Hint for Exercise 1.4.8

Prove the statement by induction on ψ. In each inductive step (e.g. the negation $\psi = \neg\psi'$), by IH, we have

$$\vdash (\phi_1 \leftrightarrow \phi_2) \to (\psi'[\phi_1/p] \leftrightarrow \psi'[\phi_2/p]).$$

Since $\psi[\phi_1/p] = \neg(\psi'[\phi_1/p])$ and $\psi[\phi_2/p] = \neg(\psi'[\phi_2/p])$, it suffices to show

$$\vdash (\phi_1 \leftrightarrow \phi_2) \to \big(\neg(\psi'[\phi_1/p]) \leftrightarrow \neg(\psi'[\phi_2/p])\big).$$

Let $\xi = (\phi_1 \leftrightarrow \phi_2)$, $\theta_1 = \psi'[\phi_1/p]$, $\theta_2 = \psi'[\phi_2/p]$, $\sigma_1 = \neg(\psi'[\phi_1/p])$, and $\sigma_2 = \neg(\psi'[\phi_2/p])$.

Then you will prove

$$\vdash \big(\xi \to (\theta_1 \leftrightarrow \theta_2)\big) \to \big(\xi \to (\neg\theta_1 \leftrightarrow \neg\theta_2)\big).$$

Let $\mathcal{D}_1$ be a derivation for this statement and $\mathcal{D}_2$ be a derivation for $(\phi_1 \leftrightarrow \phi_2) \to (\psi'[\phi_1/p] \leftrightarrow \psi'[\phi_2/p])$.

By 'concatenating' $\mathcal{D}_1$ and $\mathcal{D}_2$, you will produce a new derivation for $(\phi_1 \leftrightarrow \phi_2) \to \big(\neg(\psi'[\phi_1/p]) \leftrightarrow \neg(\psi'[\phi_2/p])\big)$ and this is what we want.