New Feynman rules for quantum mechanical nonlinear sigma models

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The subject of path integrals in curved space is arcane, complicated and controversial. In two recent articles we have considered this problem in one dimension (quantum mechanical nonlinear sigma models) and found a complete solution [1,2]. Our results have been confirmed by complete two-loop calculations in several supersymmetric and nonsupersymmetric models. In fact, it was precisely because inconsistencies between the heat kernel approach and the path integral approach surfaced at the two-loop level [3], that we came back to this problem and decided on a rigorous attack, starting from scratch. The big surprise (which solved all inconsistencies) is that *the Feynman rules which follow from the path integral differ from what one would assume to be correct.*

The main differences between our new rules and those often assumed in the literature are the following

(i) propagators contain factors $\theta(\sigma-\tau)$ and $\delta(\sigma-\tau)$ where $-1 \leq \sigma, \tau \leq 0$. Terms with more than one $\delta(\sigma-\tau)$ cancel in the integrands due to the presence of new "Lee-Yang ghosts". These ghosts come from exponentiating the factors $\det g_{ij}$ which result if one integrates out the momenta from $g^{ij}p_i p_j$, and were first introduced in [4].

(ii) products of several $\theta(\sigma-\tau)$ and one $\delta(\sigma-\tau)$ must be evaluated by taking the integrand exactly at $\sigma=\tau$. Hence $\delta(\sigma-\tau)$ remains a Kronecker delta in the continuum theory. For example

$$\int_{-1}^{0}\int_{-1}^{0} d\sigma d\tau \theta(\sigma-\tau)\theta(\sigma-\tau)\delta(\sigma-\tau) = \frac{1}{4} \tag{1}$$

If one were to argue that $\delta(\sigma-\tau)=\partial_\sigma\theta(\sigma-\tau)$ (even at the regularized level) one would find the integral of $\frac{1}{3}\partial_\sigma[\theta(\sigma-\tau)]^3$ which would be $1/3$ and which would be incorrect. In ref [5] an example is worked out of such an integral which is regulated by "mode regularization", and it is explicitly shown that (and explained why) *mode regularization gives incorrect results.* This is certainly at first sight surprising, as mode regularization is commonly thought to be the most reliable, although tedious, regularization scheme.

(iii) equal-time contractions are nonzero, and not equal to the limit of $\sigma_1 \to \sigma_2$ of an unequal-time contraction. Of course, in quantum field theory, equal-time contractions are in general ill-defined and one needs a symmetry principle (for example a Ward identity) to fix them, but in quantum mechanics everything should be well-defined and unambiguous.

(iv) there is a nontrivial measure in the path integral. The particular form of the measure depends on the way one splits the action into a free part and an interacting part, but forgetting the measure leads to incorrect results. We stress that everything, including the measure, follows deductively from our starting point (to be discussed). Other authors have invoked geometrical or symmetry arguments to give meaning to the path integral . We have not relied on such "half-way" arguments, but rather deduce everything in a mathematically rigorous way from the starting point. (Note that also the measure is needed to resolve the two-loop problems mentioned above).
Most of the literature on path integrals (even in quantum mechanics) is more focused on general issues and formal arguments than explicit calculations, but we would like to stress that we doubt that the results we have obtained could ever have been anticipated without an explicit *higher-loop* calculation.

Our starting point is the transition element

$$T(z,\bar{\eta};y,\eta;\beta) = < z,\bar{\eta}| \exp -\frac{\beta}{\hbar}\hat{H}|y,\eta > \tag{2}$$

where the kets $|y,\eta>$ are eigenstates of the position operators $\hat{x}^i (i=1,\ldots,n)$ and fermionic annihilation operators $\hat{\psi}^a (a=1,\ldots,n$ or $a=1,\ldots,[n/2]$, see [2]), while the bras $< z,\bar{\eta}|$ are eigenstates of $\hat{x}^i$ and $\hat{\psi}^\dagger_a$. So, we use fermionic coherent states. We assume that the operator ordering in $\hat{H}(\hat{x}^i,\hat{p}_i,\hat{\psi}^a,\hat{\psi}^\dagger_a)$ has been fixed a priori. (In practice, $\hat{H}$ is 1-1 related to the regulator $\hat{R}$ of a corresponding quantum field theory, and $\hat{R}$ is unique once one has specified which symmetries at the quantum level should be preserved [6]). The matrix element T can be evaluated in an unambiguous way without encountering infinities, order by order in β counting $z-y$ as of order $\sqrt{\beta}$, by inserting a complete set with momentum eigenstates and fermionic coherent states

$$T = \int < z,\bar{\eta} \mid \exp -\frac{\beta}{\hbar}\hat{H} \mid p,\xi > e^{-\bar{\xi}_a \xi^a} < p,\bar{\xi} \mid y,\eta > d^n p d\xi^a d\bar{\xi}_a \tag{3}$$

Pulling all $\hat{x}^i, \hat{\psi}^\dagger_a$ to the left and all $\hat{p}_i$ and $\hat{\psi}^a$ to the right, keeping track of (anti)commutators, one finds an unambiguous answer. We want to find a path integral representation for T, which yields the same answer in a loop expansion (on the worldline) order-by-order in β. This defines what we call "the correct path integral". Of course, there are hundreds of other definitions, some with beautiful geometrical aspects, but if they do not agree with T we call them incorrect.

The actual derivation of the path-integral is rather conventional; new is that we have pushed the usual approaches further and found *exact results for finite* N which allow us carefully to trace what happens if one takes the limit of $N \to \infty$. We insert $N-1$ complete sets of x-eigenstates, N sets of p eigenstates and N sets of fermionic coherent states. Then we Weyl-order $\hat{H}$. We do not add further terms, but only *rewrite* $\hat{H}$ as $\hat{H} = \hat{H}_W +$ "more". The terms in "more" are of order $\hbar$ and $\hbar^2$, and were already found by Schwinger long ago in a different context. They are usually of the form of a curvature term R and a product of two Christoffel symbols, but the precise forms of "more" depends on the operator ordering chosen in $\hat{H}$. Next we use a theorem which is an extension to nonlinear sigma models of the well-known Trotter formula for linear sigma models. It states that in

$$\int < x_k, \bar{\eta}_k \mid \exp -\frac{\epsilon}{\hbar}\left(\hat{H}_W + \text{“more”}\right) \mid \xi_k, p_k > e^{-\bar{\xi}_k \xi_k} < p_k, \bar{\xi}_k \mid x_{k-1}, \eta_{k-1} > d^n p d\xi_k d\bar{\xi}_k, \tag{4}$$

with $\epsilon = \beta/N$ one may replace the operators in $\hat{H}_W +$ "more" by their midpoint rules

$$\hat{x} \to \frac{1}{2}(x_k + x_{k-1}), \hat{p} \to p_k, \hat{\psi}^\dagger \to \bar{\eta}_k, \hat{\psi} \to \frac{1}{2}(\xi_k + \eta_{k-1}) \tag{5}$$

As we have proven in [5], the error is of higher order than ϵ, and vanishes therefore in the limit $N \to \infty$. This theorem was first derived by Berezin for linear sigma models. Our proof for nonlinear sigma models is based on perturbation theory, and we make no claims about nonperturbative exactness.

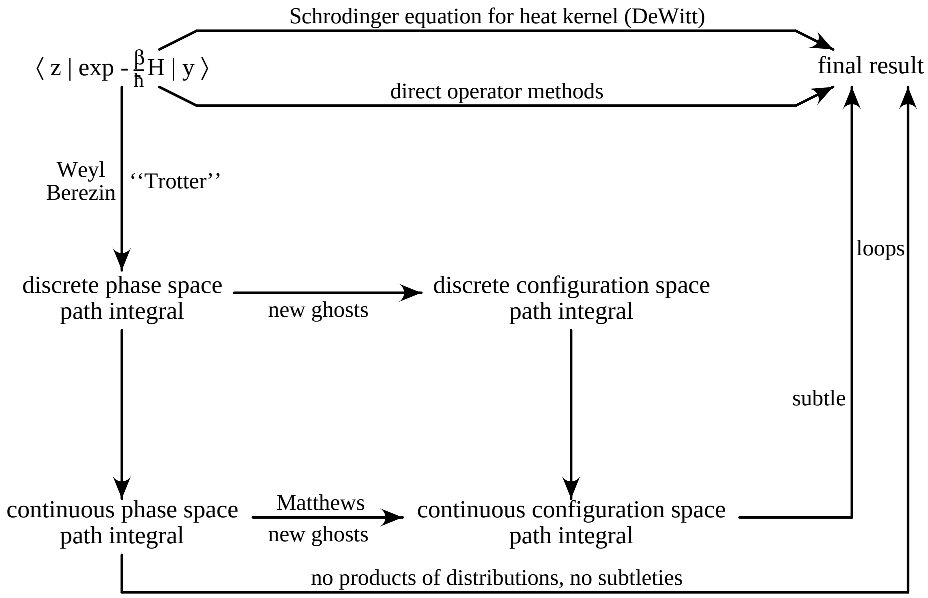
We then decompose the action into a free part and an interacting part, decompose the fields into background part and quantum parts, couple the quantum fields to discretized external sources, complete squares and find discretized expressions for the propagators and vertices *in closed form.* In ref [5] details are given; it involves heavy but elementary use of goniometric series. Finally, the limit $N \to \infty$ can be taken, with the results mentioned at the beginning.

We have done this analysis both in phase-space (with $p(\tau)$) present and in configuration space (where $p(\tau)$ is integrated out). Obviously there are no Lee-Yang ghosts in phase-space, and since there are no $\delta(\sigma - \tau)$ in the propagators in phase space, no problems are present, and mode regularization, or any other scheme, gives the same results. In fact, in phase-space one does not even need a regularization scheme at all. Again, the equal-time contractions are uniquely determined by our discretized approach, and again they are not equal to the limit of unequal-time contractions.

Of course, the phase space and configuration space path integrals should give the same final result for T. This is the content of Matthews' theorem. The propagators $< pp >$ differ from $< \dot{x}\dot{x} >$ by a $\delta(\sigma - \tau)$, and also the vertices in phase space and configuration space are very different, but if one includes the Lee-Yang ghosts, both approaches are equivalent.

Let us conclude by noting that both the action and the Feynman rules are noncovariant, even if the original $\hat{H}$ is 'covariant' (i.e. commutes with some symmetry generators). This noncovariance is a direct consequence of the midpoint rule (the midpoint $\frac{1}{2}(x_k + x_{k-1})$ is not a covariant concept in curved space). The final result, though, is covariant if $\hat{H}$ was covariant.

In [2], these results were applied to a rigorous rederivation of the results of Alvarez-Gaumé and Witten [7] on anomalies, and also some new results concerning anomalies were obtained. Also a possible extension to higher dimensions, to decide whether canonical and Faddeev-Popov quantization are equivalent, was discussed in [2].



The flow chart above summarizes this summary.

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