Meadows

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Abstract

A meadow is a commutative ring with an inverse operator satisfying two equations and in which $0^{-1} = 0$. All fields and products of fields can be viewed as meadows. After reviewing alternate axioms for inverse, we start the development of a theory of meadows. We give a general representation theorem for meadows and find, as a corollary, that the equational theory of meadows coincides with the equational theory of zero totalized fields. We also prove representation results for meadows of finite characteristic.

1 Introduction

A *meadow* is a commutative ring with unit equipped with a total unary operation x^{-1}, named inverse, that satisfies these additional equations:

$$(x^{-1})^{-1} = x \tag{1}$$

$$x \cdot (x \cdot x^{-1}) = x. \tag{2}$$

The first equation we call *Ref,* for *reflection,* and the second equation *Ril*, for *restricted inverse law.*

Meadows provide an analysis of division which is more general than the classical theory of fields. Meadows are total algebras in which $0^{-1} = 0$. We have used algebras with

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such zero totalized division in developing elementary algebraic specifications for several algebras of rational numbers in our previous paper [3] and its companions [1, 4]. Given the usefulness of zero totalized division for specification purposes, we will develop the theory of meadows from a perspective of computer science. Clearly, since meadows are commutative rings they are not without pure mathematical interest.

Let us discuss the raison d'être of meadows and survey our results. The primary algebraic properties of the rational, real and complex numbers are captured by the operations and axioms of *fields*. The field axioms consist of the equations that define commutative rings and, in particular, two axioms that are *not* equations that define the inverse operator and the distinctness of the two constants. Traditionally, fields are *partial* algebras because their inverse operations are undefined at 0. The class of fields does *not* possess an equational axiomatisation.

In [3, 4, 1], various fields were investigated using the elementary specification methods of abstract data type theory, namely total many-sorted algebras, equations, initial algebras and term rewriting. The specifications are based on *zero totalized division*: a *zero totalized field* has its inverse operator made total by setting $0^{-1} = 0$. In [3], an equational specification under initial algebra semantics of the zero totalized field of rational numbers was presented, and specifications for other zero totalized fields were developed in [4] and [1].

In [3] meadows were isolated by exploring alternate *equational* axioms for inverse. Specifically, 12 equations were found; a set *CR* of 8 equations for commutative rings was extended by a set *SIP* of 3 equations for inverse, including *Ref*, and by *Ril*. The single sorted finite equational specification $CR + SIP + Ril$ has all zero totalized fields among its models and, in addition, a large class of structures featuring zero divisors. A model of $CR + SIP + Ril$ was baptized a *meadow* in [3]. Because meadows are defined by equations, finite and infinite products of zero totalized fields are meadows as well.

Our first result will be that two of the equations from $CR + SIP + Ril$ can be derived from the other ones. This establishes the subset Md, consisting of 10 equations of the 12 equations, including the 8 equations for *CR* and the equations *Ref* and *Ril* mentioned earlier.

Our task is to start to make a classification of meadows up to isomorphism. We prove the following general representation theorem:

Theorem *Up to isomorphism, the non-trivial meadows are precisely the subalgebras of products of zero totalized fields.*

As a consequence, the equational theory of meadows is exactly the equational theory of fields with zero totalized division; this strengthens a result for closed equations in [3].

Next, we examine the relationship between fields and meadows of finite characteristic. The characteristic of a meadow is the smallest natural number $n \in \mathbb{N}$ such that $n.1 = 1 + 1 + \ldots + 1 = 0$. A prime meadow is a meadow without a proper submeadow and without a proper non-trivial homomorphic image.

Given a positive natural number k, and writing $\underline{k}$ for the numeral for k, we can define Md_k for the initial algebra of $Md + \{\underline{k} = 0\}$, i.e.,

$$Md_k \cong I(\Sigma, Md \cup \{\underline{k} = 0\}).$$

The following results are obtained:

Theorem *For k a prime number, Md_k is the zero totalized prime field of characteristic k.*

Theorem *For k a square free number, Md_k has cardinality k.*

The roots of this investigation lie in the theory of equational specifications for computable data types (see e.g., [2]) which shows that *any* computable data type possesses a range of equational specifications with nice properties. The equational specification of particular structures, such as algebras of rational numbers, is nevertheless an interesting and necessary task, because of the challenge to perfect the properties of these specifications beyond what is delivered by the general theory. Only recently, Moss found in [11] that there exists an equational specification of the ring of rationals (i.e., without division or inverse) with just *one* unary hidden function. In [3] we proved that there exists a finite equational specification under initial algebra semantics, *without* hidden functions, but making use of an inverse operation, of the field of rational numbers. In [4], the specification found for the rational numbers was extended to the complex rationals with conjugation, and in [1] a specification was given of the algebra of rational functions with field and degree operations that are all total. Full details concerning the background of this work can be found in [3].

We assume the reader is familiar with the basics of ring theory (e.g., [10, 12]), algebraic specifications (e.g., [15]), universal algebra (e.g., [14, 9]) and term rewriting (e.g., [13]).

2 Axioms for fields and meadows

We will add to the axioms of a commutative ring various alternative axioms for dealing with inverse and division. The starting point is a signature Σ_{CR} for commutative rings with unit:

signature Σ_{CR}
sorts *ring*
operations
$0\colon \ \to ring$;
$1\colon \ \to ring$;
$+\colon ring \times ring \to ring$;
$-\colon ring \to ring$;
$\cdot\colon ring \times ring \to ring$
end

To the signature Σ_{CR} we add an inverse operator $^{-1}$ to form the primary signature Σ, which we will use for both fields and meadows:

signature Σ
import Σ_{CR}

operations
$^{-1}: ring \to ring$
end

2.1 Commutative rings and fields

The first set of axioms is that of a *commutative ring with* 1, which establishes the standard properties of $+$, $-$, and $\cdot$.

equations CR

$$\begin{aligned}
(x+y)+z &= x+(y+z) && (3)\\
x+y &= y+x && (4)\\
x+0 &= x && (5)\\
x+(-x) &= 0 && (6)\\
(x\cdot y)\cdot z &= x\cdot(y\cdot z) && (7)\\
x\cdot y &= y\cdot x && (8)\\
x\cdot 1 &= x && (9)\\
x\cdot(y+z) &= x\cdot y + x\cdot z && (10)
\end{aligned}$$

end

These axioms generate a wealth of properties of $+, -, \cdot$ with which we will assume the reader is familiar. We will write $x - y$ as an abbreviation of $x + (-y)$.

2.1.1 Axioms for meadows

Having available an axiomatization of commutative rings with unit (such as the one above), we define the equational axiomatization of meadows by

$$Md = (\Sigma, CR + Ref + Ril).$$

2.1.2 Axioms for fields

On the basis of the axioms CR for commutative rings with unit there are different ways to proceed with the introduction of division. The orthodoxy is to add the following two axioms for fields: let *Gil* (*general inverse law*) and *Sep* (*separation axiom*) denote denote the following two axioms, respectively:

$$\begin{aligned}
x \neq 0 \implies x\cdot x^{-1} &= 1 && (11)\\
0 &\neq 1 && (12)
\end{aligned}$$

Let (Σ, T_{field}) be the axiomatic specification of fields, where $T_{field} = CR + Gil + Sep$. About the status of 0^{-1} these axioms say nothing. This may mean that the inverse is:

(1) a partial function, or

(2) a total function with an unspecified value, or

(3) omitted as a function symbol but employed pragmatically as a useful notation in some "self-explanatory" cases.

Case 3 arises in another approach to axiomatizing fields, taken in many text-books, which is not to have an operator symbol for the inverse at all and to add an axiom *Iel* (*inverse existence law*) as follows:

$$x \neq 0 \implies \exists y(x \cdot y = 1).$$

Each Σ algebra satisfying T_{field} also satisfies *Iel*. In models of $(\Sigma_{CR}, CR + Iel + Sep)$ the inverse is implicit as a single-valued definable relation, so we call this theory the *relational theory of fields RTF*.

2.1.3 Totalized division in fields

In field theory, if the decision has been made to use a function symbol for inverse the value of 0^{-1} is either left undefined, or left unspecified. However, in working with elementary specifications, which we prefer, operations are total. This line of thought leads to totalized division.

The class $Alg(\Sigma, T_{field})$ is the class of all possible *total* algebras satisfying the axioms in T_{field}. For emphasis, we refer to these algebras as *totalized fields*.

Now, for all totalized fields $A \in Alg(\Sigma, T_{field})$ and all $x \in A$, the inverse x^{-1} is defined. Let 0_A be the zero element in A. In particular, 0_A^{-1} is defined. The actual value $0_A^{-1} = a$ can be anything but it is convenient to set $0_A^{-1} = 0_A$ (see [3], and compare, e.g., Hodges [8], p. 695).

Definition 2.1. *A field A with $0_A^{-1} = 0_A$ is called zero totalized.*

This choice gives us a nice equation to use, the *zero inverse law Zil*:

$$0^{-1} = 0.$$

With *ZTF*, an extension of T_{field}, we specify the class of zero totalized fields:

$$ZTF = T_{field} + Zil = CR + Gil + Sep + Zil.$$

Let $Alg(\Sigma, ZTF)$ denote the class of all zero totalized fields.

Lemma 2.2. *Each Σ_{CR} algebra satisfying $CR + Iel + Sep$ can be expanded to a Σ algebra with a unique inverse operator that satisfies ZTF.*

Proof. To see this notice that if $x \cdot y = 1$ and $x \cdot z = 1$ it follows by subtraction of both equations that $x \cdot (y - z) = 0$. Now:

$$y - z = 1 \cdot (y - z) = (x \cdot y) \cdot (y - z) = x \cdot (y - z) \cdot y = 0 \cdot y = 0,$$

which implies that $y = z$ and that the inverse is unique. Let x^{-1} be the function that produces this unique value (for non-zero arguments). Choose 0^{-1} to be 0 and a zero totalized field has been built. □

2.1.4 Equations for zero totalized division

Following [3], one may replace the axioms *Gil* and *Sep* by other axioms for division, especially, the three equations in a unit called SIP for *strong inverse properties.* They are considered "strong" because they are equations involving $^{-1}$ *without any guards*, such as $x \neq 0$. These three equations were used already by Harrison in [7].

equations $SIP1, SIP2$ and $SIP3$

$$
\begin{aligned}
(-x)^{-1} &= -(x^{-1}) && (13)\\
(x \cdot y)^{-1} &= x^{-1} \cdot y^{-1} && (14)\\
(x^{-1})^{-1} &= x && (15)
\end{aligned}
$$

end

The following was proven in [3]:

Proposition 2.3. *Laws for zero*
1. $CR \cup SIP \vdash 0^{-1} = 0$
2. $CR \vdash 0 \cdot x = 0.$

2.2 Meadows and *Ril*

In [3] we add to $CR + SIP$ the equation *Ril* (*restricted inverse law*):

$$x \cdot (x \cdot x^{-1}) = x$$

which, using commutativity and associativity, expresses that $x \cdot x^{-1}$ is 1 in the presence of x. We may write $x \cdot x^{-1}$ as 1_x, in which case we have the following alternative formulations of *Ril*,

$$1_x \cdot x = x \text{ and } 1_x \cdot x^{-1} = x^{-1},$$

and also $1_x = 1_{x^{-1}}$. Following [3] we define:

Definition 2.4. *A model of* $CR + SIP + Ril$ *is called a* meadow.

Shortly, we will demonstrate that this definition is equivalent to the definition of a meadow given in the introduction. A meadow satisfying *Sep* is called *non-trivial.*

Example All zero totalized fields are clearly non-trivial meadows but not conversely. In particular, the zero totalized prime fields $\mathbb{Z}_p$ of prime characteristic are meadows. That the initial algebra of $CR + SIP + Ril$ is not a field follows from the fact that $(1 + 1) \cdot (1 + 1)^{-1} = 1$ cannot be derivable because it fails to hold in the prime field $\mathbb{Z}_2$ of characteristic 2 which is a model of these equations as well.

Whilst the initial algebra of *CR* is the ring of integers, we found in [3] that

Lemma 2.5. *The initial algebra of CR + SIP + Ril is a computable algebra but it is not an integral domain.*

2.3 Derivable properties of meadows

We will now derive some equational facts from the specification *Md* or relevant subsets of it.

Proposition 2.6.
$CR + Ril \vdash x \cdot x^{-1} = 0 \leftrightarrow x = 0.$

Proof. Indeed, we have $x \cdot x^{-1} = 0 \Longrightarrow x \cdot x^{-1} \cdot x = 0 \cdot x$, by multiplication. Thus, $x = 0$ by applying *Ril* to the LHS and simplifying the RHS. The other direction is immediate from $0 \cdot x = 0$. □

To improve readability we denote x^{-1} by $\overline{x}$ and use $1_x = x \cdot x^{-1}$. Recall that $1_x = 1_{\overline{x}}$.

Proposition 2.7. *Implicit definition of inverse:*
$CR + Ril \vdash x \cdot y = 1 \rightarrow x^{-1} = y$

Proof. $\overline{x} = 1 \cdot \overline{x} = x \cdot y \cdot \overline{x} = 1_x \cdot y = (1_x + 0) \cdot y = (1_x + 0 \cdot \overline{x}) \cdot y = (1_x + (x - x) \cdot \overline{x}) \cdot y = (1_x + (x \cdot 1 - x \cdot x \cdot \overline{x}) \cdot \overline{x}) \cdot y = (1_x + (x \cdot x \cdot y - x \cdot x \cdot \overline{x}) \cdot \overline{x}) \cdot y = (1_x + x \cdot x \cdot (y - \overline{x}) \cdot \overline{x}) \cdot y = (1_x + x \cdot (y - \overline{x})) \cdot y = (1_x + x \cdot y - x \cdot \overline{x}) \cdot y = x \cdot y \cdot y = 1 \cdot y = y$ □

Proposition 2.8. *Derivability of SIP1 and SIP2:*
1. $Md \vdash (xy)^{-1} = x^{-1}y^{-1}$
2. $Md \vdash (-x)^{-1} = -(x^{-1})$

Proof. 1. First we show that $1_{xy} = 1_x \cdot 1_y$. Indeed we have: $1_{xy} \cdot 1_x \cdot 1_y = x \cdot y \cdot \overline{xy} \cdot x \cdot \overline{x} \cdot y \cdot \overline{y}$ Applying *Ril* twice we have $x \cdot y \cdot x \cdot \overline{x} \cdot y \cdot \overline{y} = x \cdot y$, and therefore $1_{xy} \cdot 1_x \cdot 1_y = x \cdot y \cdot \overline{xy} = 1_{xy}$. On the other hand applying *Ril* once we have $x \cdot y \cdot \overline{xy} \cdot x \cdot y = x \cdot y$ and therefore $1_{xy} \cdot 1_x \cdot 1_y = x \cdot y \cdot \overline{x} \cdot \overline{y} = 1_x \cdot 1_y$ This proves the auxiliary equation. Now: $\overline{xy} = \overline{xy} \cdot 1_{xy} = \overline{xy} \cdot 1_x \cdot 1_y = \overline{xy} \cdot x \cdot \overline{x} \cdot y \cdot \overline{y} = 1_{xy} \cdot \overline{x} \cdot \overline{y} = 1_x \cdot 1_y \cdot \overline{x} \cdot \overline{y} = \overline{x} \cdot \overline{y}$.

2. The fact that $\overline{-1} = -1$ follows by an application of Proposition 2.7 to $(-1) \cdot (-1) = 1$ which is a consequence of *CR*. We now conclude with the help of 1: $\overline{-x} = \overline{(-1) \cdot x} = \overline{(-1)} \cdot \overline{x} = (-1) \cdot \overline{x} = -\overline{x}$ □

Thanks to Proposition 2.8 we obtain:

Corollary 2.9. *Md axiomatizes the meadows, i.e. Md is equivalent to CR + SIP + Ril.*

Proposition 2.10.
1. $CR + Ril + SIP2 \vdash x^2 = x \rightarrow x = x^{-1}$
2. $Md \vdash x^3 = x \rightarrow x = x^{-1}$*, and*
3. $Md \vdash x^4 = x \rightarrow x = x^{-2}$.

Proof.
1. $x = x \cdot x \cdot x^{-1} = x \cdot x^{-1} = x \cdot (x \cdot x)^{-1} = x \cdot x^{-1} \cdot x^{-1} = x^{-1}$.
2. From the assumption we obtain $x^3 \cdot x^{-1} = x \cdot x^{-1}$ and then $x \cdot x = x \cdot x^{-1}$. Thus $x \cdot x \cdot x^{-1} = x \cdot x^{-1} \cdot x^{-1}$ whence $x = ((x \cdot x^{-1} \cdot x^{-1})^{-1})^{-1} = (x^{-1} \cdot x \cdot x)^{-1} = x^{-1}$.
3. From the assumption we obtain $x^4 \cdot x^{-1} = x \cdot x^{-1}$ and then $x^3 = x \cdot x^{-1}$, from which we get $x^3 \cdot x^{-1} = x \cdot x^{-1} \cdot x^{-1}$ and $x^2 = x^{-1}$. □

2.4 Meadows and von Neumann regular rings with unit

A commutative von Neumann regular ring (e.g., see [5]) is a Σ_{CR} algebra that satisfies *CR* and which in addition satisfies the following axiom *regular ring* (*RR*):

$$\forall x.\exists y.(x \cdot y \cdot x = x).$$

A value y which satisfies $x \cdot y \cdot x = x$ is called a pseudoinverse of x. Because *Ril* indicates that x^{-1} is a pseudoinverse of x, every meadow expands a commutative von Neumann regular ring. As it turns out a converse is true in an unambiguous form. We acknowlege Robin Chapman (Exeter UK) for pointing out to us the following observation:

Lemma 2.11. *Every commutative regular von Neumann ring can be expanded to a meadow. Moreover, this expansion is unique.*

First we notice a lemma that holds for any commutative ring.

Lemma 2.12. *Given an x, any y with $x \cdot x \cdot y = x \wedge y \cdot y \cdot x = y$ is unique.*

Proof. To see this assume that, in addition, $x \cdot x \cdot z = x \wedge z \cdot z \cdot x = z$. By subtracting the first equations of both pairs we find $x \cdot x \cdot (y - z) = 0$ which implies $x \cdot x \cdot (y - z) \cdot y = 0 \cdot y$ and thus $x \cdot (y - z) = 0$, and as a consequence $x \cdot y = x \cdot z$. Now together with $y \cdot y \cdot x = y$ this yields $y \cdot z \cdot x = y$ and in combination with $z \cdot z \cdot x = z$ it yields $z \cdot y \cdot x = z$ which together imply $y = z$. □

Proof. Then we proceed with the proof of Lemma 2.11. Suppose thatΣ_{CR} algebra A satsifies *RR*. First expand the A to an algebra A' with an operator $i : ring \to ring$ that satisfies $x \cdot i(x) \cdot x = x$. This function y need not be unique, because $i(0)$ can take any value in A. However, if $j(x)$ is another function on the domain of A such that for all x, $x \cdot j(x) \cdot x = x$, then for all x, $i(x) \cdot x \cdot i(x) = j(x) \cdot x \cdot j(x)$. To see this one writes $p(x) = i(x) \cdot x \cdot i(x)$ and $q(x) = j(x) \cdot x \cdot j(x)$. Now $x \cdot x \cdot p(x) = x \cdot x \cdot i(x) \cdot x \cdot i(x) = x \cdot x \cdot i(x) = x$ and $p(x) \cdot p(x) \cdot x = i(x) \cdot x \cdot i(x) \cdot i(x) \cdot x \cdot i(x) \cdot x = i(x) \cdot x \cdot i(x) \cdot i(x) \cdot x = x \cdot i(x) \cdot i(x) = p(x)$. An application of Lemma 2.12 establishes that $p(x) = q(x)$. It follows that $p(-)$ is independent of the choice of i.

Then expand A' to the Σ algebra A'' by introducing an inverse operator as follows: $x^{-1} = p(x)$. We will show that both *Ril* and *Ref* are satisfied. For *Ril* and making use of the equations just derived for $p(-)$ we find: $x \cdot x \cdot x^{-1} = x \cdot x \cdot p(x) = x$.

Now *Ref* has to be established for the proosed inverse operator. In order to prove $(u^{-1})^{-1} = u$ write $x = u^{-1}$, $y = x^{-1}$ and $z = u$.

Then using straightforward calculations we obtain: $x \cdot x \cdot y = x$, $y \cdot y \cdot x = y$, $x \cdot x \cdot z = x$ and $z \cdot z \cdot x = z$. It follows by Lemma 2.12 that $y = z$ which is the required identity.

To see that the expansion is unique suppose that two unary functions $p(-)$ and $q(-)$ both satisfy *Ref* and *Ril*. Using Lemma 2.8 both functions satisfy $p(x \cdot y) = p(x) \cdot p(y)$ and $q(x \cdot y) = q(x) \cdot p(y)$, respectively. Given an arbitrary x we find: $x \cdot x \cdot p(x) = x$ by assumption on $p(-)$. Applying $p(-)$ on both sides we find $p(x \cdot x \cdot p(x)) = p(x)$, which using *SIP2* implies $p(x) \cdot p(x) \cdot p(p(x)) = p(x)$. Then, using *Ref* we have $p(x) \cdot p(x) \cdot x = p(x)$. Similarly we find $x \cdot x \cdot q(x) = x$ and $q(x) \cdot q(x) \cdot x = q(x)$. By means of Lemma 2.12 this yields $p(x) = q(x)$. □

The uniqueness of inverse as an expansion of commutative rings satisfying *Ref* and *Ril* indicates that the inverse operation can be implicitly defined on a commutative von Neumann regular ring. The Beth definability theorem implies the existence of an explicit definition for inverse. In this case the application of Beth definability is inessential, however, because from the proof of Lemma 2.11 an explicit definition can be inferred for $y = x^{-1}$:

$$\exists z.(x \cdot z \cdot x = x \,\&\, y = z \cdot x \cdot z).$$

3 The embedding theorem

Because the theory of meadows is equational we know from universal algebra (see [9, 14]) that:

Theorem 3.1. *The class of meadows is closed under subalgebras, direct products and homomorphic images.*

Thus, every subalgebra of a product of zero totalized fields is a meadow. Our main task is to show that every non-trivial meadow is isomorphic to a subalgebra of a product of zero totalized fields. First, we recall some basic properties of commutative rings, which can be found in many textbooks (e.g., [10]).

3.1 Preliminaries on rings

Let R be a commutative ring. An *ideal* in a ring R is a subset I with 0, and such that if $x, y \in I$ and $z \in R$, then $x + y \in I$, and $z \cdot x \in I$. R itself and $\{0\}$ are the trivial ideals. Any other ideal is a *proper ideal.*

The ideal $R \cdot x = \{y \cdot x| \;\; y \in R\}$ is *the principal ideal* generated by x. Since R has a unit, the generator $x = x \cdot 1$ is in $R \cdot x$. This is the smallest ideal that includes x.

If I is an ideal then the following relation is a Σ_{CR} congruence:

$$x \equiv y \quad \textit{iff} \quad x - y \in I$$

The set of classes R/I is a ring. The *quotient map* maps every element a of R to its equivalence class, which is denoted by $a + I$ or by a/I. The quotient map is a Σ_{CR} homomorphism from R onto R/I (an epimorphism). It is clear what it means that I is a maximal ideal in R.

Lemma 3.2. *Every ideal is contained in (at least one) maximal ideal.*

Proof. The union of a chain of ideals containing I and not 1 does not include 1. Therefore, by Zorn's lemma there is a maximal such ideal. □

Lemma 3.3. *I is a maximal ideal iff R/I is a field.*

Proof. If x is not in I then the ideal generated by I and x is R. Hence for some i in I and y in R we have $1 = i + xy$. It follows that the classes of x and of y are inverse to each other. Since x is arbitrary outside I, every class except for the class 0 (i.e, the set I) has an inverse. □

Recall that $e \in R$ is called an *idempotent* if $e \cdot e = e$.

Proposition 3.4. *Let $e \in R$ be an idempotent and $e \cdot R$ the principal ideal that it generates. Then*
1. e is a unit in the ring $e \cdot R$,
2. the mapping $H(a) = e \cdot a$ is a Σ_{CR} homomorphism from R onto the ring $e \cdot R$.
3. For every $x \in R$: $x \in e \cdot R$ iff $e \cdot x = x$

Proof.
1. Note that $e = e \cdot 1$ and therefore $e \in e \cdot R$. For every element $e \cdot a$ in $e \cdot R$ we have $e \cdot (e \cdot a) = e \cdot a$, by associativity, and because $e \cdot e = e$. Therefore e is a unit in $e \cdot R$.
2. H is a Σ_{CR} homomorphism since:
$e \cdot 0 = 0$ and $e \cdot 1 = e$, so that zero is mapped to zero, and the unit is mapped to the unit.
$e(a + b) = e \cdot a + e \cdot b$ and $e \cdot (-a) = -e \cdot a$, so that $+$ and $-$ are preserved.
$e(f \cdot g) = (e \cdot e)(f \cdot g) = (e \cdot f)(e \cdot g)$ so that multiplication is preserved.
3. If $x \in e \cdot R$ then $e \cdot x = x$ by (a). And if $x = e \cdot x$ then the right side testifies that it is an element of $e \cdot R$. □

3.2 Principal ideals in a meadow

Let R be a non-trivial meadow, and $x \in R$ a non zero element. Note that by Ril, 1_x is an idempotent.

Proposition 3.5. *The principal ideal $x \cdot R$ has the following properties:*
(a) $1_x \cdot R = x \cdot R$, and $x, 1_x$ and x^{-1} are all in $x \cdot R$.
(b) $x \cdot R$ is a ring with a unit, x is invertible in the ring and $H(y) = 1_x \cdot y$ is a Σ_{CR} homomorphism from R onto $x \cdot R$.

Proof. (a) $Now 1_x = x^{-1} \cdot x$ hence $1_x \in x \cdot R$, and $x = x \cdot 1_x$ hence $x \in 1_x \cdot R$. Therefore, $x \cdot R = 1_x \cdot R$. Consequently, both x and 1_x belong to the ideal that they generate, and since $x^{-1} = 1_x \cdot x^{-1}$, x^{-1} is also in $1_x \cdot R$.

(b) Since 1_x is an idempotent, this is Proposition 3.4. Note that x is invertible since $x \cdot x^{-1}$ is the unit in this ring, and x^{-1} is also in it. □

Proposition 3.6. *Let R be a meadow. For every non-zero $x \in R$ there is a Σ_{CR} homomorphism $H_x : R \to F_x$ from R onto a zero totalized field F_x with $H_x(x) \neq 0$.*

Proof. Let $x \neq 0$ be given, and let I be a maximal ideal in the ring $1_x \cdot R$. Then R/I is a field, and the mapping $H_x(y) = (y \cdot 1_x)/I$ is a Σ_{CR} homomorphism as it is the composition of two Σ_{CR} homomorphisms. Now $H_x(x) = x/I$ and $H_x(x) \neq 0$ because if an invertible element of $1_x \cdot R$ is mapped to 0 by the quotient map, then $1 = 0$ in the quotient R/I. □

Proposition 3.7. *If $H : R \to F$ is a Σ_{CR} homomorphism from a meadow R into a zero totalized field F then H preserves inverses and so is a Σ homomorphism.*

Proof. If $H(x) = 0$ then $H(1_x) = H(x \cdot x^{-1}) = H(x) \cdot H(x^{-1}) = 0$ so that also implies $H(x^{-1}) = H(1_x \cdot x^{-1}) = H(1_x) \cdot H(x^{-1}) = 0 = H(x)^{-1}$. The latter holds because F is zero totalized. Secondly, we consider the case that $H(x) \neq 0$. Then $H(x) = H(1_x \cdot x) = H(1_x) \cdot H(x)$ which proves that $H(1_x) = 1$, by cancellation in fields. In other words $1 = H(x \cdot x^{-1}) = H(x) \cdot H(x^{-1})$, which proves that $H(x^{-1}) = H(x)^{-1}$ using Proposition 2.7. □

The image of H is subfield of F, so it follows that given R and non-zero $x \in R$ a meadow homomorphism onto a field F can be found which maps x to a non-zero element of F. Using these preparations, we can prove the embedding theorem:

Theorem 3.8. *A Σ structure is a non-trivial meadow if and only if it is a Σ-substructure of a product of zero totalized fields.*

Proof. By Theorem 3.1 a Σ subalgebra of a product of zero totalized fields is always a meadow.

Let R be a meadow. Combining Propositions 3.6 and 3.7, for each nonzero x in R there is a field F_x and a Σ homomorphism $H_x : R \to F_x$, such that $H_x(x) \neq 0$.

We define the product of fields: $K = \prod_{x \in R} F_x$. K is a meadow with the operations defined at each coordinate. We define the map H from R to the product as follows: for every z in R, $H(z)$ is the vector that has $H_x(z)$ in the place x. Since H_x is a Σ-homomorphism with respect to all meadow operations, following the principles of universal algebra, the same is true for H as well.

If $z \neq 0$ then $H_z(z) \neq 0$ and consequently $H(z) \neq 0$. Therefore H is a Σ-monomorphism, which concludes the proof. □

Corollary 3.9. *A finite non-trivial meadow R is a Σ-substructure of a finite product of finite fields.*

3.3 Equational theory of zero totalized fields

The equational theory of zero totalized fields and of meadows are the same. More precisely:

Theorem 3.10. *For every Σ-equation e, $Alg(\Sigma, ZTF) \models e \Leftrightarrow Alg(\Sigma, Md) \models e$.*

Proof. Let e be an equation that holds in every zero totalized field, then it holds also in every product of fields and in every Σ subalgebra of a product of fields, and therefore, by the embedding theorem, also in every non-trivial meadow. Evidently, every equation holds in the trivial meadow as well.

The other way around, that equations true for all meadows hold in all zero totalized fields, is obvious because zero totalized fields are a subclass of meadows. □

3.4 Conditional equational theory of zero totalized fields

As an application of Theorem 3.10 we proceed with a strengthening concerning conditional equations.The conditional equational theory of zero totalized fields and of meadows are the same. More precisely:

Theorem 3.11. *For every conditional* Σ*-equation* e*,* $Alg(\Sigma, ZTF) \models e \Leftrightarrow Alg(\Sigma, Md) \models e$*.*

Proof. Let $t_1^1 = t_2^1 \,\&\, \ldots \,\&\, t_1^i = t_2^i \,\&\, \ldots \,\&\, t_1^n = t_2^n \rightarrow t_1 = t_2$ be an equation that holds in every zero totalized field. Without loss of generality it may be assumed that each right-hand side equals 0, using $r = s \Leftrightarrow r - s = 0$. So we assume that $t_1 = 0 \,\&\, \ldots \,\&\, t_i = 0 \,\&\, \ldots \,\&\, t_n = 0 \rightarrow t_1 = 0$ holds in all zero totalized fields. If $n = 0$ the case reduces to that of equations and the conclusion follows from Theorem 3.10. Let the Σ term $C(-,-)$ be given by

$$C(x, y) = (1 - \frac{x}{x}) \cdot y.$$

Now by inspection of zero totalized fields one has:

$$Alg(\Sigma, ZTF) \models t_1 = 0 \rightarrow t = 0 \Leftrightarrow Alg(\Sigma, ZTF) \models C(t_1, t) = 0.$$

As a consequence $Alg(\Sigma, Md) \models C(t_1, t) = 0$. Now $Md \cup \{C(t_1, t) \vdash t_1 = 0 \rightarrow t = 0$ and consequently $Md \vdash t_1 = 0 \rightarrow t = 0$ and of course $Md \models t_1 = 0 \rightarrow t = 0$.

In the case of $n = 2$ we assume that all zero totalized fields satisfy $t_1 = 0 \,\&\, t_2 = 0 \rightarrow t = 0$. We will make use of the following fact which holds in all meadows:

$$x = 0 \,\&\, y = 0 \Leftrightarrow \frac{x \cdot y}{x \cdot y} - \frac{x}{x} - \frac{y}{y} = 0$$

Here "$\Rightarrow$" is immediate and to see "$\Leftarrow$" multiply both sides with x thus obtaining:

$$\frac{x \cdot x \cdot y}{x \cdot y} - \frac{x \cdot x}{x} - \frac{x \cdot y}{y} = x \cdot 0$$

and using Md

$$\frac{x \cdot y}{y} - x - \frac{x \cdot y}{y} = 0$$

which implies $x = 0$. Similarly one derives $y = 0$. We write $U(x, y) = \frac{x \cdot y}{x \cdot y} - \frac{x}{x} - \frac{y}{y}$. Now using $U(x, y) = 0 \Leftrightarrow x = 0 \,\&\, y = 0$ we find:

$$Alg(\Sigma, ZTF) \models t_1 = 0 \,\&\, t_2 = 0 \rightarrow t = 0 \Leftrightarrow Alg(\Sigma, ZTF) \models C(U(t_1, t_2), t) = 0.$$

Using 3.10 we find that $Md \models C(U(t_1, t_2), t) = 0$ and from this fact using the known properties of $U(-)$ and $C(-,-)$ one easily derives $Md \models t_1 = 0 \,\&\, t_2 = 0 \rightarrow t = 0$. The cases $n = 3$ and further require a repeated nested use of $U(-)$. The straightforward details have been omitted and we only illustrate the encoding of conditional equations into equations in the case $n = 3$:

$$Alg(\Sigma, ZTF) \models (\bigwedge_{i=1}^{i=3} t_i = 0) \rightarrow t = 0 \Leftrightarrow Alg(\Sigma, ZTF) \models C(U(U(t_1, t_2), t_3), t) = 0.$$

$\square$

4 Finite meadows

The characteristic of a meadow is the smallest natural number $k \in \mathbb{N}$ such that $k > 0$ and $k.1 = 1 + 1 + \ldots + 1 = 0$. As usual, we will define $\underline{0}$ as 0 and $\underline{k+1} = \underline{k} + 1$ and given a positive natural number k we define the equation Z_k by

$$\underline{k} = 0.$$

We recall that a natural number k is called *squarefree* if its prime factor decomposition is the product of distinct primes.

Lemma 4.1. *Let M be a meadow of finite characteristic $k > 0$. Then k is squarefree.*

Proof. Let $M \models \underline{k} = 0$. Suppose k has two repeated prime factors, $k = p \cdot p \cdot q$. Then, using *Ril* we have

$$\underline{p} \cdot \underline{q} = (\underline{p} \cdot \underline{p} \cdot \underline{p}^{-1}) \cdot \underline{q} = (\underline{p} \cdot \underline{p} \cdot \underline{q}) \cdot \underline{p}^{-1} = \underline{k} \cdot \underline{p}^{-1} = 0 \cdot \underline{p}^{-1} = 0.$$

Thus, k is not the characteristic (= least summand that is 0) which is a contradition. □

Thus, from Lemma 4.1, the possible finite characteristics have the form $k = p_1 \ldots p_n$ where the p_i are all distinct primes. All finite meadows have finite characteristic. It follows that if a finite meadow M consists of an initial segment of the numerals $\underline{0}$, ..., $\underline{\text{k-1}}$ (like the prime fields of positive characteristic) its cardinality $\#(M) = k$ can only be a product of different primes.

Definition 4.2. *Let Md_k be the initial algebra of $Md \cup \{Z_k\}$.*

What are the initial algebras? Clearly, Md_k has finite characteristic $\leq k$. Notice the following:

Lemma 4.3. *If l divides k then the $Md + Z_l \vdash Z_k$. Thus, if l divides k then there is a Σ epimorphism $\phi\colon Md_k \to Md_l$, i.e., Md_l is a homomorphic image of Md_k.*

Thus, we have that for $k = p_1 \ldots p_n$ where the p_i are all distinct primes we have a Σ epimorphism $\phi\colon Md_k \to Md_{p_i}$. Furthermore, can be seen that for p a prime number, Md_p is the zero totalized prime field $\mathbb{Z}_p$ of characteristic p. To see this notice that for each x different from 0 there is an y with $x \cdot y = 1$. It follows that the zero totalized prime field mod p satisfied WIP and for that reason it is a meadow. As a consequence we have a Σ epimorphism $\phi\colon Md_k \to \mathbb{Z}_{p_i}$.

Theorem 4.4. *If k is squarefree then Md_k has k elements.*

Proof. If $k = p_1 \ldots p_n$ is a product of different primes that is no prime factor appears twice then we first show that Md_k has at least k elements. To see this notice that for each prime factor p of k the prime field $\mathbb{Z}_p$ of characteristic p is a model of Md_k (as the equation Z_p implies Z_k). Because that structure is a quotient of the additive group of Md_k its number of elements is a divisor of the cardinality $\#(Md_k)$ of Md_k. As a consequence $\#(Md_k)$ is a multiple of all factors of k and because k contains all of them only once $\#(Md_k) \geq k$.

In order to prove that $\#(Md_k) = k$ it suffices to find an inverse (in the sense of a meadow) for each $\underline{n}$ for $n < k$ of the form $\underline{m}$ for $m < k$. We may assume that $k > 0$ otherwise the inverse is obvious. To find the inverse consider the power series $\underline{n}^0 (= 1), \underline{n}^1, \underline{n}^2 ...$ Each value in this series is of the form $\underline{m}$ for $m < k$ because arithmetic is done modulo k. Therefore there are k and l with $k > l+1 > 0$ such that $Md_k \models \underline{n}^k = \underline{n}^l$. Let $k - 1 - l = i$. Notice that $i \geq 0$. Working in Md_k by *SIP2* we have $\underline{n}^{-k} = \underline{n}^{-l}$, and thus $\underline{n}^{-1} = \underline{n}^{-k} \cdot \underline{n}^{k-1} = \underline{n}^{-l} \cdot \underline{n}^{k-1} = \underline{n}^{k-1-l} = \underline{n}^i$. This demonstrates that the inverse is a numeral (modulo k) as required. □

It follows from the proof that the interpretation of inverse is unique in a minimal finite meadow. Recall that an algebra is minimal when it has no subalgebras or, equivalently, is generated by elements named in its signature. By Lemma 4.4, if k is a product of different primes then Md_k is the minimal meadow of characteristic k. It also follows from the proof that Md_k consists of $0, \ldots k - 1$.

Example 1. Concrete examples can be easily given, for instance Md_6 has the following inverse function: $0^{-1} = 0, 1^{-1} = 1, 2^{-1} = 2, 3^{-1} = 3, 4^{-1} = 4$, and $5^{-1} = 5$. Md_6 is the smallest non-trivial minimal meadow which is not a field.

Example 2. In Md_{10} the inverse function is given by: $0^{-1} = 0, 1^{-1} = 1, 2^{-1} = 8, 3^{-1} = 7, 4^{-1} = 4, 5^{-1} = 5, 6^{-1} = 6, 7^{-1} = 3, 8^{-1} = 2$, and $9^{-1} = 9$.

Example 3. Consider Md_4. This is a non-minimal meadow because its size of four elements exceeds its characteristic. The inverse function is the identity function. Md_4 is the smallest non-trivial meadow which is not a field.

Lemma 4.5. *Let M be a meadow of finite characteristic $k > 0$. Then there is a Σ monomorphism $\psi \colon Md_k \to M$.*

Proof. If M has characteristic k then $M \models \underline{k} = 0$. Thus, by initiality, there is a Σ homomorphism $\psi \colon Md_k \to M$. If this map were not injective then M would have characteristic lower then k. □

Lemma 4.6. *Let M be a minimal meadow of finite characteristic $k > 0$. Then Md_k and M are Σ isomorphic.*

Proof. If M has characteristic k then $M \models \underline{k} = 0$. Thus, following the previous lemma there is a Σ monomorphism $\psi \colon Md_k \to M$. Because M is minimal, ψ is surjective as well. □

Lemma 4.7. *Let M be a meadow of prime cardinality p. Then M is the zero totalized prime field of cardinality p.*

Proof. If M has characteristic k then $k > 0$ is the cardinality of the smallest additive subgroup of M which contains 1. Thus k divides p and hence $k = p$ which implies that M is minimal. Following Lemma 4.6 there is a monomorphism Md_k is isomorphic with M. At the same time the zero totalized prime field of cardinality p is a meadow and according to Lemma 4.6 it is also isomorphic to Md_k. □

Lemma 4.8. *All finite and minimal meadows are of the form Md_k for some positive natural number k.*

Proof. Let M be a finite meadow. Then M has a finite characteristic, say k. By 4.7, there is a homomorphism $\phi\colon Md_k \to M_l$. By Lemma 4.6, there is also a homomorphism $\psi\colon Md_k \to M$. □

If its non-zero characteristic is not a prime, a finite meadow has proper zero-divisors and fails to be an integral domain and, of course, it is no field either.

Lemma 4.9. *If $k = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ then $Md_k \cong Md_{p_1 \dots p_n}$. Therefore, if k and l have the same set of prime factors then $Md_k \cong Md_l$.*

Proof. Using the same argument as in Lemma 4.1, we can show that for $p_1, \dots, p_n$ be any primes and $k = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ we have $Md_k \cong Md_{p_1 \dots p_n}$. Suppose that $k = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ and $l = p_1^{\beta_1} \dots p_n^{\beta_n}$. Then by the first part of the lemma, $Md_k \cong Md_{p_1 \dots p_n}$ and $Md_l \cong Md_{p_1 \dots p_n}$ and hence $Md_k \cong Md_l$.

□

5 Concluding remarks and further questions

At the heart of the theory of meadows is the idea to use a totalized form of division. We do not claim that division by zero in the rationals or reals is possible, but we do we hold that zero totalized division is algebraically and computationally useful. In fact, we expect that for some application areas zero totalized division, based on equations and rewriting, is appropriate because it is conceptually simpler than the conventional concept of partial division. These areas include elementary school algebra as well as specifying and understanding calculators and spreadsheets. Our theory of meadows is a theory of zero totalized division and constitutes a generalization of the theory of fields.

There are many opportunities for the further development of the theory of meadows. Leaving aside questions that may emerge from the perspective of pure algebra, where the properties of invertibility and symmetry are central, we conclude with some computational and logical open questions that add to the questions posed in [3]:

Is the equational theory of meadows decidable. Is its conditional equational theory decidable?

Does Md, or a useful extension of it, admit Knuth-Bendix completion?

For a minimal meadow of finite characteristic it is interesting to know what fraction of non properly invertible elements satisfies $x = x^{-1}$.

Returning to the equational theory of meadows, following [3], let $Z(x) = 1 - x \cdot x^{-1}$. For $n > 0$, let L_n be the equation: $Z(1 + x_1^2 + ... + x_n^2) = 0$. Clearly from CR it follows that L_k implies L_n when $k > n$. All L_n are valid in the zero totalized field of rational numbers. From [3] and Proposition 2.8, it follows that $Md + L_4$ constitutes an initial algebra specification of the zero totalized field of rational numbers, which indicates the relevance of L_4. Now, conversely, the question arises if $Md + L_n$ proves L_k (again assuming $k > n$).

A related problem is to characterize the initial algebras of $Md + L_n$ for $n = 1$, $n = 2$, and $n = 3$.

A restricted version of Theorem 3.10 for equations between closed terms only, was shown in [3]. That proof is longer and more syntactic in style and uses a normal form result and straightforward induction, in spite of the fact that the result is weaker. However, it provides the additional information that the initial algebra of *Md* is a computable algebra. The proof given here uses the maximal ideal theorem, which is weaker than the axiom of choice, but still independent of the axiom system *ZF* for set theory. It is unlikely that the equality or inequality between the equational theories of zero totalized fields and meadows is independent of *ZF*. A more elementary proof of Theorem 3.10 should be sought.

Finally Theorem 3.10 and the results leading up to it may be investigated in the non-commutative case. At present the authors do not know which generalizations are possible in this direction. The terminology used in this paper is geared towards the commutative case. Some comments on the non-commutative case are in order, however. A *skew meadow* is an expansion of a non-commutative ring with an inverse operator that satisfies the axioms *Ref* and the *pseudo inverse law* (*Pil*): $x \cdot (x^{-1} \cdot x) = x$, as well as some further equations valid for all zero totalized skew fields. In the commutative case *Pil* and *Ril* are equivalent but in the non-commutative case they differ and as rewrite rules the should be distinguished as well. The precise axiomatization of skew meadows, which may conceivably involve *SIP1 SIP2* as well as the generalization of the theory of meadows as presented in this paper is left for future research.

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