ADDENDUM TO "SEMISTABLE REDUCTION OF G-BUNDLES ON CURVES"

Abstract. In this note we give a variation of the last step of the proof the semistable reduction theorem of [1], which allows to remove the assumptions on the characteristic of the ground field.

In the article [1] on semistable reduction of principal bundles on curves we had to make some assumptions on the characteristic of the ground field. The assumptions were needed only in the proof of Corollary 5.6 of [1], in order to assure Behrend's conjecture. The aim of this note is to give an alternative proof of this corollary, valid over fields of arbitrary characteristic. This implies, that the semistable reduction theorem of [1] holds in any characteristic.
The starting point of our argument is to note that a statement much weaker than Behrend's conjecture holds for simple reasons:

Remark 1. Let G be a semisimple group $P \subset G$ a parabolic subgroup and $L \subset P$ a Levi-subgroup. Then the connected component $Z(L)^\circ$ of the center of L is a non-trivial torus, acting on the Lie-algebras $\mathfrak{l}, \mathfrak{p}, \mathfrak{g}$ of L, P and G. This action induces a direct sum decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{g}/\mathfrak{p}$. In particular given a L-bundle $\mathcal{L}$ on a smooth projective curve C the sequences

$$0 \to H^0(C, \mathcal{L} \times^L \mathfrak{p}) \to H^0(C, \mathcal{L} \times^L \mathfrak{g}) \to H^0(C, \mathcal{L} \times^L \mathfrak{g}/\mathfrak{p}) \to 0$$

and

$$0 \to H^1(C, \mathcal{L} \times^L \mathfrak{p}) \to H^1(C, \mathcal{L} \times^L \mathfrak{g}) \to H^1(C, \mathcal{L} \times^L \mathfrak{g}/\mathfrak{p}) \to 0$$

are split exact.

In [1] we applied Behrend's conjecture in order to compare cohomology classes of some higher order deformations. We have to consider these more carefully, so we recall some deformation theory: The diagram

$$\begin{tikzcd} k[t]/t^n & k \arrow[l] \\ k[t]/t^{n+1} \arrow[u] & k[\epsilon]/\epsilon^2 \arrow[l, "\epsilon \mapsto t^n"'] \arrow[u] \end{tikzcd}$$

is a cartesian diagram of rings. This induces an equivalence of categories:

$$\mathrm{Bun}_G(k[\epsilon]/\epsilon^2) \cong \mathrm{Bun}_G(k[t]/t^{n+1}) \times_{\mathrm{Bun}_G(k[t]/t^n)} \mathrm{Bun}_G(k).$$

To rephrase this, let us introduce some notation. Given a $G-$bundle $\mathcal{G}$ on $C \times \mathrm{Spec}(k[t]/t^l)$ (or on $C \times \mathrm{Spec}(k[t])$ we denote by $\mathcal{G}_n$ the restriction of $\mathcal{G}$ to $C \times \mathrm{Spec}(k[t]/t^{n+1})$, for $n < l$. Using this notation we can rephrase the above equivalence of categories as follows:

Lemma 2. *Let $\mathcal{G}_0$ be a G-bundle on C. The category of tuples $(\mathcal{G}_n, \phi)$ where $\mathcal{G}_n$ is a G-bundle over $C \times \mathrm{Spec}(k[t]/t^{n+1})$ and ϕ is an isomorphism $\phi : \mathcal{G}_{n-1} \xrightarrow{\cong} \mathcal{G}_0 \times_k \mathrm{Spec}(k[t]/t^n)$ is equivalent to the category of first order deformations of $\mathcal{G}_0$. The objects of this category are therefore parametrized canonically by $H^1(C, \mathcal{G}_0 \times^G \mathfrak{g})$ and the automorphisms of objects are given by $H^0(C, \mathcal{G}_0 \times^G \mathfrak{g})$.*

Remark 3. The class of $(\mathcal{G}_n, \phi)$ depends on ϕ.

Corollary 4. *Let $\mathcal{P}$ be a family of P-bundles over $C \times \mathrm{Spec}(k[t])$ such that $\mathcal{P}_0$ is induced from an L-bundle. Then the deformation $\mathcal{G}_l := \mathcal{P} \times^P G|_{C\times \mathrm{Spec}(k[t]/t^{l+1})}$ of the G-bundle $\mathcal{G}_0 = \mathcal{P}_0 \times^P G$ is trivial if and only if the deformation $\mathcal{P}_l$ is trivial.*

Proof. By the previous lemma the obstruction to the existence of a lifting of a trivialization of an $n-$th order deformation of $\mathcal{P}_0$ to a trivialization of a deformation of order $n+1$ is an element in $H^1(C, \mathcal{P}_0 \times^P \mathfrak{p})$. Since this group is a direct summand in $H^1(C, \mathcal{P}_0 \times^P \mathfrak{g}) = H^1(C, \mathcal{G}_0 \times^G \mathfrak{g})$ (Remark 1) the obstruction vanishes if and only if the obstruction to lift the trivialization of the induced G-bundle vanishes. □

Now we can pass to the situation of [1] Corollary 5.6:
Let G be a semisimple group, $P \subset G$ a parabolic subgroup. Choose a Levi-subgroup $L \subset P$ and denote by P^- the corresponding opposite parabolic. Further, choose a 1-parameter subgroup $\lambda : \mathbb{G}_m \to Z(L)$ of the center of L such that P is defined by λ (i.e. $g \in G$ lies in P iff, $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists).
Denote by $U^- \subset P^-$ the unipotent radical of P^-. We have a canonical filtration $U^- = U_1^- \supset U_2^- \supset \dots U_k^-$ together with chosen isomorphisms $U_i^-/U_{i+1}^- \cong \mathrm{Lie}(U_i^-/U_{i+1}^-) = \mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-$. We denote by $LU_i^- \subset P^-$ the subgroup generated by L and U_i^-.
The following data was constructed on [1] Proposition 4.4 in form of the glueing cocycle $\overline{g}'$:

(1) $\mathcal{L}$ a semi-stable L-torsor on C such that $\mathcal{P} := \mathcal{L} \times^L P$ is the canonical reduction of the G-torsor $\mathcal{G} := \mathcal{L} \times^L G$.
(2) $\mathcal{P}^-$ a P^- torsor (induced from a LU_i^--torsor) carrying an isomorphism $\mathcal{P}^-/U^- \cong \mathcal{L}$.
(3) The class $[\mathcal{P}^-] \in H^1(C, {}^{\mathcal{L}} U_i^-/{}^{\mathcal{L}} U_{i+1}^-) \cong H^1(C, {}^{\mathcal{L}} (\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-))$ is non-zero.

We used this to construct a deformation $\mathcal{P}_\lambda^- := \mathcal{P}^- \times^{P^-} (\mathbb{A}^1 \times P^-)$ of $\mathcal{P}^-$ into the bundle $\mathcal{L}\times^L P^-$. Here an element $p \in P^-$ acts on (t, q) as $p.(t, q) = (t, \lambda^{-1}(t)p\lambda(t).q)$. Further, we computed that this deformation is canonically trivial over $\mathrm{Spec}(k[t]/t^l)$ and the class given by the deformation over $\mathrm{Spec}(k[t]/t^{l+1})$ is a non-zero element in $H^1(C, {}^{\mathcal{L}} \mathfrak{u}^-) \subset H^1(C, {}^{\mathcal{L}} \mathfrak{p}^-)$ ([1] Lemma 5.5).
Now we can state our improvement of the main result (Corollary 5.6) of loc.cit.. Note that in our original argument we passed to another family $\mathcal{Q}_{\overline{\lambda}}^-$, which we avoid here.

Proposition 5. *For any field k the bundle $\mathcal{P}^- \times^{P^-} G$ does not lie in the Harder-Narasimhan stratum of the bundle $\mathcal{P}$.*

Proof. Assume the contrary. In this case all fibres of the family $\mathcal{G}_\lambda := \mathcal{P}_\lambda^- \times^{P^-} G$ lie in the same Harder-Narasimhan stratum. Furthermore, the family being constant over $\mathbb{G}_m$ ([1] Lemma 5.1), there exists a canonical reduction $\mathcal{P}_\lambda^{\mathrm{can}}$ of $\mathcal{G}_\lambda|_{\mathbb{G}_m}$ which extends to a reduction over $\mathbb{A}^1$ by semi-continuity ([1] Proposition 2.2). Thus we get a commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Bun}_P \arrow[d, "\times^P G"] \\
\mathbb{A}^1 \arrow[r, "\mathcal{P}_\lambda^-"'] \arrow[urr, "\mathcal{P}_\lambda^{\mathrm{can}}"] & \mathrm{Bun}_{P^-} \arrow[r, "\times^{P^-} G"] & \mathrm{Bun}_G
\end{tikzcd}$$

We claim that this leads to a contradiction, because the deformation classes of these maps yield different elements in $H^1(C, \mathcal{L} \times^L \mathfrak{g})$:
Again we denote by $\mathcal{G}_n := \mathcal{G}_\lambda|_{k[t]/t^{n+1}}$ the n-th order deformation defined by the family $\mathcal{G}_\lambda$ and similarly for $\mathcal{P}_\lambda^-$ and $\mathcal{P}_\lambda^{\mathrm{can}}$.

The deformation $\mathcal{P}^-_{l-1}$ is trivial, so $\mathcal{G}_{l-1}$ is a trivial deformation and by Corollary 4 we find that the deformation $\mathcal{P}^{\mathrm{can}}_{l-1}$ is trivial.
Over $C\times\mathrm{Spec}(k[t]/t)$ the bundle $\mathcal{P}^-_0$ is induced from the L-bundle $\mathcal{L}$, so $\mathcal{G}_0 = \mathcal{L}\times^L G$ has an induced reduction to P, which coincides with its canonical reduction, i.e., $\mathcal{P}^{\mathrm{can}}_0 = \mathcal{L}\times^L P$. In particular on the tangent space at $\mathcal{P}^{\mathrm{can}}_0$ the map $\times^P G$ is given by the inclusion $H^1(C,\mathcal{L}\times^L \mathfrak{p}) \to H^1(C,\mathcal{L}\times^L \mathfrak{g})$ and similarly this also holds for the map $\times^{P^-} G$.
Now we inductively lift the trivialization of the deformations $\mathcal{G}_n, \mathcal{P}^-_n$ and $\mathcal{P}^{can}_n$: Assume we have chosen trivializations $\mathcal{P}^{\mathrm{can}}_{n-1} \cong \mathcal{P}^{\mathrm{can}}_0 \times \mathrm{Spec}(k[t]/t^n)$ and $\mathcal{P}^-_{n-1} \cong \mathcal{P}^-_0 \times \mathrm{Spec}(k[t]/t^n)$ such that the induced trivializations of $\mathcal{G}_{n-1}$ coincide.
By Lemma 2 we know that for $n < l$ we can lift the trivializations of $\mathcal{P}^-_{n-1}$ and $\mathcal{P}^{can}_{n-1}$ to $\mathrm{Spec}(k[t]/t^{n+1})$. By the same lemma the two induced trivializations of $\mathcal{G}_n$ differ by an element of $H^0(C,\mathcal{L}\times^L \mathfrak{g})$. Since

$$H^0(C,\mathcal{G}\times^G \mathfrak{g}) = H^0(C,\mathcal{P}_L\times^L \mathfrak{p}) \oplus H^0(C,\mathcal{P}_L\times^L \mathfrak{u}^-)$$

we can modify the trivialization of $\mathcal{P}^-_n$ by an element $\psi^- \in H^0(C,\mathcal{L}\times^L \mathfrak{u}^-) \subset H^0(C,\mathcal{L}\times^L \mathfrak{p}^-)$ and the trivialization of $\mathcal{P}^{can}_n$ by an element of $H^0(C,\mathcal{L}\times^L \mathfrak{p})$ such that the induced trivializations of $\mathcal{G}_n$ coincide.
By induction this implies that we can find trivializations $\phi^{\mathrm{can}}, \phi^-$ of the deformations $\mathcal{P}^-_{l-1}$ and $\mathcal{P}^{can}_{l-1}$ inducing the same trivialization ϕ of $\mathcal{G}_{l-1}$.
Choosing the trivializations thus obtained, we find that the pair $(\mathcal{G}_l,\phi)$ defines an element $[g_l] \in H^1(C,\mathcal{L}\times^L \mathfrak{g})$, which lies in the image of $H^1(C,\mathcal{L}\times^L \mathfrak{p}) \subset H^1(C,\mathcal{L}\times^L \mathfrak{g})$, because it is induced from the pair $(\mathcal{P}^{\mathrm{can}}_l, \phi^{\mathrm{can}})$. On the other hand $(\mathcal{P}^-_l,\phi^-)$ defines a non-trivial element $[\mathcal{P}^-_l,\phi^-] \in H^1(C,\mathcal{L}\times^L \mathfrak{p}^-)$, because the family $\mathcal{P}^-_l$ is a non-trivial deformation by assumption. Moreover the image of $[\mathcal{P}^-_l]$ in $H^1(C,\mathcal{L}\times^L (\mathfrak{p}^-/\mathfrak{u}^-))$ is 0, because the family $\mathcal{P}^-_\lambda \times^{P^-} L$ is constant. Thus we find that $[\mathcal{P}^-_l,\phi^-] \in H^1(C,\mathcal{L}\times^L \mathfrak{u}^-) \subset H^1(C,\mathcal{L}\times^L \mathfrak{g})$ is non-zero and equal to $[g_l]$. This is a contradiction, because

$$H^1(C,\mathcal{G}\times^G \mathfrak{g}) = H^1(C,\mathcal{P}_L\times^L \mathfrak{p}) \oplus H^1(C,\mathcal{P}_L\times^L \mathfrak{u}^-).$$

□

References

[1] J. Heinloth, *Semistable reduction for G-bundles on curves*, J. Alg. Geom. **17** (2008), 167–183.