Master's Thesis

Glue Functions in High-T_c Superconductors and the Search for a Gravity dual

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Understanding the pairing mechanism in high-T_c superconductors is one of the most important unsolved problems in physics. There is evidence that this paring mechanism is caused by strongly-coupled interactions with unidentified bosons. The number of available tools to analyze systems at strong coupling is very limited. However, the AdS/CFT correspondence provides a new approach to study strongly-coupled systems. In this thesis, we review a model that describes strongly-coupled interactions between electrons and phonons. The electron self-energy in this model can be written as a product of two terms: a kernel and an electron-phonon spectral function. The kernel contains all the information about the electrons and all the temperature dependence, while the electron-phonon spectral function contains all the information about the phonons. Since the bosons that are responsible for the pairing between the electrons in high-T_c superconductors are unknown, the electron-phonon spectral function is generalized to a glue function. We discuss possibilities to translate the aforementioned self-energy by using the AdS/CFT correspondence. When the dual description of this self-energy has the same structure as the original description, then one can generalize to a generic electron-boson interaction (the glue function), so that, after stripping off the kernel in the dual system, insight into the glue function can be obtained. We find a similarity between the kernel and a Green's function of a bosonic field in a BTZ black hole background. To provide background material, a phenomenological overview of how the AdS/CFT correspondence follows from String Theory has been included, together with some applications.

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Contents

1 Introduction

In 1911, *superconductivity* was discovered by Kamerlingh Onnes, when he noted that the electrical resistivity of Mercury suddenly drops to zero as the temperature is lowered below a certain critical temperature T_c [1]. A complete microscopic description of this phenomenon was given by Bardeen, Cooper and Schrieffer in 1957 [2]. This theory is known as *BCS theory*. In 1986, however, a new class of superconducting materials was discovered, which could not be described by the BCS theory [3]. This class of superconductors is based on copper and oxygen. Because these materials have a surprisingly high critical temperature, they are called *high-T_c superconductors*. At the time of writing, the highest T_c for this class is about 135 K at atmospheric pressure [4].

The BCS theory describes the pairing of electrons mediated by interactions with *phonons*, which are quasiparticles associated with the lattice vibrations of a crystalline solid. In this theory, the coupling between the electrons and phonons is weak. On the other hand, the pairing mechanism in high-T_c superconductors is still unknown. Understanding this pairing mechanism is one of the most prominent open questions in physics today. It is commonly believed that understanding these materials could enable us to synthesize room-temperature superconductors.

There is accumulating evidence that the pairing mechanism between electrons in high-T_c superconductors is primary caused by strongly-coupled interactions with bosons. For a recent review discussing the experimental evidence for mediation by strongly-coupled phonons, see [5]. For systems at weak coupling, one is able to perform calculations using perturbation theory, but for strongly-coupled systems this is not possible. For systems at strong coupling, there are only a very limited number of tools available to perform calculations on them. However, in 1997, Juan Maldacena conjectured a duality between a strongly-coupled gauge theory and a gravity theory in a weakly-curved space [6]. This duality is called the AdS/CFT correspondence or gauge/gravity duality, and follows from string theory. The AdS/CFT correspondence provides a tool to perform calculations on strongly-coupled systems. During the last few years, the study of high-T_c superconductors using the AdS/CFT correspondence has become a very active area of research. For reviews, see e.g. [7, 8].

In this thesis, it will be reviewed that the self-energy of an electron that is strongly coupled to phonons at finite temperature, can be written as (an integral over) a product of a temperature-independent function that contains all the information about the phonons, and a function that contains all the information about the electrons and thermal occupation factors. The former is the so-called *electron-boson spectral function*, and the latter is the kernel. As the electron-boson spectral function contains all the information about the phonons, it can be generalized to a *glue function* to describe the pairing between some unspecified species of bosons and electrons. A remarkable feature of the glue function is that it is (almost) temperature independent.

We will investigate possibilities to apply the AdS/CFT correspondence to the self-energy of the electron that is strongly coupled to phonons. The hope is that, when this self-energy in the dual description also can be written as a product of a kernel and a spectral function, after generalizing to coupling to unknown bosons, the kernel can stripped off and information about the glue function can be obtained, so that, ultimately, the pairing mechanism between electrons in high-T_c superconductors can be understood.

Organization of this thesis. To acquaint the reader with the AdS/CFT correspondence, an introduction is given in chapter 2. This chapter is primary meant to give the reader an idea how this correspondence follows from string theory, and is aimed at a reader that has at least a firm basic knowledge of String Theory. In section 2.1, there is an overview of the subjects covered in this chapter. In order to keep this chapter compact, there are many things that just have been stated without motivation. When a sentence ends with a reference, this is probably the case. Section 2.8.3 contains a basic dictionary for mappings in the AdS/CFT correspondence, and this section is the most important one for what follows in the subsequent chapters.

In chapter 3, some applications of the AdS/CFT correspondence with a view towards condensed-matter physics are given. The first three sections of this chapter serve to illustrate how the AdS/CFT correspondence works in a more or less practical context. Section 3.1 describes a system where the AdS spacetime —also called the *bulk*— only contains the graviton field, which is the simplest possible system in the correspondence. In section 3.2, it will be shown that when the AdS spacetime is deformed to include a black hole, the dual field theory is placed at finite temperature. While in section 3.3 it is shown, that adding a Maxwell field to the bulk, amounts to placing the field theory at a finite chemical potential.

The last three sections of chapter 3 have relevance for the final chapter, where ideas for applying the correspondence to the aforementioned electron self-energy are discussed. In section 3.4, a scalar field is added to the bulk, which has a corresponding dual scalar operator in the field theory. Finally, in sections 3.5 and 3.6, an introduction to linear response theory and properties of retarded Green's functions are discussed, respectively.

In the final chapter 4, ideas will be discussed for applying the AdS/CFT correspondence to the self-energy of an electron that interacts strongly with bosons in a system at finite temperature. Firstly, a model that describes the interactions between electrons and phonons will be discussed in section 4.1. Section 4.2 discusses Green's functions at finite temperature, and in the subsequent section 4.3, the finite-temperature Green's function and self-energy for an electron interacting with bosons will be considered. Subsequently, the electron self-energy will be described for a strongly-coupled system in section 4.4. This chapter concludes with a generalization to a strongly-coupled electron-*boson* interaction, and ideas for a translation of the electron self-energy using the AdS/CFT correspondence are discussed in section 4.5.

A word on notation. Throughout this thesis 'natural' units are used, which are defined by $\hbar = c = k_B = 1$. There is only one dimensionful unit: mass = energy, and 'dimension' refers to mass dimension. The metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$ always refers to the Minkowski metric. Finally, relevant notation has been listed in the introduction of the chapters.

2
AdS/CFT Correspondence

In quantum field theory (QFT), calculations can be performed using perturbation theory only at weak coupling. It is well-known that many interesting phenomena in QFT are non-perturbative in nature, such as Quantum Chromodynamics (QCD) at low energies. There is an approach to do calculations in QCD at strong coupling by using numerical simulations on a lattice, but, generally, for systems at strong coupling, only a very limited number of tools is available.

In 1997, Juan Maldacena conjectured a duality between a strongly-coupled gauge theory and a gravity theory in a weakly-curved space [6]. This conjecture has become known as the Anti-de Sitter-Conformal Field Theory correspondence, or *AdS/CFT correspondence* for short. The AdS/CFT correspondence provides important tools for studying systems at strong coupling, and it is one of the most significant results string theory has produced.

The core idea of the AdS/CFT correspondence goes back to attempts to understand the Bekenstein bound, which asserts that the maximum physically possible entropy for any system in a region of space is proportional to the area of the boundary of that region [9]. So, the number of degrees of freedom inside some region scales as the area of the boundary of the region, and not like its volume. This scaling behavior is not possible in standard quantum field theories. In 1993, 't Hooft conjectured that a consistent quantum theory of gravity must be *holographic*, i.e., in a quantum theory of gravity, all physics within a given volume can be described in terms of some theory on the boundary [10]. This idea was subsequently discussed by Susskind [11].

Before the advent of string theory, holography was a little more than an accounting mechanism, since there was no actual microscopic description of gravity systems like black holes. In string theory, branes are taken to form the bulk of the mass of a black hole, and by observing fluctuations of these branes, a microscopic description of black holes is obtained. It was shown by Strominger and Vafa that the thermodynamics emerging from the state-counting of those fluctuations corresponds exactly to the holographic thermodynamics of Bekenstein and Hawking [12].

There are numerous reviews of the AdS/CFT correspondence available. In this chapter, there has been made an attempt to represent the correspondence at a more or less phenomenological level to give the reader an idea how this duality follows from string theory. It is by no means a detailed derivation, but it is intended to act as a guide for studying the AdS/CFT correspondence in other sources. See for example [13, 14, 15], from which most of the material of this chapter has been taken. The content and support for the AdS/CFT correspondence are presented at two levels. First an outline of the duality is given in pictorial form, which is followed by a more detailed account of the conjecture.

The notation used in this chapter is as follows. Spacetime indices are denoted with μ, ν, while α, β denote indices of world-sheet coordinates. The indices a, b are for coordinates longitudinal to a brane, while I, J denote the transversal ones. Finally, lowercase indices i, j denote light-cone coordinates.

In the next section, a diagrammatic overview of the AdS/CFT correspondence is given. In the remaining of this chapter, each part will be worked out.

2.1 Outline of the AdS/CFT correspondence

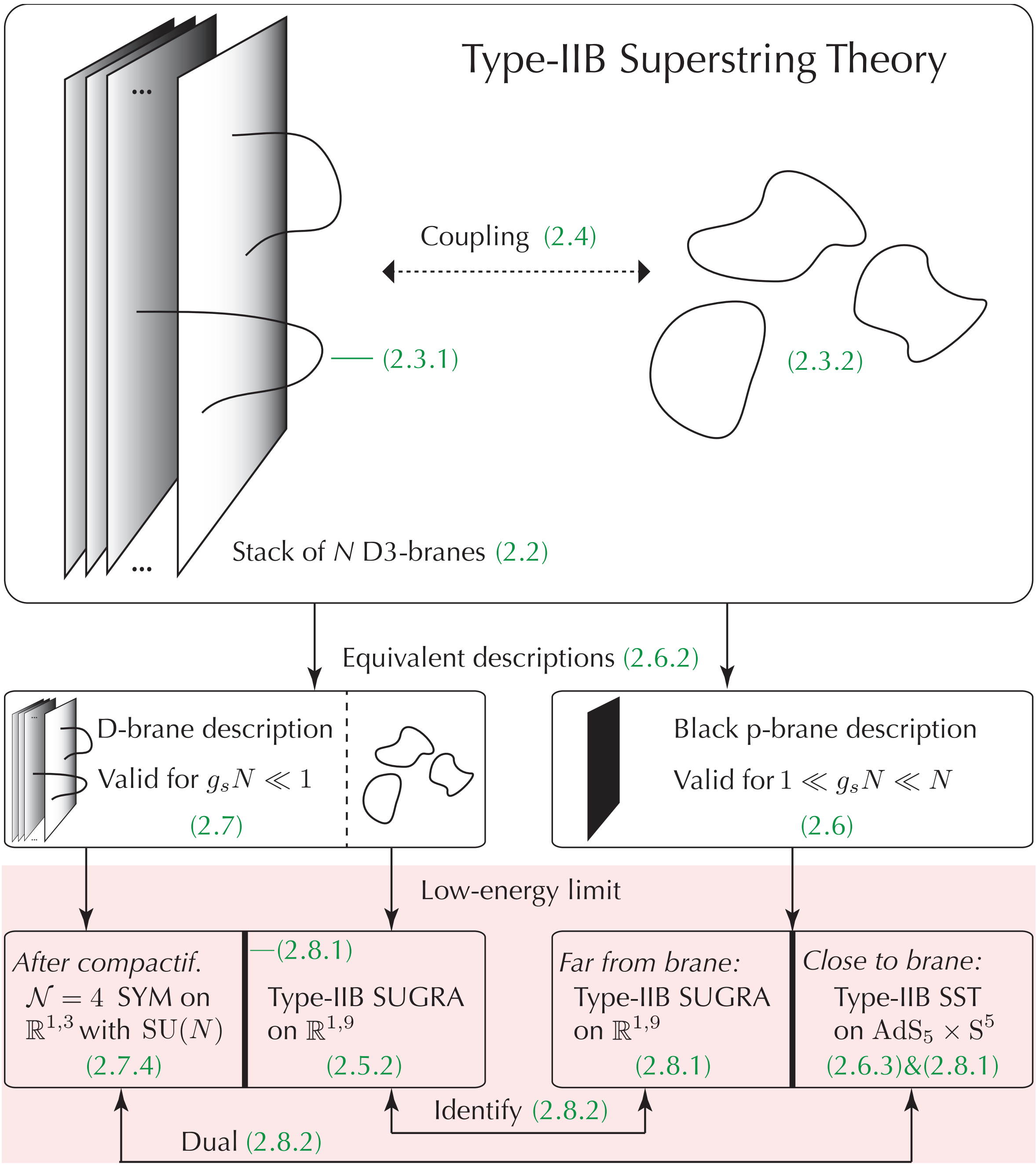

Figure 2.1: Outline of AdS/CFT correspondence. The numbers between parentheses correspond to the sections where the subject is covered.

2.2 D-branes

In this section a short introduction to D-branes is presented in order to show that open strings ending on a stack of N coinciding D-branes give rise to a $\mathrm{U}(N)$ gauge theory. For a more extended introduction to D-branes the reader can consult the lecture notes on String Theory of Tong [16], and [18, 20] for detailed accounts including superstrings.

The action describing a bosonic string which propagates in a flat D-dimensional spacetime is the *Nambu-Goto action*, which is proportional to the surface area of the world sheet swept out in spacetime by a moving string,

$$S_{\mathrm{NG}} = -T \int d^2\sigma \, \sqrt{-\det h_{\alpha\beta}} \quad \text{with} \quad h_{\alpha\beta} = \eta_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{2.1}$$

Here $\alpha = 0, 1,\ \mu = 0, \ldots, D-1$, and T the string tension, which is related to the Regge slope α' by $T = 1/2\pi\alpha'$. The metric $h_{\alpha\beta}$ is the induced metric on the world sheet from the embedding in spacetime.

In practice it is hard to work with the Nambu-Goto action due to the square root. Instead, one works with the *Polyakov action*,

$$S_{\mathrm{P}} = -\frac{T}{2} \int d^2\sigma \, \sqrt{-\gamma}\gamma^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu, \tag{2.2}$$

where $\gamma_{\alpha\beta}$ is an auxiliary world-sheet metric. The Polyakov action is classically equivalent to the Nambu-Goto action upon elimination of the auxiliary metric $\gamma_{\alpha\beta}$ using its equations of motion. From the point of view of the world-sheet, the Polyakov action describes D massless scalars coupled covariantly to the metric $\gamma_{\alpha\beta}$.

Using reparametrization and Weyl invariance one can go to *conformal gauge*, where one fixes the world-sheet metric to $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$,

$$S_{\mathrm{P}} = -\frac{T}{2} \int d^2\sigma \, \partial_\alpha X_\mu \partial^\alpha X^\mu. \tag{2.3}$$

Choose the world-sheet coordinate σ to have the range $0 \leq \sigma \leq \pi$ for convenience. By varying this action with $X^\mu \to X^\mu + \delta X^\mu$, one obtains an extra boundary term for open strings,

$$\delta S_{\mathrm{P,boundary}} = -T \int d\tau \, \delta X^\mu \partial_\sigma X_\mu \big|_{\sigma=0}^{\sigma=\pi},$$

which must vanish independently. There are two possibilities,

- Neumann boundary conditions: $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$. Because there is no restriction on δX^μ, this condition allows the end of the string to move freely and preserves translational invariance.

- Dirichlet boundary conditions: $\delta X^\mu = 0$. In this case the end points are fixed in space, $X^\mu\big|_{\sigma=0} = c^\mu$ and $X^\mu\big|_{\sigma=\pi} = d^\mu$, where c^μ and d^μ are constants. Dirichlet boundary conditions thus break translational invariance.

It is possible to choose $p+1$ Neumann boundary conditions and $D-p-1$ Dirichlet boundary conditions,

$$\begin{aligned} &\partial_\sigma X^a = 0 && \text{for } a = 0, \ldots, p \\ &X^I\big|_{\sigma=0} = c^I, \quad X^I\big|_{\sigma=\pi} = d^I && \text{for } I = p+1, \ldots, D-1. \end{aligned}$$

This fixes the end points of the string to lie in $(p+1)$-dimensional hypersurfaces in spacetime, located at c^I and d^I in the space transverse to the hypersurfaces.

In 1989, Dai, Leigh and Polchinski showed that this hypersurface is dynamical and that it has degrees of freedom living on it [21]. The wall was then called a D-brane, or Dp-brane when one wants to specify its spatial dimensions. D-branes are thus objects at which open strings can end. They are infinitely extended in space, but it is possible to define finite D-branes by specifying closed surfaces at which strings can end. The dynamics of D-branes are reviewed in section 2.7.

In the presence of a flat, infinite Dp-brane, the symmetry group of spacetime, which is the $\mathrm{SO}(1, D-1)$ Lorentz group, is broken into

$$\mathrm{SO}(1, D-1) \to \mathrm{SO}(1, p) \times \mathrm{SO}(D-p-1). \tag{2.4}$$

The $\mathrm{SO}(1, p)$ group is the Lorentz group of the D-brane world-volume, while the group $\mathrm{SO}(1, D-1)$ is a global symmetry of the D-brane theory. This means that a Dp-brane solution remains invariant under the action of either $\mathrm{SO}(1, p)$ or $\mathrm{SO}(D-p-1)$.

2.2.1 Spectrum of open strings on D-branes

To determine the spectrum of bosonic open strings ending on Dp-branes, it is convenient to work in light-cone gauge, where one sets $X^+ \sim \tau$, with the spacetime light-cone coordinate chosen to lie within the brane, $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^p)$. After canonical quantization, the mass M at level N is then given by [16],

$$M^2 = \frac{|d^I - c^I|^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'}(N-1), \tag{2.5}$$

where the first term is due to the stretching between the branes.

The ground state is defined as

$$\alpha_n^i |0, k\rangle = 0, \quad \text{for} \quad n > 0,$$

where $i = 1, \ldots, p-1, p+1, \ldots, D-1$, and α_n^i are the coefficients in the mode expansion of the solution to the equations of motion following from (2.3). Here k is a quantum number which is an eigenvalue of the momentum operator, $p^\mu |0, k\rangle = k^\mu |0, k\rangle$. From (2.5) it follows that when the branes coincide, the first term drops out, and the ground state becomes *tachyonic*, i.e., a state with imaginary mass.

String states can be split in oscillators longitudinal and transversal to the branes. The first excited states are the massless states:

- Longitudinal: $\alpha_{-1}^i |0, k\rangle, i = 1, \ldots, p-1$. The indices i lie within the brane, so this state transforms as a vector under the $\mathrm{SO}(1, p)$ Lorentz group. Since it has $p-1$ components, it is a massless spin-1 particle. One can associate a gauge field A^a, $a = 0, \ldots, p$, which lives on the brane and whose quanta are identified with this state.

- Transversal: $\alpha_{-1}^I |0, k\rangle, I = p+1, \ldots, D-1$. These are scalars under the $\mathrm{SO}(1, p)$ Lorentz group, and can be considered as arising from $D-p-2$ massless scalar fields X^I, sometimes called Nambu-Goldstone bosons, since they have emerged from breaking of a symmetry in spacetime. Under the remaining $\mathrm{SO}(D-p-1)$ rotation group they transform as vectors. As seen from the world volume of the brane this appears as a global symmetry. These bosons can be interpreted as fluctuations of the brane in the transverse directions.

2.2.2 Chan-Paton factors

When there are N parallel D-branes in the theory, then the end points of open strings can have a label $i \in \{1, \dots, N\}$, which is called a Chan-Paton charge, that corresponds to a label of the D-brane on which the string ends. This new degrees of freedom are characterized by vanishing Hamiltonian, and as a result they are strictly static terms, which means that strings retain the same Chan-Paton state. Ignoring the Fock-space label, an open string state can be written as

$$|k; ij\rangle,$$

where k is the momentum, and $i, j = 1, \dots, N$ denote the end points of the string.

From (2.5) it is obvious that when the branes are coinciding, there are N^2 massless modes, since each string end can lie on N different branes. So, there are N^2 different particles of each type. The N^2 Hermitian $N \times N$ matrices λ^r, $r = 1, \cdots, N^2$, normalized to $\mathrm{Tr}(\lambda^r \lambda^s) = \delta^{rs}$, form a complete set of states for the two endpoints, so that an open string state can be written as [17]

$$|k; r\rangle = \sum_{i,j=1}^{N} |k; ij\rangle \lambda^r_{ij}.$$

These matrices are representation matrices of $\mathrm{U}(N)$ and are called Chan-Paton factors.

In the case of oriented strings, it can be shown that the non-dynamical nature of the Chan-Paton degrees of freedom forces each open string scattering amplitude into trace-like structures [17, 20],

$$\lambda^1_{ij}\lambda^2_{jk} \cdots \lambda^n_{mi} = \mathrm{Tr}\left(\lambda^1 \lambda^2 \cdots \lambda^n\right),$$

because the two connecting ends of the string must always be in the same Chan-Paton state. All such amplitudes are invariant under the $\mathrm{U}(N)$ transformation

$$\lambda^r \to U \lambda^r U^{-1},$$

under which the end points of the string transform. There is thus an additional gauge symmetry in the theory.

There are now two sets of N^2 massless fields corresponding to the massless modes of open strings ending on a stack of coinciding D-branes. They can be packaged as $N \times N$ Hermitian matrices

$$(A^a)^i_j, \qquad \left(X^I\right)^i_j, \tag{2.6}$$

where the components, $i, j = 1, \dots, N$, of the matrices indicate from which string the field originates. Written in this way, the gauge field has the form of a $\mathrm{U}(N)$ gauge connection, and the scalar fields transform under the adjoint representation of the $\mathrm{U}(N)$ gauge group [16, 17]. So, by adding N coinciding D-branes to the theory, the $U(1)$ gauge field A^a becomes a $\mathrm{U}(N)$ gauge field, making the theory non-Abelian. So the theory becomes a $\mathrm{U}(N)$ Yang-Mills theory coupled to adjoint scalars. For an introduction to Yang-Mills theory, see [19] and [15].

In the next section an overview of superstrings is given, in order to include the field content corresponding to the massless modes in the description of the dynamics of D-branes in section 2.7.

2.3 RNS Superstrings

In this section, a short overview of the Ramond-Neveu-Schwarz (RNS) formalism is presented, in order to describe the massless field content of superstring theory. The RNS formalism is supersymmetric on the world sheet. For a short introduction to supersymmetry see [19]. In flat, ten-dimensional Minkowski space the RNS formalism is equivalent to the Green-Schwarz (GS) formalism, which is supersymmetric in spacetime. The GS formalism will not be discussed here. For details the reader can consult chapter four of [22], from where most of the material of this section has been taken.

To the action for a bosonic string (2.3) which describes D massless bosons, a Dirac term can be added to include D free massless world-sheet fermions. In conformal gauge and flat background this looks like

$$S = -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right), \tag{2.7}$$

where ψ^μ are two-component spinors on the world sheet and vectors under Lorentz transformations, and ρ_α, with $\alpha = 0, 1$, represent two-dimensional Dirac matrices which obey the Dirac algebra, $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$. Consistency of the theory requires $D = 10$, so $\mu = 0, \ldots, 9$. This action is invariant under the supersymmetry transformation,

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \qquad \delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon,$$

where ϵ is a constant infinitesimal Majorana spinor that consist of Grassmann numbers.

By choosing a basis in which the Dirac matrices are purely real,

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the spinors $\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$ become Majorana fermions, with $\psi^*_\pm = \psi_\pm$, where ψ_+ describes a left mover and ψ_- a right mover. Using $\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}$, the fermionic part of the action (2.7) then takes the form

$$S_{\text{ferm}} = iT\int d^2\sigma \left(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\right),$$

where $\partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right)$ refers to light-cone world-sheet coordinates. Under the variation $\psi \to \psi + \delta\psi$, the boundary terms of this action are

$$\delta S \sim -\int d\tau \left(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\right)\Big|_{\sigma=0}^{\sigma=\pi}.$$

For the action to be an extremum, the boundary terms have to vanish. This will give conditions for both open and closed strings.

2.3.1 Open strings

For open strings the two terms must vanish separately, so at each end of the string we have the condition $\psi^\mu_+ = \pm\psi^\mu_-$. When we set $\psi^\mu_+|_{\sigma=0} = \psi^\mu_-|_{\sigma=0}$ there are two options, which are called Neveu-Schwarz (NS) or Ramond (R) boundary conditions:

Neveu-Schwarz boundary conditions: $\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi}$
The mode expansion in the NS sector looks like

$$\psi_\pm^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau\pm\sigma)}.$$

From the fact that $\psi_\pm$ are Majorana spinors, it follows that $(b_r^\mu)^\dagger = b_{-r}^\mu$. Canonical quantization gives $\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}$.

In light-cone gauge, where again $X^+ \sim \tau$, with $X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^9\right)$, the mass M of a state $|\phi\rangle$ is given by

$$\alpha' M^2 = N - a_{NS},$$

where a_{NS} is the normal-ordering constant which is required to be $1/2$ to retain Lorentz invariance of the theory, and N is the number operator,

$$N = \sum_{n\geq 1} \alpha_{-n}^i \alpha_n^i + \sum_{r\geq\frac{1}{2}} r\, b_{-r}^i b_r^i$$

which gets replaced by the eigenvalue for the state $|\phi\rangle$. The α_n^i, $i = 1,\ldots,8$, are the coefficients of the Fourier modes in the mode expansion of the bosonic string.

The ground state $|0,k\rangle_{NS}$ in this sector is defined by

$$\alpha_n^i|0,k\rangle_{NS} = b_r^i|0,k\rangle_{NS} = 0, \qquad \text{for } n,r > 0.$$

The full Fock space of states is obtained by acting with the negative modes on the vacuum. Acting with the momentum operator α_0^μ gives

$$\alpha_0^\mu|0,k\rangle_{NS} = \sqrt{2\alpha'}\, k^\mu|0,k\rangle_{NS}.$$

The ground state is unique, so it is a spacetime scalar. Open strings with Neveu-Schwarz boundary conditions thus give rise to spacetime bosons.

Ramond boundary conditions: $\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi}$
The mode expansion in the R sector is given by

$$\psi_\pm^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau\pm\sigma)}.$$

Again, since $\psi_\pm$ are Majorana spinors, it follows that $(d_n^\mu)^\dagger = d_{-n}^\mu$. Canonical quantization gives $\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}$.

In light-cone gauge the mass M of a state $|\phi\rangle$ is given by

$$\alpha' M^2 = \Big(\sum_{n\geq 1} \alpha_{-n}^i \alpha_n^i + \sum_{n\geq 1} n\, d_{-n}^i d_n^i\Big) - a_R \equiv N - a_R,$$

where a_R is the normal-ordering constant which is required to be zero, and N gets replaced by its eigenvalue of $|\phi\rangle$.

Since now $m,n \in \mathbb{Z}$, it follows that d_0^μ commutes with the number operator, so it can act without changing the mass of a state. Therefore the ground state of the R sector is degenerate. To account for this degeneracy, the ground state gets an extra label, $|a;0,k\rangle_R$. The ground states, with $N=0$, are massless, and obey

$$\alpha_n^i|a;0,k\rangle_R = d_n^i|a;0,k\rangle_R = 0, \qquad \text{for } n>0,$$

as well as the massless Dirac equation.

The zero modes satisfy $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, which is the Dirac algebra $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$ up to a factor of two. The ground state of the R sector is thus a spinor,

$$d_0^\mu|a;0,k\rangle_R = \frac{1}{\sqrt{2}}\Gamma^\mu_{ab}|b;0,k\rangle_R,$$

where Γ^μ are ten 32×32 matrices since a general Dirac spinor has $2^{D/2}$ components. So, the ground state in the Ramond sector is a spacetime fermion.

GSO projection in the NS sector

The ground state in the NS sector is tachyonic, as $\alpha' M^2 = -1/2$, and such states are unwanted in the theory, since they violate causality. Furthermore, since this ground state is a spacetime boson, there should be a fermionic counterpart with the same mass as this tachyon, in order for the theory to possess spacetime supersymmetry, and this particle is not present in the spectrum.

The problem is solved by using *Gliozzi-Scherk-Olive* (GSO) projection, where one only keeps states with positive *G-parity*, i.e., states which have

$$G = (-1)^{F_{NS}+1} = 1,$$

with $F_{NS} = \sum_{r\geq\frac{1}{2}} b^i_{-r}b^i_r$ the number of world-sheet fermion excitations. Since now only odd values for F_{NS} are allowed, the new ground state becomes

$$b^i_{-\frac{1}{2}}|0,k\rangle_{NS},$$

which is a (transverse) vector in spacetime, having eight degrees of freedom, times a spacetime scalar. This corresponds to a gauge field, and therefore the ground state in the Neveu-Schwarz sector is a gauge field with eight degrees of freedom.

GSO projection in the R sector

As in ordinary quantum field theory, one can construct a chiral projection operator

$$P_\pm = \frac{1}{2}\left(1 \pm \Gamma_{11}\right),$$

with $\Gamma_{11} = \Gamma_0\Gamma_1\ldots\Gamma_9$, which can be regarded as the analog of γ_5 in four dimensions. These projection operators project out spinors with definite chirality, called Weyl spinors. The ground state of the R sector can then be decomposed into a direct sum of positive and negative chirality parts

$$|a;0,k\rangle_R = P_+|a;0,k\rangle_R \oplus P_-|a;0,k\rangle_R \equiv |a,+;0,k\rangle_R \oplus |a,-;0,k\rangle_R,$$

where the two Weyl states have $\Gamma_{11}|a,\pm;0,k\rangle_R = \pm|a,\pm;0,k\rangle_R$.

The G-parity operator in the Ramond sector is defined as

$$G = \Gamma_{11}(-1)^{F_R},$$

where $F_R = \sum_{n\geq 1} d^i_{-n} d^i_n$. One can choose the chirality of the ground state in the R sector by keeping states with positive or negative G-parity. So in the Ramond sector, GSO projection reduces the ground state from a Dirac spinor to a Weyl spinor. In ten dimensions, it is possible to impose the Weyl condition together with the Majorana condition. Therefore, the ground state in the R sector becomes a Majorana-Weyl spinor. A Dirac spinor in ten dimensions has 32 components but after applying the Majorana and Weyl conditions, 16 real components remain. After applying the Dirac equation, the ground state in the R sector is a Majorana-Weyl spinor with eight real degrees of freedom. This degrees of freedom match that of the ground state of the NS sector.

In conclusion, the massless states of open strings in superstring theory give rise to a massless scalar field A^μ and a chiral spinor χ in ten-dimensional spacetime, both having eight degrees of freedom. The two fields can be combined into a ten-dimensional $\mathcal{N} = 1$ supersymmetric vector multiplet. The RNS superstring theory, which was supersymmetric on the world sheet, has become supersymmetric in spacetime after applying the GSO projection. As mentioned in section 2.2, open strings end on Dp-branes. If the D-brane is taken to be $3+1$ dimensional, then, when dimensional reducing the six transverse dimensions on a torus, the $\mathcal{N} = 1$ multiplet becomes an $\mathcal{N} = 4$ multiplet, and the gauge field splits into a four-dimensional massless gauge field A^a, which lives on the Dp-brane, and six scalar fields X^I. For an example how this works, see [15].

2.3.2 Closed strings

For closed strings the condition $\psi^\mu_\pm(\sigma) = \pm\psi^\mu_\pm(\sigma+\pi)$ must be fulfilled. For fields with Ramond boundary conditions the mode expansions are given by

$$\psi^\mu_-(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)}; \qquad \psi^\mu_+(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)},$$

whereas for fields with Neveu-Schwarz boundary conditions they are given by

$$\psi^\mu_-(\sigma,\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir(\tau-\sigma)}; \qquad \psi^\mu_+(\sigma,\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)}.$$

The coefficients in the expansions obey the same commutation relations as the ones in section 2.3.1.

There are now four ways to combine left and right movers, giving rise to four sectors: NS-NS, R-R, NS-R, and R-NS. As in case of open strings, when GSO projecting in the R sector, it is possible to choose the chirality of the ground state. This choice of chirality can be done independently for the left or right movers, which yields two distinct type-II string theories. In type-IIB string theory, left- and right-moving R-sector ground states are defined to have the same chirality, chosen to be positive, which are denoted by $|+\rangle_R$. After eliminating the tachyon in the NS ground state using GSO projection, the massless states of closed strings in type-IIB string theory are given by (left $\otimes$ right mover):

$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes b^j_{-\frac{1}{2}}|0\rangle_{NS}; \qquad |+\rangle_R \otimes |+\rangle_R;$$
$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R; \qquad |+\rangle_R \otimes b^j_{-\frac{1}{2}}|0\rangle_{NS},$$

where the momentum labels have been suppressed. The upper two states are *spacetime bosons*, while the lower ones, being a tensor product of a vector and a spinor, are *spacetime fermions*. Each sector has $8 \times 8 = 64$ degrees of freedom.

The massless spectrum of the bosonic fields can be decomposed as

$$\overbrace{\underbrace{35}_{G_{\mu\nu}} + \underbrace{28}_{B_{\mu\nu}} + \underbrace{1}_{\Phi}}^{\text{NS-NS}} + \overbrace{\underbrace{1}_{\mathbf{C}_0} + \underbrace{28}_{\mathbf{C}_2} + \underbrace{35}_{\mathbf{C}_4}}^{\text{R}_+\text{-R}_+} = 128,$$

where: $G_{\mu\nu}$ is a symmetric traceless field, called *graviton*; $B_{\mu\nu}$ is an antisymmetric gauge field, called *Kalb-Ramond* field; Φ is the trace part of the NS-NS ground state, called *dilaton* field; and C_0, C_2, C_4 are *antisymmetric tensor* fields. Note that the indices i, j have been replaced by μ, ν. This can be done because both $G_{\mu\nu}$ and $B_{\mu\nu}$ enjoy a spacetime gauge symmetry which allows the removal of the extra modes [23].

The fermionic fields of type-IIB string theory can naturally be decomposed into

$$\overbrace{\underbrace{8_-}_{\lambda_a^1} + \underbrace{56_+}_{\hat{\psi}_a^{1i}}}^{\text{NS-R}_+} + \overbrace{\underbrace{8_-}_{\lambda_a^2} + \underbrace{56_+}_{\hat{\psi}_a^{2i}}}^{\text{R}_+\text{-NS}} = 128,$$

where the traceless parts of ψ_a^i, defined as $\hat{\psi}_a^i$ with $(\gamma_i \hat{\psi}^i)_a = 0$, are *gravitinos* with the same (positive) chirality, and $\lambda_a = \gamma_i \psi_a^i$ are *dilatinos* with negative chirality. Note that $i = 1, \ldots, 8$ is the vector index, while a is the spinor index.

In summary, type-IIB string theory contains two Majorana-Weyl gravitinos and dilatinos coming from the NS-R and R-NS sectors, and the bosonic fields from the NS-NS and R-R sectors. These fields match precisely with the field content of type-IIB supergravity, which will be reviewed in section 2.5.2. The closed-string modes form background fields in which other strings propagate. However, there is a technical obstruction for coupling the R-R sector fields to the string world sheet in RNS formalism. In order to overcome this obstruction, one could use the Green-Schwarz formalism, but this will not be discussed in this thesis. For this reason, R-R sector background fields will be ignored until section 2.6. The coupling to the NS-NS sector background fields, $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, will be explained in the next section.

2.4 Coupling to string backgrounds

In the previous section, it was shown that the massless states in the NS-NS sector of type-IIB string theory give rise to a graviton $G_{\mu\nu}$, a Kalb-Ramond field $B_{\mu\nu}$, and a dilaton Φ. In this section arguments will be given that these fields generate backgrounds in which strings propagate. For reasons of simplicity only the bosonic part S_{bos} of the action (2.7) is considered, an account including the fermionic part can be found in [18]. The material of this and the next section has mainly been taken from [16] and [17]. In this section and later ones, differential form notation and the notion of pull-back are used. Very accessible introductions to these subjects can be found in respectively [24] and Appendix A of [25]. A short overview of differential forms is given in Appendix A.

Coupling to the graviton field

Curvature of spacetime can be regarded as a coherent background of gravitons. The $G_{\mu\nu}$ of section 2.3.2 were called gravitons, because these fields are in the symmetric traceless second-rank representation of SO(8) by construction, and are thus massless spin-2 particles. From now on, for $G_{\mu\nu}$ the more common notation $g_{\mu\nu}$ for the metric will be used. The action (2.7) can be generalized to describe strings moving in a curved spacetime, which for the bosonic part looks like

$$S_{\mathrm{bos,G}} = -\frac{T}{2}\int d^2\sigma\,\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \tag{2.8}$$

where $g_{\mu\nu}$ are the $G_{\mu\nu}$ from section 2.3.2.

Since the spacetime metric is generated by a coherent background of closed string states, it is natural to include the backgrounds of the other massless string states as well. Note that from this viewpoint, massless states of closed strings are regarded as fluctuations of the background geometry.

Coupling to the Kalb-Ramond field

Based on the requirement of diffeomorphism and Weyl invariance, the coupling to the anti-symmetric Kalb-Ramond field $B_{\mu\nu}$ has the form

$$S_{\mathrm{bos,B}} = \frac{T}{2}\int d^2\sigma\,\sqrt{-\gamma}\varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}, \tag{2.9}$$

where $\varepsilon^{\alpha\beta}$ is the Levi-Civita tensor, normalized as $\sqrt{-\gamma}\varepsilon^{12} = +1$.

This action is the two-dimensional analog of the (electrical) coupling of a charged point particle to a background gauge potential A_μ,

$$S_A = q\int d\tau\, A_\mu(X)\frac{dX^\mu}{d\tau} = q\int_{\Sigma_1} A_1, \tag{2.10}$$

which has the geometrical interpretation of being the pull-back of the one-form $A_1 = A_\mu \mathrm{d}X^\mu$ in spacetime onto the world-line of the particle Σ_1. This is possible because the world-line is one dimensional.

In case of two-dimensional string world sheets, the analogous coupling should be a two-form gauge field in spacetime, which is antisymmetric by definition. Thus the integrand in (2.9) is nothing more than the pull-back of the two-form gauge field B_2 onto the string world sheet, i.e., $\frac{T}{2}\int B_2$ in differential form notation. The coefficient in front of the integral says that the two-form charge is equal to the string tension, which is required by supersymmetry.

The action (2.9) is invariant under the gauge transformation,

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu, \tag{2.11}$$

with C_μ an one-form, under which it changes by a total derivative. This is similar to the gauge invariance of the point particle of (2.10) under $A_\mu \to A_\mu + \partial_\mu \Lambda$, where Λ is a scalar. In electromagnetism there is a two-form gauge field strength defined by $F_2 = \mathrm{d}A_1$, which is gauge invariant. Similarly, a three-form field strength $H_3 = \mathrm{d}B_2$ can be defined,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \tag{2.12}$$

which is invariant under the gauge transformation (2.11).

Coupling to the dilaton field

The coupling to the dilaton field is given by

$$S_{\text{bos},\Phi} = \frac{T}{2} \int d^2\sigma \sqrt{-\gamma} \alpha' \Phi \mathcal{R}^{(2)}(\gamma), \tag{2.13}$$

where $\mathcal{R}^{(2)}(\gamma)$ is the Ricci scalar of the two-dimensional world sheet. This term vanishes on a flat world sheet, and it does not respect Weyl invariance. However, for a constant dilaton $\Phi(X) = \Phi_0$ it does, and in that case the action becomes

$$S_{\text{bos},\Phi} = \Phi_0 \frac{1}{4\pi} \int d^2\sigma \sqrt{-\gamma} \mathcal{R}^{(2)}(\gamma) = \Phi_0 \chi, \tag{2.14}$$

where χ is the Euler characteristic of the world sheet, which is a topologically invariant and integer. For a world sheet without boundary, the Euler characteristic is related to the number of handles h on the world sheet by $\chi = 2 - 2h$.

When calculating the partition function in string theory, one has to include summations over world sheets with different topologies. For example, in bosonic string theory the action (2.2) can be augmented by

$$S_{\text{string}} = S_{\text{P}} + \lambda \chi,$$

with λ a real number and small, so that the (Euclidean) string partition function becomes

$$\mathcal{Z} = \sum_{\substack{\text{topologies} \\ \text{metrics}}} e^{-S_{\text{string}}} \sim \sum_{\text{topologies}} e^{-\lambda \chi} \int [\mathcal{D}X][\mathcal{D}\gamma] \, e^{-S_{\text{P}}[X,\gamma]},$$

which can be regarded as an expansion in e^{λ}, and hence the *string coupling constant* can be defined as $g_s = e^{\lambda}$. Comparing with the action for the dilaton (2.14), learns that the string coupling constant is determined by $g_s = e^{\Phi_0}$.

Non-linear sigma model

As mentioned in section 2.3.2, there is a technical obstruction for coupling the R-R background to the string world sheet in the RNS formalism, and so they will be ignored. So, the action describing a bosonic string coupled to a background generated by the massless modes of closed strings is given by,

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\gamma} \left[\left(\gamma^{\alpha\beta} g_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \Phi \mathcal{R}^{(2)} \right]. \tag{2.15}$$

This is called the non-linear sigma model. Since the backgrounds in this action depend on the scalars X^μ, it is a interacting, two-dimensional theory, which generally cannot be solved exactly. For low energies one can describe the same system using an alternative action, which has the same field content, interactions and symmetries. This is the subject of the next section.

2.5 Effective actions and supergravity

In flat space and conformal gauge, the Polyakov action (2.3) describes a free theory. When including the backgrounds generated by the massless modes as in (2.8), or more general (2.15), the theory becomes interacting. This can be seen by expanding around a classical solution of a string sitting at point x^μ, $X^\mu(\sigma) = x^\mu + \sqrt{\alpha'}Y^\mu(\sigma)$, where the factor $\sqrt{\alpha'}$ has been inserted for dimensional reasons. The interaction terms then come from powers of Y^μ in the Taylor expansion of the metric,

$$g_{\mu\nu}(X) \simeq g_{\mu\nu}(x) + \sqrt{\alpha'}Y^\omega\partial_\omega g_{\mu\nu}(x) + \frac{\alpha'}{2}Y^\omega Y^\rho\partial_\omega\partial_\rho g_{\mu\nu}(x) + \cdots .$$

The derivatives of $g_{\mu\nu}$ at x can be regarded as coupling constants. In a target space with radius of curvature R, derivatives of the metric are of order $1/R$, and consequently the coupling constant is given by $\sqrt{\alpha'}/R$, which is dimensionless. This coupling constant is called *world-sheet* loop-expansion parameter to distinguish it from the *string* loop-expansion parameter g_s of the previous section [17]. The application of perturbation theory to study (2.8) or (2.15) is valid when the coupling constants are small, i.e., when $\sqrt{\alpha'} \ll R$, so when the radius of curvature is large compared to the string length $\ell_s = \sqrt{\alpha'}$.

Low-energy limit

In the previous sections, only the massless modes were considered, which corresponds to the $\alpha' \to 0$ limit, i.e., large string tension, or point-particle limit. In this limit the masses of the massive states become very large and decouple. In a Minkowski space background the only dimensionless parameter is $\alpha' E^2$, so this corresponds to the *low-energy limit*. Note that when considering the massless modes only, the condition for a large radius of curvature, $\sqrt{\alpha'} \ll R$, is implicit.

2.5.1 β-Functions

The free theory without background fields (2.2) enjoys Weyl-transformation invariance of the world sheet. At a classical level, the theories (2.9) and (2.13) also respect Weyl invariance. However, when perturbatively expanding the string partition function in $\sqrt{\alpha'}/R_c$, loops occur due to interactions, and an UV-cutoff has to be introduced in order to render the theory finite. After renormalization, the fields $g_{\mu\nu}$ and $B_{\mu\nu}$ can be dependent on a scale μ, and therefore they have β-functions associated with them. Additionally, as mentioned in section 2.4, the dilaton term in (2.15) does not respect Weyl invariance, even at classical level.

The breakdown of Weyl invariance is reflected in the non-vanishing of the trace of the stress-energy tensor $\langle T^\alpha{}_\alpha\rangle$, which turns out to be equal to [26]

$$\langle T^\alpha{}_\alpha\rangle = \sum_i \beta^i \langle\mathcal{O}_i\rangle,$$

where $\mathcal{O}_i$ are the graviton, B-field, dilaton and other massless vertex operators. Vertex operators will not be discussed here; they are described in [17]. The β-functions can be calculated in an one-loop α' expansion. For the NS-NS fields they are [26],

$$\begin{aligned}
\beta^g_{\mu\nu} &= \alpha'\mathcal{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\kappa\lambda}H_\nu{}^{\kappa\lambda} + \mathcal{O}(\alpha'^2),\\
\beta^B_{\mu\nu} &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2),\\
\beta^\Phi &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2),
\end{aligned}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor derived from $g_{\mu\nu}$, and H_3 is the three-form field strength coming from the two-form gauge field, $H_3 = \mathrm{d}B_2$.

The key point is that Weyl invariance is restored when these β-functions vanish,

$$\beta^g_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0, \tag{2.16}$$

which are equations in terms of the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Therefore, these equations will give constraints on the background fields, and they can be interpreted as the equations of motion for the backgrounds in which the string propagates.

For low energies, instead of working with the action (2.15) and the equations following from the β-functions, one can construct a target space action which has the same symmetries as the original action, and which reproduces the same equations of motion for the fields corresponding to the massless modes of closed strings. This turns out to be a supergravity theory, and in particular, when starting with type-IIB superstring theory, type-IIB supergravity theory.

2.5.2 Dynamics of closed strings - type-IIB supergravity

In section 2.3.2, it was shown that the massless spectrum of closed strings in type-IIB superstring theory includes two Majorana-Weyl gravitinos and two Majorana-Weyl dilatinos coming from the NS-R$_+$ and R$_+$-NS sectors. Further there are the bosonic fields, $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ from the NS-NS sector, and C_0, C_2 and C_4 from the R-R sector.

In ten dimensions, type-IIB supergravity with $\mathcal{N} = 2$ supersymmetry is a unique supergravity having the same field content. Also, the type-IIB supergravity action yields the same equations of motions for the fields as the ones from the β-equations (2.16). So, the low-energy effective action of type-IIB superstring theory is type-IIB supergravity theory in ten dimensions. In other words, string theory provides higher-order α' corrections to the supergravity equations of motion.

Supergravity can be considered as a theory of local supersymmetry, or equivalently, as a supersymmetric theory of gravity. For an entry-level introduction to supergravity see [13], while a more advanced version can be found in [15]. More detailed introductions to the subject can be found in [27, 28, 29].

Ignoring the fermionic fields, the ten-dimensional type-IIB supergravity action, describing the dynamics of the bosonic fields corresponding to massless excitations of closed strings, is given by [18]:

$$S_{\mathrm{IIB}} = S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}}, \tag{2.17}$$

where the action containing the NS-NS fields is given by

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\Phi}\left(\mathcal{R} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right), \tag{2.18}$$

where $\mathcal{R}$ is the Ricci scalar derived from $g_{\mu\nu}$, and $|H_3|^2 = \frac{1}{3!}H_{\mu\nu\lambda}H^{\mu\nu\lambda}$. The gravitational coupling constant κ is given by

$$\kappa^2 = g_s^{-2}\kappa_{10}^2, \quad \text{and} \quad 2\kappa_{10}^2 = 16\pi G_{10},$$

where G_{10} is Newton's constant in ten dimensions,

$$16\pi G_{10} = (2\pi)^7 g_s^2 \ell_s^8.$$

Note that κ does not contain any powers of g_s.

The dynamics of the fields from the R-R sector, C_0, C_2 and C_4, split up in a Maxwell-type kinetic term,

$$S_{\rm R} = -\frac{1}{4\kappa^2}\int d^{10}x\,\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right),$$

and a Chern-Simons coupling

$$S_{\rm CS} = -\frac{1}{4\kappa^2}\int C_4 \wedge H_3 \wedge F_3.$$

In these formulas $F_{n+1} = \mathrm{d}C_n$, $|F_p|^2 = \frac{1}{p!}F_{\mu_1\cdots\mu_p}F^{\mu_1\cdots\mu_p}$, and

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3.$$

In type-IIB string theory, the five-form $\tilde{F}_5$ is self dual, and the above action has to be supplemented with the following condition that has to be imposed on the solutions:

$$\star\,\tilde{F}_5 = \tilde{F}_5, \tag{2.19}$$

where $\star$ denotes the hodge dual; see Appendix A.

In the next section, a certain class of solutions to the equations of motion following from this type-IIB supergravity action are given.

2.6 Black p-branes

The type-IIB supergravity action (2.17) gives rise to equations of motion which turn out to be non-linear partial differential equations, and one can thus expect a large variety of solutions [23]. Among them, there is a class of solutions that preserve the subgroup $SO(1,p) \times SO(9-p)$ of the Poincaré group $SO(1,9)$ and half of the supersymmetries. These solutions will be reviewed in this section. Details can be found in [14, 15], and a step-by-step derivation of the results of this section can be found in [29].

R-R background coupling

In section 2.4, it was shown that the NS-NS sector fields become backgrounds in which other strings propagate. Also it was mentioned that there are technical obstructions for the coupling of R-R backgrounds to the string world sheet in the RNS formalism. However, there are objects which extend in p space dimensions, whose $(p+1)$-dimensional world volume Σ_{p+1} couples electrically (cf. (2.10)) to a Ramond-Ramond $(p+1)$-form C_{p+1} as [18]

$$\int_{\Sigma_{p+1}} C_{p+1},$$

where C_{p+1} is the pull-back onto the world volume, and p odd in case of type-IIB theory. These objects are called p-branes.

In electromagnetism in four dimensions, the electric charge of a point-like configuration is given by the integration of the electric field over a two sphere S^2 that surrounds the particles

$$Q_{\text{electric}} = \int_{S^2} E \cdot dS = \int_{S^2} \star F_2.$$

In an analog way, the electric R-R charge (or flux) N of a p-brane can be obtained by integrating the field strength over an $(8-p)$ sphere that completely surrounds the brane,

$$N = \int_{S^{8-p}} \star F_{p+2}, \tag{2.20}$$

with the field strength given by $F_{p+2} = \mathrm{d}C_{p+1}$. Here the assumption has been made that the metric is spherically symmetric in $10-p$ dimensions.

2.6.1 p-Brane supergravity action and solution

On p-branes the NS-NS two-form vanishes, so the relevant part of the supergravity action is given by

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\Phi} \left(\mathcal{R} + 4\partial_\mu \Phi \partial^\mu \Phi \right) - \frac{1}{2} |F_{p+2}|^2 \right].$$

In the special case of $p = 3$, an extra factor $1/2$ should be inserted in the F_5 term, and the condition (2.19) has to be imposed on the solutions.

By making an ansatz that possesses an $\mathrm{SO}(1,p)\times$ $\mathrm{SO}(9-p)$ isometry and preserves half of the supersymmetries, the solution to the equations of motion following from this action can be shown to be of the form [30]

$$ds^2 = H_p(r)^{-1/2} \Big(-dt^2 + \sum_{i=1}^{p} dx^i dx^i \Big) + H_p(r)^{1/2} \left(dr^2 + r^2 d\Omega_{8-p}^2 \right) \tag{2.21}$$

where the harmonic function H_p is given by

$$H_p(r) = 1 + \left(\frac{r_p}{r} \right)^{7-p}, \qquad r_p^{7-p} = d_p g_s N \ell_s^{7-p}, \tag{2.22}$$

with N given by (2.20) and d_p some constant. The coordinates along the branes are denoted by $t, x^1, \ldots, x^p$, and $dr^2 + r^2 d\Omega_{8-p}^2$ is the Euclidean metric in the $9-p$ directions transverse to the branes, with r denoting the radial coordinate.

This solution has a horizon for $r \to 0$, so p-branes are higher-dimensional equivalents of four-dimensional classical black hole solutions. Hence the name *black* p-branes. As seen from the $9-p$ transverse directions, these p-branes look like point-like singularities which enjoy a $\mathrm{SO}(9-p)$ symmetry. From this viewpoint, classical black holes can be regarded as 0-branes.

A p-brane carries electric charge with respect to the R-R form C_{p+1}, and in order for the solution to have the $\mathrm{SO}(1,p)$ isometry, it must be extremal, what means that the mass M of the solution must satisfy a lower bound with respect to the charge N of the solution [14]. The above solution thus describes an *extremal* charged black p-brane. The fact that these solutions preserve half of the supersymmetries means that p-branes are BPS objects; see e.g. [32].

Type-II p-branes were originally found as classical solutions to supergravity field equations, however they are expected to extend to solutions of the full type-II string equations, in which the metric and other fields will be subject to α' corrections. The extremal solution is then thought to correspond to the ground state of the black p-brane for a given charge N.

The solutions of the dilaton and $(p+1)$-form gauge field are given by

$$e^{\Phi} = g_s H_p^{(3-p)/4}, \qquad C_{p+1} = \big(H_p(r)^{-1} - 1\big)\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^p.$$

In the above formulas g_s is the string coupling constant at infinity, since when taking the limit $r \to \infty$, $H_p \to 1$ and the dilaton approaches a constant.

Since (2.21) gives a solution for any function H_p that is harmonic in the $(9-p)$ dimensions transverse to the brane, multiple p-brane solutions can be superimposed by using

$$H_p(\vec{r}) = 1 + \sum_{i=1}^{k} \frac{r_{(i)p}^{7-p}}{|\vec{r} - \vec{r_i}|^{7-p}}, \qquad r_{(i)p}^{7-p} = d_p g_s N_i \ell_s^{7-p}, \tag{2.23}$$

which is called a *multicenter solution*. Such a solution represents k parallel extremal p-branes, each located at position $\vec{r_i}$ in the space transverse to the branes, with N_i units of the R-R charge.

Special case of p=3 and validity of solution

For the special case $p = 3$ the harmonic function H_p (2.22) becomes

$$H_3(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv r_3^4 = d_3 g_s N \ell_s^4 = 4\pi g_s N \alpha'^2, \tag{2.24}$$

and the dilaton $e^{\Phi} = g_s$ becomes a constant regardless of the value of r. As will be shown in section 2.6.3, for $p = 3$ the solution does not have a singularity.

The p-brane solution is a *classical* solution to supergravity, which is appropriate when closed string loops can be neglected. As was shown in section 2.4, the string loop-expansion parameter is g_s, so this condition is met when $g_s \ll 1$. Since for $p = 3$ the value of the dilaton is constant, the string coupling can be made small everywhere in the geometry.

In addition, as was mentioned in section 2.5, application of perturbation theory is valid when the world-sheet loop-expansion parameter is small, that is, when $R \gg \ell_s$. In this context R characterizes the curvature of the black p-brane solution, so this condition is fullfilled when the p-brane curvature is small in comparison to the string scale.

By using the ten-dimensional Planck length $l_P = g_s^{1/4} \ell_s$, the condition for weak coupling can be expressed as $l_P \ll \ell_s$. Then the conditions for the applicability of classical supergravity can be combined to $l_P \ll \ell_s \ll R$, and by using $R^4 = 4\pi g_s N \ell_s^4$, this is equivalent to [14]

$$1 \ll g_s N \ll N. \tag{2.25}$$

2.6.2 Equivalence of p-branes and D-branes

In 1995, it has been shown by Polchinski, that when N Dp-branes are put on top of each other, the $(p+1)$-dimensional hyperplane carries N units of the R-R $(p+1)$-form charge, which coincides with the charge of the p-brane under this R-R form [31]. It is believed that Dp-branes and extremal p-branes in supergravity are in fact two different descriptions of the same object. The reasoning goes as follows.

In section 2.4, it was argued that the dilaton is related to the string coupling constant by $g_s = e^{\Phi_0}$. For weak coupling, i.e., for small g_s, one can make a genus expansion in string world sheets. Besides the low-energy supergravity description, the other well-understood approximation in string theory is the weak-coupling limit, where $g_s \to 0$.

In this weak-coupling limit, the $r_{(i)p}$ in (2.23) go to zero, and $H_p(\vec{r}) \to 1$, except for $|\vec{r} - \vec{r}_i| = 0$, where the metric seems to be singular. Thus, in the weak-coupling limit, the metric becomes flat everywhere, except on the p-branes itself, which become localized defects in flat spacetime.

Strings that propagate in this background therefore move in flat spacetime, except where the string touches a brane. This will give boundary conditions on the string dynamics, which turn out to be Dirichlet boundary conditions in the directions transversal to the brane and Neumann boundary conditions parallel to the brane. These boundary conditions precisely match that of the Dp-branes described in section 2.2. Also, it turns out that the isometries, supersymmetry and tension match, and hence they are believed to be different manifestations of the same object.

Validity of D-brane description

In case of a stack of N coinciding D-branes, each open string boundary loop ending on the D-branes has the Chan-Paton factor N together with the coupling constant g_s, since for every D-brane which is added to the theory, another boundary is added to the problem [32]. Therefore, the effective loop expansion parameter is $g_s N$ instead of g_s. This means that the D-brane description is valid when

$$g_s N \ll 1.$$

Comparing this with (2.25), learns that p-branes and Dp-branes describe two complementary regimes of the same object.

2.6.3 Geometry of a stack of D3-branes

As explained before, a stack of N coincident D3-branes can be described by the extremal 3-brane solution in supergravity, which is given by (2.21) and (2.22),

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right), \tag{2.26}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $R^4 = 4\pi g_s N \alpha'^2$.

To study the metric close to the branes, i.e., for $r \to 0$, a new coordinate $z = R^2/r$ can be introduced, so that the metric takes the form

$$ds^2 = \left(1 + \frac{R^4}{z^4}\right)^{-1/2} \frac{R^2}{z^2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^4}{z^4}\right)^{1/2} R^2 \left(\frac{dz^2}{z^2} + d\Omega_5^2\right).$$

Taking the near-horizon limit, $z \to \infty$, the metric becomes

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) + R^2 d\Omega_5^2. \tag{2.27}$$

The first part of this metric is a five-dimensional anti-de-Sitter spacetime (AdS_5) in Poincaré coordinates with radius R, while the second part is S^5, also with radius R. For a short description of AdS spacetimes, see Appendix B. So, the geometry close to (the horizon of) a stack of N coincident D3-branes is given by $AdS_5 \times S^5$.

On the other hand, far away from the D3-branes, that is, for $r \to \infty$, the metric becomes flat ten-dimensional Minkowski metric. It is shown in Appendix B, that, more generally, the conformal boundary of an asymptotically AdS_{d+1} spacetime is equal to a conformally compactified d-dimensional Minkowski spacetime.

Now that in the previous sections the coupling to backgrounds generated by closed string modes, low-energy effective actions, and p-branes have been discussed, where the latter turned out to be equivalent to Dp-branes, in the next section the dynamics of Dp-branes will be considered. This is a continuation of the end of section 2.2.

2.7 D-brane dynamics

As noted in section 2.2, D-branes are objects which have dynamics of their own. They can fluctuate and interact with strings and other branes. In this context massless excitations of open strings correspond to fluctuations of D-branes, analogous to the regarding of massless closed string modes as deformations of empty space. Most of the results in this section will just be stated; details can be found in [16, 17, 18].

It was shown in the aforementioned section, that massless open string states of strings ending on Dp-branes give rise to a gauge field A^a and $(D-p-2)$ massless scalar fields X^I. For the case of superstrings, it was shown in section 2.3.1, that the massless modes coming from the NS sector give rise to a massless gauge field A^μ, which, after compactification of six directions, becomes a massless gauge field A^a living on a D3-brane and six massless bosons X^I. These six bosons can be interpreted as coming from fluctuations of the brane in transverse directions. The action describing these dynamics will be covered first.

2.7.1 Dirac action

The action describing the transverse fluctuations of D-branes is just a higher dimensional extension of the Nambu-Goto action (2.1), called the Dirac action. The form of the Dirac action is dictated by Lorentz and reparametrization invariance and is given by

$$S_{\mathrm{D}} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma_{ab}}.$$

Here, ξ^a, $a = 0, \ldots, p$, are the world-volume coordinates of the brane, γ_{ab} is the pull-back of the spacetime metric onto the world volume of the brane,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}, \tag{2.28}$$

and T_p is the D-brane tension,

$$T_p = \frac{2\pi}{g_s(2\pi\ell_s)^{p+1}}. \tag{2.29}$$

2.7.2 Born-Infeld action

As mentioned, open strings ending on D-branes also give rise to a U(1) gauge field A_a living on the D-brane. The low-energy effective action describing the dynamics of this gauge field is the Born-Infeld action,

$$S_{\mathrm{BI}} = -T_p \int d^{p+1}\xi \sqrt{-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)},$$

where T_p is again the brane tension, ξ^a the world-volume coordinates of the brane, and F_{ab} is the field strength of the gauge potential A_a, $F = \mathrm{d}A$. The gauge potential is to be thought of as a function of the world-volume coordinates, $A_a = A_a(\xi)$. This action is valid for slowly-varying field strengths, so that terms involving derivatives of the field strength can be neglected.

Expanding this action to quadratic order for small field strengths, $F_{ab} \ll 1/\alpha'$, gives

$$S_{\mathrm{BI}} \simeq -T_p \int d^{p+1}\xi \left(1 + \frac{(2\pi\alpha')^2}{4} F_{ab}F^{ab} + \cdots\right),$$

which leading-order term is the Maxwell action. So, for small field strengths, the dynamics of the gauge field on the D-brane are governed by Maxwell's equations. As the field strengths are increased, non-linear corrections to the dynamics become important and are captured by the Born-Infeld action.

2.7.3 Dirac-Born-Infeld action

The combined action of the dynamics of the transverse fluctuations of the D-branes in flat space and the gauge fields living on them, is described by a mixture of the Dirac action and the Born-Infeld action. This is called the Dirac-Born-Infeld (DBI) action,

$$S_{\mathrm{DBI}} = -T_p \int d^{p+1}\xi \sqrt{-\det\left(\gamma_{ab} + 2\pi\alpha' F_{ab}\right)},$$

where γ_{ab} is again the pull-back of the spacetime metric onto the world-volume of the brane (2.28), and the embedding coordinates $X^\mu(\xi)$, $\mu = 0, \ldots, D-1$, are considered to be dynamical fields. The DBI action is valid for slowly-varying field strengths, so that corrections from derivatives of the field strength can be neglected.

The DBI action has a reparametrization invariance, which removes the longitudinal degrees of freedom from X^μ, so that only the transversal ones remain. This reparametrization invariance can be used to go to *static gauge*. For an infinite, flat D-brane, it is useful to set $X^a = \xi^a$, $a = 0, \ldots, p$, so that the pull-back of the metric only depends on the transverse fluctuations X^I,

$$\gamma_{ab} = \eta_{ab} + \frac{\partial X^I}{\partial \xi^a}\frac{\partial X^J}{\partial \xi^b}\delta_{IJ}.$$

Up to a constant term, to leading order, the expansion of the DBI action for small field strengths F_{ab} and small derivatives $\partial_a X^I$ is given by

$$S_{\mathrm{DBI}} \simeq -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^I\partial^a\phi^I + \cdots\right),$$

where the transverse coordinates have been rewritten as

$$\phi^I = \frac{X^I}{2\pi\alpha'}, \tag{2.30}$$

to emphasize they are massless scalar fields. Note that for $p = 3$, the α' in the factors in front of this action precisely cancel. This action describes a free Maxwell theory coupled to free massless scalar fields ϕ^I.

DBI action with background fields

The DBI action in a background generated by the closed string modes of section 2.3.2 is given by

$$S_{\text{DBI}} = -T_p \int d^{p+1}\xi \, e^{-\tilde{\Phi}} \sqrt{-\det\left(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab}\right)}. \tag{2.31}$$

The coupling to the background metric $g_{\mu\nu}$ appears in the pull-back metric γ_{ab},

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu}.$$

The gauge field B_{ab} is the pull-back of the two-form gauge field to the world-volume,

$$B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}.$$

Its appearance in the DBI action is required by gauge invariance. The combination

$$B_{ab} + 2\pi\alpha' F_{ab},$$

is gauge invariant under the transformation

$$A_a \to A_a - \frac{1}{2\pi\alpha'} C_a, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu,$$

where C_μ is an one-form. The fact that the gauge-invariant field strength involves a combination of both F_{ab} and B_{ab}, is related to the fact that strings in spacetime are charged under $B_{\mu\nu}$ and the ends of open strings are charged under the gauge field A_a. This means that open strings deposits B charge on the brane, where it is converted into A charge.

The dilaton can be decomposed in a constant piece and a varying piece, $\Phi = \Phi_0 + \tilde{\Phi}$, where the constant piece is given by the vacuum expectation value (VEV) of the dilaton, $\Phi_0 = \langle\Phi\rangle$. As mentioned in section 2.4, the constant part of the dilaton governs the asymptotic string coupling, $g_s = e^{\Phi_0}$. This constant part sits implicitly in front of the action via (2.29). The varying part remains explicitly in the action. Physically this means that the effective string coupling at a point X in spacetime depends on the local value of the dilaton field and is given by $g_s^{eff} = e^{\Phi(X)} = g_s e^{\tilde{\Phi}(X)}$.

By expanding the action (2.31) around a flat background, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{10} h_{\mu\nu}$, it can be shown that D$p$-branes couple to gravity with strength $\kappa_{10}^2 T_p/\ell_s^{7-p}$, where the gravitational coupling constant was given in section 2.5.2, $\kappa_{10}^2 \sim g_s^2 \ell_s^8$, and T_p is given by (2.29), $T_p \sim g_s^{-1}\ell_s^{-(p+1)}$. This means that for $p < 7$ the interactions between the background coming from the closed string modes and the D-branes vanish in the low-energy limit.

Other fields and couplings

It was mentioned in section 2.6, that p-branes couple to $(p+1)$-form Ramond-Ramond potentials C_{p+1}, and by the equivalence of extremal p-branes and D-branes, the latter also should couple to them. The part of the action describing the coupling of D-branes to these potentials is given by a Chern-Simons term [22],

$$S_{\mathrm{CS}} = T_p \int_{\Sigma_{p+1}} \sum_n C_n e^{(2\pi\alpha' F + B)}.$$

As described in section 2.3.1, there are also fermionic degrees of freedom coming from open strings ending on D-branes, which, like the bosonic ones, can be regarded as transverse excitations of the D-brane. The inclusion of these degrees of freedom in a DBI-like action is not discussed in this thesis, but a description can be found in chapter 6 of [22].

2.7.4 Action for N coincident D-branes

In section 2.2.2, it was shown that the massless fields on the D-brane could be written as $N \times N$ Hermitian matrices, with the element of the matrix corresponding to which brane the end points terminate on (2.6). The massless excitations of N coincident branes are a U(N) gauge field $(A^a)^i_j$, together with scalars $(\phi^I)^i_j$ (cf. (2.30)) which transform in the adjoint representation of the U(N) gauge group. The action describing the interactions of these fields, would be a non-Abelian generalization of the DBI action, but such an action is not known.

However, the low-energy limit corresponding to small field strengths, is known. As mentioned above, in this limit, the interactions between the closed string modes and the brane vanish, and as a consequence one can use a flat background space. It can be shown, by insisting on gauge invariance and using supersymmetry, that the action describing the dynamics of N coincident Dp-branes is up to a constant term [18],

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \, \mathrm{Tr}\Big(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\mathcal{D}_a\phi^I\mathcal{D}^a\phi^I - \frac{1}{4}\sum_{I\neq J}[\phi^I,\phi^J]^2 + \cdots\Big), \tag{2.32}$$

where the trace is over the Chan-Paton matrix indices, which have been omitted, and the dots denote fermionic terms. All higher-order α' terms have been suppressed by the $\alpha \to 0$ limit. For an account including α' corrections see [33]. The field strength is now non-Abelian and is given by

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b].$$

The derivatives in the kinetic term for ϕ^I,

$$\mathcal{D}_a\phi^I = \partial_a\phi^I + i[A_a, \phi^I],$$

reflect the fact that these fields transform in the adjoint representation of the gauge group.

The action (2.32) can be recognized as the bosonic part of the U(N) super Yang-Mills action; see e.g. [15]. The coefficient in front of this Yang-Mills action, $1/2g^2_{\mathrm{YM}}$, can be read off from this action,

$$\frac{1}{2g^2_{\mathrm{YM}}} = \frac{1}{4}(2\pi\alpha')^2 T_p \implies g^2_{\mathrm{YM}} = 2(2\pi)^{p-2}\ell_s^{p-3} g_s,$$

where for the second identity (2.29) has been used. This yields for $p = 3$ the relation

$$g_{\text{YM}}^2 = 4\pi g_s.$$

It was mentioned in section 2.3.1, that the field content of massless open string excitations in type-IIB superstring theory can be combined into an $\mathcal{N} = 1$ super multiplet, which becomes an $\mathcal{N} = 4$ multiplet when the brane is a D3-brane and the six transverse directions have been compactified. So, the theory describing the dynamics of a stack of N coinciding D3-branes becomes $\mathrm{U}(N)$ $\mathcal{N} = 4$ super Yang-Mills theory, which is a conformal field theory (CFT) in $3 + 1$ dimensions. For a short overview of conformal field theories, see Appendix C.

A note on the $\mathrm{U}(N)$ group. This group is basically equivalent to $\mathrm{U}(1) \times \mathrm{SU}(N)$. In this case the symmetry group $\mathrm{U}(1)$ corresponds to the center of mass of the D-branes and this decouples from the theory in most circumstances [14]. So, the theory on the world volume of the stack of D3-branes essentially $\mathrm{SU}(N)$ $\mathcal{N} = 4$ super Yang-Mills in $3 + 1$ dimensions.

At this point, all the ingredients of the AdS/CFT correspondence have been described. In the next section, all the pieces will be assembled together in order to give a motivation for the correspondence.

2.8 Maldacena conjecture

The starting point for giving a description of the AdS/CFT correspondence is a stack of N coincident D3-branes embedded in flat, ten-dimensional Minkowski spacetime in type-IIB superstring theory. As argued in section 2.6.2, Dp-branes and extremal p-branes are descriptions of the same object pertaining to different regimes. In this section, the stack of branes first will be described from both viewpoints, in order to identify the free supergravity part of them and state the Maldacena conjecture [6]. Details of this section can be found in [14].

There are two types of excitations, namely, open and closed strings. At low energies, only the massless modes remain, which were described in sections 2.3.1 and 2.3.2 for open and closed strings respectively. In addition, at low energies, the actions describing the dynamics of these massless modes could be replaced by effective actions, which for closed strings turned out to be type-IIB supergravity, and for open strings the DBI action.

2.8.1 Two equivalent viewpoints

D-brane viewpoint

From the D-brane perspective, the combined effective action describing the dynamics of the massless modes of both open and closed strings schematically looks like,

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}},$$

where S_{brane} is the non-Abelian DBI action which describes the open string excitations (2.32), S_{bulk} is the type-IIB supergravity action with α' corrections describing the dynamics of the closed string states as presented in section 2.5.2, and S_{int} describes the interaction between the brane modes and the bulk modes.

As was touched upon in section 2.7.3, in the low-energy limit, the interactions between the brane and bulk modes vanish. In an analogous way, it can be shown by expanding the action which describes the dynamics of the (bosonic) background fields (2.18) around a flat metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{10} h_{\mu\nu}$, that the interaction terms for the background fields vanish in the $\alpha' \to 0$ limit [14]. So, the dynamics of the background fields are described by *free* type-IIB supergravity with the α' corrections suppressed. Finally, in the low-energy limit and compactification of the six transverse directions, the brane action becomes the SU(N) $\mathcal{N} = 4$ super Yang-Mills action.

Concluding, from the point of view of the DBI-action, the stack of N D3-branes gives rise to two decoupled systems: SU(N) $\mathcal{N} = 4$ super Yang-Mills theory on the branes, and free type-IIB supergravity in the bulk.

Extremal 3-brane viewpoint

At the same time, as was claimed in section 2.6.2, the stack of N D3-branes could be described by the extremal black 3-brane solution with R-R charge N, where one regard D-branes, being massive and charged, as the sources for the various supergravity fields. It was shown in section 2.6.3, that this solution possesses a horizon at $r = 0$, and that the background becomes flat Minkowski far away from the horizon, $r \to \infty$, which is called the *asymptotic region*, while close to the horizon the geometry is given by $\mathrm{AdS}_5 \times \mathrm{S}^5$ (cf. (2.27)).

For an observer far away from the horizon, $r \to \infty$, there are two types of low-energy excitations in this geometry. Firstly, massless particles propagating in the bulk, which in the low-energy limit becomes free type-IIB supergravity as was argued in the section about the D-brane viewpoint. Secondly, any string excitations in the vicinity of the horizon due to a large redshift.

This second type of low-energy excitations can be seen as follows. Suppose an observer at infinite distance from the brane, where the geometry is flat, measures an energy E for an excitation at a point r close to the brane. This observed energy is lower than the energy E_p measured at the point r itself. This follows from

$$E_p = p^0 = m\frac{dx^0}{d\tau} = m\frac{1}{\sqrt{-g_{00}}}\frac{dx^0}{dt} = \frac{1}{\sqrt{-g_{00}}}E = \left(1 + \frac{R^4}{r^4}\right)^{1/4} E, \tag{2.33}$$

where in the third and last equality (2.26) has been used. For small r, one can write

$$E = \left(1 + \frac{R^4}{r^4}\right)^{-1/4} E_p \sim \frac{r}{\alpha'}(E_p\sqrt{\alpha'}) \equiv U(E_p\sqrt{\alpha'}). \tag{2.34}$$

Where $E_p\sqrt{\alpha'}$ is the energy in string units, which is dimensionless. Therefore, $U = r/\alpha'$ has dimensions of energy. This scale will be interpreted in 2.8.3. When taking the $\alpha' \to 0$ limit, one can keep U fixed by letting $r \to 0$. As a consequence, in the low-energy limit, $\alpha' \to 0$, the energy observed from infinity, E, goes to zero for any finite E_p close to the horizon. So, as seen by an *asymptotic observer*, which is at infinite distance from the horizon, any string excitation close to the horizon has low energy. In section 2.6.3, it was shown that the near-horizon geometry is given by $\mathrm{AdS}_5 \times \mathrm{S}^5$, so the low-energy excitations close to the horizon are given by type-IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

In the low-energy limit, the two types of excitations again decouple, because in this limit, the cross-section for a supergravity wave of frequency ω from the asymptotic region to be absorbed by the near-horizon region is given by [34]

$$\sigma_{\mathrm{AdS}} \sim \omega^3 R^8,$$

which goes to zero for $\omega \to 0$. This can be understood from the fact that the wavelength of particles in this limit becomes much larger than the gravitational size of the brane. At the same time, the very close to the horizon the gravitational potential is very steep, and excitations cannot escape to the asymptotic region.

From the extremal 3-brane point of view, the theory consists again of two decoupled systems: free type-IIB supergravity in the asymptotic region where the geometry is flat Minkowski, and type-IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

2.8.2 The conjecture

In both viewpoints there are two decoupled systems of which one is type-IIB supergravity in flat Minkowski space. It is very natural to identify these two supergravity systems.

This leads to the following conjecture due to Maldacena [6]:

> Type-IIB superstring theory compactified on $\mathrm{AdS}_5 \times \mathrm{S}^5$, [1] where the radii of AdS_5 and S^5 are given by $R^4 = 4\pi g_s N \alpha'$, and the electric R-R flux through S^5 is given by $N = \int_{S^5} \star F_5$ (cf. (2.20)),
>
> is dual to
>
> $\mathcal{N} = 4$ superconformal Yang-Mills theory in $3+1$ dimensions with gauge group $\mathrm{U}(N)$, and the Yang-Mills coupling $g_{\mathrm{YM}}^2 = 4\pi g_s$.

The conjecture is a *duality*, because the two different viewpoints pertain to different regimes. As was shown in section 2.6.1, the extremal p-brane viewpoint is valid when $1 \ll g_s N \ll N$, while in section 2.6.2 it was argued that the D-brane viewpoint is valid for $g_s N \ll 1$. The viewpoints therefore are perfectly incompatible. The AdS/CFT correspondence describes a duality in the sense that, when the Yang-Mills side is strongly coupled, the supergravity side is weakly coupled and vice versa.

Both type-IIB superstring theory and $\mathcal{N} = 4$ super Yang-Mills are invariant under an $\mathrm{SL}(2,\mathbb{Z})$ symmetry, under which $g_s \to 1/g_s$. This is called *Montonen-Oliven* in the context of Yang-Mills theory, and strong-weak or *S-duality* in the context of super string theory. The duality strong-weak of the AdS/CFT correspondence has some resemblance to these dualities.

The AdS/CFT correspondence is a *conjecture*, because the AdS_5 geometry came from a classical supergravity solution, which means that it did not contain g_s or α' corrections. To prove the correspondence, one should study string theory on a curved background non-perturbatively, which is at present not well understood.

Versions of the duality

There are several forms of the conjecture depending on what limits are taken, but in order to describe them, it is convenient to introduce *'t Hooft parameter*:

$$\lambda \equiv g_{\mathrm{YM}}^2 N = 4\pi g_s N.$$

With this parameter it is possible to re-express the condition for the validity of the supergravity description as $\lambda \gg 1$ (cf. (2.25)), which needs to be supplemented with the condition of large N, since by using an S-duality transformation, g_s can be mapped to $1/g_s$. With this parameter the different forms of the conjecture are:

[1] Actually, the spacetime only needs to be asymptotic to $\mathrm{AdS}_5 \times \mathrm{S}^5$ [14]. See Appendix B for a definition of asymptotically AdS.

Weakest: valid for large λ with $N \to \infty$. Here α' and g_s corrections might not agree on both sides. The gravity side is classical type-IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Less weak: called 't Hooft limit, where λ is kept fixed and $N \to \infty$. In this case α' corrections agree, but the g_s corrections might not. The gravity side is classical type-IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Strong: valid for all λ and N. The two theories are exactly the same for all values of g_s and N. The gravity side is full type-IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The weak version already is very useful, since large λ corresponds to large g_{YM}, which is the coupling constant of the Yang-Mills theory. So, the large-λ limit corresponds to the strong-coupling limit on the gauge theory side, and as a consequence supergravity can be used to study the properties of a strongly-coupled gauge theory. Stated slightly different: the AdS/CFT correspondence can be used to study systems at strong coupling.

2.8.3 Basic dictionary of AdS/CFT

As mentioned before, the scale $U(r)$ in (2.34) has dimensions of energy and was defined for an asymptotic observer. Now the dual field theory has been defined to live at the boundary[2] of AdS, this scale, or, actually (cf. (2.33)),

$$u(r) \equiv \frac{1}{\sqrt{\alpha'}} \left(1 + \frac{R^4}{r^4}\right)^{-1/4},$$

can be interpreted as the energy scale in field theory, and, therefore, different regions of AdS space correspond to physics at different energy scales in the field theory.

The 't Hooft parameter can be related to the radius R of AdS_5 and S^5 by using (2.24),

$$\lambda = g_{\mathrm{YM}}^2 N = 4\pi g_s N = \frac{R^4}{\alpha'^2}.$$

The mappings in the AdS/CFT correspondence which have been made hitherto are listed in table 2.1.

Super Yang-Mills (CFT) side		Supergravity (bulk) side	
λ	't Hooft coupling	R	radius of AdS_5 and S^5
g_{YM}	Yang-Mills coupling	g_s	string coupling
N	rank of gauge group (number of colors)	N	R-R flux through S^5
u(r)	energy scale	r	radial direction

Table 2.1: Mappings in the AdS/CFT correspondence.

Since a conformal field theory is scale invariant, and therefore does not have an S-matrix, the natural objects to consider are operators and their correlation functions. In Euclidean

[2] Note that even though this terminology is common, it is dangerous. The field theory actually describes all the physics that is going on inside AdS. From the AdS point of view, it is incorrect to consider the field theory as an additional theory that lives at the same time at the boundary [14].

formulation, the *generating functional* for an operator $\mathcal{O}_\Delta$ of conformal dimension Δ is given by

$$\mathcal{Z}_{\rm CFT}[\phi_{(0)}] = \langle e^{\int d^4x\, \phi_{(0)}(x)\mathcal{O}_\Delta(x)}\rangle_{\rm CFT},$$

so that an n-point function can be obtained by

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle = \frac{\delta}{\delta\phi_{(0)}(x_1)}\frac{\delta}{\delta\phi_{(0)}(x_2)}\cdots\frac{\delta}{\delta\phi_{(0)}(x_n)} W_{\rm CFT}[\phi_{(0)}]\big|_{\phi_{(0)}=0}, \tag{2.35}$$

where $W_{\rm CFT} = -\ln \mathcal{Z}_{\rm CFT}$ has been used to cancel the factor $\mathcal{Z}_{\rm CFT}[0]$.

A precise correspondence between correlators on the Yang-Mills side and the supergravity side was proposed by Witten [35]. This was done for the Euclidean version of AdS_5, with the Yang-Mills living on $\mathbb{R}^4$. In the next chapter, some of the subtleties involving the Lorentzian version will be discussed. The correlators are obtained by the prescription

$$\mathcal{Z}_{\rm string}[\phi]\big|_{\phi(x,z)|_{z=0}=\phi_{(0)}(x)} = \mathcal{Z}_{\rm CFT}[\phi_{(0)}] = \langle e^{\int d^4x\, \phi_{(0)}(x)\mathcal{O}_\Delta(x)}\rangle_{\rm CFT}, \tag{2.36}$$

where $\phi_{(0)}(x)$ is the supergravity field ϕ which is restricted to the *boundary of AdS*, $\phi_{(0)}(x) = \phi(x,z)|_{z=0}$. In table 2.2, the duals for various fields are listed for easy reference.

In the classical supergravity limit, the partition function $\mathcal{Z}_{\rm string}[\phi]$ becomes

$$\mathcal{Z}_{\rm string}[\phi] = e^{-S_{\rm SUGRA}[\phi]},$$

so that (2.36) gives

$$\langle e^{\int d^4x\, \phi_{(0)}(x)\mathcal{O}_\Delta(x)}\rangle_{\rm CFT} = e^{-S_{\rm SUGRA}[\phi]}\big|_{\phi(x,z)|_{(z=0)}=\phi_{(0)}(x)}. \tag{2.37}$$

Here, $S_{\rm SUGRA}[\phi]\big|_{\phi(x,z)|_{(z=0)}=\phi_{(0)}(x)}$ is the supergravity action evaluated on a solution to the equations of motion subject to the boundary condition $\phi(x,z)|_{(z=0)} = \phi_{(0)}(x)$. By differentiating with respect to $\phi_{(0)}$, as in (2.35), n-point functions can be obtained. The interactions in the bulk can be calculated using Feynman diagrams whose external legs correspond to the boundary values $\phi_{(0)}$.

Super Yang-Mills (CFT) side		Supergravity (bulk) side		Section
$T_{\mu\nu}$	energy momentum tensor	g_{MN}	graviton	3.1.1
J_μ	global current	A_M	Maxwell field	3.3.2
	global symmetry		gauged symmetry	3.3.2
$\mathcal{O}_B$	scalar operator	ϕ	scalar field	3.4.2
$\mathcal{O}_F$	fermionic operator	ψ	fermionic field	3.4.7

Table 2.2: Basic dictionary of the AdS/CFT correspondence with section numbers where the dualities are covered. Indices M, N, denote $d+1$ bulk spacetime indices.

The mass m of a scalar field ϕ living in the bulk is related to the conformal dimension Δ of the operator $\mathcal{O}_B$ on the boundary CFT by [35]

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2m^2}.$$

Similarly, the relation between the conformal dimension of a fermionic operator $\mathcal{O}_F$ and the mass m of a Dirac fermion ψ is given by

$$\Delta = \frac{d}{2} + Rm.$$

Relations for various other fields can be found in section 3.3.1 of [14] and references therein.

2.8.4 Support for the conjecture

A strong piece of evidence for the duality comes from the fact that the symmetry groups on both sides match. Both type-IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ super Yang-Mills theory have the following symmetries [14]:

1. The 32 supersymmetries of the superconformal group, which are left unbroken by the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry;
2. The $\mathrm{SO}(2,4)$ conformal group, corresponding to the isometries of AdS_5;
3. The $\mathrm{SU}(4)$ R-symmetry group, corresponding to the isometries of S_5, i.e., $\mathrm{SO}(6)$;
4. The $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Oliven group, corresponding to the S-duality group of type-IIB string theory.

In fact, the AdS/CFT correspondence is a precise implementation of what is called holography, to which have been alluded to in the introduction of this chapter. As was mentioned there, holography was useful for understanding the Bekenstein bound.

The Bekenstein bound can be understood as follows. The entropy of a black hole is proportional to the area of its horizon, $S_{\mathrm{BH}} = \frac{A}{4G_N}$. Suppose now, that in a given volume V with boundary A, there is a configuration with entropy S, which is larger than the entropy of the largest possible black hole that fits into V, $S > S_{\mathrm{BH}}$. Suppose further that this configuration has less energy than the black hole. Then, by throwing in an object with high entropy, which necessarily carries energy, one can make a black hole. However, a black hole carries less entropy, and therefore the second law of thermodynamics has been violated. This means, that in a theory including gravity, a black hole is the most entropic configuration possible within a given volume.

The entropy of a QFT on the same space is however much larger than the area of its boundary. This is the issue which is resolved by holography. The number of degrees of freedom for a theory including gravity must scale like that of a QFT in a smaller number of dimensions.

Another piece of evidence for the conjecture is that a large-N gauge theory is equivalent to a string theory, shown by 't Hooft [36]. This will not be discussed in this thesis, but an accessible discussion can be found in section 1.2 of [14]. Furthermore, additional evidence for the duality is discussed in section 3.2 of that article.

3
Applied AdS/CFT

After a somewhat technical and succinct introduction to the AdS/CFT correspondence, in this chapter the duality of section 2.8 is going to be applied to various systems. The material of this chapter has mainly been taken from the lecture notes on this subject of Hartnoll [37] and McGreevy [38].

In section 2.8.3, it was stated that the AdS/CFT correspondence relates an operator $\mathcal{O}$ in a conformal field theory to the boundary value of a dynamical field ϕ living in the bulk gravitational theory, through the relation (2.36). Witten proposed this relation only for the Euclidean version. However, under certain conditions it is possible to use the *Lorentzian* version,

$$\langle e^{i\int d^4x\,\sqrt{-g_{(0)}}\,\phi_{(0)}(x)\mathcal{O}_\Delta(x)}\rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}}[\phi]\big|_{\phi(x,z)|_{z=0}=\phi_{(0)}(x)}, \tag{3.1}$$

where also a static background metric $g_{(0)\mu\nu}$ for the field theory has been introduced. The conditions will be discussed in section 3.4.3. In the limit where the gravity theory becomes classical, the path integral of the right hand side can be done by a *saddle point* approximation, where the string partition function is expressed as the classical action, evaluated on a solution of the equations of motion with boundary condition $\phi(x,z)|_{z=0} = \phi_{(0)}(x)$ (cf. (2.37)). The CFT side of this equality can be interpreted as the perturbation of the scale-invariant field theory action by adding an operator coupled to some source. This viewpoint will be elaborated in section 3.4.4.

The notation used throughout this chapter is as follows. Bulk spacetime coordinates are denoted with indices M, N, while the coordinates of the CFT on the boundary have indices μ, ν. For the sake of flexibility, the field theory side lives in d dimensions, unless stated otherwise.

As a warm-up, in the first section the dual of a simple system is considered, namely a system where the bulk only includes the graviton field.

3.1 Simple system

The simplest possible action to consider in the bulk is the *Einstein-Hilbert* action with a negative cosmological constant

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int d^{d+1}x\,\sqrt{-g}\left(\mathcal{R} + \frac{d(d-1)}{R^2}\right), \tag{3.2}$$

where $\mathcal{R}$ is the Ricci scalar, R a length scale, and κ the gravitational constant which was mentioned in section 2.5.2, $2\kappa^2 = g_s^{-2}16\pi G_{d+1}$. The equation of motion that describes the dynamics of the bulk field g_{MN} can be obtained by varying this action with respect to the metric,

$$\mathcal{R}_{MN} + \frac{d}{R^2} g_{MN} = 0. \tag{3.3}$$

This equation of motion is just Einstein's equation with negative cosmological constant and zero energy-momentum tensor. The most symmetric solution to these equations of motion is given by AdS_{d+1} spacetime,

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^i dx^i + dz^2\right). \tag{3.4}$$

Here z is the additional coordinate the gravitational bulk has with respect to the boundary CFT (cf. (2.27)), and $i = 1, \ldots, d-1$. From this solution, it is clear that the length scale R corresponds to the radius of curvature of the AdS spacetime.

The full symmetry group of AdS_{d+1} is given by the group $\mathrm{SO}(2, d)$, which equals the conformal group in d dimensions, as required for the validity of the AdS/CFT correspondence. In particular, the scale invariance (or dilatation) symmetry of the CFT acts on the spacetime as

$$\{t, x^i, z\} \to \{\lambda t, \lambda x^i, \lambda z\}, \tag{3.5}$$

with $i = 1, \ldots, d-1$, under which (3.4) is obviously invariant.

3.1.1 Field theory dual of the metric

The linearized bulk field g_{MN} should be dual to an operator $\mathcal{O}$ in the field theory. The metric g_{MN} is present in all classical theories of gravity, so one expect that the dual operator should be present in all dual field theories. Furthermore, one expects that the dual operator has spin two, like the graviton. The natural guess, then, for the dual operator is the energy-momentum tensor $T_{\mu\nu}$ of the field theory.

Another way to see this is as follows. When restricted to the boundary, the metric $g_{MN}(z)$ of some asymptotically AdS space tends to a certain value $g_{(0)\mu\nu}$,

$$g_{\mu\nu}(z) = \frac{R^2}{z^2} g_{(0)\mu\nu} + \cdots \quad \text{as} \quad z \to 0, \tag{3.6}$$

where the metric has been pulled back to the boundary to eliminate the g_{zz} component. In particular, for the metric (3.4)

$$ds^2 = \frac{R^2}{z^2} g_{(0)\mu\nu} dx^\mu dx^\nu + R^2 \frac{dz^2}{z^2},$$

where $g_{(0)\mu\nu} = \eta_{\mu\nu}$. The metric $g_{(0)\mu\nu}$ can be regarded as the background metric of the field theory, and since the z dependence has been factored out, this metric is non-dynamical. Note that it is the combination $\frac{R^2}{z^2}$ that has been factored out, so that the boundary metric is dimensionful. Furthermore, it is not problematic for the field theory to have a non-flat metric, as long as it is not dynamical.

As suggested by the relation (3.1), the metric $g_{(0)\mu\nu}$ should be a source for an operator in the field theory. The object that couples to the metric $g_{(0)\mu\nu}$ in the field theory is the energy-momentum tensor $T^{\mu\nu}$, since in that context it is defined as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g_{(0)}}} \frac{\delta S}{\delta g_{(0)\mu\nu}}. \tag{3.7}$$

So, the field theory dual of the bulk metric is given by the energy-momentum tensor, i.e.,

$$\boxed{T_{\mu\nu} \longleftrightarrow g_{MN}},$$

where $\leftrightarrow$ means 'dual to'. A possible point of confusion is that in a classical gravity system one has

$$S = S_{\text{EH}} + S_{\text{matter}},$$

such that

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2\kappa^2}\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}.$$

If now -2 times the second term is identified with the energy-momentum tensor, analog to (3.7), one obtains Einstein's equation. In the field theory however, there is no Einstein-Hilbert part in the action, and we can write (3.7) with the full action.

Note that the AdS solution (3.4) corresponds to the vacuum of a CFT. This can be seen by differentiating relation (3.1) with respect to $g_{(0)\mu\nu}$ to obtain $\langle T^{\mu\nu}\rangle$ in the field theory, while the 'string-theory' side vanishes on-shell.

3.1.2 Counter terms

A field ϕ is said to be *normalizable*, if the action in Euclidean spacetime is finite, $S[\phi] < \infty$, since then it has a finite contribution to the partition function $Z[\phi] = \sum_\phi e^{-S[\phi]}$. Modes with boundary conditions which force $S[\phi] = \infty$ would not contribute.

To isolate possible divergences on the boundary $z \to 0$, one can restrict the range of integration of the variable z to $z \geq \epsilon$, and evaluate boundary terms at $z = \epsilon$. This action is called the *regulated* action S_{reg}. After subtracting terms which diverge as $\epsilon \to 0$, one take the limit $\epsilon \to 0$:

$$S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{sub}} \equiv \lim_{\epsilon\to 0}(S_{\text{reg}} + S_{\text{ct}}), \tag{3.8}$$

where S_{sub} is the *subtracted* action, S_{ren} is the *renormalized* action, and S_{ct} are (minus) the divergent terms of S_{reg}. See [39] for a more detailed description of this technique in the context of AdS/CFT.

Because AdS spacetime has a boundary, actually two boundary terms must be added to the action (3.2). The first counter term is a constant (intrinsic) boundary counter term, which must be added to (3.2) in order to obtain a finite action [40],

$$S_{\text{ct}} = -\frac{1}{2\kappa^2}\int_{z=\epsilon} d^dx\,\sqrt{-\gamma}\,\frac{2(d-1)}{R}.$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary $z = \epsilon$:

$$ds^2\big|_{z=\epsilon} \equiv \gamma_{\mu\nu}dx^\mu dx^\nu = \frac{R^2}{\epsilon^2}g_{(0)\mu\nu}dx^\mu dx^\nu, \tag{3.9}$$

with $g_{(0)\mu\nu} = \eta_{\mu\nu}$ in case of AdS spacetime.

The second counter term is a so-called *Gibbons-Hawking-York* term,

$$S_{\text{GHY}} = -\frac{1}{2\kappa^2}\int_{z=\epsilon} d^dx\,\sqrt{-\gamma}\,(2K)\,,$$

where K is the trace over the extrinsic curvature of the boundary, $K = \gamma^{\mu\nu}\nabla_\mu n_\nu$, with n_ν an outward-pointing unit normal to the boundary. This term affects which boundary conditions are imposed on the metric. Without this term, integration by parts of the Einstein-Hilbert term to obtain the equations of motion, produces a boundary term proportional to variations of *derivations* of the metric, which is incompatible with imposing a Dirichlet boundary condition on the metric, i.e., specifying $g_{(0)\mu\nu}$ [41].

In the next section, it will be shown how a finite temperature in the field theory can be obtained.

3.2 Nonzero temperature

Since any field theory can be placed at finite temperature, it should not be necessary to add extra ingredients to the bulk in order to get a finite temperature. Therefore, the action (3.2) should still be valid as a gravity dual for a system at finite temperature.

3.2.1 Breaking scale invariance

Placing the field theory at finite temperature, would break the scale invariance. As it was mentioned in the previous section, the scale invariance symmetry of the field theory corresponds to the scale invariance of AdS spacetime. It is possible to break the scale invariance of the bulk spacetime, since the bulk spacetime only needs to be asymptotically AdS, as was mentioned in section 2.8 and Appendix B.

If the scaling symmetry of the bulk solution (3.4) is relaxed, while invariance under spatial rotations and spacetime translations is maintained, the metric will look like

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dx^i dx^i + h(z)dz^2\right), \tag{3.10}$$

where the functions $f(z)$, $g(z)$ and $h(z)$ are to be determined, and $i = 1, \dots, d-1$. Note that Lorentz invariance is also broken when $f \neq g$, which is to be expected for a system at finite temperature.

There is a certain gauge freedom, which enables free choice of one of the functions by a rescaling $z \to \hat{z}(z)$, and we can exploit this freedom to set $g(\hat{z}) = 1$. If we plug this ansatz for the metric into the equation of motion (3.3), and solve for f and h, we find the *Schwarzschild-AdS* black hole solution

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + dx^i dx^i + \frac{dz^2}{f(z)}\right), \tag{3.11}$$

with

$$f(z) = 1 - \left(\frac{z}{z_H}\right)^d,$$

which is sometimes called *emblackening factor*. This solution is called black hole, since there is a horizon for $z = z_H$, because $g_{tt} \to 0$ for $z \to z_H$. That means that light emitted to an asymptotic observer becomes infinitely redshifted, as can be seen from the relation $E = \sqrt{-g_{tt}}E_{z_H}$ (cf. (2.33)), with E the energy measured by the asymptotic observer, and E_{z_H} the energy at the horizon. Note that the horizon is planar, since it is of the form $\mathbb{R}^{d-1}$.

On the boundary this solution reduces to AdS, since $f(z) \to 1$ as $z \to 0$. Hence the solution is asymptotically AdS as required.

3.2.2 Analytic continuation of the solution

In its original form, the Witten prescription (2.36) relates the Euclidean version of AdS with Yang Mills on a Euclidean spacetime. Using this relation, the field theory partition function can be found by evaluating the partition function of the bulk theory on the (Euclidean) saddle $g_\star$,

$$\mathcal{Z}_E = e^{-S_{\mathrm{E}}[g_\star]}. \tag{3.12}$$

By analytic continuation of the Schwarzschild-AdS black hole (3.11) to Euclidean signature, i.e., by setting $\tau = it$,

$$ds_\star^2 = \frac{R^2}{z^2}\left(f(z)d\tau^2 + dx^i dx^i + \frac{dz^2}{f(z)}\right), \quad \text{with} \quad f(z) = 1 - \left(\frac{z}{z_H}\right)^d, \tag{3.13}$$

such a saddle is obtained, as can be checked by substituting this solution into the equation of motion (3.3).

Due to its Euclidean signature, the metric (3.13) does not make sense inside the horizon, since $f(z)$ is negative for $z > z_H$, making its signature non-Euclidean. The near-horizon ($z \sim z_H$) expansion of $f(z)$ is given by

$$f(z) \simeq f(z_H) + (z - z_H)\frac{df(z)}{dz}\Big|_{z=z_H} + O(z-z_H)^2 = |f'(z_H)|(z_H - z) + O(z-z_H)^2,$$

which is written in this way, because $z_H > z$. Substituting this expansion into the metric (3.13) yields

$$ds^2_{\star,\mathrm{N.H.}} \simeq \frac{R^2}{z_H^2}\left(|f'(z_H)|(z_H - z)d\tau^2 + dx^i dx^i + \frac{dz^2}{|f'(z_H)|(z_H - z)}\right). \tag{3.14}$$

By substituting further $\rho^2 = \frac{R^2}{z_H^2}\kappa(z_H - z)$, with $\kappa \equiv \frac{4}{|f'(z_H)|} = \frac{d}{2z_H}$, and $\phi = \frac{2}{\kappa}\tau = \frac{|f'(z_H)|}{2}\tau$, the near-horizon metric becomes

$$ds^2_{\star,\mathrm{N.H.}} \simeq d\rho^2 + \rho^2 d\phi^2 + \frac{R^2}{z_H^2}dx^i dx^i, \tag{3.15}$$

which looks like $\mathbb{R}^{d-1}$ times a Euclidean plane in polar coordinates $\{\rho, \phi\}$.

In order the metric (3.13) to be regular, ϕ must be periodic. However, for a general period $\phi \in [0, 2\pi - \Delta]$, with $\Delta \neq 0$, the $\{\rho, \phi\}$ part of the metric (3.15) describes a cone with a singularity at $\rho = 0$. This conical singularity is absent when ϕ has period 2π. So,

$$\phi \sim \phi + 2\pi \Longleftrightarrow \tau \sim \tau + \frac{4\pi}{|f'(z_H)|}, \tag{3.16}$$

with $|f'(z_H)| = \frac{d}{z_H}$. The necessity of the periodicity in ϕ can also be seen when one transforms the $\{\rho, \phi\}$ part of the metric (3.15) into flat Euclidean spacetime by the substitution $T = \rho \sin\phi$ and $X = \rho\cos\phi$, so that $ds_\star^2 \simeq dT^2 + dX^2 + \frac{R^2}{z_H^2}dx^i dx^i$.

3.2.3 Appearance of temperature

The periodic identification of the Euclidean time in the field theory corresponds to placing the theory at a finite temperature. This can be understood as follows. In quantum mechanics, the transition amplitude between two states $|q_i, t_i\rangle$ and $|q_f, t_f\rangle$ is given by

$$\langle q_f, t_f | q_i, t_i\rangle = \langle q_f | e^{-iH(t_f - t_i)} | q_i\rangle,$$

which also can be written as a path integral,

$$\langle q_f, t_f | q_i, t_i\rangle = \int_{\substack{q(t_f)=q_f \\ q(t_i)=q_i}} [\mathcal{D}q(t)]\, e^{iS[q(t)]}.$$

After performing a Wick rotation by setting $\tau = it$, and defining $\beta = i(t_f - t_i) = \tau_f - \tau_i$, the combination of the two preceding formulas has the form

$$\langle q_f, \tau_f | q_i, \tau_i\rangle = \langle q_f | e^{-\beta H} | q_i\rangle = \int_{\substack{q(\tau_f)=q_f \\ q(\tau_i)=q_i}} [\mathcal{D}q(\tau)]\, e^{-S_E[q(\tau)]}.$$

In statistical mechanics, the partition function of a system at temperature $T = 1/\beta$ is given by

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta H},$$

where the trace runs over a complete set of states. So, for a closed path $q_f = q_i$ we have

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta H} = \langle q | e^{-\beta H} | q\rangle = \int_{q(\tau_f)=q(\tau_i)=q} [\mathcal{D}q(\tau)]\, e^{-S_E[q(\tau)]},$$

and the generalization to quantum field theory is straightforward:

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta H} = \int_{\phi(x,\tau_f)=\phi(x,\tau_i)=\phi(x,\tau)} [\mathcal{D}\phi]\, e^{-S_E[\phi]}.$$

In the path integral the Euclidean time has period β, since $\phi(x, \tau_f) = \phi(x, \tau_i + \beta) = \phi(x, \tau_i)$ are identified. Therefore, the Euclidean path integral with periodically identified time equals a partition function at finite temperature T. From (3.16) the relation between the temperature and the parameter z_H can be read off,

$$T = \frac{1}{\beta} = \frac{|f'(z_H)|}{4\pi} = \frac{d}{4\pi z_H}. \tag{3.17}$$

Due to the scale invariance of the CFT, there are only two temperatures possible: zero and nonzero. This is because the parameter z_H can be eliminated from the metric (3.11) by the scaling (3.5) with $\lambda = z_H$. From the partition function (3.12) standard thermodynamic quantities can be calculated, e.g. the free energy $F = -T \log \mathcal{Z}_E = TS_E[g_\star]$, and the entropy $S = -\partial F/\partial T$. The results are listed in [37, 38].

In the coming two sections, structure is going to be added to the bulk theory.

3.3 Bulk Maxwell field and chemical potential

In the previous section, the most universal deformation away from scale invariance was discussed by placing the field theory at a finite temperature. Another important deformation away from scale invariance is to place the system at a finite chemical potential. In this section, it will be shown, that this is achieved by adding a Maxwell field to the bulk.

3.3.1 Adding a Maxwell field to the bulk

Adding a bulk Maxwell field to the Einstein-Hilbert action, gives the *Einstein-Maxwell* action:

$$S_{\text{EM}} = \int d^{d+1}x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(\mathcal{R} + \frac{d(d-1)}{R^2} \right) - \frac{1}{4g_F^2} F^2 \right],$$

where $F = \mathrm{d}A$ is the electromagnetic field strength, and g_F the Maxwell coupling. The equation of motion for the bulk field g_{MN} is given by

$$\mathcal{R}_{MN} - \frac{R}{2} g_{MN} - \frac{d(d-1)}{2R^2} g_{MN} = \frac{\kappa^2}{2g_F^2} \left(2F_{MP}F_N{}^P - \frac{1}{2} g_{MN} F_{PQ} F^{PQ} \right),$$

and for the Maxwell field A it is given by

$$\nabla_M F^{MN} = 0,$$

where the covariant derivative appears due to the curvature of the background.

To solve these equations of motion, we make an ansatz for a Maxwell field with vanishing magnetic field,

$$A = A_t(z)\mathrm{d}t. \tag{3.18}$$

It turns out that there is a unique solution to the above equations of motion of this form and the form (3.10). This solution describes a black hole carrying an electric charge, called a *Reissner-Nordström-AdS* black hole [42, 43],

$$ds^2 = \frac{R^2}{z^2} \left(-f(z)dt^2 + dx^i dx^i + \frac{dz^2}{f(z)} \right).$$

It has the same form as the Schwarzschild-AdS black hole (3.11), but with a different emblackening factor,

$$f(z) = 1 - \left(1 + \frac{z_H^2 \mu^2}{\gamma^2} \right) \left(\frac{z}{z_H} \right)^d + \frac{z_H^2 \mu^2}{\gamma^2} \left(\frac{z}{z_H} \right)^{2(d-1)}, \tag{3.19}$$

where

$$\gamma^2 = \frac{(d-1)g_F^2 R^2}{(d-2)\kappa^2},$$

which is a dimensionless measure of the relative strengths of the gravitational and Maxwell forces. The nonzero Maxwell gauge potential in (3.18) is given by

$$A_t(z) = \mu \left[1 - \left(\frac{z}{z_H} \right)^{d-2} \right]. \tag{3.20}$$

The constant term is necessary, otherwise the one form A will not be well defined at the horizon. As in the case of the Schwarzschild-AdS black hole, it is necessary to periodically identify the Euclidean time. When an integral around a closed time loop of the one form will be nonzero on the boundary, it will imply that A is singular, as the time circle shrinks to zero on the boundary. Therefore, the time component A_t must vanish on the boundary.

Note that for $d \geq 3$, it is not necessary to add a boundary term to the Einstein-Maxwell action for the Maxwell field. For the above solution of the Maxwell potential (3.20), the only nonvanishing components of the Maxwell field strength are $F_{zt} = \partial_z A_t = -(d-2)\mu z_H^{2-d} z^{d-3} = -F_{tz}$, which for $d \geq 3$ go to zero or a constant near the boundary $z \to 0$.

3.3.2 Field theory dual of a bulk Maxwell field

As with the metric in (3.6), a bulk Maxwell potential tends to a certain value $A_{(0)\mu}$ when restricted to the boundary,

$$A_\mu(z) = A_{(0)\mu} + \cdots \quad \text{as} \quad z \to 0,$$

where the potential has been pulled back to the boundary to eliminate the A_z component. From (3.20), it is clear that $A_{(0)t}$ is given by μ, which is interpreted as the chemical potential; this will be explained shortly. The other components are zero, because there is no magnetic field in the ansatz (3.18).

The relation (3.1) suggests that the boundary value of the bulk Maxwell field should be a source for an operator in the field theory. In [37], it is motivated that the gauged U(1) symmetry of the bulk corresponds to a *global* U(1) symmetry on the boundary. Associated with this global symmetry there is a conserved current, which in this case is the electromagnetic current J^μ. It is this current to which the boundary background field couples to. So, the bulk U(1) gauge field A_M is dual to a global current J_μ in the field theory,

$$\boxed{J_\mu \longleftrightarrow A_M}\,.$$

From this relation, it is clear that the time component of the background field $A_{(0)t}$, couples to the time component of the current J_0.

The correspondence between a local U(1) symmetry in the bulk and a global U(1) symmetry on the boundary, is part of a more general duality between a gauge group in the bulk and a global group in the field theory. This is motivated by the fact that gauge symmetries include the subgroup of so-called *large* gauge symmetries, which are symmetries which act non-trivially as global symmetries on the boundary of spacetime. This global subgroup of the bulk symmetry group, is in the AdS/CFT correspondence identified with the global symmetry group of the field theory on the boundary.

3.3.3 Chemical potential

The interpretation of the time component of the boundary gauge potential $A_{(0)t} = \mu$ as chemical potential needs some explanation.

In the grand canonical ensemble, the partition function is given by

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta(H-\mu N)},$$

where N is the number operator, which gives the total number of particles in the system, and $\beta = 1/T$.

For a Lagrangian (density) of the form

$$\mathcal{L} = -\frac{1}{4g_F^2}F^2 + A_\mu J^\mu,$$

the corresponding Hamiltonian with the potential (3.18) reads

$$\mathcal{H} = \frac{\delta\mathcal{L}}{\delta(\partial_t A_\mu)}\partial_t A_\mu - \mathcal{L} = F_{t\mu}\partial_t A_\mu + \frac{1}{4g_F^2}F^2 - A_\mu J^\mu = \frac{1}{4g_F^2}F^2 - A_\mu J^\mu.$$

Therefore, the change in the Hamiltonian due to a variation of the potential, $A \to A + \mathrm{d}\Lambda$, is given by

$$\delta\mathcal{H} = -\partial_\mu \Lambda J^\mu,$$

so that a shift $A_{(0)t} = \mu$, that is, $\Lambda = \mu t$, gives

$$\delta H = -\int d^d x\, \delta A_{(0)t} J^t = -\mu \int d^d x\, J^t. \tag{3.21}$$

3.3.4 Temperature

The temperature for the dual field theory can be determined in the same way as was done for the Schwarzschild-AdS black hole in section 3.2.2. The periodicity of the Euclidean time is again given by (3.16), but now $f(z)$ is given by (3.19), so that

$$T = \frac{1}{4\pi z_H}\left(d - \frac{(d-2)z_H^2\mu^2}{\gamma^2}\right).$$

There are two scales in this solution: the chemical potential μ and the radius of the horizon z_H. As was shown in section 3.2.3, it was possible to eliminate the z_H dependence in the case of zero chemical potential. Using the same scaling, together with a rescaling of $\mu \to z_H\mu$, z_H can again be scaled out (cf. (3.19)). However, the scale μ remains. Since the field theory is scale invariant, the only non-trivial dependence on chemical potential and temperature can be in the dimensionless ratio T/μ. As opposed to the case of zero chemical potential, where there was only zero or nonzero temperature, the ratio T/μ can be varied continuously.

For $d = 3$, it is also possible to consider the ansatz for the vector potential with a nonzero magnetic field. In this case, the solution turns out to be a *dyonic* Reissner-Nordström-AdS$_4$ solution [37].

In the next section, the case of a bulk scalar field is going to be considered.

3.4 Bulk scalar field

In the previous two sections, the scale invariance of the field theory was broken by placing the theory at finite temperature and nonzero chemical potential. It was shown that in the bulk this amounts to introduce a Schwarzschild-AdS black hole and a Maxwell field, respectively. Scale invariance of the bulk spacetime can also be broken by adding a relevant operator to the field theory, which will be shown in this section.

3.4.1 Adding a scalar field to the bulk

Adding a bulk scalar field to the Einstein-Hilbert action yields the *Einstein-scalar* action

$$S = \int d^{d+1}x \sqrt{-g}\left[\frac{1}{2\kappa^2}\left(\mathcal{R} + \frac{d(d-1)}{R^2}\right) - \frac{1}{2}\left(\nabla\phi\right)^2 - V(\phi)\right], \tag{3.22}$$

where $(\nabla\phi)^2 = g^{MN}\partial_M\phi\partial_N\phi$, and $V(\phi)$ a potential. The scalar field has to fall of sufficient quick near the boundary, which means that it has to go to zero or a constant field. If not, the metric on the boundary will *backreact*, what means that the metric will change due to the matter term in this action, and the spacetime will no longer be asymptotically AdS.

One can proceed as in the previous sections by determining the equations of motion from this action and solve them. By adding a scalar field to the bulk, one expect that the solution for the metric retains Lorentz invariance, while scale invariance is broken. In (3.10) Lorentz invariance in d dimensions is retained for $f = g$. Moreover, one can use gauge invariance to set $h(\hat{z}) = 1$, so that a suitable ansatz for the metric would be

$$ds^2 = \frac{R^2}{z^2}\Big(f(z)\left(-dt^2 + dx^i dx^i\right) + dz^2\Big). \tag{3.23}$$

As the metric should be asymptotically AdS, $f(z) \to 1$ near the boundary $z \to 0$. Solutions for the graviton will not be shown in this thesis, but a calculation can be found in [45].

The equation of motion for the scalar field has the form of the Klein-Gordon equation,

$$\Box_g \phi - \frac{\partial V(\phi)}{\partial \phi} = 0, \quad \text{with} \quad \Box_g \equiv \frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\right). \tag{3.24}$$

Near the boundary, where the scalar field falls off fast, the potential has the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \cdots \quad \text{as} \quad z \to 0, \tag{3.25}$$

with m the mass of the field. From (3.23) follows that near the boundary

$$g_{MN} = \frac{R^2}{z^2}\eta_{MN}, \quad \text{and} \quad \sqrt{-g} = \left(\frac{R}{z}\right)^{d+1},$$

so that one gets after some manipulations (again, near the boundary)

$$\Box_g = \frac{1}{R^2}\Big(-(d-1)z\partial_z + z^2\partial_z^2 + z^2\partial_\mu\partial^\mu\Big). \tag{3.26}$$

The relation (3.1) again suggests that the boundary value of the scalar field should be a source for an operator in the field theory. The bulk scalar field restricted to the boundary, $\phi_{(0)}$, couples naturally to a scalar operator. Therefore, the dual operator should be a Lorentz scalar operator $\mathcal{O}_B$, i.e.,

$$\boxed{\mathcal{O}_B \longleftrightarrow \phi}.$$

Using the equation of motion (3.24), the conformal (or scaling) dimension of this dual operator $\mathcal{O}_B$ in the field theory can be determined.

3.4.2 Conformal dimension of the bosonic operator

Focussing on the z dependence of ϕ only, $\phi = \phi(z)$, the equation of motion (3.24) becomes with (3.26)

$$z^2\partial_z^2\phi - (d-1)z\partial_z\phi = (Rm)^2\phi.$$

When we insert the ansatz

$$\phi(z) = \left(\frac{z}{R}\right)^{d-\Delta}\phi_{(0)} + \cdots \quad \text{as} \quad z \to 0, \tag{3.27}$$

for the near-boundary behavior of the scalar field into this equation, then

$$(Rm)^2 = \Delta(\Delta - d).$$

This equation has two solutions:

$$\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2m^2} \quad \text{and} \quad \Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + R^2m^2} = d - \Delta_+. \tag{3.28}$$

Note that we equally well could have started with Δ instead of $d - \Delta$ in the ansatz. Writing $\Delta = \Delta_+$, so that $\Delta_- = d - \Delta$, and restoring the x dependence of the scalar field, near the boundary one has

$$\phi(z, x) = \left(\frac{z}{R}\right)^{d-\Delta} \left(\phi_{(0)}(x) + O(z^2)\right) + \left(\frac{z}{R}\right)^{\Delta} \left(\phi_{(1)}(x) + O(z^2)\right) + \cdots . \tag{3.29}$$

By writing the near-boundary behavior of the scalar field in this way, the first term is leading-order, since for $z \to 0$ it falls off slower than the second, because $d - \Delta_+ \leq \Delta_+$. The $\phi_{(0)}$ is called *source* for obvious reasons, and $\phi_{(1)}$ is called *response* for reasons explained in section 3.4.6. Since $\Delta = \Delta_+$ in (3.29) is larger than zero for any value of the mass m, the second term vanishes on the boundary of AdS.

Since the field $\phi(z, x)$ is a scalar field, it should be invariant under the scaling (3.5), $\{t, x^i, z\} \to \lambda\{t, x^i, z\}$. Therefore, under this scaling the boundary value of the bulk scalar field should transform as $\phi_{(0)}(x) \to \phi_{(0)}(\lambda x) = \lambda^{\Delta - d}\phi_{(0)}(x)$. So the scaling dimension of $\phi_{(0)}(x)$ is $d - \Delta$. The combination $\int d^dx \sqrt{-g_{(0)}}\, \phi_{(0)}(x)\mathcal{O}_B(x)$ in (3.1) should be dimensionless in order to preserve Lorentz invariance, and as a result, the scaling dimension of $\mathcal{O}_B$ should be Δ:

$$\dim[\mathcal{O}_B] = \Delta.$$

In order for the boundary conformal field theory to retain unitary, the conformal dimension Δ of the operator $\mathcal{O}_B$ must obey a so-called *unitarity bound* [46], which in this case is

$$\Delta_\pm \geq \frac{d-2}{2}. \tag{3.30}$$

3.4.3 Boundary conditions for the scalar field

AdS spacetimes often possess a horizon. It was shown explicitly in section 3.2, this is the case when the field theory has a finite temperature. However, the g_{tt} component also goes to zero in case of the pure AdS spacetime (3.4) as $z \to \infty$, which is the so-called *Poincaré horizon*, because (3.4) only describes the Poincaré patch of global AdS spacetime. Note that the pure AdS metric (3.4) can be regarded as the zero-temperature limit (i.e., $z_H \to \infty$ in (3.17)) of the Schwarzschild-AdS metric (3.11), which is called an *extremal* Schwarzschild-AdS black hole.

When the anti-de-Sitter spacetime is endowed with a horizon, in addition to the condition $\phi(x, z)|_{z=0} = \phi_{(0)}(x)$ for the boundary behavior of the bulk field, there is another boundary condition to be imposed: the bulk field should be regular on this horizon. In the Euclidean formulation, these conditions are sufficient to uniquely determine the solution. However, in the Lorentzian case one needs a more refined interior boundary condition, since there are two (oscillatory) solutions which are both regular. This can be seen as follows.

For a potential of the form (3.25), $V(\phi) = \frac{1}{2}m^2\phi^2$, and a Schwarzschild-AdS black hole metric (3.11), the scalar wave equation (3.24) looks like

$$\frac{1}{\sqrt{-g}}\partial_z\left(\sqrt{-g}g^{zz}\partial_z\phi\right) + g^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0,$$

which has to be solved for a fixed boundary condition at $z = \epsilon$. Using the Fourier representation, the solution can be written as

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x} f_k(z)\phi_{(0)}(k) \quad \text{with} \quad k_\mu \equiv (-\omega, k^i),$$

where $\phi_{(0)}(k)$ is determined by the boundary condition

$$\phi(z,x)|_{z=\epsilon} = \int \frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x}\phi_{(0)}(k),$$

and f_k is the solution to the scalar wave equation

$$\frac{1}{\sqrt{-g}}\partial_z\left(\sqrt{-g}g^{zz}\partial_z f_k\right) - (g^{\mu\nu}k_\mu k_\nu + m^2)f_k = 0, \tag{3.31}$$

with boundary condition $f_k(\epsilon) = 1$.

By using the Lorentzian version of (3.14), the $\{t,z\}$ part of the near-horizon metric of the Schwarzschild-AdS black hole (3.11) can be written as

$$\begin{aligned} ds^2_{\{t,z\},\mathrm{NH}} &= -a(z_H - z)dt^2 + \frac{b}{z_H - z}dz^2 \\ &= a(z_H - z)\left(-dt^2 + dz_\star^2\right), \qquad (z \sim z_H) \end{aligned}$$

where $z_\star \equiv \sqrt{b/a}\log(z_H - z)$, $a = (R^2/z_H^2)|f'(z_H)|$, and $b = (R^2/z_H^2)|f'(z_H)|^{-1}$. Spacetimes of this form are called *Rindler* spacetimes. With this metric, and setting the mass of the scalar field ϕ to zero for the sake of simplicity, (3.31) becomes

$$\begin{aligned} 0 &= -\frac{1}{\sqrt{ab}}\partial_z\left(\sqrt{\frac{a}{b}}(z_H - z)\partial_z\right)f_k - \frac{1}{a(z_H - z)}\omega^2 f_k \\ &= -\partial^2_{z_\star} f_k - \omega^2 f_k, \end{aligned}$$

which has solutions of the form

$$f_k \sim e^{\mp i\omega z_\star} = e^{\mp i\omega\sqrt{\frac{b}{a}}\log(z_H - z)} = e^{\mp\frac{i\omega}{4\pi T}\log(z_H - z)},$$

where in the last equality $\sqrt{b/a} = 1/|f'(z_H)|$ and (3.17) have been used. Incorporating the time dependence gives

$$e^{-i\omega t} f_k \sim e^{-i\omega(t \pm z_\star)}.$$

So, there are two linearly independent solutions. Depending on the sign in the exponent, the wavefront moves towards the horizon ('+') or away from the horizon ('−') as time advances. For we do not want the black hole to radiate and disturb the system, we impose ingoing boundary conditions ('+'). Note that ingoing boundary conditions correspond to setting boundary conditions on the future horizon, and breaks time-reversal symmetry. In the presence of a future horizon, it is possible to have dissipation, since energy passing the horizon is lost to an asymptotic observer.

One can proceed in an analogous way for the case of zero temperature, but now one has to expand the emblackening factor $f(z)$ to second order (cf. (3.14)). The $\{t,z\}$ part of the near-horizon metric looks like

$$ds^2_{\{t,z\},\mathrm{NH}} = -\frac{(z_H - z)^2}{R_2^2}dt^2 + \frac{R_2^2}{(z_H - z)^2}dz^2 = \frac{(z_H - z)^2}{R_2^2}\left(-dt^2 + dz^2_{\star 0}\right),$$

where $z_{\star 0} \equiv R_2^2(z_H - z)^{-1}$, and R_2 is the radius of the near-horizon region. Note that this spacetime is AdS_2. Note that global factors can be left out, since it is the ratio of metric components that matters in the solution. The solution to (3.31) for this metric has the form

$$e^{-i\omega t} f_k \sim e^{-i\omega(t \pm z_{\star 0})} = e^{-i\omega(t \pm R_2^2/(z_H - z))},$$

where now the ('$-$') corresponds to the ingoing choice, since $z_{\star 0}$ increases as z increases.

Although imposing ingoing boundary conditions at the horizon solves one of the problems with working in spacetimes with Lorentzian signature, there remains another problem: the retarded Green's function becomes real. The details and a workaround for this problem is discussed in [47].

3.4.4 Relevant or marginal deformation of the field theory

In section 2.8.3, it was shown that the energy scale $U(r)$ of the field theory corresponds to the extra dimension $r = R^2/z$ of the bulk AdS spacetime. This extra dimension can be regarded as the *renormalization group* (RG) scale of the field theory, where the boundary corresponds to the UV of the field theory, and the deep bulk ($z \to \infty$) corresponds to the IR. In this context, the scale invariance of the CFT is interpreted as an UV fixed point, and the theory can be deformed away from scale invariance by adding a relevant operator. The renormalization group then flows from the UV fixed point to an IR fixed point.

As mentioned in the introduction of this chapter, the CFT side of (3.1) can be interpreted as the perturbation of the scale-invariant field theory by the addition of the term

$$\int d^d x \sqrt{-g_{(0)}}\, \phi_{(0)}(x) \mathcal{O}_B(x)$$

to the CFT action. When calculating n-point functions from (3.1), instead of setting $\phi_{(0)}$ to zero after performing the variation, now one puts $\phi_{(0)}$ to a nonzero value. This perturbation generates a renormalization group flow, where one can regard the source $\phi_{(0)}$ as a coupling constant for the operator $\mathcal{O}_B$.

Keeping the boundary z at a finite value ϵ before taking the limit $\epsilon \to 0$ for regulating divergences of the action, amounts to keeping a finite UV cutoff $\Lambda = 1/\epsilon$ in the theory. Under an RG transformation towards the IR, that is, $\Lambda \to \Lambda/b$ with $b > 1$, the 'coupling constant' will transform as

$$\phi_{(0)} \to b^{d-\Delta} \phi_{(0)}.$$

Therefore, when $\Delta < d$, the deformation by the addition of the operator $\mathcal{O}_B$ will be strong in the IR and weak in the UV, a so-called *relevant* deformation. Hence, a relevant deformation will not destroy the UV critical point. For $\Delta = d$, the deformation is called *marginal*, and this does not break conformal invariance to leading order in the deformation.

The condition $d - \Delta \geq 0$ coincides with the condition for the scalar field to go to a constant or zero on the boundary, as can be seen from (3.27). Therefore, a relevant or marginal operator can be added to the field theory, without destroying the asymptotically AdS region of the metric.

When $\Delta > d$, the effect becomes stronger as the energy increases, and this is called an *irrelevant* deformation. Adding an irrelevant deformation to the theory will change the UV, and therefore the theory would require a new description for this regime. However, it makes more sense to start with a theory that has the correct UV description and then study the flow into the IR, as is commonly done. Furthermore, as can be seen from (3.29), the spacetime would no longer be asymptotically AdS, since the field ϕ does not fall off near the boundary inducing a backreaction of the metric.

Negative mass

As can be seen from (3.28), the condition for the operator $\mathcal{O}_B$ to be relevant, $\Delta < d$, is only possible for negative m^2. As long as

$$m^2 > -\left(\frac{d}{2R}\right)^2 \equiv -|m_{\mathrm{BF}}|^2,$$

the Δ_+ and Δ_- in (3.28) are real. Particles with mass that obey this bound are allowed in AdS spacetimes. These particles are called *Breitenlohner-Freedman tachyons* [48].

Usually, imaginary mass means instability of the vacuum at $\phi = 0$, where the field is at a local maximum instead of a local minimum of its potential energy, where it causes a normalizable mode to grow in time without a source. However, for $m^2 < 0$ the field $\phi(z, x)$ vanishes on the boundary, that is in the UV, since its leading-order behavior is $z^{d-\Delta}$ (cf. (3.29)). Thus, it is possible to have particles with negative mass squared in AdS spacetimes, while in flat spacetimes this is impossible.

3.4.5 Counter terms

As was mentioned in section 3.1.2, a counter term is needed in order to obtain a finite on-shell Einstein-Hilbert action, since AdS spacetime has a boundary. For the Einstein-scalar action (3.22) this is also necessary. Supplementary to the counter terms rendering the Einstein-Hilbert part finite, there are two cases for the scalar field part of the action.

By integrating by parts the scalar field part of the Einstein-scalar action (3.22), in addition to the Klein-Gordon equation (3.24), one gets the boundary term

$$S_{\mathrm{reg}} = \frac{1}{2}\int_{z=\epsilon} d^d x\, \sqrt{-\gamma}\frac{R}{\epsilon} g^{zz}\phi\partial_z\phi, \tag{3.32}$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary, which is equal to (3.9), since the space is asymptotically AdS. Further, an additional minus sign has been introduced, since the boundary is at $z = 0$, which is the lower limit of integration. The term R/ϵ comes from $\sqrt{-g}|_{z=\epsilon} = R/\epsilon\sqrt{-\gamma}|_{z=\epsilon}$ (cf. (3.9)).

When the scalar field has the form (3.27), i.e., $\Delta = \Delta_+$, which is valid for $d - \Delta < \Delta$ (cf. (3.28)), i.e., $\Delta > d/2$, the counter term canceling the boundary term (3.32) is given by

$$S_{\mathrm{ct}} = -\frac{d-\Delta}{2R}\int_{z=\epsilon} d^d x\, \sqrt{-\gamma}\phi^2. \tag{3.33}$$

For masses in the range

$$-\frac{d^2}{4R^2} < m^2 < -\frac{d^2}{4R^2} + 1,$$

Klebanov and Witten have shown that by adding a boundary term to the action, the bound $\Delta > d/2$ can be relaxed, and the source $\phi_{(0)}$ and response $\phi_{(1)}$ in (3.29) can be interchanged [49, 38], which is equivalent to setting $\Delta = \Delta_-$. Depending on the choice of the boundary condition, one has two different field theories in AdS, which correspond to two different CFTs. The boundary term has the form

$$S_{\text{bdy}} = -\int_{z=\epsilon} d^d x\, \sqrt{-\gamma}\phi\, n^\mu \partial_\mu \phi$$

The addition of this term makes us to impose Neumann boundary conditions on ϕ, instead of Dirichlet conditions. In this case, the counter term including the term canceling the divergences of the action will be

$$S_{\text{ct}} = -\int_{z=\epsilon} d^d x\, \sqrt{-\gamma}\left(\phi\, n^\mu \partial_\mu \phi + \frac{\Delta}{2R}\phi^2\right),$$

where n is the outward-pointing unit normal to the boundary. Demanding that ϕ is normalizable (see section 3.1.2), gives the bound [38]

$$\Delta \geq \frac{d-2}{2},$$

which is lower than the bound $\Delta > d/2$ of the first case, and is precisely the unitarity bound (3.30). For a systematic treatment including the range of masses which are allowed in both cases, see [50].

If there are terms between the leading and sub-leading fall offs in the asymptotic expansion (3.29), then one has to include other counter terms as well, see for example [39] for a systematic treatment.

3.4.6 Expectation value for the scalar operator

Since $\phi_{(0)}$ in (3.1) is a source, n-point functions of $\mathcal{O}_\Delta$ can be obtained by differentiating (3.1) with respect to it. In particular, the expectation value of $\mathcal{O}_\Delta$ is given by (cf. (2.35)):

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta W_{\text{CFT}}[\phi_{(0)}]}{\delta \phi_{(0)}(x)}\Big|_{\phi_{(0)}=0} = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S_{\text{ren}}[\phi_{(0)}]}{\delta \phi_{(0)}(x)}\Big|_{\phi_{(0)}=0}, \tag{3.34}$$

with $W_{\text{CFT}} = -i \log \mathcal{Z}_{\text{CFT}}$. In the second equality, a saddle point approximation has been made (cf. (2.37)), where S_{ren} is the renormalized on-shell bulk action (3.8).

To actually calculate the expectation value of an operator $\mathcal{O}_\Delta$, pertaining to a bulk field $\phi(z, x)$ that has an asymptotic expansion like (3.27), with $d - \Delta = \Delta_-$, one has to evaluate

$$\langle \mathcal{O}_\Delta(x) \rangle_s = \lim_{\epsilon \to 0} \left(\left(\frac{R}{\epsilon}\right)^{d-\Delta_-} \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{sub}}}{\delta \phi(\epsilon, x)} \right), \tag{3.35}$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary (3.9), and $S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}$ (3.8). The factor $(R/\epsilon)^{d-\Delta_-}$ comes from rewriting of the metric and the functional derivative. The subscript s denotes the presence of sources, which afterwards can be removed by setting $\phi_{(0)} = 0$. Now, the functional derivative of the subtracted action S_{sub} can be rewritten using a Hamilton-Jacobi like method.

Hamilton method

Consider the dynamics of a classical particle in one dimension that is described by the action

$$S[q] = \int_{t_i}^{t_f} dt\, \mathcal{L}\left(q(t), \dot{q}(t)\right).$$

Variation of one of the end-points of the solution to the equation of motion $\underline{q}$, e.g. the endpoint, $\underline{q}(t_f) \to \underline{q}(t_f) + \delta\underline{q}(t_f)$, will change the action by an amount

$$\delta S[q] = \int_{t_i}^{t_f} dt\, \delta\underline{q} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \delta\underline{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\bigg|_{t_i}^{t_f} = \delta\underline{q}(t_f)\Pi(t_f) \tag{3.36}$$

where the *canonical momentum* is defined as

$$\Pi(t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}(t),$$

and the term between parentheses disappeared because $\underline{q}$ is on-shell. Using (3.36), the canonical momentum can be written as

$$\Pi(t) = \frac{\delta S_{\text{on-shell}}}{\delta q(t)}.$$

It has been argued in [51] that the above can be generalized to

$$\Pi(z,x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} = -\frac{\delta S_{\text{reg, on-shell}}}{\delta \phi(z,x)},$$

where $\Pi(z,x)$ is the *bulk-field momentum* with z regarded as time, and again a minus sign has been introduced, because the boundary is at $z=0$ (cf. (3.36)). Then (3.35) becomes

$$\langle \mathcal{O}_\Delta(x)\rangle_s = \lim_{\epsilon\to 0}\left[\left(\frac{R}{\epsilon}\right)^{d-\Delta_-}\frac{1}{\sqrt{-\gamma}}\left(-\Pi(\epsilon,x) + \frac{\delta S_{\text{ct}}}{\delta\phi(\epsilon,x)}\right)\right]. \tag{3.37}$$

For the scalar field considered in this section (with action (3.22)), that has near-boundary behavior (3.29) and counter term (3.33), we find

$$\frac{1}{\sqrt{-\gamma}}\left(-\Pi(\epsilon,x) + \frac{\delta S_{\text{ct}}}{\delta\phi(\epsilon,x)}\right) = \frac{\epsilon}{R}\partial_\epsilon\phi(\epsilon,x) - \frac{d-\Delta}{R}\phi(\epsilon,x),$$

so that,

$$\langle \mathcal{O}_\Delta(x)\rangle_s = \frac{2\Delta - d}{R}\phi_{(1)}. \tag{3.38}$$

Here, we see that the sub-leading fall off of the field ϕ encodes the expectation value of the dual operator, which justifies naming $\phi_{(1)}$ the response. This formula can be applied to other types of fields as well, as long as the kinetic term in the action is brought in the same form as the kinetic term for ϕ in (3.22).

3.4.7 Bulk fermionic field

For a discussion of the addition of a Dirac fermionic field to the bulk, we refer to [52]. A detailed sample calculation can be found in Appendix A of [53]. Finally, boundary terms for spinor fields are discussed in [54].

In the coming section, the response of the boundary system to a small spacetime dependent perturbation of a bulk field is going to be considered.

3.5 Linear response theory

In the previous sections, it was shown how a system responds to the addition of a black hole in the bulk, a bulk Maxwell field, and a bulk scalar field, giving rise to a finite temperature, chemical potential, and vacuum expectation value for the operator in the field theory, respectively. All these backgrounds in the field theory were time-independent and homogeneous.

In this section, the equilibrium of the boundary field theory will be disturbed by a small space and time dependent perturbation. Because the perturbation is small, it is reasonable to assume that the response of the system is linear. This is what linear response theory deals with. First, an succinct overview of averaging over ensembles will be given; for details, see for example [55] or [56].

3.5.1 Averaging over ensembles

In quantum mechanics, the time evolution of the state vector is governed by the Schrödinger equation,

$$i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{3.39}$$

Solutions to this equation, may be represented in terms of a unitary *time-evolution operator* $U(t,t_0)$, which transforms the state $|\psi(t_0)\rangle$ at some initial time t_0, to the state $|\psi(t)\rangle$ at time t,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle. \tag{3.40}$$

Substituting this expression into the Schrödinger equation gives an operator equation for U. This equation has the solution

$$U(t,t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} dt' H(t')},$$

where $\mathcal{T}$ denotes time ordering, which is necessary because H depends on time.

The *ensemble average* of an operator $\mathcal{O}_A$ for a system in a mixed state in equilibrium is given by

$$\langle \mathcal{O}_A\rangle_p(t) = \mathrm{Tr}\left(\rho(t)\mathcal{O}_A\right), \tag{3.41}$$

where $\rho(t)$ is the density operator of the system. The density operator has a spectral decomposition in terms of a complete set of states $\{|\psi_n(t)\rangle\}$ which form an orthonormal basis of the Hilbert space,

$$\rho(t) = \sum_n p_n|\psi_n(t)\rangle\langle\psi_n(t)| = \sum_n p_n U(t,t_0)|\psi_n(t_0)\rangle\langle\psi_n(t_0)|U^{-1}(t,t_0), \tag{3.42}$$

where p_i is the probability for the system to be in state $|\psi_i(t)\rangle$, and in the last equality (3.40) has been used. Therefore, the time-dependent density matrix can be written as

$$\rho(t) = U(t,t_0)\rho_0\, U^{-1}(t,t_0),$$

with $\rho_0 \equiv \rho(t_0)$ is the equilibrium density matrix. By differentiating this equation with respect to time, one obtains the so-called *von Neumann equation*, which is an equation of motion for $\rho(t)$:

$$i\partial_t \rho(t) = [H(t), \rho(t)]. \tag{3.43}$$

Interaction picture

In the Schrödinger picture, states evolve with time as in (3.40), while operators are time-independent, and in the Heisenberg picture it is the other way around. The interaction picture is an intermediate between those two pictures.

Suppose the Hamiltonian consists of a time-independent part and a small time-dependent part, $H(t) = H_0 + \delta H(t)$. In the interaction representation, the time-evolution operator is written as

$$U(t, t_0) = U_0(t, t_0)U'(t, t_0), \tag{3.44}$$

where U_0 and U' are the time-evolution operators pertaining to H_0 and δH,

$$U_0(t, t_0) = e^{-iH_0(t-t_0)} \quad \text{and} \quad U'(t, t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} dt' \delta H^I(t')},$$

with

$$\delta H^I(t) \equiv U_0^{-1}(t)\delta H(t)U_0(t),$$

which follows from substituting (3.44) and U_0 into the Schrödinger equitation (3.39). Note that the time dependence due to H_0 has been absorbed into the operator $\delta H^I(t)$.

Linear response

Introducing an x dependence, the expectation value (3.41) can with (3.44) be rewritten in the interaction representation

$$\begin{aligned}
\langle \mathcal{O}_A \rangle_p(t, x) &= \mathrm{Tr}\left(\rho(t)\mathcal{O}_A(x)\right) \\
&= \mathrm{Tr}\left(U_0(t, t_0)U'(t, t_0)\rho_0\, U'^{-1}(t, t_0)U_0^{-1}(t, t_0)\, \mathcal{O}_A(x)\right) \\
&= \mathrm{Tr}\left(\rho_0\, U'^{-1}(t, t_0)\mathcal{O}_A^I(t, x)U'(t, t_0)\right),
\end{aligned}$$

where again the time dependence due to H_0 has been absorbed into the operator,

$$\mathcal{O}_A^I(t, x) \equiv U_0^{-1}(t, t_0)\mathcal{O}_A(x)U_0(t, t_0). \tag{3.45}$$

In the last equality, the cyclic invariance of the trace has been exploited. Expanding $U'(t)$ to linear order in $\delta H(t)$ gives

$$\begin{aligned}
\langle \mathcal{O}_A \rangle_p(t, x) &\simeq \mathrm{Tr}\left(\rho_0 \left(1 + i\int_{t_0}^{t} dt' \delta H^I(t')\right) \mathcal{O}_A^I(t, x) \left(1 - i\int_{t_0}^{t} dt' \delta H^I(t')\right)\right) \\
&\simeq \mathrm{Tr}\left(\rho_0\, \mathcal{O}_A^I(t, x)\right) - i\,\mathrm{Tr}\left(\rho_0 \int_{t_0}^{t} dt' \left[\mathcal{O}_A^I(t, x), \delta H^I(t')\right]\right),
\end{aligned} \tag{3.46}$$

which is of the form $\langle \mathcal{O}_A \rangle_p(t, x) = \langle \mathcal{O}_A^I \rangle_0(t, x) + \delta\langle \mathcal{O}_A \rangle(t, x)$. The second term represents the *linear response* of the system to the addition of the time-dependent perturbation $\delta H(t)$ to the Hamiltonian H_0. When the time-independent equilibrium density matrix ρ_0 has the form (3.42), the first term is independent of time due to the cyclicity of the trace:

$$\langle \mathcal{O}_A^I \rangle_0(t, x) = \mathrm{Tr}\left(\rho_0 \mathcal{O}_A(x)\right) \equiv \langle \mathcal{O}_A \rangle(x). \tag{3.47}$$

3.5.2 Response and retarded Green's function

Consider a time-dependent perturbation of the Hamiltonian of the form

$$\delta H(t) = \int d^{d-1}x\, \delta\phi_{B(0)}(t,x)\mathcal{O}_B(x). \tag{3.48}$$

Writing $\mathrm{Tr}\,(\rho_0 A) = \langle A\rangle$ throughout and setting $t_0 = -\infty$, the change in expectation value for $\mathcal{O}_A$ due to this perturbation is given by (3.46):

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(t,x) &= -i\int_{-\infty}^{t} dt'\,\langle[\mathcal{O}_A^I(t,x), \delta H^I(t')]\rangle \\ &= -i\int_{-\infty}^{t} dt' \int d^{d-1}x'\,\langle[\mathcal{O}_A^I(t,x), \mathcal{O}_B^I(t',x')]\rangle\delta\phi_{B(0)}(t',x'). \end{aligned}\tag{3.49}$$

Taking a Fourier transformation of this change in average gives

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(\omega,k) &= \int_{-\infty}^{\infty} dt d^{d-1}x\, e^{i\omega t - ik\cdot x}\delta\langle\mathcal{O}_A\rangle(t,x) \\ &= -i\int_{-\infty}^{\infty} dt d^{d-1}x\, e^{i\omega t-ik\cdot x}\int_{-\infty}^{\infty} dt' d^{d-1}x'\,\theta(t-t')\times \\ &\qquad\qquad \times\langle[\mathcal{O}_A^I(t,x), \mathcal{O}_B^I(t',x')]\rangle\delta\phi_{B(0)}(t',x'),\end{aligned}\tag{3.50}$$

where the Heaviside step function $\theta(t-t')$ has been introduced to extend the upper limit of the t' integral.

The *retarded Green's function* (RGF) is defined as

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(t-t', x-x') \equiv -i\,\theta(t-t')\langle[\mathcal{O}_A^I(t,x), \mathcal{O}_B^I(t',x')]\rangle, \tag{3.51}$$

which depends only on the difference of the coordinates, since the system is spacetime translational invariant.[1] Exploiting this, and subsequently using the convolution theorem, the linear response (3.50) becomes

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(\omega,k) &= \left(\int_{-\infty}^{\infty} dt d^{d-1}x\, e^{i\omega t-ik\cdot x} G^R_{\mathcal{O}_A\mathcal{O}_B}(t,x)\right)\delta\phi_{B(0)}(\omega,k) \\ &= G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\,\delta\phi_{B(0)}(\omega,k).\end{aligned}\tag{3.52}$$

As a result, we can conclude that the linear relation between the perturbation by a source $\delta\phi_{B(0)}(\omega,k)$ and the response of the system $\delta\langle\mathcal{O}_A\rangle(\omega,k)$, is given by the retarded Green's function $G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)$. For that reason, the retarded Green's function can be considered as a kind of *susceptibility*.

Using (3.38), the retarded Green's function in terms of bulk fields becomes

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) = \frac{\delta\langle\mathcal{O}_A\rangle_s(\omega,k)}{\delta\phi_{B(0)}(\omega,k)}\bigg|_{\phi_{A(0)}=0} = \frac{2\Delta_A - d}{R}\frac{\delta\phi_{A(1)}(\omega,k)}{\delta\phi_{B(0)}(\omega,k)}. \tag{3.53}$$

For $\langle\mathcal{O}_A\rangle_s$ also (the Fourier transform of) (3.37) can be substituted. Note that this result matches nicely with a two-point function obtained from (3.1) by functional differentiating with respect to $\phi_{A(0)}$ and subsequently to $\phi_{B(0)}$ (cf. (3.34)).

[1] Note that the retarded Green's function for fermions is defined with an anti-commutation relation.

Boundary conditions for the field ϕ_B

In order to have a perturbation of the form (3.48), the entire bulk field needs to be perturbed, $\phi_B \to \phi_B + \delta\phi_B$, and this perturbed bulk field has to obey the equations of motion, as well as the boundary conditions. The boundary conditions for a bulk scalar field were discussed in section 3.4.3.

In the above calculation we have set $t_0 = -\infty$ in (3.46), which yielded the retarded Green's function in (3.50). Setting $t_0 = +\infty$ in (3.46) would yield the *advanced Green's function* (AGF),

$$G^A_{\mathcal{O}_A\mathcal{O}_B}(t-t', x-x') \equiv i\,\theta(t'-t)\langle[\mathcal{O}^I_A(t,x), \mathcal{O}^I_B(t',x')]\rangle.$$

This is closely related to which kind of boundary conditions we impose in the interior for ϕ_B; see, e.g. [57]. For the ingoing boundary condition, we obtain the retarded Green's function, whereas for the outgoing, the advanced. The retarded Green's function and ingoing boundary conditions both describe things that happen, rather than unhappen.

Retarded Green's functions play an important role in calculations in AdS/CFT contexts, and therefore, in the next section some properties of retarded Green's functions are listed.

3.6 Properties of retarded Green's functions

Retarded Green's functions enjoy several important properties, which are considered in this section. First of all, the RGF is *causal*. This follows directly from the step function in (3.51). Further, from the momentum space representation of the RGF (cf. (3.52)), it is clear that the RGF has the following symmetry property:

$$G^{R\ *}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(-\omega, -k). \tag{3.54}$$

3.6.1 Analyticity and Cauchy representation

Causality implies the following property for the RGF:

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) \quad \text{is analytic in } \omega \text{ for } \mathrm{Im}\,\omega > 0.$$

This can be seen by considering the inverse frequency Fourier transformation of the retarded Green's function

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(t, k) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k),$$

which for $t < 0$ can be evaluated by closing the ω contour in the upper half plane of complex frequencies. At the same time, the causality property implies that for $t < 0$ this integral equals zero (cf. (3.51) with $t' = 0$), so the RGF is analytic in ω for $\mathrm{Im}\,\omega > 0$.

Suppose on the contrary, that the RGF has, say, a simple pole at $\omega = \omega_\star$ in the upper half plane. Then, for $t < 0$ this leads to an exponentially growing mode,

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(t, k) \sim e^{-i\omega_\star t} \sim e^{\mathrm{Im}\,|\omega_\star| t},$$

which indicates that the vacuum in which the Green's function has been calculated is unstable. Therefore, whenever a pole in the retarded Green's function appears, we should be careful.

Analyticity in the upper half plane of complex frequencies implies that the RGF can be represented using *Cauchy's integral formula*

$$G^R(z) = \oint_\gamma \frac{d\omega'}{2\pi i} \frac{G^R(\omega')}{\omega' - z}, \tag{3.55}$$

where the contour γ runs along the real axis and closes in the upper half plane, and the k dependence has been ignored.

3.6.2 Kramer-Kronig relation

From the Cauchy representation one can obtain the Kramers-Kronig relation, which relates the real and imaginary parts of a function. Substituting into (3.55) an ω that is slightly shifted above the real axis, $z = \omega + i0$, gives

$$\begin{aligned} G^R(\omega + i0) &= \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{G^R(\omega')}{\omega' - \omega} + i\pi \operatorname*{Res}_{\omega'=\omega} \frac{1}{2\pi i} \frac{G^R(\omega')}{\omega' - \omega} \\ &= \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{G^R(\omega')}{\omega' - \omega} + \frac{1}{2} G^R(\omega), \end{aligned}$$

where $\mathcal{P}$ denotes the *Cauchy principal value*, which, roughly speaking, excludes the point from the interval where the integrand becomes infinite. The second term is the contribution from the integral over an infinitesimal semi-circle under the pole. So, we obtain the relation

$$G^R(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi i} \frac{G^R(\omega')}{\omega' - \omega}.$$

Substituting $G^R(\omega) = \operatorname{Re} G^R(\omega) + \operatorname{Im} G^R(\omega)$ into this expression, gives the *Kramers-Kronig relation*

$$\begin{aligned} \operatorname{Re} G^R(\omega) &= \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\operatorname{Im} G^R(\omega')}{\omega' - \omega}, \\ \operatorname{Im} G^R(\omega) &= -\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\operatorname{Re} G^R(\omega')}{\omega' - \omega}. \end{aligned}$$

3.6.3 Spectral representation

It is useful to express the retarded Green's function in its spectral representation. Let $\{|n\rangle\}$ be a complete set of eigenstates of the Hamiltonian H_0 and momentum operator P, with eigenvalues E_n and k_n, respectively. This is possible when the momentum is conserved, i.e., when $[H_0, P] = 0$. Assume that ρ_0 is diagonal in this basis, with matrix elements $\langle m|\rho_0|n\rangle = \rho_{0,n}\delta_{mn}$ (cf. (3.42)). Then the *spectral decomposition* of the retarded Green's function is given by

$$\begin{aligned} G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega + i0, k) &= \sum_{m,n} \rho_{0,n} \left(\frac{A_{nm}B_{mn}(2\pi)^{d-1}\delta^{(d-1)}(k_{nm} - k)}{E_n - E_m + \omega + i0} - (n \leftrightarrow m) \right) \\ &= \sum_{m,n} (\rho_{0,n} - \rho_{0,m}) \frac{A_{nm}B_{mn}(2\pi)^{d-1}\delta^{(d-1)}(k_{nm} - k)}{E_n - E_m + \omega + i0} \end{aligned} \tag{3.56}$$

where $A_{mn} \equiv \langle m|\mathcal{O}_A(0,0)|n\rangle$, $B_{mn} \equiv \langle m|\mathcal{O}_B(0,0)|n\rangle$, $k_{mn} \equiv k_m - k_n$, and ω has been shifted slightly into the upper half plane, $\omega \to \omega + i0$, in order to get a finite result. The derivation of this expression is given in Appendix D.1.

The spectral decomposition of the retarded Green's function consists of a series of delta-function spikes, weighted by the density matrix ρ_0, and matrix elements A_{nm} and B_{mn}. The delta functions of momenta indicate values of momenta corresponding to particle excitations. Furthermore, all the singularities are simple poles, and each pole corresponds to a definite excitation energy.

Quasi-normal modes

The spectral decomposition of the retarded Green's function (3.56) is roughly of the form

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) \sim \sum_{\omega_\star} \frac{c_\star}{\omega - \omega_\star},$$

where $\omega_\star$ are simple poles. These poles are called the *quasi-normal frequencies* or *resonance frequencies* of the retarded Green's function. In the case of a bulk scalar field, the retarded Green's function (3.53) has clearly a pole for

$$\phi_{B(0)}(\omega_\star, k) = 0. \tag{3.57}$$

From the bulk point of view, quasi-normal modes are solutions to the equations of motion satisfying ingoing boundary conditions at the horizon, together with boundary condition (3.57). They describe the decay of small perturbations for black holes at equilibrium in asymptotically AdS spacetimes.

It was shown in [58], that for a BTZ black hole[2], the quasi-normal modes of this black hole are precisely the poles of the retarded Green's function. This idea was in [47] extended to general black holes in asymptotically AdS spacetimes.

3.6.4 Spectral function

The anti-Hermitian part of the retarded Green's function is called the *spectral function* or *spectral density*, and is given by

$$\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) \equiv i\left(G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) - G^{R\,\dagger}_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\right), \tag{3.58}$$

with $G^{R\,\dagger}_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) = G^{R\,*}_{\mathcal{O}_B\mathcal{O}_A}(\omega,k)$. It is shown in Appendix D.2, that by using the spectral representation of the retarded Green's function (3.56), this spectral function can be brought in the form

$$\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) = (2\pi)^d \sum_{m,n} (\rho_{0,n} - \rho_{0,m})\delta(E_n - E_m + \omega)\delta^{(d-1)}(k_{nm} - k)A_{nm}B_{mn}. \tag{3.59}$$

By inserting the identity $\int d\omega' \delta(E_n - E_m + \omega')$ into the second equality of (3.56), the retarded Green's function can be written as

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega + i0, k) = \int \frac{d\omega'}{2\pi} \frac{\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega',k)}{\omega - \omega' + i0}, \tag{3.60}$$

[2] BTZ black holes are solutions of Einstein equations of the form (3.2) in $2+1$ dimensions. They possess an inner and outer horizon and have angular momentum.

which is called the *(Källén-)Lehmann spectral representation* of the retarded Green's function; see e.g. [59]. From this representation, it is clear that the retarded Green's function is a superposition of free particle propagators with different frequencies ω'. The weight of each contribution is given by the spectral density. It is also clear from this representation, that a pole in the RGF corresponds to a delta function in the spectral density.

From this representation of the RGF, it is clear that the spectral density can be considered as a generalization of the density of states, which is the case when there is only one source:

$$\mathcal{A}(\omega, k) = -2\,\mathrm{Im}\, G^R_{\mathcal{O}_A\mathcal{O}_A}(\omega, k), \tag{3.61}$$

so that the retarded Green's function has the form

$$G^R_{\mathcal{O}_A\mathcal{O}_A}(\omega + i0, k) = \int \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G^R_{\mathcal{O}_A\mathcal{O}_A}(\omega, k)}{\omega' - \omega - i0}.$$

The spectral density can be interpreted as the probability for the operator $\mathcal{O}_A$ to have energy ω and momentum k. For a non-interacting operator, there is only one ω for each k and vice versa, so that the spectral density is a delta function. By taking the limit $\omega \to 0$ in this expression for the RGF, we obtain the so-called *thermodynamic sum rule*,

$$\chi \equiv \lim_{\omega \to 0} G^R_{\mathcal{O}_A\mathcal{O}_A}(\omega + i0, k) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G^R(\omega')}{\omega' - i0},$$

which is called a sum rule, since it relates a static (i.e., $\omega \to 0$) quantity to an integral over frequencies. With (3.53), χ is the response function for an external perturbation with zero frequency, therefore it is the static susceptibility.

For the case that the equilibrium density matrix is given by the *canonical ensemble*, i.e., $\rho_0 = \mathcal{Z}^{-1} e^{-\beta H_0}$, it is shown in Appendix D.2.1 that the spectral function (3.59) can be written as

$$\begin{aligned}\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = 2\mathcal{Z}^{-1} \sinh\left(\frac{\omega}{2T}\right) \sum_{m,n} e^{-(E_n+E_m)/2T}\, 2\pi\delta(E_n - E_m + \omega)\times \\ \times (2\pi)^{d-1} \delta^{(d-1)}(k_{nm} - k) A_{nm} B_{mn}.\end{aligned}$$

From this relation, it follows that the diagonal components of the spectral function multiplied by ω, are nonnegative, since $A_{nm}A_{mn} = |\langle n|\mathcal{O}_A|m\rangle|^2$.

3.6.5 Dissipation

The time-varying external source $\delta\phi_{B(0)}(t, x)$ in (3.48) does work on the system. It will now be shown that the time-averaged rate of change of the the total energy, is, to leading order in the external source, measured by the spectral function.

The rate dW/dt at which the external source does work on the system, or the *dissipated power*, is equal to the rate of change of the total energy of the system:

$$\frac{dW}{dt} = \frac{d}{dt} \mathrm{Tr}\left(\rho H\right) = \mathrm{Tr}\left(\partial_t \rho\, H\right) + \mathrm{Tr}\left(\rho\, \partial_t H\right),$$

where in the first equality (3.41) has been used. Assume that the density matrix is of the form (3.42). By using the Schrödinger equation for $\rho(t)$, (3.43), the first term of the second equality vanishes:

$$\mathrm{Tr}\left(\frac{d\rho}{dt} H\right) = \frac{1}{i} \mathrm{Tr}\left([H, \rho] H\right) = \frac{1}{i} \mathrm{Tr}\left(\rho [H, H]\right) = 0.$$

With $H = H_0 + \delta H(t)$, so that $\partial_t H = \partial_t \delta H(t)$, and with $\delta H(t)$ given by (3.48) with $B \to A$, the second term becomes

$$\mathrm{Tr}\left(\rho\, \partial_t H\right) = \int d^{d-1}x \ \mathrm{Tr}\left(\rho(t)\mathcal{O}_A(x)\right)\partial_t \delta\phi_{A(0)}(t,x).$$

Using (3.46), the rate of change of the total energy to linear order in $\delta H(t)$, or to linear order in $\delta\phi_{A(0)}$ by (3.48), has the form

$$\frac{dW}{dt} = \int d^{d-1}x \left(\langle \mathcal{O}^I_A\rangle_0(t,x) + \delta\langle\mathcal{O}_A\rangle(t,x)\right)\partial_t\delta\phi_{A(0)}(t,x). \tag{3.62}$$

The power dissipated averaged over one cycle of the external field, is defined as

$$\overline{\frac{dW}{dt}} \equiv \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, \frac{dW}{dt}.$$

It is shown in Appendix D.3, that for an external field oscillating at a single frequency ω,

$$\delta\phi_{A(0)}(t,x) = \mathrm{Re}\left(\phi_{A(0)}(x)e^{-i\omega t}\right) = \frac{1}{2}\left(\phi_{A(0)}(x)e^{-i\omega t} + \phi^*_{A(0)}(x)e^{i\omega t}\right),$$

the averaged dissipated power is given by

$$\begin{aligned}\overline{\frac{dW}{dt}} &= \frac{\omega}{4}\int d^{d-1}x d^{d-1}x' \phi^*_{A(0)}(x)\, i\left(G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, x-x') - G^{R\,*}_{\mathcal{O}_B\mathcal{O}_A}(\omega, x-x')\right)\phi_{B(0)}(x') \\ &= \frac{1}{4}\int d^{d-1}x d^{d-1}x' \phi^*_{A(0)}(x)\, \omega\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, x-x')\phi_{B(0)}(x').\end{aligned}$$

Therefore, the dissipation of the system is captured by ω times the spectral function, which was (minus) the anti-Hermitian part of the retarded Green's function, and is nonnegative when there is only one source.

4 Superconductors and electron-phonon interaction

Superconductivity was discovered in 1911 by Onnes, when he noted that the (DC) electrical resistivity of Mercury suddenly drops to zero as the temperature is lowered below a critical temperature T_c [1]. During the years after its discovery, many metals were found to exhibit this phenomenon. Furthermore, at temperatures lower than the critical temperature, these materials exhibit the *Meissner effect*: external magnetic fields are completely expelled. This perfect diamagnetism does not follow from the perfect conductivity, and is an independent property.

In 1935, a phenomenological description of both of these properties was given by the London brothers [60]. Years later, in 1950, Landau and Ginzburg described superconductivity in terms of a second-order phase transition, whose order parameter is a complex scalar field [61]. Finally, a microscopic theory of superconductivity was given by Bardeen, Cooper and Schrieffer in 1957 [2]. This theory is known as *BCS* theory.

BCS theory describes the pairing of two electrons —or, actually, quasiparticles derived from electrons— with opposite spin into a charged boson caused by interactions with *phonons*, which are quasiparticles associated with the lattice vibrations of a crystalline solid. These electron pairs are called *Cooper pairs*. When the temperature is lower than the critical temperature T_c, these bosons condense and the DC electrical resistivity drops to zero. The Cooper pairs are not very tightly bound and have a size which is typically much larger than the lattice spacing.

In 1986, a new class of superconductors was discovered [3]: the so-called *high T_c superconductors* (HTSC), or, simply, high-temperature superconductors. Conventional theories have failed to describe these systems. This class of materials consists of two-dimensional layers of CuO_2, and, therefore, they are called *cuprates*. These two-dimensional layers are sandwiched between so-called block layers. At the time of writing, the highest T_c for this class of superconductors is about 135 K at atmospheric pressure [4]. There is evidence that superconductivity in these high-T_c superconductors is still caused by the forming of electron pairs, but the pairing mechanism is not well understood, since the coupling is believed to be strong. For an up-to-date review, see e.g. [5].

In this chapter, an expression for the self-energy of an electron that interacts with phonons in a system at finite temperature will be studied. It will be shown, that this self-energy can be written as an integral over a product of a term that contains all the information about the phonons, and a term that contains the electronic and thermal information. At the end of this chapter, we will consider options for translating a generalized expression for the electron self-energy using the AdS/CFT correspondence. But first a model that describes the interactions between electrons and phonons will be reviewed.

4.1 Electron-phonon interaction

The interaction between electrons and lattice degrees of freedom plays an important role in understanding the properties of many materials. An electron that is coupled to phonons is called a *polaron*, introduced by Landau in 1933. In a more general context, a polaron describes a charged quantum particle interacting with a bosonic environment. The particle gets dressed by the bosonic modes, changing its properties like the energy and effective mass. Besides dressing charge carriers, phonons also induce an effective attraction between them. Even a weak attraction is enough to make electrons form pairs in momentum space, close to the Fermi level. These are the aforementioned Cooper pairs.

In this section, we will consider a model that describes the interaction of electrons with a vibrating lattice of ions. After quantization, the vibrating lattice gives rise to phonons, which are considered as harmonic oscillators. So, the model in this section describes the electron-phonon interaction, which is commonly abbreviated to *EPI*. Most of the material of this section can be found in [62] and [63].

4.1.1 EPI model

The Hamiltonian describing electrons coupled to phonons is assumed to have the form

$$H = H_{\text{el}} + H_{\text{ph}} + H_{\text{el-ion}} + H_{\text{el-el}}, \tag{4.1}$$

where the first term describes the kinetic energy of the electrons,

$$H_{\text{el}} = \sum_{\mathbf{p},\sigma} \frac{p^2}{2m} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma},$$

with $c^\dagger_{\mathbf{p},\sigma}$ ($c_{\mathbf{p},\sigma}$) the creation (annihilation) operator of an electron state with momentum $\mathbf{p}$ and spin σ. The second term describes the lattice, which is considered as a set of linear, independent oscillators with frequency $\omega_{\mathbf{q},\lambda}$

$$H_{\text{ph}} = \sum_{\mathbf{q},\lambda} \omega_{\mathbf{q},\lambda} \left(a^\dagger_{\mathbf{q},\lambda} a_{\mathbf{q},\lambda} + \frac{1}{2} \right)$$

where $a^\dagger_{\mathbf{q},\lambda}$ ($a_{\mathbf{q},\lambda}$) is the creation (annihilation) operator of a phonon with momentum $\mathbf{q}$ and *branch index* λ, which labels the polarization of the phonon. If the solid is d dimensional and has N unit cells that contain L ions, then the ions can vibrate in dL ways; see figure 4.1a. Of these modes, there are d *acoustical modes*, which correspond to displacing all the ions in a unit cell by nearly the same amount in the same direction, which costs arbitrarily low energy. The $d(L-1)$ remaining modes are called *optical modes*. An example is shown in figure 4.1b. Due to their low energies, acoustical modes are dominant at low temperatures. Phonons are *bosons*, since their creation and annihilation operators obey bosonic commutation relations.

Additionally, the electron-ion interaction term has in position space the form

$$H_{\text{el-ion}} = \sum_{i=1}^{N_e} \tilde{V}(\mathbf{r}_i), \quad \text{with} \quad \tilde{V}(\mathbf{r}_i) = \sum_j V(\mathbf{r}_i - \mathbf{R}_j),$$

where N_e is the total number of electrons, and $V(\mathbf{r}_i - \mathbf{R}_j)$ describes the potential of the crystal with $\mathbf{R}_j$ the position of the j-th ion. Assume that the actual position of the j-th ion

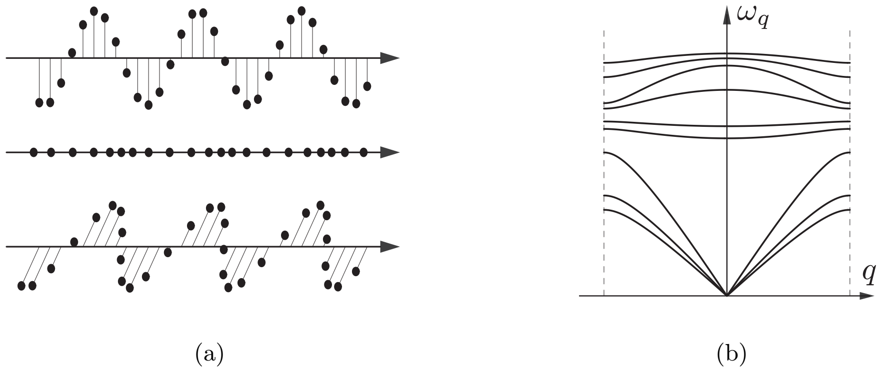

Figure 4.1: Phonon modes for a unit cell containing three ions. (a) Three examples of polarization in phonon modes. (b) Phonon spectrum with three acoustical modes, and six optical modes. Figures taken from [62].

is given by $\mathbf{R}_j = \mathbf{R}_j^0 + \mathbf{Q}_j$, where the latter is the deviation from the equilibrium position $\mathbf{R}_j^0$, which is taken to be small. Expansion of the crystal potential gives

$$V(\mathbf{r}_i - \mathbf{R}_j) \simeq V(\mathbf{r}_i - \mathbf{R}_j^0) + \mathbf{Q}_j \cdot \nabla_{\mathbf{R}} V(\mathbf{r}_i - \mathbf{R}_j^0). \tag{4.2}$$

The first term gives rise to the interaction of electrons with a periodic potential. Together with the electronic kinetic term in (4.1), this term will produce non-interacting *Bloch states*, which in second-quantization formalism can be written as

$$H_{\text{el}} = \sum_{\mathbf{p},\sigma} \varepsilon_{\mathbf{p},\sigma}\, c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma},$$

where $\varepsilon_{\mathbf{p},\sigma}$ is the Bloch energy, with $\varepsilon_{\mathbf{p}_F,\sigma} = \mu$, and $c^\dagger_{\mathbf{p},\sigma}$ ($c_{\mathbf{p},\sigma}$) creates (annihilates) an electron in a Bloch state of crystal momentum $\mathbf{p}$ and spin σ.

Electron-phonon interaction term

The second term in (4.2) results in the electron-phonon interaction term. After taking the Fourier transformation, the electron-phonon interaction term assumes the form [62]

$$H_{\text{el-ph}} = \frac{1}{\mathcal{V}^{\frac{1}{2}}} \sum_{\mathbf{q},\lambda} g_{\mathbf{q},\lambda} \left(a_{\mathbf{q},\lambda} + a^\dagger_{-\mathbf{q},\lambda}\right) \sum_{\mathbf{p},\sigma} c^\dagger_{\mathbf{p}+\mathbf{q},\sigma} c_{\mathbf{p},\sigma},$$

where $\mathcal{V}$ is a normalization constant related to the system's size, $\int d\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} = \mathcal{V}\delta_{\mathbf{p},0}$. The electron-phonon interaction can be interpreted as the scattering of an electron from an initial state $|\mathbf{p},\sigma\rangle_{\text{el}}$ to the final state $|\mathbf{p}+\mathbf{q},\sigma\rangle_{\text{el}}$, either by absorbing a phonon of state $|\mathbf{q},\lambda\rangle_{\text{ph}}$ or by emitting a phonon of state $|-\mathbf{q},\lambda\rangle_{\text{ph}}$. The *electron-phonon coupling* is given by $g_{\mathbf{q},\lambda}$, which is temperature independent.

The electron-phonon interaction term can be rewritten slightly in the form

$$H_{\text{el-ph}} = \frac{1}{\mathcal{V}^{\frac{1}{2}}} \sum_{\mathbf{p},\sigma} \sum_{\mathbf{q},\lambda} g_{\mathbf{q},\lambda}\, A_{\mathbf{q},\lambda} c^\dagger_{\mathbf{p}+\mathbf{q},\sigma} c_{\mathbf{p},\sigma}, \quad \text{with} \quad A_{\mathbf{q},\lambda} \equiv a_{\mathbf{q},\lambda} + a^\dagger_{-\mathbf{q},\lambda}. \tag{4.3}$$

Here, the phonon operator $A_{\mathbf{q}}$ can be interpreted as removing momentum $\mathbf{q}$ from the phonon system, either by annihilating a phonon with momentum $\mathbf{q}$, or by creating a phonon with momentum $-\mathbf{q}$.

Since the ion lattice is periodic, there is also a lattice in momentum space (MS). This lattice is called the *reciprocal lattice* (RL), and is defined as $\mathrm{RL} = \{\mathbf{G} \in \mathrm{MS} | e^{i\mathbf{G}\cdot\mathbf{R}_j} = 1\}$, where $\mathbf{R}_j$ are the equilibrium positions of the ions. A closely related concept is the *first Brillouin zone* (FBZ), which is defined as $\mathrm{FBZ} = \{\mathbf{q} \in \mathrm{MS} \,|\, |\mathbf{q}| < |\mathbf{q} - \mathbf{G}|, \forall \mathbf{G} \neq 0\}$. Any phonon wave vector $\mathbf{k}$ can be decomposed into $\mathbf{k} = \mathbf{q} + \mathbf{G}$, where $\mathbf{q} \in \mathrm{FBZ}$ and $\mathbf{G} \in \mathrm{RL}$. Processes involving phonons with wave vectors restricted to the first Brillouin zone ($\mathbf{G} = 0$), are called *normal* processes. *Umklapp* processes, on the other hand, are processes with $\mathbf{G} \neq 0$.

It was mentioned before that acoustical modes are dominant. Therefore, only normal processes in the electron-phonon interaction (4.3) are considered. So, we restrict the phonon wave vectors to the first Brillouin zone, $\mathbf{q} \in \mathrm{FBZ}$.

Remaining terms

The electron-electron term in (4.1) is usually taken to be the Coulomb interaction. The effect of this term is to renormalize the electron-phonon coupling constant, $g \to \bar{g}$, and the phonon frequencies $\omega_\lambda(\mathbf{q})$ [63]. Finally, ion-ion interactions are ignored, for we assume the ions to be sufficiently far apart.

Altogether, the full Hamiltonian of the system is given by

$$H = \sum_{\mathbf{p},\sigma} \varepsilon_{\mathbf{p},\sigma}\, c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma} + \sum_{\mathbf{q},\lambda} \omega_{\mathbf{q},\lambda} \left(a^\dagger_{\mathbf{q},\lambda} a_{\mathbf{q},\lambda} + \frac{1}{2} \right) + \frac{1}{\mathcal{V}^{\frac{1}{2}}} \sum_{\mathbf{p},\sigma} \sum_{\mathbf{q},\lambda} \bar{g}_{\mathbf{q},\lambda}\, A_{\mathbf{q},\lambda} c^\dagger_{\mathbf{p}+\mathbf{q},\sigma} c_{\mathbf{p},\sigma}, \tag{4.4}$$

where $\bar{g}$ is the renormalized electron-phonon coupling constant due to screening, which is not discussed further here, since only the structure of the Hamiltonian is important for our purposes.

Because lattice vibrations occur at finite temperature, in order to describe the Green's functions for this system, it necessary to review first some properties of Green's functions at finite temperature.

4.2 Green's functions at finite temperature

In this section, Green's functions at finite temperature are reviewed, first for electrons, and then for phonons. The connection between the periodicity in imaginary time and temperature was already discussed in section 3.2.3. It is convenient to describe Green's functions at finite temperature in imaginary time, where $t \to \tau = it$, since, as will be shown, the imaginary time evolution operator and the Boltzmann factor can be treated on an equal footing and a single perturbative expansion suffices.

We will work in the grand canonical ensemble, where the number of particles is variable, and the total Hamiltonian is given by

$$K = K_0 + V = H_0 - \mu N + V, \quad \text{and} \quad H = H_0 + V, \tag{4.5}$$

with H_0 the Hamiltonian of the unperturbed system, μ the chemical potential, N the particle number operator, and V describes the interactions. We assume that the unperturbed

Hamiltonian does not commute with the interactions, $[H_0, V] \neq 0$. Furthermore, we assume that the number operator commutes with the Hamiltonian,

$$[H, N] = 0, \qquad [H_0, N] = 0,$$

so that we can define simultaneous eigenstates of H and N, and of H_0 and N.

4.2.1 Electron Matsubara Green's function

The electron Green's function in imaginary time, or electron *Matsubara Green's function* (MGF), is given by

$$\mathcal{G}_\sigma(\tau - \tau', \mathbf{p}) = -\langle \mathcal{T}_\tau c_{\mathbf{p},\sigma}(\tau) c^\dagger_{\mathbf{p},\sigma}(\tau') \rangle, \quad \text{with} \quad -\beta \leq \tau - \tau' \leq \beta,$$

where $\mathcal{T}_\tau$ denotes ordering in imaginary time, which arranges operators with smallest τ to the right. We will represent Matsubara Green's functions with a script capital, e.g. $\mathcal{G}$ for the electron MGF, while real-time Green's functions are denoted with a normal capital, e.g. G. The bracket $\langle \cdot \rangle$ means taking the thermodynamic average:

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\left(\rho\, \mathcal{O}\right) = \frac{1}{\mathcal{Z}} \mathrm{Tr}\left(e^{-\beta K} \mathcal{O}\right), \quad \text{with} \quad \mathcal{Z} = \mathrm{Tr}\left(e^{-\beta K}\right).$$

For the bracket with a subscript $\langle \cdot \rangle_0$, we take the thermodynamic average over the unperturbed system, i.e., with $K = K_0$.

Interaction picture

In the interaction picture, which was introduced in section 3.5.1, the imaginary-time dependence due to the free Hamiltonian H_0 can be absorbed into the operators, so that (cf. (3.45))

$$c_{\mathbf{p}}(\tau) = U^{-1}(\tau, 0) c_{\mathbf{p}}(0) U(\tau, 0) = U'^{-1}(\tau, 0) c^I_{\mathbf{p}}(\tau) U'(\tau, 0),$$

where operators in the imaginary time interaction representation are denoted by

$$c^I_{\mathbf{p}}(\tau) = U_0^{-1}(\tau, 0) c_{\mathbf{p}}(0) U_0(\tau, 0). \tag{4.6}$$

The imaginary-time evolution operator $U(\tau, 0)$ is defined as (cf. (3.44))

$$U(\tau, 0) = U'(\tau, 0) U_0(\tau, 0), \quad \text{with} \quad U_0(\tau, 0) = e^{-K_0 \tau},$$

and $U'(\tau, 0)$ is the imaginary-time evolution operator pertaining to V,

$$U'(\tau, 0) \equiv \mathcal{T}_\tau e^{-\int_0^\tau d\tau' V^I(\tau')}, \quad \text{with} \quad V^I(\tau) \equiv U_0^{-1}(\tau, 0) V(0) U_0(\tau, 0), \tag{4.7}$$

which has this form because H and V do not commute, and can be obtained in an analogous way as the real-time-evolution operator described in section 3.5.1. For convenience, we define the *S-matrix* for the scattering of a state at 'time' τ_1 to a state at 'time' τ_2,

$$S(\tau_1, \tau_2) \equiv \mathcal{T}_\tau e^{-\int_{\tau_1}^{\tau_2} d\tau' V^I(\tau')} = U'(\tau_1, 0) U'^{-1}(\tau_2, 0).$$

Electron MGF in interaction picture

Since the electron Green's function in imaginary time only depends on the difference in imaginary time, one can shift $\tau - \tau' \to \tau$, and, together with the above definitions, the Green's function becomes

$$\mathcal{G}_\sigma(\tau, \mathbf{p}) = -\frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\beta K} \mathcal{T}_\tau \left(U'^{-1}(\tau,0) c^I_{\mathbf{p},\sigma}(\tau) U'(\tau,0)\right) c^{I\dagger}_{\mathbf{p},\sigma}(0)\right).$$

In the imaginary-time formalism, the 'time' evolution of the density operator ρ can be written in terms of the time-evolution operator U', since the operator $e^{-\beta K}$ can be rewritten as

$$e^{-\beta K} = e^{-\beta K_0} e^{\beta K_0} e^{-\beta K} = e^{-\beta K_0} U_0^{-1}(\beta,0) U(\beta,0) = e^{-\beta K_0} U'(\beta,0), \tag{4.8}$$

where in the last equality (4.7) has been used. Using further the identities $U'(\tau,0) = S(\tau,0)$ and $U'(\beta,0)U'^{-1}(\tau,0) = S(\beta,\tau)$, the electron MGF becomes

$$\begin{aligned}
\mathcal{G}_\sigma(\tau, \mathbf{p}) &= -\frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\beta K_0} \mathcal{T}_\tau S(\beta,\tau) c^I_{\mathbf{p},\sigma}(\tau) S(\tau,0) c^{I\dagger}_{\mathbf{p},\sigma}(0)\right) \\
&= -\frac{1}{\mathcal{Z}} \operatorname{Tr}\left(e^{-\beta K_0} \mathcal{T}_\tau S(\beta,0) c^I_{\mathbf{p},\sigma}(\tau) c^{I\dagger}_{\mathbf{p},\sigma}(0)\right) \\
&= -\frac{\langle \mathcal{T}_\tau S(\beta,0) c^I_{\mathbf{p},\sigma}(\tau) c^{I\dagger}_{\mathbf{p},\sigma}(0)\rangle_0}{\langle S(\beta,0)\rangle_0},
\end{aligned}$$

where for the last equality the operator $S(\tau,0)$ could be moved to the left due to the presence of the imaginary-time ordering operator, and $\mathcal{Z}$ was rewritten by using (4.8). Also, the identity $\mathcal{Z}_0/\mathcal{Z}_0$ has been inserted.

4.2.2 Dyson's equation for the electron MGF

The expression for the electron MGF can be evaluated by expanding the S-matrices

$$\mathcal{G}_\sigma(\tau, \mathbf{p}) = -\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \langle \mathcal{T}_\tau c^I_{\mathbf{p},\sigma}(\tau) V^I(\tau_1) \cdots V^I(\tau_n) c^{I\dagger}_{\mathbf{p},\sigma}(0)\rangle_0}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \langle \mathcal{T}_\tau V^I(\tau_1) \cdots V^I(\tau_n)\rangle_0}.$$

By applying Wick's theorem in imaginary time, the n-point functions in the numerator can be written as a sum over connected diagrams and disconnected diagram, where the latter are of the form

$$\langle \mathcal{T}_\tau V^I(\tau_1) \cdots V^I(\tau_j) c^I_{\mathbf{p}}(\tau) c^{I\dagger}_{\mathbf{p}}(0)\rangle_0 \langle \mathcal{T}_\tau V^I(\tau_j) \cdots V^I(\tau_n)\rangle_0$$

The denominator gives just the vacuum polarization terms, which cancel the disconnected parts from the numerator, so that

$$\mathcal{G}_\sigma(\tau, \mathbf{p}) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \langle \mathcal{T}_\tau c^I_{\mathbf{p},\sigma}(\tau) V^I(\tau_1) \cdots V^I(\tau_n) c^{I\dagger}_{\mathbf{p},\sigma}(0)\rangle_{0,\text{connected}}.$$

One can get rid of the $1/n!$ factor by summing over *different* connected diagrams. This is because the expansion by Wick's theorem gives $n!$ diagrams that have the same topology, i.e., diagrams that can be brought in the same form by permuting their internal time arguments. So,

$$\mathcal{G}_\sigma(\tau,\mathbf{p}) = -\sum_{n=0}^{\infty}(-1)^n \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \, \langle \mathcal{T}_\tau c^I_{\mathbf{p},\sigma}(\tau) V^I(\tau_1)\cdots V^I(\tau_n) c^{I\dagger}_{\mathbf{p},\sigma}(0)\rangle_{0,\text{diff-conn}}. \quad (4.9)$$

The Fourier transform of the electron Matsubara Green's function reads [62]:

$$\mathcal{G}_\sigma(ip_n,\mathbf{p}) = \int_0^\beta d\tau \, e^{ip_n\tau}\mathcal{G}_\sigma(\tau,\mathbf{p}), \quad \text{with} \quad p_n \equiv \frac{(2n+1)\pi}{\beta}, \quad (4.10)$$

where p_n are called fermion *Matsubara frequencies*, which follows from the anti-periodicity of the electron MGF with respect to its 'time' argument, $\mathcal{G}_\sigma(\tau,\mathbf{p}) = -\mathcal{G}_\sigma(\tau+\beta,\mathbf{p})$, for $-\beta < \tau < 0$.

After taking the Fourier transform, one obtains a series with self-energy diagrams that can be collected into a Dyson's equation

$$\begin{aligned}\mathcal{G}_\sigma(ip_n,\mathbf{p}) &= \mathcal{G}^{(0)}_\sigma(ip_n,\mathbf{p}) + \mathcal{G}^{(0)}_\sigma(ip_n,\mathbf{p})\Sigma_\sigma(ip_n,\mathbf{p})\mathcal{G}_\sigma(ip_n,\mathbf{p}) \\ &= \frac{\mathcal{G}^{(0)}_\sigma(ip_n,\mathbf{p})}{1-\mathcal{G}^{(0)}_\sigma(ip_n,\mathbf{p})\Sigma_\sigma(ip_n,\mathbf{p})},\end{aligned}$$

with $\mathcal{G}^{(0)}(ip_n,\mathbf{p})$ the unperturbed electron Matsubara Green's function. Further, $\Sigma(ip_n,\mathbf{p})$ is the electron self-energy, which a sum over one-particle irreducible diagrams[1], as shown in figure 4.2. In section 4.3, an example for the electron-phonon interaction is given.

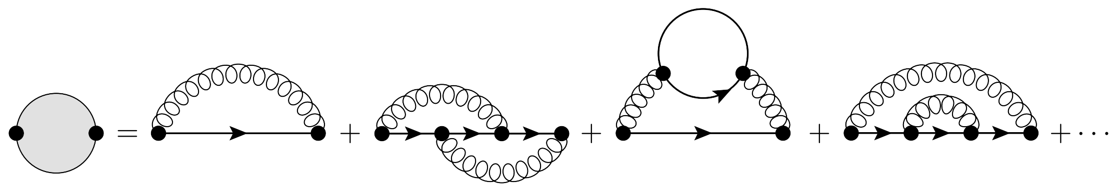

Figure 4.2: Electron self-energy diagrams. The dots at the vertices represent the electronically screened electron-phonon coupling.

4.2.3 Phonon Matsubara Green's function

The above also holds for the phonon Matsubara Green's function, which is defined as

$$\mathcal{D}_\lambda(\tau-\tau',\mathbf{q}) = -\langle \mathcal{T}_\tau A_{\mathbf{q},\lambda}(\tau)A_{-\mathbf{q},\lambda}(\tau')\rangle, \quad \text{with} \quad -\beta \le \tau-\tau' \le \beta,$$

where

$$A_{\mathbf{q},\lambda}(\tau) = e^{H\tau}A_{\mathbf{q},\lambda}(0)e^{-H\tau} \quad \text{and} \quad \langle\mathcal{O}\rangle = \mathrm{Tr}\left(e^{-\beta H}\right)^{-1}\mathrm{Tr}\left(e^{-\beta H}\mathcal{O}\right),$$

because phonons have no chemical potential, since one can make an arbitrary number of them.

[1]Diagrams involving tadpoles have been ignored, since in the context of EPI, described in the next section, they would require phonons with zero wave vector. Phonons with zero wave vector do not exist in our model, since they correspond to a translation of the crystal.

The Fourier transform of the phonon MGF has the form [62]:

$$\mathcal{D}_\lambda(i\omega_n, \mathbf{q}) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\mathcal{D}_\lambda(\tau, \mathbf{q}), \quad \text{with} \quad \omega_n \equiv \frac{2n\pi}{\beta}, \tag{4.11}$$

where ω_n are called boson Matsubara frequencies, which follows from the periodicity of the phonon Matsubara Green's function, $\mathcal{D}_\lambda(\tau, \mathbf{q}) = \mathcal{D}_\lambda(\tau + \beta, \mathbf{q})$, for $-\beta < \tau < 0$. The Dyson's equation for the phonon Matsubara Green's function has the form

$$\mathcal{D}_\lambda(i\omega_n, \mathbf{q}) = \frac{\mathcal{D}_\lambda^{(0)}(i\omega_n, \mathbf{q})}{1 - \mathcal{D}_\lambda^{(0)}(i\omega_n, \mathbf{q})\Pi_\lambda(i\omega_n, \mathbf{q})},$$

where $\Pi(i\omega_n, \mathbf{q})$ is the phonon self-energy, and $\mathcal{D}^{(0)}(i\omega_n, \mathbf{q})$ is the unperturbed phonon MGF.

In the next section, the finite-temperature Green's functions are going to be considered in the context of the electron-phonon interaction model.

4.3 Electron Matsubara Green's function for EPI

By combining the two preceding sections, the Matsubara Green's function for an electron interacting with phonons can be determined. In this section, this will be done for a weakly coupled system, that is, for small values of $\bar{g}_{\mathbf{q},\lambda}$ in (4.4).

In case of the electron-phonon interaction described in section 4.1.1, the interaction V in (4.5) is given by the electron-phonon interaction term (4.3). For this interaction, only the terms with n even in the sum over different connected diagrams (4.9) remain. This can be seen as follows. In an unperturbed system the electronic and phononic degrees of freedom decouple, and since the thermal average in (4.9) is taken with respect to the free system, the thermal average splits into a product of a phononic and an electronic thermal average. The terms with n odd contain factors with $\langle \mathcal{T}_\tau A_{\mathbf{q}}(\tau_1)\cdots A_{\mathbf{q}}(\tau_n)\rangle_0$, and these vanish since they contain an odd number of $a_{\mathbf{q}}$ or $a^\dagger_{-\mathbf{q}}$ (cf. (4.3)).

The electron MGF can then schematically be written as

$$\mathcal{G}_\sigma(\tau, \mathbf{p}) = \mathcal{G}_\sigma^{(0)}(\tau, \mathbf{p}) + \mathcal{G}_\sigma^{(2)}(\tau, \mathbf{p}) + \mathcal{G}_\sigma^{(4)}(\tau, \mathbf{p}) + \cdots,$$

where the first term is the free-electron Green's function. By using Wick's theorem, the term for $n = 2$ in (4.9) gives six terms, but there is only one different connected term of the form

$$\mathcal{G}_\sigma^{(2)}(\tau, \mathbf{p}) = -\frac{1}{\mathcal{V}}\sum_{\mathbf{p},\sigma}\sum_{\mathbf{q},\lambda}\bar{g}^2_{\mathbf{q},\lambda}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\, \mathcal{D}_\lambda^{(0)}(\tau_1 - \tau_2, \mathbf{q})\, \mathcal{G}_\sigma^{(0)}(\tau - \tau_1, \mathbf{p})\times$$
$$\times\, \mathcal{G}_\sigma^{(0)}(\tau_1 - \tau_2, \mathbf{p} - \mathbf{q})\mathcal{G}_\sigma^{(0)}(\tau, \mathbf{p}).$$

By taking the Fourier transform of the imaginary time (cf. (4.10)), this can be cast in the form

$$\mathcal{G}_\sigma^{(2)}(ip_n, \mathbf{p}) = \mathcal{G}_\sigma^{(0)}(ip_n, \mathbf{p})^2\, \Sigma_\sigma^{(1)}(ip_n, \mathbf{p}), \tag{4.12}$$

where

$$\Sigma_\sigma^{(1)}(ip_n, \mathbf{p}) = -\frac{1}{\beta\mathcal{V}}\sum_{\mathbf{q},\lambda}\sum_{i\omega_n}\bar{g}^2_{\mathbf{q},\lambda}\, \mathcal{D}_\lambda^{(0)}(i\omega_n, \mathbf{q})\, \mathcal{G}_\sigma^{(0)}(ip_n - i\omega_n, \mathbf{p} - \mathbf{q}), \tag{4.13}$$

which is the contribution to the electron self-energy from one phonon, and is sometimes called the lowest-order polaron self-energy. The Feynman diagram of this interaction is the first diagram on the right hand side in figure 4.2.

Since we are in the weak-coupling regime, higher order terms in the self-energy can be neglected, because all higher-order diagrams contain higher powers of the coupling constant. Thus,

$$\Sigma \simeq \Sigma^{(1)}.$$

Unperturbed Matsubara Green's functions

It is shown in Appendix E.1 that the unperturbed electron Matsubara Green's function for the Hamiltonian (4.1) is given by

$$\mathcal{G}^{(0)}_{\sigma}(ip_n, \mathbf{p}) = \frac{1}{ip_n - \xi_{\mathbf{p},\sigma}}, \quad \text{with} \quad \xi_{\mathbf{p},\sigma} \equiv \varepsilon_{\mathbf{p},\sigma} - \mu.$$

Here, the energy of the electron $\xi_{\mathbf{p},\sigma}$ is measured with respect to the Fermi surface[2], i.e., $\xi_{\mathbf{p}_F,\sigma} = \varepsilon_{\mathbf{p}_F,\sigma} - \mu = 0$. It is also shown in Appendix E.1, that for the unperturbed phonon MGF one has

$$\mathcal{D}^{(0)}_{\lambda}(i\omega_n, \mathbf{q}) = \frac{2\omega_{\mathbf{q},\lambda}}{(i\omega_n)^2 - \omega^2_{\mathbf{q},\lambda}}.$$

4.3.1 The electron self-energy for weak coupling

Substituting the free Green's functions into the expression for the self-energy (4.13) gives

$$\Sigma^{(1)}_{\sigma}(ip_n, \mathbf{p}) = -\frac{1}{\beta\mathcal{V}} \sum_{\mathbf{q},\lambda} \bar{g}^2_{\mathbf{q},\lambda} \sum_{i\omega_n} \frac{2\omega_{\mathbf{q},\lambda}}{(i\omega_n)^2 - \omega^2_{\mathbf{q},\lambda}} \frac{1}{ip_n - i\omega_n - \xi_{\mathbf{p}-\mathbf{q},\sigma}}. \tag{4.14}$$

In Appendix E.2, it is shown that evaluating the sum over the phonon Matsubara frequencies by using contour integration, gives

$$\Sigma_{\sigma}(ip_n, \mathbf{p}) = \frac{1}{\mathcal{V}} \sum_{\mathbf{q},\lambda} \bar{g}^2_{\mathbf{q},\lambda} \left(\frac{n_B(\omega_{\mathbf{q},\lambda}) + n_F(\xi_{\mathbf{p}-\mathbf{q},\sigma})}{ip_n + \omega_{\mathbf{q},\lambda} - \xi_{\mathbf{p}-\mathbf{q},\sigma}} + \frac{n_B(\omega_{\mathbf{q},\lambda}) + 1 - n_F(\xi_{\mathbf{p}-\mathbf{q},\sigma})}{ip_n - \omega_{\mathbf{q},\lambda} - \xi_{\mathbf{p}-\mathbf{q},\sigma}} \right),$$

where n_B and n_F are the bosonic and fermionic distributions,

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1} \quad \text{and} \quad n_F(\xi) = \frac{1}{e^{\beta\xi} + 1}, \tag{4.15}$$

which are the Bose-Einstein distribution and the Fermi-Dirac distribution, respectively. As a remark, at zero temperature the Bose-Einstein distribution vanishes and the Fermi-Dirac distribution becomes a unit step function,

$$n_B(\omega) \xrightarrow{T\to 0} 0, \quad \text{and} \quad n_F(\xi) \xrightarrow{T\to 0} \theta(\mu - \xi).$$

Furthermore, as mentioned in section 4.1.1, the electron-phonon coupling function is temperature independent.

[2] Note that for not too high temperatures, the chemical potential μ and the Fermi energy ε_F are more or less equal.

Taking the system's size to infinity, $\mathcal{V} \to \infty$, so that

$$\lim_{\mathcal{V}\to\infty} \frac{1}{\mathcal{V}} \sum_{\mathbf{q}} f(\mathbf{q}) = \int \frac{d\mathbf{q}}{(2\pi)^3} f(\mathbf{q}),$$

the expression for the one-phonon self-energy of the electron assumes the form

$$\Sigma_\sigma(ip_n, \mathbf{p}) = \sum_\lambda \int \frac{d^3\mathbf{q}}{(2\pi)^3} \bar{g}_\lambda(\mathbf{q})^2 \left(\frac{n_B(\omega_\lambda) + n_F(\xi_\sigma)}{ip_n + \omega_\lambda(\mathbf{q}) - \xi_\sigma} + \frac{n_B(\omega_\lambda) + 1 - n_F(\xi_\sigma)}{ip_n - \omega_\lambda(\mathbf{q}) - \xi_\sigma} \right), \tag{4.16}$$

where ξ_σ is a shorthand for $\xi_\sigma(\mathbf{p} - \mathbf{q})$.

This concludes the section of weakly-coupled theory. In the next section, the theory for strong coupling is considered.

4.4 Strong-coupling theory

In the previous section, the electron self-energy for weak coupling was discussed. Because of the weak coupling, it was possible to ignore higher-order corrections to the electron self-energy. In this section, the electron self-energy in a strongly-coupled regime will be considered. When the coupling between electrons and phonons is strong, it is not possible to ignore higher order terms.

The electron self-energy contains diagrams of the form given in figure 4.2. Diagrams like the second one on the right hand side of this figure, contain phonon corrections to the electron-phonon vertex like the left diagram in figure 4.3. However, for metals in the *normal state*, that is, the non-superconducting state, Migdal argued in 1958, that the renormalization due to phonons of the electron-phonon vertex, is suppressed at least by a factor $\sqrt{m/M} \sim 10^{-2}$, where m and M are the masses of the electron and ion [64]; see figure 4.3. Since corrections in the electron-phonon vertex are suppressed, one only has to consider diagrams of the form displayed in figure 4.4 to calculate the electron self-energy.

The proof builds on the assumption of the presence of a regular Fermi surface, and the renormalization of high-frequency phonons to low-frequency acoustical phonons by electron screening processes.[3] This agrees with experiments, where only the latter were observed. For an explicit calculation showing this renormalization, see Chapter 17 of [62].

The generalization of Migdal's argument to superconducting metals was given by Eliashberg in 1960 [66], who presented an improvement of the BCS theory, because he took into account the retardation of the electron-phonon interaction. A further generalization by including Coulomb interactions was given in [67], from which the theory of this section was mainly taken.[4] A slightly modified form of this version can be found in [63].

4.4.1 The electron self-energy

Instead of using the unperturbed matsubara Green's functions in the electron self-energy (4.13), one needs to include the fully dressed Green's functions in strong-coupling theory, as

[3]In [65], it has been shown that Migdal's argument also holds when the limiting energy of the phonons is much larger than the Fermi energy instead. Here, also a compact reformulation of Migdal's original discussion is presented.

[4]In case one has doubts about the validity of Migdal's argument in the strong-coupling limit: it has been shown in [70], that in a large-N approach, where N is the number of fermionic flavors, vertex corrections are of order $1/N$.

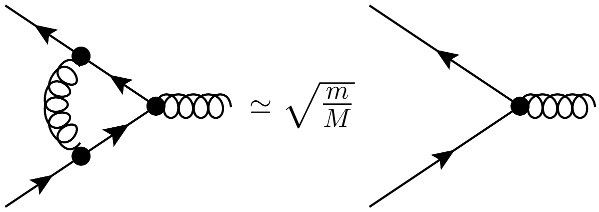

Figure 4.3: Migdal's argument for electron-phonon vertex corrections.

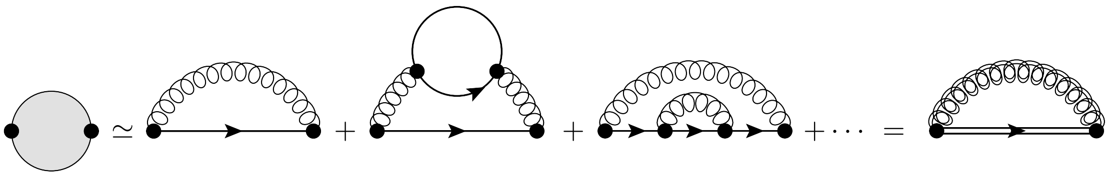

Figure 4.4: Reduced electron self-energy due to ignoring vertex corrections. The double lines in the diagram on the right denote the fully dressed electron and phonon propagators.

can be seen from the rightmost diagram in figure 4.4. This diagram has the expression

$$\Sigma_\sigma(ip_n, \mathbf{p}) = -\frac{1}{\beta}\sum_\lambda \int \frac{d^3\mathbf{q}}{(2\pi)^3}\,\bar{g}_\lambda(\mathbf{q})^2 \sum_{i\omega_n} \mathcal{D}_\lambda(i\omega_n, \mathbf{q})\mathcal{G}_\sigma(ip_n - i\omega_n, \mathbf{p}-\mathbf{q}). \tag{4.17}$$

By performing a calculation as in Appendix D.1, one obtains the spectral decomposition, and by using the *electron spectral density* $\mathcal{A}_\sigma(\omega, \mathbf{p}) = -2\,\mathrm{Im}\,G^R_\sigma(\omega, \mathbf{p})$ (cf. (3.59)), the electron Matsubara Green's function can be written in its Lehmann representation (cf, (3.60)),

$$\mathcal{G}_\sigma(ip_n - i\omega_n, \mathbf{p}-\mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi}\,\frac{\mathcal{A}_\sigma(\varepsilon', \mathbf{p}-\mathbf{q})}{ip_n - i\omega_n - \varepsilon'}. \tag{4.18}$$

Likewise, using the *phonon spectral density* $\mathcal{B}_\lambda(\omega, \mathbf{q}) = -2\,\mathrm{Im}\,D^R_\lambda(\omega, \mathbf{q})$, the phonon Matsubara Green's function has the Lehmann representation

$$\mathcal{D}_\lambda(i\omega_n, \mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\,\frac{\mathcal{B}_\lambda(\omega', \mathbf{q})}{i\omega_n - \omega'} = \int_0^{\infty} \frac{d\omega'}{2\pi}\,\mathcal{B}_\lambda(\omega', \mathbf{q})\left(\frac{1}{i\omega_n - \omega'} - \frac{1}{i\omega_n + \omega'}\right),$$

where in the second equality the antisymmetry of the spectral function with respect to ω' has been exploited, which follows from (D.3).

By comparing the Lehmann representations of the Matsubara Green's function (4.18) and the retarded Green's function (3.60), it is clear that the retarded Green's functions can be obtained from the Matsubara Green's functions by an analytic continuation,

$$\mathcal{G}_\sigma(ip_n \to \xi + i0, \mathbf{p}) = G^R_\sigma(\xi, \mathbf{p}), \quad \text{and} \quad \mathcal{D}_\lambda(i\omega_n \to \omega + i0, \mathbf{q}) = D^R_\lambda(\omega, \mathbf{q}). \tag{4.19}$$

With the Lehmann representations of the Matsubara Green's functions, the self-energy has the form

$$\Sigma_\sigma(ip_n, \mathbf{p}) = \sum_\lambda \int \frac{d^3\mathbf{q}}{(2\pi)^3}\bar{g}_\lambda(\mathbf{q})^2 \int_0^{\infty} \frac{d\omega'}{2\pi}\,\mathcal{B}_\lambda(\omega', \mathbf{q}) \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi}\,\mathcal{A}_\sigma(\varepsilon', \mathbf{p}-\mathbf{q})\times$$
$$\times \left(S_0(\omega', \varepsilon', ip_n) - S_0(-\omega', \varepsilon', ip_n)\right),$$

where

$$S_0(\omega', \varepsilon', ip_n) \equiv -\frac{1}{\beta} \sum_{i\omega_n} \frac{1}{i\omega_n - \omega'} \frac{1}{ip_n - i\omega_n - \varepsilon'} = \frac{n_B(\omega') + n_F(\varepsilon')}{ip_n + \omega' - \varepsilon'}$$

is the sum over the product of the non-interacting Green's functions. This sum over Matsubara frequencies has been evaluated by employing (E.1).

4.4.2 Eliashberg spectral function

By using $n_B(-\omega') = -(1 + n_B(\omega'))$, the expression for the self-energy becomes

$$\Sigma_\sigma(ip_n, \mathbf{p}) = \int_0^\infty d\omega' \int \frac{d^3\mathbf{q}}{(2\pi)^3} \, \alpha^2 F(\omega', \mathbf{q}) \, K_\sigma(ip_n, \mathbf{p} - \mathbf{q}, \omega', T), \tag{4.20}$$

where the *kernel* is defined as

$$K_\sigma(ip_n, \mathbf{p}', \omega', T) \equiv \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi} \mathcal{A}_\sigma(\varepsilon', \mathbf{p}') \left(\frac{n_B(\omega') + n_F(\varepsilon')}{ip_n + \omega' - \varepsilon'} + \frac{n_B(\omega') + 1 - n_F(\varepsilon')}{ip_n - \omega' - \varepsilon'} \right), \tag{4.21}$$

which describes the thermal excitations of the bosons and electrons. In addition, the *electron-phonon spectral function* is defined as

$$\alpha^2 F(\omega', \mathbf{q}) \equiv \frac{1}{2\pi} \sum_\lambda \bar{g}_\lambda(\mathbf{q})^2 \, \mathcal{B}_\lambda(\omega', \mathbf{q}). \tag{4.22}$$

This electron-phonon spectral function is also called the *Eliashberg spectral function*, and it is a measure for the effectiveness of scattering electrons from a state $\mathbf{p}$ on the Fermi surface to a state $\mathbf{p}-\mathbf{q}$ on the Fermi surface. Explicit temperature dependence of this spectral function has been dropped, since the temperature dependence of the boson spectral density is very weak [68]. Note that all the information about the phonons is contained in this electron-phonon spectral function.

It is worthwhile to mention that the only approximations that have been made up to this point, are the disregarding of vertex corrections by Migdal's argument, and ignoring short-range screened Coulomb repulsions. The latter will not be discussed here, but for a treatment see [67]. The reason is that, besides giving a renormalization, they will only play a role in the anomalous Green's functions, which are defined at the end of this section.

In the literature, the Eliashberg spectral function is frequently found averaged over the Fermi surface

$$\alpha^2 F(\omega) = \langle \alpha^2 F(\omega, \mathbf{q}) \rangle_{\text{FS}} = \int_{\text{FS}} \frac{d^2 q}{v_F} \, \alpha^2 F(\omega, \mathbf{q}) \Big/ \int_{\text{FS}} \frac{d^2 q}{v_F}.$$

This function can be interpreted as a measure for the effectiveness of scattering electrons from any state on the Fermi surface to any other state on the Fermi surface. An example of such an averaged spectral function is given shortly.

The self-energy in the weak-coupling regime (4.16) can be obtained by substituting the unperturbed spectral functions $\mathcal{A}_\sigma^{(0)}(\varepsilon', \mathbf{p}) = 2\pi\delta(\varepsilon' - \xi_\sigma(\mathbf{p}))$ and $\mathcal{B}_\lambda^{(0)}(\omega', \mathbf{q}) = 2\pi\delta(\omega' - \omega_\lambda(\mathbf{q}))$ into (4.20).

4.4.3 Reduction to a single variable

It is possible to reduce the integrals in (4.20) to an integral over a single variable. The following description is adapted from [63] and [67]. We drop spin indices throughout this section.

We assume that the Fermi sphere is spherical, which is the case for an isotropic system. Since we have assumed that only acoustical phonons are present, the phonon frequencies will be lower than the Debye frequency ω_D, which is the maximum frequency phonons in a simplified model can have. This Debye frequency is much smaller than the Fermi energy ε_F. Therefore, only electrons with momenta close to the Fermi surface will be susceptible to scattering by phonons. In addition, since we have assumed that the Fermi surface is spherical, the $\mathbf{p}$ dependence of the self-energy is unimportant. Hence, it will have arguments (ip_n, p_F), where p_F is the Fermi momentum.

The first step is to rewrite the phonon wave vector integral:

$$\int d^3\mathbf{q} \to \int_0^\infty dq\, q^2 \int_{-1}^1 d\theta' \int_0^{2\pi} d\phi, \quad \text{with} \quad \theta' = \cos\theta,$$

where $q = \sqrt{\mathbf{q}\cdot\mathbf{q}}$, and θ is the angle between $\mathbf{p}$ and $\mathbf{q}$; see figure 4.5. Note that q would normally run over the first Brillouin zone, however by including Umklapp processes, the integral has been extended to infinity.

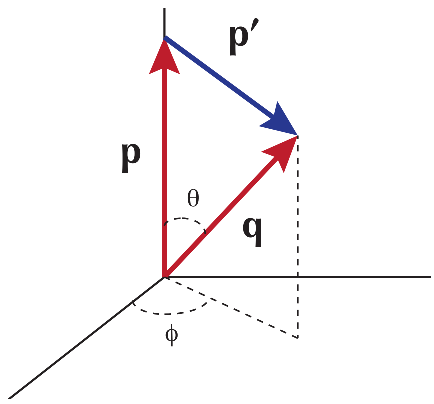

Figure 4.5: Coordinate system for carrying out the wave vector integral.

Next, define $\mathbf{p}' \equiv \mathbf{q} - \mathbf{p}$, which is the wave vector for the electron in the intermediate state in figure 4.4. Then, for $p' = \sqrt{\mathbf{p}'\cdot\mathbf{p}'}$, we find $p'dp' = pq\,d\theta'$, and we can write

$$\int d^3\mathbf{q} \to \frac{1}{p}\int_0^\infty dq\, q \int_{|p-q|}^{p+q} dp'\, p' \int_0^{2\pi} d\phi.$$

The second step is to define

$$m\, d\xi_{p'} \equiv p'\, dp'.$$

so that the phonon wave-vector integral can be written as

$$\int d^3\mathbf{q} \to \frac{m}{p}\int_0^\infty dq\, q \int_{\frac{(p-q)^2}{2m}-\varepsilon_F}^{\frac{(p+q)^2}{2m}-\varepsilon_F} d\xi_{p'} \int_0^{2\pi} d\phi,$$

where ε_F is the Fermi energy. Since everything happens around the Fermi surface, we can replace p by p_F, and $m/p \to m/p_F = 1/v_F$, where v_F is the Fermi velocity.

The last step is to decouple the limits of integration over $\xi_{p'}$. Employing the above argument, the dominant contribution to this integral comes from the region $|\xi_{p'}| \leq \omega_D$. As a result, the limits of the $\xi_{p'}$ integral can be extended:

$$\int d^3\mathbf{q} \to \frac{1}{v_F} \int_0^{2p_F} dq\, q \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\xi_{p'}. \tag{4.23}$$

In addition, the upper limit of the q integral has been reduced, since the wave vectors $\mathbf{p}$ and $\mathbf{p}'$ are restricted to lie close to the Fermi surface, which limits $q \leq 2p_F$ for a spherical Fermi surface. The maximum momentum transfer $2p_F$ happens when an electron with momentum $\mathbf{p}_F$ scatters to a state with momentum $-\mathbf{p}_F$. The integrals over $q\, dq$ and $d\phi$ are understood to be over the spherical Fermi surface, from a point $\mathbf{p}_F$ on the Fermi surface to all other points $\mathbf{p} - \mathbf{q}$ on the Fermi surface.

Substituting (4.23) for the phonon wave vector integral, the electron self-energy becomes

$$\Sigma(ip_n, p_F) = \int_0^{\infty} d\omega'\, \alpha^2 F(\omega', p_F) K(ip_n, \omega', T), \tag{4.24}$$

where the kernel is now given by

$$K(ip_n, \omega', T) = \int_{-\infty}^{\infty} d\xi_{p'} \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi} \mathcal{A}(\varepsilon', \xi_{p'}) \left(\frac{n_B(\omega') + n_F(\varepsilon')}{ip_n + \omega' - \varepsilon'} + \frac{n_B(\omega') + 1 - n_F(\varepsilon')}{ip_n - \omega' - \varepsilon'} \right),$$

and the Eliashberg spectral function has the form

$$\alpha^2 F(\omega', p_F) = \frac{1}{v_F (2\pi)^4} \int_0^{2p_F} dq\, q \int_0^{2\pi} d\phi \sum_{\lambda} \bar{g}_{\lambda}(\mathbf{q})^2\, \mathcal{B}_{\lambda}(\omega', \mathbf{q}), \tag{4.25}$$

which is an integral over the Fermi surface. The dependence on the Fermi momentum p_F will not be written henceforth. A slightly more formal separation of the 'energy' and 'angular' parts of the phonon wave vector integration is given in [69].

Further reduction for the normal state

For metals in the normal state, Migdal argued further that one is allowed to replace the electron Green's function by the non-interacting Green's function. This is called *Migdal's theorem* [64]. This theorem is not (known to be) valid in the superconducting state. Moreover, vertex corrections are vital for superconductivity in the BCS theory [63].

Using Migdal's theorem, we are allowed to replace the electron spectral density by its unperturbed one, $\mathcal{A}(\varepsilon', \xi_{p'}) = \mathcal{A}^{(0)}(\varepsilon', \xi_{p'}) = 2\pi\delta(\varepsilon' - \xi_{p'})$ [64]. In that case, the kernel of (4.24) is given by

$$K(ip_n, \omega', T) = \int_{-\infty}^{\infty} d\varepsilon \left(\frac{n_B(\omega') + n_F(\varepsilon)}{ip_n + \omega' - \varepsilon} + \frac{n_B(\omega') + 1 - n_F(\varepsilon)}{ip_n - \omega' - \varepsilon} \right), \tag{4.26}$$

where we have changed the integration variable to ε.

From (4.17) and (4.19), it is clear that the *retarded* self-energy can be obtained by making the substitution $ip_n \to \xi + i\delta$:

$$\Sigma^R(\xi) = \int_0^\infty d\omega' \, \alpha^2 F(\omega') K^R(\xi, \omega', T), \tag{4.27}$$

where $K^R(\xi, \omega', T) \equiv K(ip_n \to \xi + i0, \omega', T)$.

The integral over ε in the kernel (4.26) can be performed analytically. In Appendix B of [69], it is shown that this gives the result

$$K^R(\xi, \omega', T) = -i\pi \coth\left(\frac{\xi}{2T}\right) + \Psi\left(\frac{1}{2} + i\frac{\xi - \omega'}{2\pi T}\right) - \Psi\left(\frac{1}{2} - i\frac{\xi + \omega'}{2\pi T}\right), \tag{4.28}$$

where $\Psi(x)$ is the digamma function, $\Psi(x) = d \ln \Gamma(x)/dx$.

Superconductivity state

It is worthwhile to mention that for metals in the superconducting state one also has to include

$$\mathcal{F}(\tau - \tau', \mathbf{p}) = \langle \mathcal{T}_\tau c_{-\mathbf{p},\downarrow}(\tau) c_{\mathbf{p},\uparrow}(\tau') \rangle, \quad \text{with} \quad -\beta \leq \tau - \tau' \leq \beta,$$

and its complex conjugate in the theory; see e.g. [63]. These are the so-called *anomalous Green's functions*, and they appear because in the superconducting state, the ground state is a superposition of electronic states containing a different number of electrons, so that it is possible to have two annihilation or creation operators between a ground-state bra and ket. Details can be found in [67] and [63].

In this section we have found an expression for the electronic self-energy and showed two reduced forms. Next, we will generalize the Eliashberg spectral function to a glue function and discuss some outcomes from experiments.

4.5 Glue function

The description of the electron-phonon interaction in the previous section included retarded effects due to electron-phonon interactions. Effects due to the Coulomb interaction, which are non-retarded, or, instantaneous, were ignored or absorbed into the electron-phonon coupling.

In the pursuit of understanding the pairing mechanism between conducting electrons in high-T_c superconductors, various sources for the pairing have been studied. These include the phonons studied in the previous sections, but also other virtual bosons, such as *magnons*, which are quantized spin waves, and *plasmons*, which are quantized collective electronic charge-density waves.

Since the Eliashberg spectral function (4.22) or (4.25) contains all the information about the phonons in the system, and the kernel contains the electronic information and thermal occupation factors, to describe pairing induced by other bosons, one just has to replace the Eliashberg spectral function in the theory. For example, in case of magnons the electron-magnon spectral function is denoted by $I^2\chi(\omega)$.

As the pairing mechanism in high-T_c superconductors is unknown, the electrons are commonly thought to be held together by some 'pairing glue' with unknown (boson-induced) constituents. In that case, one generalizes the Eliashberg spectral function to a *glue function* $\tilde{\Pi}(\omega)$, in which the contributions from various bosonic sources are lumped. The expression for the retarded electron self-energy (4.27) is then changed to

$$\Sigma^R(\xi) = \int_0^\infty d\omega\, \tilde{\Pi}(\omega) K^R(\xi, \omega, T). \tag{4.29}$$

Because the electron-phonon coupling in the Eliashberg spectral functions (4.22) and (4.25) was renormalized due to Coulomb interactions, non-retarded effects due to these interactions are also present in the glue function.

4.5.1 Measurements of the glue function

Using modern experimental techniques, such as angle-resolved photo emission (ARPES), the glue function can be measured. See, e.g., [71] for an introduction to ARPES. Additionally, the glue function can be obtained by means of optical spectroscopy from which the *optical conductivity* $\sigma(\omega)$ can be determined. The optical conductivity is the frequency-dependent conductivity in a metal, which is measured by the absorption, transport and emission of photons by electrons. From this optical conductivity, one is able to determine the single-particle electron self-energy by using a relation given by Allen [72, 73],

$$\sigma(\omega) = \frac{ine^2}{m\omega} \int_{-\infty}^{\infty} d\xi\, \frac{n_F(\xi) - n_F(\xi + \omega)}{\omega - \Sigma^R(\xi + \omega) + \Sigma^{R*}(\xi)}. \tag{4.30}$$

Here, Σ^R is the electron self-energy averaged over the Fermi surface (4.27), and e, m and n are the electron charge, mass and density, respectively. This expression is valid when anisotropies of the Fermi surface can be ignored.

In deriving the expression for the electron self-energy (4.27), and the above relation with the optical conductivity, the following assumptions have been made: (i) isotropy of the system, so that the Fermi sphere is spherical, (ii) the frequency of the bosons is of order of the Debye frequency, and (iii) vertex corrections can be ignored. These assumptions are reasonable for conventional superconductors in the *normal state*. However, vertex corrections may play a role, as they do in the BCS theory that describes superconductivity in conventional superconductors.[5] For that reason, the glue function $\tilde{\Pi}(\omega)$ is sometimes *defined* as the effective quantity that returns the exact value of $\sigma(\omega)$ for each frequency [74].

The glue function has been measured for various high-T$_c$ superconducting materials by several groups. For a comparison of these measurements of the glue function, see [5]. Recently, Heumen et al. have measured the glue functions of two high-T$_c$ cuprates at three temperatures and four doping levels [74]. They have observed a peak in the glue function in the 50–60 meV range. A remarkable thing about these peaks is, that the energies at which the peaks appear seem to be independent of temperature and doping level. In addition, the position of the peak does not seem to vary for different compounds. Finally, there is a doping-dependent continuum extending to 300–400 meV for the samples with the highest T$_c$. The results are displayed in figure 4.6

Although the position of the peaks with respect to the energy does not vary with temperature, the form of the peaks does. The temperature dependence of the glue functions that are displayed in figure 4.6, is discussed by Heumen et al. in [68]. Although the Eliashberg

[5] Vertex corrections give rise to the Cooper instability [63].

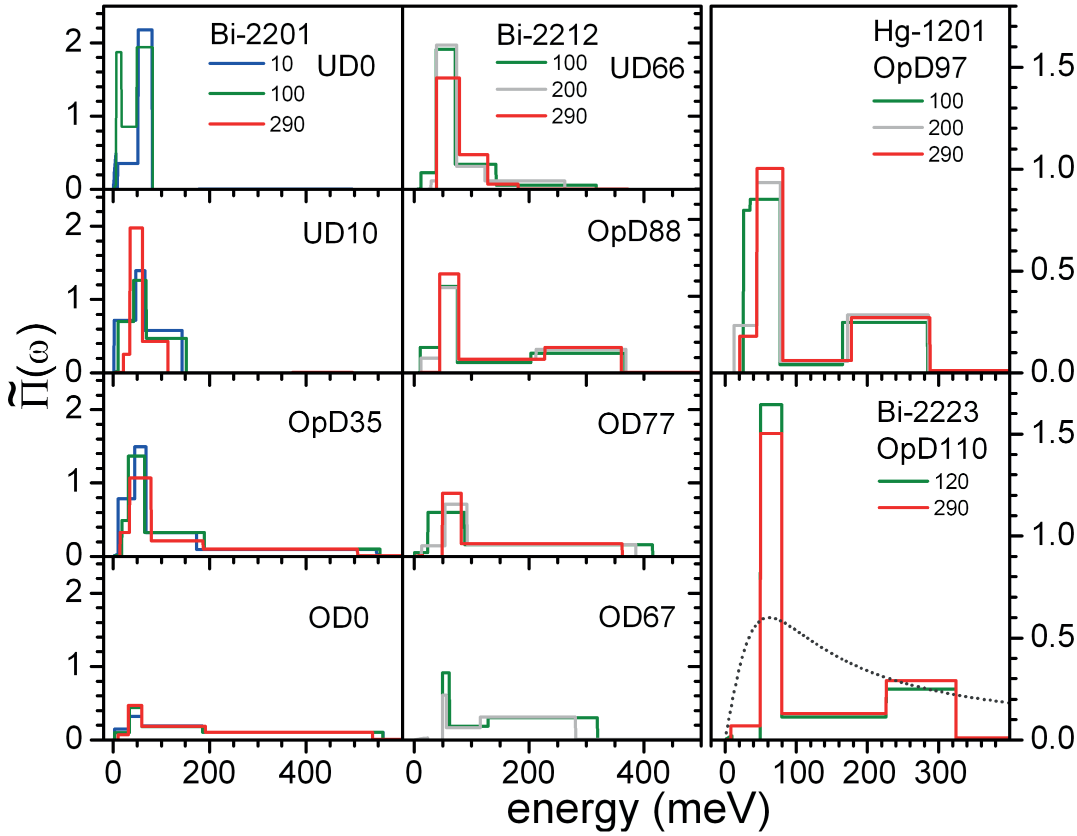

Figure 4.6: Electron-boson glue function for various doping levels and temperatures (denoted by the colors). The doping level increases from top to bottom, where UD means under doped, OD over doped, and OpD means optimally doped. The numbers in, e.g., OpD88, denote the critical temperature T_c. Note the peak in the 50–60 meV range. Figure taken from [74].

spectral function (4.25) (or (4.22)) was said to be temperature independent —which is true for phonons— there can enter some dependence through the boson spectral function $\mathcal{B}$, when the Eliashberg spectral function is generalized to a glue function to account for some unknown bosonic source. It has been argued in [70], that the temperature dependence can be relevant if the bosons derive from electronic degrees of freedom. The temperature dependence of the measured glue functions, suggest there is at least some contribution from these kind of bosons.

Now, that the electron-phonon spectral function has been generalized to a generic electron-boson spectral function, that is, the glue function $\tilde{\Pi}(\omega)$, we will consider options for translating the electron self-energy of the form (4.29) using the AdS/CFT correspondence in the next section.

4.6 Gravity dual for the electron self-energy

In this section, we will give suggestions for finding a dual description of the electron self-energy (4.29). To construct a gravity dual for this self-energy, one is immediately confronted with the question how to model phonons in the dual AdS spacetime. We will start with proposing some options for this. The ideas that are presented in this section are sketchy and have to be worked out further.

4.6.1 Ideas for calculating the electron self-energy in AdS/CFT

The phonons discussed in this chapter are quasiparticles associated with lattice vibrations. For this reason, a suggestion for modeling phonons in the gravity dual could be a lattice in AdS spacetime, and study its vibrations. Lattices in the AdS/CFT correspondence are discussed in [75]. However, we would probably not gain much information from this approach, since the dual model will somehow be an engineered copy of the original one, albeit at weak coupling.

A totally other approach would be to study the optical conductivity in AdS/CFT, since the electron self-energy in the aforementioned experiments is obtained by measuring this quantity. The optical conductivity for a fermionic system is given by the Kubo formula,

$$\sigma(\omega) = \frac{1}{i\omega}\langle J_x(\omega)J_x(-\omega)\rangle_{\text{ret}},$$

where J_x is a current density at zero spatial momentum. For example, this optical conductivity is calculated in a $2+1$-dimensional AdS spacetime in the presence of a charged extremal black hole in [53]. However, a generalization to a non-extremal black hole is necessary, in order to have a nonzero temperature. One could then try to relate this optical self-energy to (4.30).

The most natural approach seems to be a model where phonons correspond to a bosonic field in the bulk. The bulk action that describes a bosonic field that is coupled to a Dirac fermionic field, has roughly the form

$$S_{\text{bulk}} = S[\phi] + S[\psi] + g_{\text{el-ph}}\int d^{d+1}x\,\sqrt{-g}\,\phi\bar{\psi}\psi, \tag{4.31}$$

where $S[\phi]$ is the action of a free boson (see section 3.4), $S[\psi]$ the action describing a free Dirac fermion, and $g_{\text{el-ph}}$ is the fermion-boson coupling. In addition, there should be a Einstein-Hilbert term to account for the gravitons, and in order to have a finite temperature in the CFT, the metric should describe a black hole.

Because of the background's curvature, solving the equations of motions for the electron that follow from this action is not a trivial task. By imposing ingoing boundary conditions at the horizon and using an expression like (3.53), the retarded Green's for the electron can be calculated.

As the coupling constant g in the bulk is small, one can make a perturbative expansion of the retarded Green's function of the form (cf. (4.12))

$$G_R(\omega,k) = G_R^{(0)}(\omega,k) + G_R^{(0)}(\omega,k)^2\,\Sigma^{(1)}(\omega,k) + \mathcal{O}(g^4).$$

If it is possible to get a structure as in (4.29) for the electron self-energy, then by stripping off the kernel, information about the glue function can be obtained, and one can try to reproduce the observed peak. At first sight, finding a dual for the kernel seems to be a daunting task, but in the next subsection we will consider a possible candidate for it.

Another possibility would be regarding phonons as density waves, since the phonons are related to lattice vibrations. In section 3.1.1, it was shown that the energy density T_{00} is related to g_{00}. Therefore, the dual system to consider will be a model like (4.31) with the bosonic field ϕ replaced by g_{00}. Finally, a possibility would be to replace the bosonic field ϕ by a gauge field A_μ.

4.6.2 Gravity dual for the kernel

Son and Starinets [47], have calculated the retarded Green's function for a massive scalar field in a non-extremal BTZ black hole background, which is a solution of the Einstein equation with a negative cosmological constant (3.2) in $2+1$-dimensions.

As explained in section 3.4.2, there are two branches for the scalar field in the bulk. By choosing a branch, one determines whether $\phi_{(0)}$ or $\phi_{(1)}$ in (3.29) is the source for the dual operator in the CFT. Son and Starinets have calculated the Green's function for the '+' branch, which corresponds to the form (3.29) for the near-boundary behavior of the scalar field, with $\Delta = \Delta_+$. Here, Δ is the conformal dimension of the corresponding operator in the two-dimensional CFT.

For $\Delta_+ = 1/2$,[6] they have obtained an expression of the form

$$G^R(p_+, p_-) = C\left[\Psi\left(\frac{1}{2} - \frac{ip_+}{2\pi T_L}\right) + \Psi\left(\frac{1}{2} - \frac{ip_-}{2\pi T_R}\right)\right],$$

where C is some normalization constant, and $\Psi(x)$ are digamma functions. Further, the momenta $p_\pm$ are defined as

$$p_+ = \frac{\omega - k}{2} \quad \text{and} \quad p_- = \frac{\omega + k}{2},$$

and the temperatures are given by

$$T_{L,R} = \frac{\rho_+ \mp \rho_-}{2\pi},$$

where $\rho_\pm$ are the locations of the outer $(+)$ and inner $(-)$ horizons of the BTZ black hole. By substituting the momenta $p_\pm$, the Green's function can be rewritten as

$$G^R(k, \omega) = C\left[\Psi\left(\frac{1}{2} + i\frac{k - \omega}{2\pi \tilde{T}_L}\right) + \Psi\left(\frac{1}{2} - i\frac{k + \omega}{2\pi \tilde{T}_R}\right)\right], \tag{4.32}$$

with $\tilde{T}_{L,R} = 2\,T_{L,R}$.

The value $\Delta_+ = 1/2$ for the conformal dimension of the dual operator, satisfies the unitarity bound (3.30). From the relation (3.28), which expresses Δ_+ in terms of the scalar mass and AdS radius, we see that (4.32) is the Green's function for a tachyonic scalar field; see section 3.4.4.

For $\tilde{T}_L = \tilde{T}_R$ this retarded Green's function has a striking resemblance to the kernel (4.28). However, there are a few differences. Firstly, there is an extra term in (4.28). Nevertheless, this term does not depend on ω', and can be taken out of the integral in (4.29). Secondly, there is a minus sign between the digamma functions in (4.28). In spite of that, the similarity between this Green's function and the kernel (4.28), leads to the belief that it is possible to find a separation of the electron self-energy into something like the kernel and glue function in the bulk spacetime.

[6] We have made a slight modification in notation: Δ_+ here, corresponds to $\beta_+ = \Delta_+/2$ of [47].

5 Conclusion

The search for a bulk dual of a glue function is far from complete. One of the first problems one encounters in this endeavor, is the question what a useful dual description for phonons is. The most appropriate candidate for such a description of phonons is probably a bosonic bulk field. In the bulk, one can then couple a Dirac fermion field to this bosonic field. If one is able to solve the field equations for this fermionic field, the retarded Green's function of it can be calculated. The next step would be stripping off the temperature-dependent kernel that is present in the electron self-energy, so that one is left with the glue function. This may seem challenging, but the similarity between this kernel and the expression for the retarded Green's function for a bosonic field in a BTZ black hole background, suggests that it is not too hard to find a description for the kernel in the bulk. This raises the hope, that in this way it is possible to gain insight in the pairing mechanism between electrons in high-T_c superconductors.

In this thesis, we have reviewed how the AdS/CFT correspondence follows from type-IIB superstring theory. The AdS/CFT correspondence provides an important tool to study strongly-coupled systems at the field-theory side, since the correspondence asserts that this system is dual to a weakly-coupled theory in the AdS spacetime, and vice versa. Further, it was shown, that adding a black hole to the bulk, corresponds to placing the field theory at a finite temperature. Likewise, for adding a Maxwell field to the bulk, and obtaining a finite chemical potential at the field theory side. The effect and subtleties of adding a scalar field to the AdS spacetime have been discussed subsequently. A relation for the Green's function of a scalar operator in the CFT was given in terms of boundary values of this bulk field.

Properties of retarded Green's functions and spectral functions have been reviewed quite extensively. On top of that, Green's functions at finite temperature have been studied. These Green's functions were analysed for a model that describes the electron-phonon interaction. An expression for the electron self-energy was obtained for strong electron-phonon coupling. It was possible to rewrite this expression as an integral over a temperature-independent part that contains the phononic information, and a temperature-dependent part containing the electronic information. These are the Eliashberg (or, electron-phonon) spectral function and the kernel, respectively. The former has been generalized to a glue function, to account for an unknown bosonic source.

In the concluding chapter, we have tried to translate the expression for the electron self-energy involving the glue function in the AdS/CFT correspondence. Thus far, only preparatory work has been done. A first step for further research, would probably be solving the equations of motion for an electron that is coupled to a boson in the bulk. If one, in addition, is able to find an exact expression for the kernel, one can isolate the glue function, and try to reproduce the observed peaks.

A
Differential forms

p-Form fields

A p-form field is given by

$$A_p \equiv \frac{1}{p!} A_{\mu_1\mu_2\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$

where $A_{\mu_1\mu_2\cdots\mu_p}$ denotes the components with respect to a coordinate basis, and the wedge product of p basis 1-forms $\mathrm{d}x^\mu$ is given by

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \equiv \sum_\pi \mathrm{sgn}(\pi) \mathrm{d}x^{\mu_{\pi(1)}} \otimes \mathrm{d}x^{\mu_{\pi(2)}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{\pi(p)}},$$

with $\mathrm{sgn}(\pi) = +1, -1$ for even or odd permutations respectively. Note that by the way it is defined, $A_{\mu_1\mu_2\cdots\mu_p}$ is antisymmetric in its indices.

A p-form field naturally couples to geometrical objects Σ_p with spacetime dimension p via

$$S = T_p \int_{\Sigma_p} A_p.$$

This action is invariant under reparametrizations, and is invariant under the gauge transformation

$$A_p \to A_p + \mathrm{d}\Lambda_{p-1}, \tag{A.1}$$

under which S changes by a total derivative.

A p-form field A_p has a field strength F_{p+1} associated to it,

$$F_{p+1} = \mathrm{d}A_p$$

which is gauge invariant under (A.1).

Wedge product of two forms

The wedge product of a p-form A_p and a q-form B_q is defined as

$$A \wedge B \equiv \frac{1}{p!q!} A_{\mu_1\mu_2\cdots\mu_p} B_{\nu_1\nu_2\cdots\nu_q} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2} \wedge \cdots \wedge \mathrm{d}x^{\nu_q},$$

so that

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]},$$

where $A_{[\ldots}B_{\ldots]}$ denotes antisymmetrization.

Exterior derivative

The exterior derivative allows to differentiate a p-form field to obtain an $(p+1)$-form field. It is defined as

$$\mathrm{d}A_p \equiv \frac{1}{p!}\partial_\nu A_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$

so that for the components one has

$$(\mathrm{d}A_p)_{\mu_1\mu_2\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}.$$

Note that in particular $\mathrm{d}(\mathrm{d}A_p) = 0$ for any p, which is often written as $\mathrm{d}^2 = 0$.

Levi-Civita tensor

The Levi-Civita symbol $\tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n}$ equals $+1$ for even permutations of $\mu_1\mu_2\cdots\mu_n$, -1 for odd permutations, and 0 otherwise. It does not transform as a tensor under coordinate transformations. The Levi-Civita tensor, defined as

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n},$$

does transform as a tensor. Note that by acting with the (inverse) metric $g^{\mu\nu}$ indices can be raised.

Hodge dual

The Hodge duality operator maps p-forms into q-forms, where $q = n-p$ with n the dimensions of the coordinate system. It is defined as

$$\star(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) \equiv \frac{1}{q!}\varepsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q},$$

so that

$$(\star A_p)_{\mu_1\cdots\mu_q} = \frac{1}{p!}\varepsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_q}A_{\nu_1\cdots\nu_p}.$$

Applying the Hodge star twice returns plus or minus the original form,

$$\star\star A_p = (-1)^{s+pq}A_p,$$

where s is the number of minus signs in the eigenvalues of the metric.

Each p-form field A_p has a magnetic dual $A^{\mathrm{magn}}_{n-p-2}$ which is a form field of rank $n-p-2$, whose field strength is given by

$$\mathrm{d}A^{\mathrm{magn}}_{n-p-2} \equiv \star\mathrm{d}A_p.$$

B Anti-de-Sitter spacetime

In this appendix, a short overview of Anti-de-Sitter spacetimes is given. It is by no means intended to be complete. For more details, see e.g. [25], and in the context of the AdS/CFT correspondence, e.g. [14] or [81].

The Einstein-Hilbert action in $d+1$ dimensions with a cosmological term is given by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{-g} \left(\mathcal{R} - 2\Lambda \right).$$

Its equation of motion is the vacuum Einstein equation with cosmological constant,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{B.1}$$

with Ricci scalar and tensor

$$\mathcal{R} = 2\frac{d+1}{d-1}\Lambda \quad \text{and} \quad \mathcal{R}_{\mu\nu} = \frac{2\Lambda}{d-1} g_{\mu\nu}.$$

Spacetimes with maximal symmetry additionally obey

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{\mathcal{R}}{d(d+1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}).$$

Maximally symmetric spacetimes with $\mathcal{R} = 0$ are simply flat Minkowski spacetimes. For positive constant curvature, $\mathcal{R} > 0$, these spaces are called de-Sitter spacetimes, and with negative constant curvature, $\mathcal{R} < 0$, *anti-de-Sitter (AdS) spacetimes*. AdS spacetimes thus have a negative cosmological constant Λ.

B.1 Representation by embedding

The $(d+1)$-dimensional Anti-de-Sitter (AdS) spacetime can be represented as a hyperboloid

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R^2, \tag{B.2}$$

embedded in a flat $(d+2)$-dimensional spacetime with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2. \tag{B.3}$$

Equation (B.2) can be solved by introducing the coordinates

$$X_0 = R\cosh\rho\,\cos\tau, \qquad X_{d+1} = R\cosh\rho\,\sin\tau,$$

$$X_i = R\sinh\rho\,\Omega_i,\;\; i=1,\ldots,d, \qquad \sum_{i=1}^{d}\Omega_i^2 = 1.$$

For $0 \leq \rho$ and $0 \leq \tau < 2\pi$, these coordinates cover the hyperboloid exactly once, and for this reason they are called *global coordinates*. By substituting these coordinates into (B.3), one obtains the metric on AdS_{d+1}

$$ds^2 = R^2(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega^2), \tag{B.4}$$

with $d\Omega^2$ is the metric on a unit $(d-1)$-sphere, because of the restriction on the d coordinates Ω_i.

For small ρ, the metric on AdS_{d+1} has the form

$$ds^2 = R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2),$$

which is topologically $S^1 \times \mathbb{R}^{d+1}$, where S^1 represents closed timelike curves in the τ direction. The closed timelike curves can be removed by considering the *universal cover*, where one unwraps the circle S^1, i.e., one takes $-\infty < \tau < \infty$.

From the representation (B.2), it is clear that the isometry group of AdS_{d+1} is $\mathrm{SO}(2,d)$, and that it is homogeneous and isotropic. The group $\mathrm{SO}(2,d)$ has a maximal compact subgroup $\mathrm{SO}(2)\times\mathrm{SO}(d)$, where the $\mathrm{SO}(2)$ subgroup corresponds to translations in the τ direction, and $\mathrm{SO}(d)$ gives rotations of the sphere S^{d-1}.

B.2 Conformal compactification

Under a series of coordinate changes, the Minkowski spacetime $\mathbb{R}^{1,d-1}$, with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2,$$

can be transformed into the form [14]:

$$d\tilde{s}^2 = -d\tau^2 + d\theta^2 + \sin^2\!\theta\, d\Omega_{d-2}^2, \quad \text{with} \quad 0 \leq \theta < \pi,\; |\tau| + \theta < \pi, \tag{B.5}$$

where $d\tilde{s}^2$ is equal to ds^2 up to a conformal transformation, $d\tilde{s}^2 = \omega^2(\tau,\theta,\ldots)ds^2$. The metric of $\mathbb{R}^{1,d-1}$ is therefore conformally mapped into the interior of a triangle that is the right half of a square which has its corners at $\pm\pi$ at the τ and θ axes. Each point in this triangle corresponds to a $(d-2)$-dimensional sphere. The boundary of this triangle corresponds to infinity of the original coordinates, and is called the *conformal infinity*. By adding the conformal infinity to the spacetime, one obtains a bounded spacetime which is called the *conformal compactification* of spacetime.

If one spatially compactifies the Minkowski metric by including the boundary at $\theta = \pi$, and extends the range of the timelike component to $-\infty < \tau < \infty$, one obtains the (spatial) conformal compactification of d-dimensional Minkowski spacetime, which has the geometry of $\mathbb{R} \times S^{d-1}$, the *Einstein static universe*. The north pole of S^{d-1} corresponds to $\theta = 0$, and the south pole to $\theta = \pi$.

B.3 Asymptotic flatness and asymptotically AdS

The coordinates (τ, θ) in the metric (B.5) are well-defined at the asymptotical regions of the flat Minkowski spacetime. Therefore, conformal compactification is useful to define the notion of asymptotic flatness of spacetime: a spacetime is called *asymptotically flat* if it has the same boundary structure as flat space after conformal compactification [14].

For the case of Anti-de-Sitter spacetimes, by introducing the coordinate

$$\tan\theta = \sinh\rho, \quad \text{with} \quad 0 \leq \theta < \frac{\pi}{2},$$

the metric on AdS_{d+1} (B.4) can be rewritten as

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2).$$

This metric is conformally equivalent to $\mathbb{R} \times S^d$. However, the coordinate θ ranges up to $\pi/2$, rather than to π, which means that only half of the S^d is covered. Therefore, AdS_{d+1} is conformally equivalent to one half of the Einstein static universe.

By adding the boundary $\theta = \pi/2$ (which has the topology S^{d-1}), and extending the range of τ to $-\infty < \tau < \infty$, one obtains the conformally compactified AdS_{d+1} spacetime. As with asymptotic flatness, one uses conformal compactification to define what is called asymptotically AdS: a spacetime is called *asymptotically AdS*, when it can be conformally compactified into a region which has the same boundary structure as one half of the Einstein static universe.

For purposes concerning the AdS/CFT correspondence, it is very important to note that the boundary of AdS_{d+1} spacetime is topologically equivalent to $\mathbb{R} \times S^{d-1}$. This is precisely identical to the conformal compactification of d dimensional Minkowski spacetime.

B.4 Poincaré coordinates

Equation (B.2) can also be solved by introducing the coordinates [14]

$$X_0 = \frac{z}{2}\left(1 + \frac{1}{z^2}(R^2 + \sum_{i=1}^{d-1} x^i x^i - t^2)\right), \qquad X_i = \frac{R}{z}x_i, \; i = 1, \ldots, d-1,$$
$$X_d = \frac{z}{2}\left(1 - \frac{1}{z^2}(R^2 - \sum_{i=1}^{d-1} x^i x^i - t^2)\right), \qquad X_{d+1} = \frac{R}{z}t.$$

These coordinates cover half of the hyperboloid defined in (B.2). By substituting these coordinates into the metric (B.3), one obtains

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^i dx^i + dz^2\right).$$

These coordinates are called *Poincaré coordinates*, and they have the property that constant z slices are just copies of Minkowski space. This metric is a solution to the Einstein equation (B.1) with a cosmological constant $\Lambda = -\frac{d(d-1)}{2R^2}$.

C
Conformal field theory

In this appendix an introduction to conformal field theory is given. Only the parts relevant for the AdS/CFT correspondence are presented, and much of the material is taken from [6]. For an extended introduction to this subject, see for example [76] or [77].

Since symmetry principles, and in particular Poincaré invariance, play a major role in understanding quantum field theory, it is natural to look for possible generalizations of Poincaré invariance. One such generalization is the addition of a scale invariance (or dilatation) symmetry, linking physics at different scales. Scale-invariant quantum field theories are important as possible endpoints of renormalization group flows. Many interesting field theories are scale invariant, such as Yang-Mills theory in four dimensions, which is a non-Abelian gauge theory based on the group $SU(N)$. The conformal group is a simple group that includes Poincaré invariance, scale invariance, and inversion.

The space we consider in this section is $\mathbb{R}^d$ with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, which has the form $\mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$, with signature (p, q), where p and q are the number of eigenvalues -1 and $+1$ respectively, and $\mu, \nu = 0, \cdots, d-1$ correspond to the space-time coordinates. In particular we focus on Minkowski space, which has signature $(1, d-1)$, where d is the number of spacetime dimensions.

C.1 The conformal group and algebra

Under a change of coordinates, $x^\mu \to x'^\mu$, the metric $g_{\mu\nu}$ changes as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{C.1}$$

By definition, the conformal group is the subgroup of coordinate transformations which preserve the form of the metric up to an arbitrary scale change,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{C.2}$$

Note that, since the Poincaré group leaves the metric invariant, $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$, or $\Omega^2(x) = 1$, the Poincaré group is a subgroup of the conformal group.

The generators of the group can be determined by considering the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. Using this in (C.1) and comparing with (C.2) yields the condition [76]

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu},$$

which for $d > 2$ has the solution

$$\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + (b^\mu x^2 - 2x^\mu b \cdot x), \tag{C.3}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ being antisymmetric The parameters of conformal transformations are thus a^μ, ${\omega^\mu}_\nu$, λ, and b^μ, corresponding respectively to translations, Lorentz rotations, scale transformations, and a new type of transformations, called special conformal transformations. In the special case of $d = 2$ the conformal group is infinite dimensional, and thus has infinite many parameters.

To determine the generators, one can study the action of these infinitesimal conformal transformations on a space of functions of x [78]. For each transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, we can define a differential operator O_a such that

$$f(x) \to f(x) - i\delta_a O_a f(x), \tag{C.4}$$

where δ_a is a parameter characterizing the transformation. A scalar function transforms simply as $f'(x') = f(x)$, so that $f'(x') = f(x' - \epsilon) = f(x') - \epsilon^\mu \partial_\mu f(x')$. By comparing with (C.4) and using (C.3), the conformal generators are obtained,

Translations:	$P_\mu = -i\partial_\mu$
Lorentz transformations:	$M_{\mu\nu} = -i(x_\nu\partial_\mu - x_\mu\partial_\nu)$
Scaling transformations:	$D = -ix_\mu\partial^\mu$
Special conformal transformations:	$K_\mu = -i(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu)$.

Each of these operators are to be contracted with its corresponding parameter of the conformal transformation, a^μ, $\omega^{\mu\nu}$, λ, and b^μ.

By letting the commutators act on a test function $f(x)$, it is straightforward to show that the conformal algebra is given by

$$\begin{aligned}
&[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu); && [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu);\\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm perm.; && [M_{\mu\nu}, D] = 0;\\
&[D, K_\mu] = iK_\mu; && [D, P_\mu] = -iP_\mu;\\
&[P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D,
\end{aligned} \tag{C.5}$$

with all other commutators vanishing.

The number of components for the parameters of conformal transformations are respectively d, $d(d-1)/2$, 1, d, which add up to $(d+1)(d+2)/2$. By constructing the anti-symmetric $(d+2) \times (d+2)$ matrix,

$$J_{ab} = \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ -\frac{1}{2}(K_\mu - P_\mu) & 0 & -D \\ -\frac{1}{2}(K_\mu + P_\mu) & D & 0 \end{pmatrix},$$

where $a, b = 0, \cdots, d+1$, the conformal generators form a group. In case of Minkowski space, $(p, q) = (1, d-1)$, this group has a Lie algebra which has the standard form of the $SO(2, d)$ algebra,

$$[J_{ab}, J_{cd}] = i\left(\eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right)$$

with associated metric of signature $(-, +, +, \cdots, +, -)$. Note that, strictly speaking, $SO(2, d)$ is a group that only contains elements continuously connected to the identity, however the conformal group also contains the inversion $x^\mu \to \frac{x^\mu}{x^2}$, for which $\Omega^2(x) = x^2$. Apart from such subtleties, conformal invariance in flat $(1, d-1)$ dimensions $(d > 2)$ corresponds to the symmetry group $SO(2,d)$. Since the AdS/CFT correspondence will be defined in Euclidean space, we note that conformal theory in Euclidean space has the conformal group $SO(1,d+1)$.

Beside translations and Lorentz transformations, the global versions of the conformal transformations are the scale transformations,

$$x^\mu \to \lambda x^\mu, \tag{C.6}$$

and the special conformal transformations,

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}.$$

C.2 Primary fields

We will be interested in fields (or operators in case of a quantum theory) with a definite conformal dimension Δ, which under the scaling transformation (C.6) transform as

$$\phi(x) \to \phi'(x') = \lambda^\Delta \phi(\lambda x).$$

Infinitesimally, with $\lambda = 1 + \delta$ and $\phi' = (1 - i\delta D)\phi$ (C.4),

$$\phi(x) \to \phi'(x') = (1 - i\delta D)\phi(x') = \lambda^\Delta \phi(\lambda x) = (1 + \delta\Delta)\phi(x'),$$

so that these fields are eigenfunctions of the scaling operator D, with eigenvalue $-i\Delta$, where Δ is called the scaling (or conformal) dimension of the field.

The commutation relations (C.5) imply that P_μ raises the dimension of the field,

$$\begin{aligned}
[D, P_\mu]\phi(x) &= (DP_\mu - P_\mu D)\phi(x) = D(P_\mu \phi(x)) + i\Delta P_\mu \phi(x) \\
&= -iP_\mu \phi(x),
\end{aligned}$$

so that

$$D(P_\mu \phi(x)) = -i(\Delta + 1)\phi(x).$$

In the same way follows that K_μ lowers it. For any type of field there is a lower bound on its dimension which follows from unitarity. For scalar fields this is $\Delta \geq (d-2)/2$, where equality can occur only for a free scalar field [80]. Therefore, in each representation there will be a field Φ_0 of lowest dimension, which must be annihilated by K_μ (at $x = 0$). These fields are called primary fields (or operators). Representations of the conformal group are obtained from these primaries by acting successively with the 'raising operator' P_μ.

The generator of dilatations commutes with the generators of Lorentz transformations, so we should be able to assign a conformal dimension to fields that carry a representation of the Lorentz algebra. This dependence can be made explicit by writing Φ_Δ for fields with conformal dimension Δ. Since fields in this representation are eigenfunctions of D, it follows from (C.5) that they cannot in general be eigenfunctions of the Hamiltonian P_0 or of the mass operator $M^2 = P_\mu P^\mu$, which is a Casimir of the Lorentz group, but not of the full conformal group.

C.3 Correlation functions

Being a conformal theory, correlation functions of primary fields must be invariant under conformal transformations. Since the conformal group is much larger than the Poincaré group, it severely restricts the correlation functions of primary fields. By using the conformal algebra, it can be shown that 2-point functions of different dimension vanish [76], while for a single scalar field of scaling dimension Δ,

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{c_{12}}{r_{12}^{2\Delta}},$$

where $r_{12} \equiv |x_1 - x_2|$ and c_{12} is some constant. In the same way are 3-point functions restricted to be of the form

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{c_{ijk}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{31}^{\Delta_3+\Delta_1-\Delta_2}}.$$

Similar expressions arise for non-scalar fields.

C.4 Operator product expansions

In quantum field theories, there exists an *operator product expansion* (OPE), which is an expansion of the product of two operators. When two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$ are brought to the same point, their product creates a local disturbance at that point, which may be expressed as a sum of local operators acting at that point. This can generally be expressed as

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \to \sum_n C_{12}^n(x-y)\mathcal{O}_n(y),$$

where this expression should be understood as appearing inside correlation functions. The coefficient functions C_{12}^n are independent of the other operators in the correlation function. The above expression is useful when the distance to all other operators is much larger than $|x-y|$. In a conformal field theory, the form of the coefficient functions is determined by conformal invariance to be

$$C_{12}^n(x-y) = \frac{c_{12}^n}{|x-y|^{\Delta_1+\Delta_2-\Delta_n}}.$$

D

Calculations of Chapter 3

D.1 Spectral decomposition of the retarded Green's function

Let $\{|n\rangle\}$ be a complete set of eigenstates of the Hamiltonian H_0 and momentum operator P, with eigenvalues E_n and k_n, respectively. Assume that ρ_0 is diagonal in this basis, with matrix elements $\langle m|\rho_0|n\rangle = \rho_{0,n}\delta_{mn}$.

The retarded Green's function in position space is given by (3.51)

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(t,x) = -i\,\theta(t)\langle[\mathcal{O}^I_A(t,x),\mathcal{O}^I_B(0,0)]\rangle. \tag{D.1}$$

Inserting identity operators into $\langle[\mathcal{O}^I_A(t,x),\mathcal{O}^I_B(0,0)]\rangle$, and ignoring t_0, gives

$$\begin{aligned}\langle[\mathcal{O}^I_A(t,x),\mathcal{O}^I_B(0,0)]\rangle &= \mathrm{Tr}\left(\rho_0\left[U_0^{-1}(t)\mathcal{O}_A(x)U_0(t),\mathcal{O}_B(0)\right]\right)\\ &= \mathrm{Tr}\,\rho_0\Big(\sum_k |k\rangle\langle k|U_0^{-1}(t)\mathcal{O}_A(x)\sum_m |m\rangle\langle m|U_0(t)\mathcal{O}_B(0)\sum_n|n\rangle\langle n| -\\ &\quad-\sum_k|k\rangle\langle k|\mathcal{O}_B(0)U_0^{-1}(t)\sum_m|m\rangle\langle m|\mathcal{O}_A(x)U_0(t)\sum_n|n\rangle\langle n|\Big).\end{aligned}$$

Using the cyclic property of the trace, and the identity $U_0(t)|n\rangle = e^{-iE_nt}|n\rangle$, yields

$$\begin{aligned}\langle[\mathcal{O}^I_A(t,x),\mathcal{O}^I_B(0,0)]\rangle &= \sum_{k,m,n}\langle n|\rho_0|k\rangle\langle k|e^{iE_kt}\mathcal{O}_A(x)|m\rangle\langle m|e^{-iE_mt}\mathcal{O}_B(0)|n\rangle-\\ &\qquad-\langle n|\rho_0|k\rangle\langle k|\mathcal{O}_B(0)e^{iE_mt}|m\rangle\langle m|\mathcal{O}_A(x)e^{-iE_nt}|n\rangle\\ &= \sum_{m,n}\rho_{0,n}e^{i(E_n-E_m)t}\langle n|\mathcal{O}_A(x)|m\rangle\langle m|\mathcal{O}_B(0)|n\rangle-\\ &\qquad-\rho_{0,n}e^{i(E_m-E_n)t}\langle n|\mathcal{O}_B(0)|m\rangle\langle m|\mathcal{O}_A(x)|n\rangle.\end{aligned}$$

Using the identity

$$\langle m|\mathcal{O}_A(x)|n\rangle = \langle m|e^{ix\cdot P}\mathcal{O}_A(0)e^{-ix\cdot P}|n\rangle = e^{ix\cdot(k_m-k_n)}\langle m|\mathcal{O}_A(0)|n\rangle,$$

and define $A_{mn}\equiv\langle m|\mathcal{O}_A(0)|n\rangle$, $B_{mn}\equiv\langle m|\mathcal{O}_B(0)|n\rangle$, and $k_{mn}\equiv k_m-k_n$, this can be written as

$$\langle[\mathcal{O}^I_A(t,x),\mathcal{O}^I_B(0,0)]\rangle = \sum_{m,n}\rho_{0,n}\Big(e^{i(E_n-E_m)t}e^{ix\cdot k_{nm}}A_{nm}B_{mn} - e^{i(E_m-E_n)t}e^{ix\cdot k_{mn}}B_{nm}A_{mn}\Big).$$

We want to calculate the Fourier transform of $-i\theta(t)$ times this expression, which is the Fourier transform of (D.1), i.e., $G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)$. The Fourier transform of $\exp(ix \cdot k_{nm})$ is given by

$$\int d^{d-1}x\, e^{-ik\cdot x} e^{ix\cdot k_{nm}} = \int d^{d-1}x\, e^{-i(k-(k_n-k_m))\cdot x} = (2\pi)^{d-1}\delta^{d-1}(k_{nm} - k).$$

Further, the Fourier transform of the time-dependent part is

$$\begin{aligned} -i\int_{-\infty}^{\infty} dt\, e^{izt}\,\theta(t)e^{i(E_n-E_m)t} &= -i\int_0^{\infty} dt\, e^{izt}\, e^{i(E_n-E_m)t} \\ &= -i\left(\frac{e^{i(z+E_n-E_m)t}}{i(z+E_n-E_m)}\right)\Bigg|_{t=0}^{t=\infty} \\ &= \frac{1}{E_n - E_m + z} \qquad\qquad \text{for } \operatorname{Im} z > 0. \end{aligned}$$

Altogether, the spectral decomposition of the retarded Green's function becomes

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega + i0, k) = \sum_{m,n} \rho_{0,n}\left(\frac{A_{nm}B_{mn}(2\pi)^{d-1}\delta^{(d-1)}(k_{nm}-k)}{E_n - E_m + \omega + i0} - (n \leftrightarrow m)\right).$$

D.2 Spectral function

From the spectral representation of the retarded Green's function (3.56) follows

$$G^{R\,*}_{\mathcal{O}_B\mathcal{O}_A}(\omega + i0, k) = \sum_{m,n} \rho_{0,n}\left(\frac{B_{mn}A_{nm}(2\pi)^{d-1}\delta^{(d-1)}(k_{nm}-k)}{E_n - E_m + \omega - i0} - (n \leftrightarrow m)\right),$$

where the identity $A^*_{nm} = A_{mn}$ has been used. Then, the spectral density (3.58) becomes

$$\begin{aligned} &\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = i\left(G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) - G^{R\,\dagger}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)\right) \\ &= i(2\pi)^{d-1}\sum_{m,n} \rho_{0,n}\left(\frac{A_{nm}B_{mn}\delta^{(d-1)}(k_{nm}-k)}{E_n - E_m + \omega + i0} - \frac{B_{mn}A_{nm}\delta^{(d-1)}(k_{nm}-k)}{E_n - E_m + \omega - i0} \mp (n \leftrightarrow m)\right). \end{aligned} \tag{D.2}$$

The Sokhotsky-Weierstrass theorem states that (see e.g. [82]):

$$\lim_{\epsilon\to 0^+} \frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x),$$

where $\mathcal{P}$ denotes the Cauchy principal value. This expression is to be understood as a generalized function, like the Dirac function, i.e., it makes sense only under an integral,

$$\lim_{\epsilon\to 0^+} \int dx\, \frac{f(x)}{x \pm i\epsilon} = \mathcal{P}\int dx\, \frac{f(x)}{x} \mp i\pi f(0).$$

By using this identity, the spectral density (D.2) can be written as

$$\begin{aligned} \mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) &= (2\pi)^d \sum_{m,n} \rho_{0,n}\left(\delta(E_n - E_m + \omega)\delta^{(d-1)}(k_{nm}-k)A_{nm}B_{mn} - (n \leftrightarrow m)\right) \\ &= (2\pi)^d \sum_{m,n} (\rho_{0,n} - \rho_{0,m})\,\delta(E_n - E_m + \omega)\delta^{(d-1)}(k_{nm}-k)A_{nm}B_{mn}. \end{aligned} \tag{D.3}$$

The terms involving the Cauchy principal value have cancelled each other due to the relative minus sign in (D.2).

D.2.1 Spectral function for canonical ensemble

Assume that we are in the canonical ensemble, so that $\rho_0 = \mathcal{Z}^{-1} e^{-\beta H_0}$, then

$$\begin{aligned}\sum_{m,n} (\rho_{0,n} - \rho_{0,m})\delta(E_n - E_m + \omega) &= \mathcal{Z}^{-1} \sum_{m,n} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) \delta(E_n - E_m + \omega) \\ &= \mathcal{Z}^{-1} \sum_{m,n} \left(e^{-\beta(E_n+E_m-\omega)/2} - e^{-\beta(E_m+E_m+\omega)/2}\right) \delta(E_n - E_m + \omega) \\ &= 2\mathcal{Z}^{-1} \sinh \frac{\omega}{2T} \sum_{m,n} e^{-(E_n+E_m)/2T} \delta(E_n - E_m + \omega).\end{aligned}$$

With this the spectral density becomes

$$\begin{aligned}\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = 2\mathcal{Z}^{-1} \sinh\left(\frac{\omega}{2T}\right) \sum_{m,n} e^{-(E_n+E_m)/2T}\, 2\pi\delta(E_n - E_m + \omega)\times \\ \times (2\pi)^{d-1}\delta^{(d-1)}(k_{nm} - k) A_{nm} B_{mn}.\end{aligned}$$

D.3 Averaged dissipated power

As mentioned in section 3.6.5, the averaged dissipated power is given by the expression

$$\begin{aligned}\overline{\frac{dW}{dt}} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, \frac{dW}{dt} \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \int d^{d-1}x \left(\langle \mathcal{O}_A^I\rangle_0(t,x) + \delta\langle \mathcal{O}_A\rangle(t,x)\right)\partial_t \delta\phi_{A(0)}(t,x).\end{aligned}$$

where in the second equality, (3.62) has been used. The first term is given by (3.47), $\langle \mathcal{O}_A^I\rangle_0(t,x) = \langle \mathcal{O}_A\rangle_0(x)$, and this vanishes when averaging over a single cycle of the external field $\delta\phi_{A(0)}(t,x)$, since $\langle \mathcal{O}_A\rangle_0(x)$ is independent of time.

Substituting an external field oscillating at a single frequency ω,

$$\delta\phi_{A(0)}(t,x) = \mathrm{Re}\left(\phi_{A(0)}(x)e^{-i\omega t}\right) = \frac{1}{2}\left(\phi_{A(0)}(x)e^{-i\omega t} + \phi^*_{A(0)}(x)e^{i\omega t}\right), \tag{D.4}$$

into the above expression for the average dissipated power, yields

$$\overline{\frac{dW}{dt}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \int d^{d-1}x\, \delta\langle \mathcal{O}_A\rangle(t,x) \frac{i\omega}{2}\left(\phi^*_{A(0)}(x)e^{i\omega t} - \phi_{A(0)}(x)e^{-i\omega t}\right).$$

By substituting (3.49), (3.51) and (D.4), this becomes

$$\begin{aligned}\overline{\frac{dW}{dt}} = \frac{\omega}{2\pi}\frac{i\omega}{4}\int_0^{2\pi/\omega} dt \int d^{d-1}x \int dt' d^{d-1}x'\, G^R_{\mathcal{O}_A\mathcal{O}_B}(t-t', x-x')\times \\ \times \left(\phi^*_{A(0)}(x)e^{i\omega t} - \phi_{A(0)}(x)e^{-i\omega t}\right)\left(\phi_{B(0)}(x')e^{-i\omega t'} + \phi^*_{B(0)}(x')e^{i\omega t'}\right).\end{aligned}$$

For the term with $\phi^*_{A(0)}(x)\phi_{B(0)}(x')e^{i\omega(t-t')}$ one has

$$\int_{-\infty}^{\infty} dt'\, \phi^*_{A(0)}(x) G^R_{\mathcal{O}_A\mathcal{O}_B}(t-t', x-x')\phi_{B(0)}(x')e^{i\omega(t-t')} = \phi^*_{A(0)}(x)G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, x-x')\phi_{B(0)}(x'),$$

and for the term involving $\phi^*_{B(0)}(x)\phi_{A(0)}(x')e^{-i\omega(t-t')}$ we find

$$\begin{aligned}
-\int_{-\infty}^{\infty} dt'\, \phi_{A(0)}(x)G^R_{\mathcal{O}_A\mathcal{O}_B}(t-t', x-x')\phi^*_{B(0)}(x')e^{-i\omega(t-t')} = \\
&= -\int_{-\infty}^{\infty} dt'\, \phi^*_{A(0)}(x)G^R_{\mathcal{O}_B\mathcal{O}_A}(t-t', x'-x)\phi_{B(0)}(x')e^{-i\omega(t-t')} \\
&= -\phi^*_{A(0)}(x)G^R_{\mathcal{O}_B\mathcal{O}_A}(-\omega, x'-x)\phi_{B(0)}(x') \\
&= -\phi^*_{A(0)}(x)G^{R\,*}_{\mathcal{O}_B\mathcal{O}_A}(\omega, x-x')\phi_{B(0)}(x'),
\end{aligned}$$

where in the first step the labels A and B has been interchanged, and likewise for the integration variables, $x \leftrightarrow x'$. In the last equality, the symmetry property (3.54) has been exploited. When averaging over a full cycle of the external field, the terms involving $\phi_{A(0)}(x)\phi_{B(0)}(x')e^{-i\omega(t+t')}$ and $\phi^*_{A(0)}(x)\phi^*_{B(0)}(x')e^{i\omega(t+t')}$ vanish .

After collecting the terms and performing the t integral, one gets

$$\begin{aligned}
\overline{\frac{dW}{dt}} &= \frac{i\omega}{4}\int d^{d-1}x d^{d-1}x'\, \phi^*_{A(0)}(x)\Big(G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, x-x') - G^{R\,*}_{\mathcal{O}_B\mathcal{O}_A}(\omega, x-x')\Big)\,\phi_{B(0)}(x') \\
&= \frac{1}{4}\int d^{d-1}x d^{d-1}x'\, \phi^*_{A(0)}(x)\,\omega\mathcal{A}_{\mathcal{O}_A\mathcal{O}_B}(\omega, x-x')\phi_{B(0)}(x').
\end{aligned}$$

E Calculations of Chapter 4

In this Appendix, some calculation involving Matsubara Green's functions are given. These calculations are more or less standard, and can for example be found in [62] and [63].

E.1 Free electron and phonon Matsubara Green's functions

In this section, the Matsubara Green's functions for fermions and bosons for non-interacting systems are given.

Free electron MGF

For a non-interacting system in the grand canonical ensemble (cf. (4.5)), the part of the Hamiltonian (4.4) describing the free electrons reads

$$K_0 = H_0 - \mu N = \sum_{\mathbf{p},\sigma} (\varepsilon_{\mathbf{p},\sigma} - \mu) c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma} = \sum_{\mathbf{p},\sigma} \xi_{\mathbf{p},\sigma} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma}, \quad \text{with} \quad \xi_{\mathbf{p},\sigma} \equiv \varepsilon_{\mathbf{p},\sigma} - \mu.$$

The 'time' dependence of the annihilation operators can be written as (cf. (4.6))

$$c_{\mathbf{p},\sigma}(\tau) = e^{K_0 \tau} c_{\mathbf{p},\sigma} e^{-K_0 \tau} = e^{-\xi_{\mathbf{p},\sigma}\tau} c_{\mathbf{p},\sigma},$$

where $\xi_{\mathbf{p},\sigma} \equiv \varepsilon_{\mathbf{p},\sigma} - \mu$, and in the last equality the Baker-Hausdorff formula,

$$e^A \, c \, e^{-A} = c + [A, c] + \frac{1}{2!}[A, [A, c]] + \frac{1}{3!}[A, [A, [A, c]]] + \cdots ,$$

has been been used.

Ignoring spin indices, the electron Matsubara Green's function for the non-interacting system is given by

$$\begin{aligned}
\mathcal{G}^{(0)}(\tau, \mathbf{p}) &= -\langle \mathcal{T}_\tau c_{\mathbf{p}}(\tau) c^\dagger_{\mathbf{p}}(0) \rangle_0 \\
&= -\theta(\tau) e^{-\xi_{\mathbf{p}}\tau} \langle c_{\mathbf{p}} c^\dagger_{\mathbf{p}} \rangle_0 + \theta(-\tau) e^{-\xi_{\mathbf{p}}\tau} \langle c^\dagger_{\mathbf{p}} c_{\mathbf{p}} \rangle_0 \\
&= -e^{-\xi_{\mathbf{p}}\tau} \Big(\theta(\tau) \big(1 - n_F(\xi_{\mathbf{p}})\big) - \theta(-\tau) n_F(\xi_{\mathbf{p}}) \Big) \\
&= -e^{-\xi_{\mathbf{p}}\tau} \big(\theta(\tau) - n_F(\xi_{\mathbf{p}}) \big),
\end{aligned}$$

where $n_F = \langle c^\dagger_{\mathbf{p}} c_{\mathbf{p}} \rangle_0$ is the free-electron occupation number. In the second equality, a minus term in front of the second term has been introduced due to the anti-commutativity of the fermionic operators.

By taking the Fourier transform (4.10), we obtain the free fermionic Green's function in frequency representation

$$
\begin{aligned}
\mathcal{G}^{(0)}(ip_n,\mathbf{p}) &= \int_0^\beta d\tau\, e^{ip_n\tau}\mathcal{G}^{(0)}(\tau,\mathbf{p}), \qquad\qquad p_n = \frac{(2n+1)\pi}{\beta} \\
&= -\int_0^\beta d\tau\, e^{ip_n\tau}e^{-\xi_\mathbf{p}\tau}\big(\theta(\tau)-n_F(\xi_\mathbf{p})\big) \\
&= -\big(1-n_F(\xi_\mathbf{p})\big)\int_0^\beta d\tau\, e^{\tau(ip_n-\xi_\mathbf{p})} \\
&= -\big(1-n_F(\xi_\mathbf{p})\big)\frac{e^{\beta(ip_n-\xi_\mathbf{p})}-1}{ip_n-\xi_\mathbf{p}}.
\end{aligned}
$$

By using the fact that the occupation of free electrons is given by the Fermi-Dirac equation, i.e.,

$$
\langle c^\dagger_\mathbf{p} c_\mathbf{p}\rangle_0 = n_F(\xi_\mathbf{p}) = \frac{1}{e^{\beta\xi_\mathbf{p}}+1},
$$

and using $e^{\beta ip_n} = 1$ for $\beta ip_n = i(2n+1)\pi$, we obtain

$$
\mathcal{G}^{(0)}_\sigma(ip_n,\mathbf{p}) = \frac{1}{ip_n-\xi_{\mathbf{p},\sigma}}.
$$

Free boson MGF

The unperturbed phonon Matsubara Green's function is given by

$$
\mathcal{D}^{(0)}_\lambda(\tau,\mathbf{q}) = -\langle\mathcal{T}_\tau A_{\mathbf{q},\lambda}(\tau)A_{-\mathbf{q},\lambda}(0)\rangle_0.
$$

The 'time' dependence of the phonon creation and annihilation operators can be written as

$$
a_{\mathbf{q},\lambda}(\tau) = e^{H_{\mathrm{ph}}\tau}a_{\mathbf{q},\lambda}e^{-H_{\mathrm{ph}}\tau} = e^{-\omega_\mathbf{q}\tau}a_{\mathbf{q},\lambda}, \quad \text{and} \quad a^\dagger_{\mathbf{q},\lambda}(\tau) = e^{\omega_\mathbf{q}\tau}a^\dagger_{\mathbf{q},\lambda},
$$

where again the Baker-Haussdorf formula has been used.

Ignoring branch indices, the unperturbed phonon MGF can then be written as

$$
\begin{aligned}
\mathcal{D}^{(0)}_\lambda(\tau,\mathbf{q}) = -\theta(\tau)\langle(e^{-\omega_\mathbf{q}\tau}a_\mathbf{q}+e^{\omega_\mathbf{q}\tau}a^\dagger_{-\mathbf{q}})(a_{-\mathbf{q}}+a^\dagger_\mathbf{q})\rangle_0 - \\
-\theta(-\tau)\langle(a_{-\mathbf{q}}+a^\dagger_\mathbf{q})(e^{-\omega_\mathbf{q}\tau}a_\mathbf{q}+e^{\omega_\mathbf{q}\tau}a^\dagger_{-\mathbf{q}})\rangle_0.
\end{aligned}
$$

Terms like $\langle a_\mathbf{q}a_{-\mathbf{q}}\rangle_0$ and $\langle a^\dagger_\mathbf{q}a^\dagger_{-\mathbf{q}}\rangle_0$ vanish. The remaining terms can be recast by writing $\langle a^\dagger_\mathbf{q}a_\mathbf{q}\rangle_0 = n_B(\omega_\mathbf{q})$ and $\langle a_\mathbf{q}a^\dagger_\mathbf{q}\rangle_0 = 1+n_B(\omega_\mathbf{q})$, so that

$$
\begin{aligned}
\mathcal{D}^{(0)}_\lambda(\tau,\mathbf{q}) &= -\theta(\tau)\Big(\big(1+n_B(\omega_\mathbf{q})\big)e^{-\omega_\mathbf{q}\tau}+n_B(\omega_\mathbf{q})e^{\omega_\mathbf{q}\tau}\Big) - \\
&\qquad -\theta(-\tau)\Big(n_B(\omega_\mathbf{q})e^{-\omega_\mathbf{q}\tau}+\big(1+n_B(\omega_\mathbf{q})\big)e^{\omega_\mathbf{q}\tau}\Big) \\
&= -\Big(\theta(\tau)+n_B(\omega_\mathbf{q})\Big)e^{-\omega_\mathbf{q}\tau} - \Big(\theta(-\tau)+n_B(\omega_\mathbf{q})\Big)e^{\omega_\mathbf{q}\tau}.
\end{aligned}
$$

Finally, by taking the Fourier transform we obtain

$$\begin{aligned}\mathcal{D}^{(0)}(i\omega_n,\mathbf{q}) &= \int_0^\beta d\tau\, e^{i\omega_n\tau}\mathcal{D}^{(0)}(\tau,\mathbf{q}), \qquad\qquad \omega_n = \frac{2n\pi}{\beta}\\ &= -(1+n_B(\omega_\mathbf{q}))\int_0^\beta d\tau\, e^{\tau(i\omega_n-\omega_\mathbf{q})} - n_B(\omega_\mathbf{q})\int_0^\beta d\tau\, e^{\tau(i\omega_n+\omega_\mathbf{q})}\\ &= -(1+n_B(\omega_\mathbf{q}))\frac{e^{\beta(i\omega_n-\omega_\mathbf{q})}-1}{i\omega_n-\omega_\mathbf{q}} - n_B(\omega_\mathbf{q})\frac{e^{\beta(i\omega_n+\omega_\mathbf{q})}-1}{i\omega_n+\omega_\mathbf{q}}.\end{aligned}$$

By using the fact that the occupation number of phonons in an unperturbed system is given by the Bose-Einstein distribution,

$$\langle a_\mathbf{q}^\dagger a_\mathbf{q}\rangle_0 = n_B(\omega_\mathbf{q}) = \frac{1}{e^{\beta\omega_\mathbf{q}}-1},$$

and using further $e^{\beta i\omega_n}=1$ for $\beta i\omega_n = i2n\pi$, we obtain

$$\mathcal{D}_\lambda^{(0)}(i\omega_n,\mathbf{q}) = \frac{1}{i\omega_n-\omega_{\mathbf{q},\lambda}} - \frac{1}{i\omega_n+\omega_{\mathbf{q},\lambda}} = \frac{2\omega_{\mathbf{q},\lambda}}{(i\omega_n)^2-\omega_{\mathbf{q},\lambda}^2}.$$

Note that in this last expression, the phonon branch index was restored.

E.2 Summation over bosonic Matsubara frequencies

In this section, we will perform the summation over the bosonic Matsubara frequencies in (4.14),

$$\Sigma_\sigma^{(1)}(ip_n,\mathbf{p}) = -\frac{1}{\beta\mathcal{V}}\sum_{\mathbf{q},\lambda}\bar{g}_{\mathbf{q},\lambda}^2\sum_{i\omega_n}\frac{2\omega_{\mathbf{q},\lambda}}{(i\omega_n)^2-\omega_{\mathbf{q},\lambda}^2}\,\frac{1}{ip_n-i\omega_n-\xi_{\mathbf{p}-\mathbf{q},\sigma}}.$$

To evaluate the summation, consider

$$S \equiv -\frac{1}{\beta}\sum_{i\omega_n} f(i\omega_n), \quad \text{with} \quad f(i\omega_n) \equiv \frac{2\omega_\mathbf{q}}{(i\omega_n)^2-\omega_\mathbf{q}^2}\,\frac{1}{ip_n-i\omega_n-\xi_{\mathbf{p}'}},$$

where electron spin and phonon branch indices have been ignored, and $\mathbf{p}'\equiv\mathbf{p}-\mathbf{q}$. This summation is going to be performed by using contour integration. Consider therefore the integral

$$I \equiv \oint_{\mathcal{C}_\infty}\frac{dz}{2\pi i}\, f(z)n_B(z) = \sum_{\text{Res}} f(z)n_B(z),$$

where the Bose-Einstein distribution has been inserted to generate poles at $z = i\omega_n$, with n even, and the contour $\mathcal{C}_\infty$ encloses all the poles, $\mathcal{C}_\infty = \lim_{R\to\infty} Re^{i\theta}$; see figure E.1.

The poles of $f(z)n_B(z)$ and their corresponding residues R are given by

- due to $n_B(z)$:

  $$z_n = i\omega_n, \quad R_n = \operatorname*{Res}_{z=i\omega_n} f(z)n_B(z) = \frac{1}{\beta}f(i\omega_n),$$

  which follows from

  $$\operatorname*{Res}_{z=i\omega_n} n_B(z) = \lim_{z\to i\omega_n}\frac{z-i\omega_n}{e^{\beta z}-1} = \lim_{\epsilon\to 0}\frac{\epsilon}{e^{\beta i\omega_n}e^{\beta\epsilon}-1} = \frac{1}{\beta}.$$

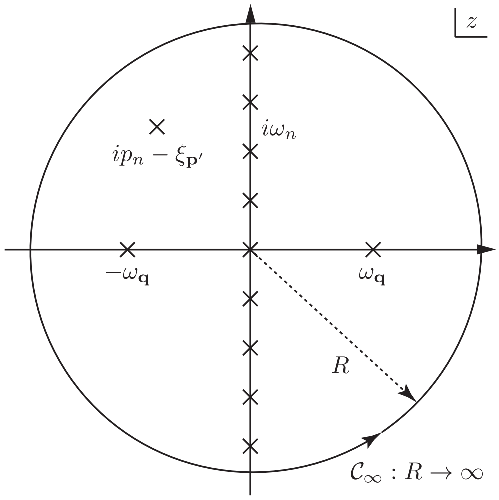

Figure E.1: Contour $\mathcal{C}_\infty$ for performing the Matsubara sum. The poles are indicated with a cross.

- due to $f(z)$:

$$z_1 = \omega_{\mathbf{q}}, \qquad R_1 = \frac{n_B(\omega_{\mathbf{q}})}{ip_n - \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}};$$

$$z_2 = -\omega_{\mathbf{q}}, \qquad R_2 = \frac{-n_B(-\omega_{\mathbf{q}})}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} = \frac{1 + n_B(\omega_{\mathbf{q}})}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}};$$

$$\begin{aligned} z_3 = ip_n - \xi_{\mathbf{p}'}, \qquad R_3 &= n_B(ip_n - \xi_{\mathbf{p}'}) \frac{2\omega_{\mathbf{q}}}{(ip_n - \xi_{\mathbf{p}'})^2 - \omega_{\mathbf{q}}^2} \\ &= (1 - n_F(\xi_{\mathbf{p}'})) \frac{2\omega_{\mathbf{q}}}{(ip_n - \xi_{\mathbf{p}'})^2 - \omega_{\mathbf{q}}^2} \\ &= \frac{n_F(\xi_{\mathbf{p}'}) - 1}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} - \frac{n_F(\xi_{\mathbf{p}'}) - 1}{ip_n - \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}}, \end{aligned}$$

  where in the second to last equality of R_3, the identity $n_B(ip_n - \xi_{\mathbf{p}'}) = 1 - n_F(\xi_{\mathbf{p}'})$ has been used, which follows from the definitions of n_B and n_F.

Altogether, the summation over the residues in the integral I becomes

$$I = \frac{1}{\beta} \sum_{i\omega_n} f(i\omega_n) + \frac{n_B(\omega_{\mathbf{q}}) + n_F(\xi_{\mathbf{p}'})}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} + \frac{n_B(\omega_{\mathbf{q}}) + 1 - n_F(\xi_{\mathbf{p}'})}{ip_n - \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}}.$$

In the limit as $R \to \infty$, the integral over the contour vanishes, $I = 0$, so that

$$\begin{aligned} -\frac{1}{\beta} \sum_{i\omega_n} f(i\omega_n) &= -\frac{1}{\beta} \sum_{i\omega_n} \frac{2\omega_{\mathbf{q}}}{(i\omega_n)^2 - \omega_{\mathbf{q}}^2} \frac{1}{ip_n - i\omega_n - \xi_{\mathbf{p}'}} \\ &= \frac{n_B(\omega_{\mathbf{q}}) + n_F(\xi_{\mathbf{p}'})}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} + \frac{n_B(\omega_{\mathbf{q}}) + 1 - n_F(\xi_{\mathbf{p}'})}{ip_n - \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}}. \end{aligned} \tag{E.1}$$

Finally, the electron self-energy becomes

$$\begin{aligned} \Sigma_\sigma^{(1)}(ip_n, \mathbf{p}) &= -\frac{1}{\beta \mathcal{V}} \sum_{\mathbf{q},\lambda} \bar{g}_{\mathbf{q},\lambda}^2 \sum_{i\omega_n} \frac{2\omega_{\mathbf{q},\lambda}}{(i\omega_n)^2 - \omega_{\mathbf{q},\lambda}^2} \frac{1}{ip_n - i\omega_n - \xi_{\mathbf{p}-\mathbf{q},\sigma}} \\ &= \frac{1}{\mathcal{V}} \sum_{\mathbf{q},\lambda} \bar{g}_{\mathbf{q},\lambda}^2 \left(\frac{n_B(\omega_{\mathbf{q}}) + n_F(\xi_{\mathbf{p}'})}{ip_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} + \frac{n_B(\omega_{\mathbf{q}}) + 1 - n_F(\xi_{\mathbf{p}'})}{ip_n - \omega_{\mathbf{q}} - \xi_{\mathbf{p}'}} \right). \end{aligned}$$

F
Nederlandstalige samenvatting

Supergeleiders zijn een belangrijk onderzoeksterrein binnen de natuurkunde. Dit zijn materialen die elektrische stroom kunnen geleiden zonder weerstand, maar de temperatuur moet daarvoor wel onder een bepaalde kritische waarde zitten. Er zijn sinds eind jaren tachtig materialen ontdekt die supergeleidend zijn bij relatief hoge temperaturen. Dit zijn de zogenaamde hoge-temperatuur supergeleiders. De theorie achter de tot dan toe waargenomen supergeleiders was sinds lange tijd bekend. Daar vormen elektronen paren ten gevolge van wisselwerkingen met trillingen van het metaalrooster waar ze in bewegen. Deze roostertrillingen worden fononen genoemd.

Hoe de paring van elektronen bij hoge-temperatuur supergeleiders werkt, is niet bekend. Er wordt vermoed, dat deze paring door soortgelijke deeltjes als fononen wordt veroorzaakt. Een belangrijk verschil is echter dat de sterkte van de wisselwerking, de zogenaamde koppeling, groot is. Maar het is lastig om sterk wisselwerkende systemen te bestuderen, omdat daarvoor weinig middelen zijn. Echter, sinds een paar jaar bestaat de zogenaamde AdS/CFT correspondentie. Dit is een nieuw stuk gereedschap dat ons in staat stelt sterk gekoppelde systemen te bestuderen, en is een resultaat uit de snarentheorie. In deze thesis proberen we deze AdS/CFT correspondentie toe te passen op een systeem van elektronen die sterk wisselwerken met bosonen, om zo meer inzicht te krijgen in het koppelingsmechanisme tussen elektronen in hoge temperatuur supergeleiders.

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