

Workshop on McKay correspondence for symplectic resolutions (following Bezrukavnikov-Kaledin)

The aim of the Workshop is to understand the proof, due to Bezrukavnikov and Kaledin (*McKay equivalence for symplectic resolutions of quotient singularities* [1]), of a generalization of the so called McKay correspondence. This proof uses a rather large variety of ideas, so we hope that the proof itself might be as interesting as the result:

Theorem 1 (Bezrukavnikov-Kaledin). *Let V be a finite dimensional, symplectic vector space (over \mathbb{C}) and $\Gamma \subset Sp(V)$ a finite subgroup. Assume that we are given a resolution $\pi : X \rightarrow V/\Gamma$ such that the symplectic form on the smooth part of V/Γ extends to a non-degenerate symplectic form on X . Then there is an equivalence:*

$$D^b(\text{Coh}(X)) \cong D^b(\text{Coh}^\Gamma(V))$$

of the derived category of X and the equivariant derived category of V (which is compatible with the $\mathcal{O}_{V/\Gamma}$ -linear structure on both categories).

This is a generalization of the so called McKay correspondence, which is the two-dimensional version of this theorem. In the original version of McKay this was considered in the case $V = \mathbb{C}^2$. In this case the minimal resolution of the surface singularity \mathbb{C}^2/Γ is the only symplectic resolution and McKay formulated the theorem as a correspondence between representations of Γ and the configuration of the projective lines in the exceptional fiber. (This fiber is a chain of \mathbb{P}^1 's such that their intersections are given by one of the Dynkin diagrams without multiple lines (the A,D,E diagrams).)

Bridgeland, King and Reid then proved in [2] that one could regard this as an equivalence of derived categories and that a generalization holds in dimension 3. (Here the Γ -Hilbertscheme is a crepant resolution). This is one precise formulation of the more general idea that the geometry of a crepant resolution should be encoded in the geometry of the quotient stack $[V/\Gamma]$ or equivalently the equivariant geometry of V .

As indicated already, the proof of Bezrukavnikov and Kaledin is a surprising combination of ideas from quite different versions of geometry and therefore the main reason to suggest this program was to try to understand the relation of these ideas. To explain this, let us give a rough overview of the strategy:

The authors start by a close examination of the two dimensional situation. Here the McKay correspondence can be viewed as a Fourier-Mukai equivalence, because one can explicitly describe the resolution X . However, general arguments on derived categories allow one to reformulate this as the existence of a particular vector bundle \mathcal{E} on the resolution. By the defining properties of \mathcal{E} , this bundle can be lifted over formal completions and this implies that one may reduce to characteristic p .

Next Bezrukavnikov and Kaledin observe that one can simplify the picture if one passes to a non-commutative version: There is a non-commutative deformation \mathcal{A} of V (i.e. of the algebra of functions on V) such that passing to the quotient modulo Γ already gives a Morita equivalence between equivariant \mathcal{A} -modules and modules under the invariant functions \mathcal{A}^Γ . Moreover this deformation (called quantization) is obtained using the symplectic structure and this is the reason to expect that there is also a deformation of the algebra of functions on X . (Roughly, this means that for the non-commutative algebras the equivariant category on V is the same as the one for the coarse quotient and this in turn can be compared to X .)

This does not seem to give information about the original correspondence, but if one passes to positive characteristics this changes! Namely, if one works over a field of positive characteristic, the non-commutative algebras used above can be viewed as Azumaya algebras (i.e. they are locally isomorphic to matrix algebras)

and can therefore be studied using algebraic geometry. (This use of Azumaya algebras was motivated by the observation that in characteristic p one can view the (infinite dimensional) algebra of differential operators on a variety as a finite dimensional algebra on the tangentbundle of the Frobenius-twist of the variety). Moreover once one knows how to prove the equivalence in this case one can deduce the original equivalence using Azumaya algebra techniques. (Here the main point is the construction of a particular Azumaya algebra on the coarse quotient V/Γ).

Finally, one can show that the resulting equivalence can be lifted back to characteristic 0.

Talk 1 (The case of dimension 2 and almost exceptional objects). To illustrate what the seminar is about briefly explain the geometry of the two-dimensional case (see beginning of section 5.1.1 and Nakajima's Lectures [5] Thm 4.1, don't give the GIT-description, this will be done later - you may just state that the Hilbert scheme is smooth in dimension 2.)

Remark that in this situation the correspondence has a direct interpretation in terms of Dynkin-Diagrams and representations of Γ , which is the classical McKay correspondence. (See e.g. Reid's Bourbaki article [6])

This should take half of the lecture. The second half should explain a way how to prove the correspondence. This is usually constructed by a Fourier-Mukai transform. (One therefore always tries to construct resolutions as moduli spaces). However in our situation this is easier:

Prove Proposition 2.2 (without the Azumaya algebra i.e., just put $\mathcal{A} = \mathcal{O}$) of [1] and illustrate why this can be applied here. With these preparations, state the main Theorem (Thm 1.1 [1]) and explain the reduction to the construction of a certain vector bundle on X (Thm 2.3.) Explain briefly that the assumption on vanishing of the Ext groups implies that the vector bundle always lifts on formal completions. And state that one can get rid of the formal completion by using the \mathbb{G}_m -action, but skip the proof.

References: To connect this to the following talks one should use Section 5.1 and Section 2 of [1]. The proof of Proposition 2.2 follows from general remarks on derived categories, a more detailed exposition can be found in Huybrechts book [3] where the term "right admissible subcategory" is used. Here we might need the dual version, using "left" subcategories.

Talk 2 (A noncommutative version - quantizations and Azumaya algebras). There is a non-commutative deformation of the situation, in which taking invariants already defines an equivalence of categories. (As Morita equivalence.) (Section 6.3). As indicated above there are two observations which make this useful for the main theorem: 1. The noncommutative version also lifts to the symplectic resolution. 2. In positive characteristics the non-commutative algebra can be treated by means of algebraic geometry, because it is an Azumaya algebra.

This seems to be the starting point of the construction. So we will first try to understand 1:

Describe the equivariant sheaves on V as modules under $\text{Sym } V^* \# \Gamma$. Define the Weyl algebra W_\hbar a deformation $\text{Sym } V^*$, our first example of "quantization" and prove Lemma 6.1.

To relate this to the geometry of the resolution, the next step is to define an analogous quantization of the algebra of functions on X . This motivates the study of quantizations. Explain Section 3.1 and 3.3. (Skip the deformation theory in 3.2, this will be done later), which explains the special properties in positive characteristics. In particular the short computation in local coordinates needed in Proposition 3.5 is essential and should not be skipped. (You should briefly remind us that an Azumaya algebra is a form of a matrix algebra, but that should be sufficient here.)

Talk 3 (The quantization in case $\dim = 2$). We return to the 2-dimensional case. Explain how to obtain the deformation of the algebra \mathcal{O}_X in this case (i.e., prove Propositions 5.3 and 5.8.). This uses a GIT-construction of the Hilbert scheme.

Talk 4 (The new proof in dimension 2 - untwisting). In the case of dimension 2, explain the equivalence Theorem 6.3. And deduce the main theorem in this case (Theorem 6.7). The key-point is the construction of an Azumaya algebra on $V^{(1)}/\Gamma$. Here again you should briefly recall the theorems on Azumaya algebras needed, but we will just use them. Note that the lifting to characteristic 0 has essentially been done already by the deformation argument in Talk 1.

Talk 5 (The quantization in arbitrary dimension). Now we are left to construct the deformation of \mathcal{O}_X in arbitrary dimension, using the fact that our X is a symplectic manifold. The idea is to begin with the deformation on the part of V/Γ , where the quotient is nonsingular. To extend this in codimension 1 one uses the two-dimensional case (Lemma 4.2 and Proposition 5.10). Finally to extend this to the whole of X one uses that one can compute the obstruction to the existence of an extension in cohomology. The (Hochschild) cohomology lemma needed is 3.2 (skip the \mathbb{G}_m -action). And the final conclusion is Proposition 5.11. (see [4] for explanations about Hochschild cohomology)

REFERENCES

- [1] R.V. Bezrukavnikov, D. Kaledin. McKay equivalence for symplectic resolutions of quotient singularities. *Proc. Steklov Inst. Math.* (246) no. 3 2004, 13–33 [arXiv:math/0401002](#)
- [2] T. Bridgeland, A. King, M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.* 14 no. 3 (2001), 535–554.
- [3] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. viii+307 pp.
- [4] D. Kaledin. Cartier isomorphism and Hodge Theory in the non commutative case. [arXiv:math/0708.157](#)
- [5] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*. University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.
- [6] M. Reid. La correspondance de McKay. [The McKay correspondence] Séminaire Bourbaki, Vol. 1999/2000, *Astérisque* 276 (2002), 53–72.