate. Thus it imbues the solution space with the structure of a phase space.

Although the constraint equations (2) are invariant under global $\text{SO}(3)$ rotations these are nonetheless (generically) degrees of freedom of the system and this is reflected in the symplectic form. If we contract (10) with the vector field that generates rotations around the 3-vector n^i (i.e. we take $\delta x_{pq}^i = \epsilon^{ijk} n^j x_{pq}^k$) then the symplectic form reduces to

$$\tilde{\Omega} \rightarrow n^i \delta J^i \quad (11)$$

where J^i are the components of the angular momentum vector defined in (6).

This is nothing more than the statement that the components J^i are the conjugate momenta associated to global $\text{SO}(3)$ rotations. In general the symplectic form on any of our phase spaces^c will have terms like the above coming from the global $\text{SO}(3)$ rotations, in addition to terms depending on other degrees of freedom. As advertised (11) implies that solution spaces with any $J^i = 0$ will have a degenerate symplectic form and will therefore not constitute a proper phase space^d.

4.1. *Quantizing the Two-center Phase Space*

The inter-center position of a two center configuration is fixed in terms of the charges and the moduli at infinity but the axis of the centers can still be rotated so, neglecting the center of mass degree of freedom, we are left with a solution space

^cThis does not hold for solutions spaces with unbroken rotational symmetries, such as solution spaces containing only collinear centers or only a single center. In these cases some $\text{SO}(3)$ rotations act trivially, do not correspond to genuine degrees of freedom and do not appear in the symplectic form.

^dAs mentioned in footnote c this does not hold in the two center case where some $\text{SO}(3)$ directions decouple. There are also potential subtleties with solution spaces where $J = 0$ at a single point.

that is topologically a two-sphere with diameter

$$x_{12} = \frac{\langle h, \Gamma_1 \rangle}{\langle \Gamma_1, \Gamma_2 \rangle}. \quad (12)$$

The symplectic form (10) is proportional to the standard volume form on the two-sphere and is entirely of the form (11) (note here that, as mentioned in footnote c, collinearity of the solution implies that one $U(1) \subset SO(3)$ decouples). In terms of standard spherical coordinates it is given by

$$\tilde{\Omega} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \sin \theta d\theta \wedge d\phi = |J| \sin \theta d\theta \wedge d\phi. \quad (13)$$

We can now quantize the moduli space using the standard rules of geometric quantization, and we find the usual $2|J| + 1$ Landau levels on the sphere. However, we have been a bit sloppy by ignoring the fermionic degrees of freedom in the theory. Including those changes the number of states to $2|J|$ ³.

4.2. Quantizing the Three-center Phase Space

The phase space of the three center case is four dimensional. Placing one center at the origin (fixing the translational degrees of freedom) leaves six coordinate degrees of freedom but these are constrained by two equations. This leaves four degrees of freedom, of which three correspond to rotations in $SO(3)$ and one of which is related to the separation of the centers. We will take as coordinates three angular variables θ, ϕ, σ on $SO(3)$ plus the norm of the angular momentum j . The symplectic form in these coordinates is (see Ref. 3 for a derivation):

$$\tilde{\Omega} = j \sin \theta d\theta \wedge d\phi - dj \wedge D\sigma \quad (14)$$

with $D\sigma = d\sigma - A$, $j = |\vec{J}|$, and $dA = \sin \theta d\theta \wedge d\phi$, so that A is a standard monopole one-form on S^2 . The gauge field A implements the non-trivial fibration of σ over the S^2 . A convenient choice for A is $A = -\cos \theta d\phi$ so that finally the symplectic form can be written as a manifestly closed two-form

$$\tilde{\Omega} = -d(j \cos \theta) \wedge d\phi - dj \wedge d\sigma. \quad (15)$$

The angular momentum spans a range $j \in [j_-, j_+]$ and the solution space is a toric Kähler manifold which turns out to be equal to the second Hirzebruch surface F_2 . Counting the number of states is straightforward using the toric polytope and after proper inclusion of the fermionic degrees of freedom we obtain

$$\mathcal{N} = (j_+ - j_-)(j_+ + j_-). \quad (16)$$

5. Applications and Conclusions

To conclude, we mention a few applications of the above results.

- Entropy Enigma: the entropy enigma¹⁰ is the puzzle that for suitable large charges, configurations consisting of bound states of two or more black holes entropically dominate single center black holes. This goes against the lore that single center black holes should dominate the entropy for large charges. After taking the decoupling limit, it turns out that the entropy enigma precisely appears whenever the radius of a single center black hole becomes less than the curvature radius of AdS_3 . One can show that there then is a supersymmetric version of the Gregory-Laflamme transition and that the black hole localizes on the S^2 , removing most of the mystery surrounding the entropy enigma².
- Wall Crossing: When the moduli of the Calab-Yau are varied, bound states of black holes can appear and disappear from the space of solutions. When this happens the moduli are at a *wall of marginal stability* and the number of boundstates jumps. The amount by which they jump is given by the *wall-crossing formula*¹⁰ which contains two components: the entropy of the black constituents that make up the bound state, plus an overall factor which is the entropy in the wave function that makes up the bound state. One can show that both in the two-centered case and the three-centered case the number of states obtained from our quantization procedure matches exactly with the wall-crossing formula, a highly non-trivial check of the consistency of the whole framework.
- Fuzzball Proposal: For suitable choices of the charges of all the centers, in particular if all the charges are those of single D6 or anti-D6 branes with only world-volume fluxes turned on, the supergravity solutions are completely smooth. The states obtained from the quantization procedure are therefore candidate fuzzball microstates for large supersymmetric black holes with the same overall charges. Though many smooth solutions and fuzzball states can be found in this way, it remains for now unclear whether there are enough to account for a significant fraction of the black hole entropy, or even whether one can construct states which could be considered typical. Clearly, much more work is needed in this direction.

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6. References

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