Giving students the run of sprinting models

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(Dated: June 2, 2009)

A biomechanical study of sprinting is an interesting and attractive practical investigation task for undergraduates and pre-university students who have already done a fair amount of mechanics and calculus. These students can work with authentic real data about sprinting and carry out practical investigations similar to the way sports scientists do research. Student research activities are viable when the students are familiar with tools to collect and work with data from sensors and video recordings, and with modeling tools for comparison of simulation results with experimental results. This article presents a multi-purpose system, named COACH, which is rather unique in the sense that it offers students and teachers a versatile integrated set of tools for learning, doing, and teaching mathematics and science in a computer-based inquiry approach. Automated tracking of reference points and correction of perspective distortion in video clips, state-of-the-art algorithms of data smoothing and numerical differentiation, and graphical system dynamics based modeling, are some of the built-in techniques that are very suitable for motion analysis. Their implementation in COACH and their application in student activities involving mathematical models of running are discussed.

I. INTRODUCTION

The following two types of mathematical models of running are distinguished in the biomechanics literature:

1. kinematic models, based on Newton's second law of motion;[1–5]
2. kinetic models, based on an energy balance for a runner.[6–13]

All models lead to differential equations for the velocity of the runner. With the exception of a few simple models, these differential equations can only be solved numerically. Computer modeling enables students, who have already done a fair amount of mechanics and calculus, to relate the theory with results from measurements. Data may come from research literature or from students' own measurements. For example, Wagner[14] has described how high school students could use a simple mathematical model to analyze both world record races of the 100-meter dash and recordings of their own sprint races.

With the availability of educational video analysis software such as VIDEOPOINT[15] and the video tool of LOGGER PRO[16] to aid experimental investigation, and with the availability of modeling programs such as STELLA[17] to facilitate the numerical solution of nonlinear differential equations with a minimum of programming effort, the study of the start of a sprint has become a viable, authentic, and attractive investigation for students in an inquiry-based physics course. Strong motives to let students carry out motion analysis and computer modeling in practical investigations are to bring them into contact with modern approaches in human movement science and to let them experience that a basic level of mathematics and physics suffices to do an interesting project similar to the work real motion scientists engage in.

However, one must realize that the research methods of motion analysis are not simple[18–21] and put high demands on the quality and ease of use of the tools that are deployed. One of the most difficult parts of experimental work is getting high-quality data in the first place. If possible, experimental conditions for video recordings are optimized to ensure the camera can take good videos that can be analyzed easily and robustly. For a viable video analysis, image processing tools are often indispensable and a robust tracking method must be available to collect data in a video recording. Once the data have been collected, rather advanced data processing tools must be available in the tool box, for example to compute useful numerical derivatives of measured quantities that are subject to noise. In the next steps of the investigation it must be easy to work with the data in conjunction with a nonlinear regression tool and a modeling tool.

Although students can use all kinds of video analysis tools and modeling tools in their study of sprinting, it is considered an advantage when these tools have been integrated into a single environment so that switching application software and the use of interfacing programs is not needed. In this way, students and teachers only need to familiarize themselves with one environment, in which components are geared with each other, and can grow in their roles of skilled users of the system during learning and teaching. A learn-once-use-often philosophy of educational tools is realizable. Such a versatile computer environment for learning and doing mathematics, science, and technology in an inquiry approach has been developed in the last twenty-five years at the AMSTEL Institute of the University of Amsterdam. This system, named COACH,[22,23] consists of integrated tools for data collection, for control of processes and devices, for processing and analyzing data, for modeling, and for authoring activities. COACH has been translated into several languages and is used in many countries. In recent years, this environment has been deployed for various practical investigations in movement science by Dutch pre-university students (age 17 to 18).[24–27]

The rest of this paper is organized as follows. The computer environment COACH is briefly described in Section II, in the light of biomechanical applications. In particular, attention is paid to the correction of perspective distortion of planes in video clips, to point tracking as a means for automated video data collection, and to numerical differentiation of noisy data, because these methods play an important role in motion analysis. Section III reviews various mathematical models of sprinting, illustrating that only basic levels of mathematics and physics are needed to understand what is going on. Some results of students' activities are presented in Section IV, illustrating what even inexperienced users can achieve with COACH. Conclusions about the strength and viability of the chosen approach are given in Section V.

II. COACH: A MULTI-PURPOSE ENVIRONMENT FOR MATHEMATICS AND SCIENCE EDUCATION

COACH has been developed for educational purposes, offering students genuine scientific experiences, but in such way that it has strong resemblance with contemporary professional tools. It can be described as a versatile learning and authoring environment for mathematics, and science education that integrates tools for

- measurement with interfaces and sensors;
- control of devices (motors, lamps, etc.) and processes (automated titration, a temperature regulator, etc.);
- data video, i.e., measurement on digital video clips and digital images (capturing of movies included);
- processing and analysis of data (differentiation and integration, data smoothing, regression analysis, etc.);
- modeling (a text-based, equation-based, and graphical approach to system dynamics);
- simulation and animation of modeled phenomena;
- representation of measured data and computed results (graphs, tables, meters, etc.);
- authoring by instructional designers, teachers, and students (texts, multimedia components, hyperlinks, etc.)

The video analysis, data smoothing and numerical differentiation facilities, and the graphical modeling tool of COACH are briefly described, in view of motion analysis.

A. Video analysis

1. Practical issues

In human movement science, analyzing video clips of human body and body part motions is a popular research method to study the kinematics and the forces involved.[21] The availability of affordable digital video and computer technology offers students the opportunity to actively engage in motion analysis. Educational projects that meet this learning goal have already been reported.[24–29] In these projects, students conduct all aspects of motion analysis: they formulate research questions about a human motion, design experiments, carry out the collection and analysis of video data, and finally interpret and report their results. The video analysis equipment used in professional motion laboratories is powerful, but complicated and expensive, certainly for studying three-dimensional kinematics. However, for educational purposes, when mostly planar motions are the subjects of study, less power and sophistication is needed, and costs can be rather low. A webcam that can record at a speed of 30 frames per second suffices for data collection in many student projects. In case it is needed, the cost of a good point-and-shoot high-speed camera, e.g., a CASIO Exilim camera[30] that can record at a speed of 1000 fps, has dropped to about 300 US$ in 2009. So, even a high-speed camera is now attainable for doing motion analysis in schools and undergraduate laboratories.

Technology used in motion analysis may change rapidly, but what remains the same in video analysis is the care needed for the video recording, the data collection, and the data analysis. Educational point-and-click video analysis software such as VIDEOPOINT[15], and the video tool of LOGGER PRO[16] and COACH[22,23] allow students to collect position and time data from a recorded video clip by using the computer mouse to click in the selected frames on the location(s) of the reference point(s) on the moving object(s). However, this manual data collection procedure can be time consuming and introduce errors into the measurements. This holds in particular for motions recorded with a high-speed camera. Automated tracking of reference points on moving objects is the only solution to this problem. This has been implemented in COACH.

The experimental setting may also cause problems in video analysis. For example, it may be impossible in practice to put the camera fronto-parallel to the plane of motion, leading to a perspective distortion of the image. In such a case, image rectification or further data processing is needed to obtain useful position data from the video measurements. COACH contains a basic set of necessary image processing tools.

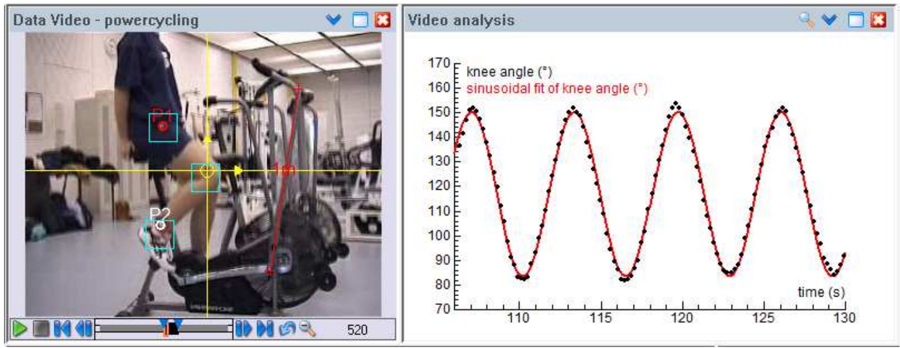


FIG. 1: A screen shot of a COACH activity about fitness cycling, with automated tracking of three points (including the origin of a moving frame of reference located at the knee joint). The diagram to the right shows the data plot of the knee joint angle calculated from the measured position data of the hip and ankle joint of the right leg, and a sine regression curve that fits the knee angle data.

Other points that may need attention during a video recording are:

- adequate illumination of the scene, especially when a high-speed camera is used and/or a short shutter time is necessary in order to capture a fast motion. A good light performance and high resolution of the image sensor is important for obtaining useful results;
- avoidance of image distortions such as geometric deformations due to the quality of the camera lens, and disturbances like motion blur, the rolling shutter effect, and errors in time intervals due to the quality and built-in technology of the camera;[31,32]
- visibility of the reference point(s) on the moving object(s). Occlusion should be avoided (if possible), and when two or more reference points are tracked, then one should take care that reference points can be distinguished from each other, even when they get close together;
- restriction of the action in a single plane of motion so that two-dimensional motion analysis is indeed applicable;
- presence of a suitable length-calibration object in the plane of motion.

The above list of obstacles and points for attention while setting up a video recorded experiment may give the impression that this experimental work is not viable in undergraduate laboratories, unless one is satisfied with rather low-quality data. But this is in reality no longer true because modern, affordable webcams and video camcorders take away many of the problems of illumination and image distortions, and the other points for attention have to do with a good experimental design. Nice examples of students' work have been reported in physics education literature.[24–29,33–35] Furthermore, high school students can effectively use webcams to collect high-quality data which are useful in their science investigations. One of the authors have even done this type of work successfully with 15 year old pupils in a vocational stream of Dutch secondary education.[24]

2. *Point tracking in video clips and image sequences*

Various methods exist for automated tracking of reference points on moving objects in a video clip: for example, they can be based on image recognition techniques and geometric properties of the objects of interest,[31,36] or they can be based on color discrimination.[37] The second method of point tracking has been implemented in COACH as a transformation filter, which can be thought of as a program that connects a source, e.g., a video clip or a streaming video of a webcam, with the video tool window, while at the same time passing the recorded position data to the computer application. The algorithm used in the tracking filter is composed of two parts: (1) finding the best match of a given model template in a subsequent frame, i.e., locating the area that resembles most the specified area; and (2) limiting the search area in order to reduce computing time or to avoid ambiguity. The template that is tracked and that is selected by the user at the start of the tracking process is an area bounded by a circle with a user-specified radius. The comparison of an area with the model template is based on pixel intensities in the three channels of the RGB color model. More specifically, the sum of squared differences of intensities between image and model template is used as the function that must be minimized in the search algorithm. This method has been selected for template matching after testing various commonly used algorithms on sample video clips. The search area for the subsequent frame is a rectangle of user specified dimensions, centered on the position that is found for the current frame (see Figure 1). Despite its simplicity, the tracking filter works well in many practical situations, and the need of special illumination of the moving objects or other special scene

FIG. 2: A video clip of moving coins on a flat surface before (left) and after (right) correction of perspective distortion.

preparation is limited. It is mostly restricted to avoidance of distortion such as rotation and deformation of the detected reference area, and to avoidance of an accidental change in the coloring of the moving objects because of changes in the incidence of light. For example, the reference points in the scene shown in Fig. 1 are just a label on the shorts of the cyclist, a black dot made with a whiteboard marker on the knee joint, and a simple homemade marker placed at the ankle. The normal lighting of the fitness center has been used and the video was created under these circumstances through a CCD webcam with a speed of 30 frames per second. There were no problems with the point tracking algorithm.

3. Correction of perspective distortion of a plane

Figure 2, taken from a study about moving coins,[38] exemplifies the way in which the perspective distortion of a plane in a digital image or a video clip can be corrected. The plane of interest is the rectangular paper on the horizontal table on which the coins move. In reality the edges of the piece of paper are two pairs of parallel lines. However, this is not the case in the video clip shown to the left in Fig. 2. The image transformation that restores this property is determined by mapping the four corners of the rectangular paper with a projective transformation to the corners of a rectangle in a new image (see the image to the right in Fig. 2). All frames in the video clip are transformed such that the overlay becomes a rectangle. In fact, the orthogonality of two pairs of parallel lines determines the image rectification up to an unknown aspect ratio. This scaling can be specified, if one wishes a rescaling to the real proportionality. The basis of the method is that any projective transformation can be uniquely written as a composition of three transformations, viz., a perspectivity P that maps a quadrilateral into a parallelogram, an affine mapping A that transforms the parallelogram into a rectangle, and finally a similarity transformation S with isotropic scaling that rotates, translates and/or scales the rectangle.[39,40] For measurements in a rectified digital image or video clip, it suffices to find appropriate transformations P and A, provided that one can freely position and rotate the coordinate system, and use different scaling of the coordinate axes. Examples of using image rectification and point tracking have already been reported.[24,27,38,41,42]

B. Data smoothing and numerical differentiation

In motion analysis, derived quantities like speed and acceleration are often needed. The numerical differentiation methods must be sophisticated enough to deal with noise in the signal. Otherwise, numerical derivatives may become extremely noisy and useless, especially when divided difference formulas for the first and second derivative are applied (See Figure 3). Popular data smoothing techniques such as moving averages, Savitsky-Golay filtering, and spline smoothing can reduce the noise in a signal. These methods are available in COACH, but special attention has been paid to the spline smoothing technique: the system provides its user a generalized cross-validatory penalized quintic spline smoothing technique. The method is outlined below; details can be found in the literature.[43–45]

Given a set of equidistant abscissae $t_1, \ldots, t_n$, with corresponding ordinates $y_1, \ldots, y_n$, a penalized spline of order $2m$ (with $2m \leq n$) is a piecewise polynomial $\hat{y}$ of degree $2m-1$ that minimizes the penalized residual sum of squares

$$\mathrm{C}_\lambda(t,y) = \sum_{i=1}^{n} |\hat{y}(t_i) - y(t_i)|^2 + \lambda \int_{t_1}^{t_n} \left(\hat{y}^{(m)}(t)\right)^2 dt, \quad (1)$$

where λ is a suitable penalization or smoothing parameter. The first $2m-2$ derivatives of each local polynomial of the spline are continuous at each value of t_i. A remarkable theorem, which can be found in advanced texts on smoothing,[46] is that the spline function that minimizes $\mathrm{C}_\lambda(t,y)$ is a natural spline, i.e., the derivatives at the corners $\hat{y}^{(j)}(t_1)$ and $\hat{y}^{(j)}(t_n)$ are zero for $j = m, \ldots, 2(m-1)$. The first term in Eq. (1) is the standard sum of squared errors that is used for fitting data by a spline. The second term, the integrated square of the mth order derivative, is a measure of the roughness of the data. The parameter λ is the smoothing parameter that controls the trade-off between fitting the data by minimizing the residual sum of squares and minimizing the roughness of the approximation. With $\lambda = 0$, only fitting of the data matters, so an interpolating spline through the data points will be obtained. As λ increases, more emphasis is placed on penalizing roughness. As $\lambda \to \infty$, then only roughness matters and the standard least squares fit of the data using a single polynomial of degree $m-1$ will be obtained.

There exist data-driven methods for automatic selection of the parameter λ. The generalized cross-validation (GCV) criterion developed by Craven and Wahba[47] has been implemented in COACH. Although the GCV method has been found to be reliable, it is useful to remember that it only offers a reasonable smoothing parameter to start exploratory work. It inevitably involves personal judgment to select a suitable value of the parameter λ. This is why a user can change or set the value of the smoothing parameter to see the effect of this action.

In COACH the value of m has been set to 3 because the quintic spline smoothing has been reported in many studies[48,49] as being most suitable in motion analysis for leading to the best possible numerical derivatives, and because infinite smoothing then leads to a quadratic regression curve with which students are familiar. Figure 3 illustrates the superiority of this differentiation method, when applied to the drop of a golf ball,[50] in comparison with divided difference formulas. The data come from Biomechanical Data Resources provided by the International Society of Biomechanics at www.isbweb.org/data.

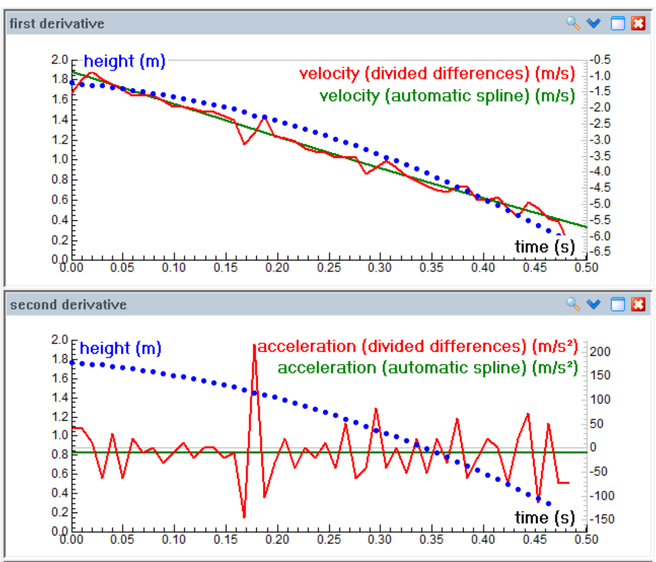


FIG. 3: The velocity and acceleration curves for the drop of a golf ball. A comparison of the method of divided differences with the generalized cross-validatory penalized quintic spline smoothing technique.

C. Graphical modeling

The modeling tool of COACH allows students to create and run numerical models, and to compare modeling results with experimental data. Three kinds of model editors are provided: text-based, equation-based, and graphical. The first type of modeling is just programming in a computer language that is dedicated to mathematics, science, and technology education. The last two types of modeling are based on a system dynamics approach that also forms the basis of a graphical, aggregate-focused system such as STELLA.[17] This type of modeling system uses aggregated amounts, i.e., quantities (commonly called stocks or levels) that change over time through physical inflows and outflows, as the core elements of a model. Not only physical flow, but also information flow determines the system's behavior over time. Information flow is best understood as an indication of dependencies or influences between variables in the model. These relations are made explicit in the form of mathematical formulas and graphical or tabular relationships. The variables involved can be stocks, flows, parameters, and auxiliary variables.

The stock/flow modeling language has a graphical representation in which a user can express his or her thoughts about the behavior of a dynamic system, and these ideas are then translated into more formal mathematical representations. The conventional symbol of stock and flow is a rectangular box and a double arrow with a valve, respectively. A single arrow commonly represents information flow. Auxiliary variables and parameters appear as circles and circles with small handles on both sides, respectively. Figure 4 is a simple illustrative example of a graphical model.

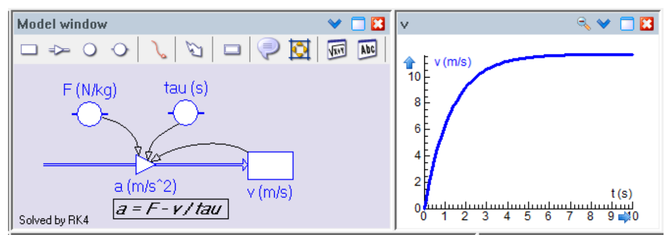


FIG. 4: A graphical model implementing the Keller model of sprinting and the calculated speed-time curve in a simulation run.

The above example also illustrates that the stock/flow metaphor of a dynamical system should not be taken too literal. The graphical model represents in fact a computer model, which provides in many cases an iterative numerical solution of a system of differential equations. Fig. 4 is a screen shot of a graphical model in COACH that implements in fact the Keller model of sprinting, given by the following initial value problem:

$$v'(t) = F - v(t)/\tau, \quad v(0) = 0, \tag{2}$$

where v is the sprinter's speed, and F and τ are constants. The rate of change of the speed, i.e., the acceleration a, is in the graphical model represented by an inflow that depends on v, F, and τ, as shown in the annotation. The diagram in Fig. 4 displays a speed-time curve that has been calculated for a certain parameter value through the 4th order Runge-Kutta method. This is one of the numerical ODE solvers present in COACH. The graphical model looks similar in other stock/flow-based modeling tools such as STELLA.

III. MATHEMATICAL MODELS OF RUNNING

Two types of mathematical models of running are discussed here, viz., models based on Newton's second law of motion and those based on an energy balance. They represent a kinematic approach and a kinetic approach to the modeling of running, respectively. Whereas a kinematic approach focuses on the physics of motion of the human body and body parts in running events, the kinetic approach aims at strengthening the physiological basis of the mathematical models of running and developing methods to quantitatively evaluate the influence of technical, physiological, and environmental variables on the performance in sports events. It will be shown that a basic knowledge of calculus and physics suffices to understand the modeling process.

A. Models based on Newton's 2nd law of motion

All models of this kind are based on

$$a(t) = F_{\text{propulsive}}(t) - F_{\text{resistive}}(t), \tag{3}$$

where $a(t)$ represents the acceleration of the runner at time t, $F_{\text{propulsive}}$ the horizontal component of the propulsive force per unit mass at time t, i.e., the normalized force that the runner applies at the given moment to be in motion, and $F_{\text{resistive}}$ the force per unit mass that the runner has to overcome at time t in order to be in motion. We have adopted in Eq. (3) the common approach in biomechanics to normalize quantities to body mass.

Keller[1,2] assumed in his model of competitive running that the propulsive term is constant for sprinting and that the dominant resistive effects from running mainly result from frictional losses within the body of the runner and can be modeled by a term that is linear in speed. This model, in which air friction is neglected, can be written for a runner who is initially at rest as:

$$v'(t) = F - v(t)/\tau, \quad v(0) = 0, \tag{4}$$

where F is the constant force per unit mass that the runner can exert in the horizontal direction, v is the velocity of the runner, and τ is a time constant. This initial value problem can be solved analytically:

$$v(t) = F\tau\Big(1 - e^{-t/\tau}\Big). \tag{5}$$

The product $F\tau$ equals the maximum speed of the sprinter. The distance $D(t)$ covered in time t is found by integration of Eq. (5) under the assumption that $D(0) = 0$:

$$D(t) = F\tau^2\Big(t/\tau + e^{-t/\tau} - 1\Big). \tag{6}$$

For small values of time t the following holds:

$$e^{-t/\tau} \approx 1 - t/\tau + \tfrac{1}{2}(t/\tau)^2. \tag{7}$$

Substitution of Eq. (7) into Eq. (6) leads to the following approximate formula valid at the start of a sprint:

$$D(t) \approx \tfrac{1}{2}Ft^2. \tag{8}$$

A good estimate of the model parameter F can be obtained via this approximation from a plot of the covered distance against time for the start of the sprint. In combination with the measured maximum speed, this value provides an estimate of the parameter τ. These estimates of the parameters can also be used as initial parameter values in an iterative regression method for the distance-time data, using Eq. (5) as regression function. Parameter values found in this manner can be compared with the values obtained by linear regression in an acceleration-speed data plot (cf., Eq. (4)), or by cubic polynomial regression with the following alternative approximate formula valid at the start of a sprint:

$$D(t) \approx \frac{Ft^2}{2} + \frac{Ft^3}{6\tau}. \tag{9}$$

It is an interesting lesson subject to discuss with students which regression method is most reliable.

Tibshirani[3] removed the unrealistic assumption that a sprinter applies a constant maximum force for the duration of a race. He chose a linear decrease of the propulsive force in time. In this case, the equation of motion is

$$v'(t) = F - ct - v(t)/\tau, \quad v(0) = 0, \tag{10}$$

for some constant c. This model can also be solved analytically:

$$v(t) = k - c\tau t - k\,e^{-t/\tau}, \tag{11}$$

where $k = F\tau + c\tau^2$. The distance $D(t)$ covered in time t is found by integration of Eq. (11) under the assumption that $D(0) = 0$:

$$D(t) = kt - \tfrac{1}{2}c\tau t^2 + k\tau\Big(e^{-t/\tau} - 1\Big). \tag{12}$$

Now, for small values of time t the following holds:

$$D(t) \approx \frac{Ft^2}{2} - \frac{(F + c\tau)t^3}{6\tau}. \tag{13}$$

In the Tibshirani model of sprinting, the mathematical formulas again play an important role in the determination of good estimates for the model parameters.

This article describes the use of a graphical modeling tool by students in a sports motion context to create representations of quantities and relationships where analytic solutions of equations are too difficult or not possible. The need for computer models is quickly felt when one wants to take several factors into account for sprinting models. For example, the effect of a head- or tailwind can be translated mathematically into an aerodynamic term derived completely from air resistance. The empirical relation for the drag force is

$$F_{\text{drag}} = \tfrac{1}{2}\rho\, C_d A(v-w)^2, \qquad (14)$$

where w represents the wind speed (positive and negative values indicating tailwind and headwind, respectively), v the speed of the sprinter, ρ the air density, C_d the drag coefficient, and A the cross-sectional area of the sprinter.

Other effects on sprint race times like altitude, air pressure, temperature, humidity, curvature of lanes at the sprint track have been discussed in the research literature.[51–55] These effects can also be easily incorporated in the computer models.

B. Models based on an energy balance

Many forms of energy besides mechanical energy play a role in the law of conservation of energy when it is applied to human motion: elastic energy, heat, chemical energy, and so on. In muscles, chemical energy is transformed into mechanical energy. Mechanical energy degrades partly to thermal energy in the case of concentric muscle contraction (when the muscle is shortened). It can also be stored in tendons and muscles as elastic energy, in the case of eccentric muscle contraction (when the muscle is extended). A power balance model based on a supply/demand approach describes the energetics of many human motions, including running and other endurance sports.[6] In other words, it means that the sum of all the rates of flow of energy into and out of the human body equals the rate of change of energy of the human body. The power balance method, which relates power production and power dissipation, can be written as

$$P_{\text{o}} = P_{\text{f}} + \frac{dE}{dt} + \frac{dH}{dt} \qquad (15)$$

where P_{o} represents the power which in principle might be used to perform movements, P_{f} the power loss due to friction, dE/dt the rate of change of external mechanical energy (kinetic, rotational and potential energy of the body), and dH/dt the power loss due to degradation of mechanical energy into thermal energy. The power production (P_{o}) is usually the sum of the power production by the aerobic (P_{aer}) and anaerobic (P_{an}) energy production systems:

$$P_{\text{o}} = P_{\text{aer}} + P_{\text{an}}. \qquad (16)$$

We discuss below two power balance models of running.

Ward-Smith[8] modeled the aerobic and anaerobic power as follows:

$$P_{\text{aer}} = R\left(1-e^{-\lambda t}\right) \quad \text{and} \quad P_{\text{an}} = P_{\max}\, e^{-\lambda t}, \qquad (17)$$

with R the maximum aerobic power, $P_{\max}$ the maximum anaerobic power at $t=0$, and λ a constant. The exponential model of the aerobic power is based on experimental work of Margaria[56] and others about oxygen kinetics in a high intensity exercise. Through the anaerobic term it is expressed that the maximal anaerobic power cannot be sustained and decays exponentially. Ward-Smith justified the mathematical expressions by theoretical arguments. In addition, he modeled the thermal power as

$$\frac{dH}{dt} = \alpha\, v(t). \qquad (18)$$

The friction term and the rate of change of external mechanical energy are given in this model by

$$P_{\text{f}} = F_{\text{drag}}\, v = \tfrac{1}{2}\rho\, C_d A\, v(v-w)^2, \qquad (19a)$$

$$\frac{dE}{dt} = \frac{d\left(\frac{1}{2}mv^2\right)}{dt} = mv\frac{dv}{dt}. \qquad (19b)$$

Substitution of Eqs. (16) to (19b) into Eq. (15) and the assumption that the constants R, $P_{\max}$,A, and α are proportional to the mass (the tilde $\sim$ denotes that a quantity is normalized to the body mass m) give

$$\begin{aligned}&\tilde{R}\left(1-e^{-\lambda t}\right) + \tilde{P}_{\max}e^{-\lambda t} = \\ &\tilde{\alpha}v + \tfrac{1}{2}\rho\, C_d\tilde{A}\, v(v-w)^2 + v\, v', \quad v(0)=0.\end{aligned} \qquad (20)$$

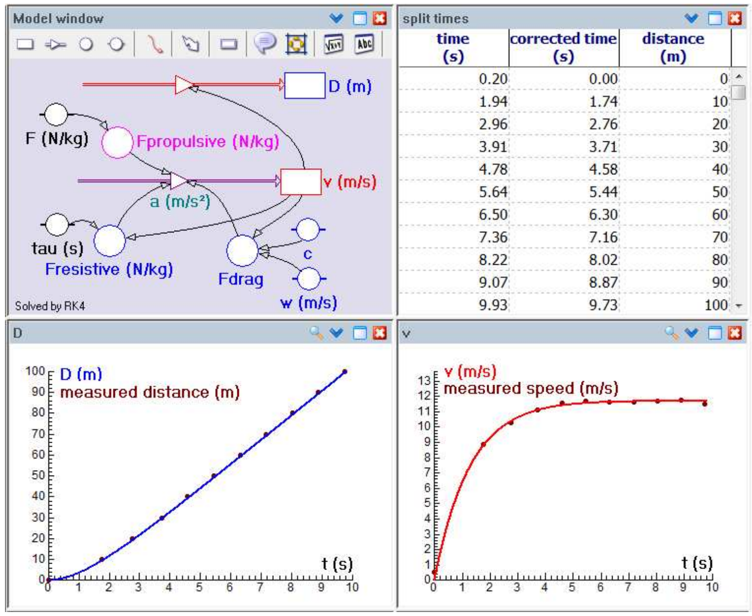
This initial value problem can be rewritten as

$$\begin{aligned} v' = \frac{1}{v}\Big(&\tilde{R}\left(1-e^{-\lambda t}\right) + \tilde{P}_{\max}e^{-\lambda t} \\ &-\tilde{\alpha}v - \tilde{K}v(v-w)^2\Big), \quad v(0)=0,\end{aligned} \qquad (21)$$

where $\tilde{K} = \frac{1}{2}\rho\, C_d\tilde{A}$. There is only one problem with this system dynamics equation: the initial velocity $v(0)=0$ gives an infinite acceleration. Rewriting it as an initial value problem of kinetic energy helps:

$$\begin{aligned}\frac{d\left(\frac{1}{2}v^2\right)}{dt} &= \tilde{R}\left(1-e^{-\lambda t}\right) + \tilde{P}_{\max}e^{-\lambda t} - \tilde{\alpha}\, v \\ &\quad -\tilde{K}v(v-w)^2, \quad v(0)=0.\end{aligned} \qquad (22)$$

Note that the aerobic and anaerobic energy productions are causally related in the Ward-Smith model by the use of the same time constant λ. More recently, Ward-Smith has introduced separate time constants for the aerobic and anaerobic energy releases, and he has even described the anaerobic metabolism with three components.[10,11] Following the approach of Péronnet and Thibault,[12] he has also replaced the assumption that the maximum aerobic power $P_{\max}$ is constant, by an exponential decay after some time interval, in order to be able to model well middle-distance and long-distance running.



time (s)	corrected time (s)	distance (m)
0.20	0.00	0
1.94	1.74	10
2.96	2.76	20
3.91	3.71	30
4.78	4.58	40
5.64	5.44	50
6.50	6.30	60
7.36	7.16	70
8.22	8.02	80
9.07	8.87	90
9.93	9.73	100

FIG. 5: A screen shot of a COACH activity showing the simulation of the wind-adapted Keller model for the 100-meter race of Carl Lewis in the finals of the World Championships of 1987. The position-time curve of the measured data has been plotted and the speed of the sprinter has been calculated and plotted against time. Parameters in the model have been chosen such that the model results match the measurement results well.

These adapted models can also be easily implemented with a graphical modeling tool.

Van Ingen Schenau and colleagues[6,7] have developed another power balance model in which they took into account the limited efficiency of converting metabolic power to external power. Their model, which can be used for various sports events like running, rowing, swimming, cycling, and speed skating, can be written as

$$\eta P_{\text{metabolic}} = \alpha\, v + k_d\, v^3 + v\, v', \tag{23}$$

where η is the efficiency coefficient, and the metabolic power $P_{\text{metabolic}}$ consists of

$$P_{\text{aer}} = R\left(1 - e^{-\lambda_1 t}\right) \quad \text{and} \quad P_{\text{an}} = E_0 \lambda_2 e^{-\lambda_2 t}, \tag{24}$$

where E_0 is the anaerobic capacity. For sprinting they reported the following differential equation:

$$\begin{aligned} &22.44e^{-0.0403t} + 4.19\left(1 - e^{-0.0384t}\right) = \\ &0.97v + 0.00366v^3 + v\, v'. \end{aligned} \tag{25}$$

They developed this kinetic model for the purpose of comparing athletes' performances in various sports events[6,7] and evaluating the influence of technical, physiological, and environmental variables on speed skating performance in a quantitative way.[57,58]

IV. EXPERIMENTAL RESULTS

This section illustrates what can be done with a computer learning environment, such as COACH, that contains integrated tools for video analysis, data analysis, and graphical system dynamics based modeling, by students with limited knowledge and experience. It is true that the video analysis presented in this paper can be done with popular programs such as VIDEOPOINT[15] and LOGGER PRO,[16] and that graphical modeling can also be done with modeling software such as STELLA.[17] But, as argued in Section II, motion analysis demands more sophistication of many tools (e.g., image processing, and advanced methods for data smoothing and numerical differentiation) and ease of use (e.g., the ease of comparing modeling results with experimental results and the interaction with the tools). In this section, some results of working with COACH[22,23] are presented for the mathematical models treated in Section III. In addition, experimental results of Dutch pre-university students (age 17-18) are discussed.

A. Graphical models of running

Figure 5 is a screen shot of the analysis of the sprint of Carl Lewis in the 100-meter final at the IAAF World Championships in Athletics of 1987 in Rome through the wind-adapted Keller model. The graphical model in the upper left corner of the screen shot of a COACH activity is a visual representation of the initial value problem

$$\begin{aligned} &v'(t) = F_{\text{propulsive}}(t) - F_{\text{resistive}}(t) - F_{\text{drag}}(t), \\ &v(0) = 0, \end{aligned} \tag{26}$$

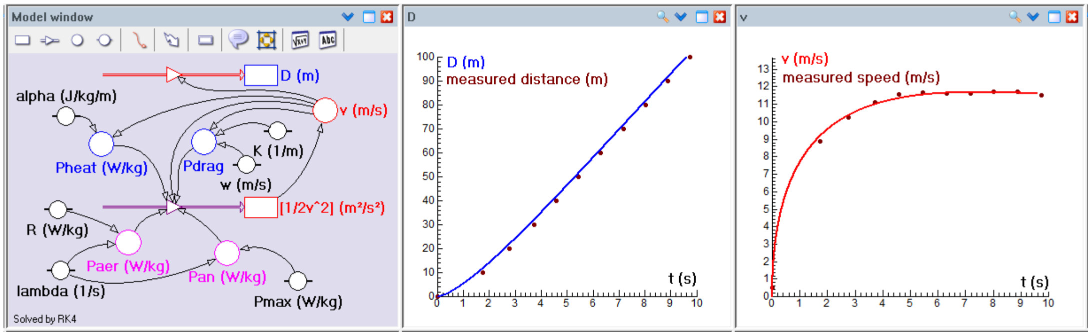


FIG. 6: A screen shot of a graphical model and simulation of the Ward-Smith model for the sprint of Carl Lewis at the World Championships in Athletics of 1987 in Rome. Model results about the sprinter's position.

where

$$\begin{aligned} &F_{\text{propulsive}}(t) = F, \quad F_{\text{resistive}}(t) = v(t)/\tau, \\ &F_{\text{drag}}(t) = c\big(v(t) - w\big)^2. \end{aligned} \tag{27}$$

Here it is assumed that the wind speed was constant during the run. The race was run in fact with a measured $+0.95\,\mathrm{m\,s^{-1}}$ tailwind. The table in the upper right corner lists the split times of the sprint of Carl Lewis.[59] Time was corrected for the relatively slow reaction time of 0.196 s at the start. Speed and acceleration data, which are shown in the diagrams as data points, were computed numerically from the split times through generalized cross-validatory penalized quintic spline smoothing. They are referred to as the measured speed and acceleration. When the computer model is run for certain values of model parameters, then distance, speed, and acceleration curves are computed and plotted. Figure 5 shows the curves that match well with the measured quantities. Values of the model parameters are:

$$F = 9.2\,\mathrm{N\,kg^{-1}}, \quad \tau = 1.343\,\mathrm{s}, \quad c = 0.00375\,\mathrm{m^{-1}}.$$

They differ a bit from the values of the Keller model without air resistance, as follows:

$$F = 8.9\,\mathrm{N\,kg^{-1}}, \quad \tau = 1.338\,\mathrm{s}\,.$$

The sprint of Carl Lewis in 1987 can also be analyzed with the Ward-Smith model expressed by Eq. (22). The following parameters were chosen for the simulation shown in Figure 6:

$$\begin{aligned} &\tilde{R} = 25\,\mathrm{W\,kg^{-1}},\ \lambda = 0.03\,\mathrm{s^{-1}},\ \tilde{P}_{\text{max}} = 48.5\,\mathrm{W\,kg^{-1}}, \\ &\tilde{\alpha} = 3.5\,\mathrm{J\,kg^{-1}\,m^{-1}},\ \tilde{K} = 0.002\,\mathrm{m^{-1}},\ w = +0.95\,\mathrm{m\,s^{-1}}. \end{aligned}$$

Although the two types of models of sprinting are qualitatively much different, they lead both to modeling results that are in good agreement with measured data. Sports scientists apparently make their choice of mathematical modeling on other grounds.

Figure 7 shows a screen shot of a simulation of the van Ingen Schenau model for a 400-meter race expressed by Eq. (25). Like the Ward-Smith model, the initial value problem has been rewritten in a system dynamics form:

$$\frac{d\left(\frac{1}{2}v^2\right)}{dt} = \eta(P_{\text{aer}} + P_{\text{an}}) - \alpha v - k_d v^3, \quad v(0) = 0. \tag{28}$$

Note that the graphical model is almost the same as that of the Ward-Smith model in Fig. 6; only the mathematical formulas hidden behind the icons are slightly different. This underpins the idea of using a graphical modeling tool with students to let them concentrate on the conceptual structure of the model instead of being completely bound up in the mathematical details.

Kinematic and kinetic models of sprinting could be used to discuss hypothetical questions like "How fast could Usain Bolt have run in the 100-meter final at the Beijing Olympics 2008?" A recent analysis,[60] which was based on data smoothing, numerical differentiation, and extrapolation reported that a race time somewhere between 9.55 and 9.61 s could have been realized. The above wind-adapted Keller model, with parameter values chosen such that the measured split times for the first 80 meters are in agreement with the results of modeling, leads to a similar race time estimation of 9.60 s.

B. Practical investigations of students

Although it is interesting to study models of sprinting of top-class athletes, there is nothing more compelling for students than investigating their own running performances. This practical work can be realized in various ways. In this paper, video recorded sprint data originate from a one-day visit of upper secondary school students to the Faculty of Human Movement Science of the VU University Amsterdam and to the University Sports Center of the University of Amsterdam. The purpose of this school visit was to give the students an impression of what movement scientists do in practice. Students were introduced at the university to the daily work of human movement scientists, to the field of exercise physiology, and to measurement of energy capacity. Some students

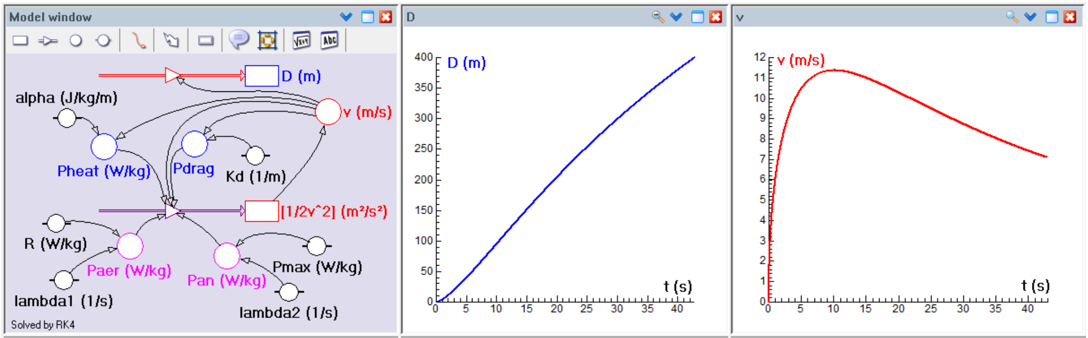


FIG. 7: A screen shot of a graphical model and simulation of the van Ingen Schenau model for a 400-meter race, showing position and speed curves.

had the opportunity to measure their oxygen consumption in a bicycle ergometer test in the laboratory. At the sports center the focus was on research methods to collect motion data. The students measured their heart rates during running at various intensity levels, and they assessed their anaerobic capacity via the Wingate cycle ergometer test, in which they pedaled at maximal speed for 30 s against a high braking force. The students also used a high-speed camera to record a soccer kick and the take-off phase of a sprint. In addition, they used a webcam to record 25-meter sprints of other students in the sports hall. Back at school they analyzed their sprint data and compared the measured and derived physical quantities with simulation results from some of the mathematical models discussed in Section III.

This was actually a viable approach because the students participating in this project already had experience with modeling in COACH. They could build the Keller model and the Tibshirani model from verbal descriptions of the model and include aerodynamic effects. The students were not expected to come up with a kinetic model themselves or build a computer model from a verbal description of such kind of mathematical model. We envisioned the Ward-Smith model as a first example of a kinetic model of running to discuss with the students. Based on the implementation of this model in the graphical modeling tool, the students would continue with the study and implementation of the van Ingen Schenau model and compare modeling results with their empirical data. The power balance models discussed in Section III could provide estimates of the aerobic and anaerobic capacity of the sprinting student. In the project however, the students finally did not apply kinetic models to their own sprint data. The main reason was that the students should have run a longer distance to really see the difference between kinetic and kinematic models when results of modeling are compared with experimental results.

Figure 8 shows the video analysis of a 25-meter sprint of a 16 years old female student running near the wall of the sports hall. Due to the experimental setting, the perspective distortion of the plane of motion could not be avoided in the video recording. But image rectification through the method outlined in Section II is possible, by using the wall as rectangular object in the plane of motion. The result is shown in the upper right corner of the screen shot. Note that the video clip has also been flipped horizontally to change the direction of motion from left to right so that the horizontal position and the distance are the same mathematical notions in a traditional coordinate system with the positive axis on the right.

Because the sprint takes about 5 s and the frame rate of the recorded video clip is 30 fps, manual data collection is too time-consuming, error prone, and tedious to do. Therefore, data collection has been done through automated tracking of the sprinter in the video clip. The result in the lower left corner of Fig. 8 shows that this method is on the one hand applicable even under non-optimal conditions, but on the other hand it may also lead to rather noisy data. In this particular case, the penalized spline smoothing method outlined in Section II helps to obtain a suitable speed-time curve (actually, by manual selection of an appropriate penalization). The speed-time data plot in the lower right corner of Fig. 8 has been approximated by an exponential regression curve according to the Keller model of sprinting with a nonzero initial velocity (cf., Eq. (5) in Section III):

$$v(t) = 6.38\big(1 - e^{-0.802t}\big) + 1.25\,. \qquad (29)$$

Thus, the time constant τ is about 1.2 s, the initial speed 1.25 m s^{-1}, and the maximum speed 7.6 m s^{-1}. These values correspond with a horizontal propulsive force F per unit mass of 6.1 N kg^{-1} in the Keller model of sprinting. These values can be used for the parameters in a computer model. A simulation of the Keller model with measured data points in the background is shown in Figure 9. The descriptive power of the Keller model seems to be good for the measured sprint data. But when it is used to compute the expected result of a 100-meter sprint of this student, the race time of 14 s seems a bit unrealistic for a non-athlete, who is not expected to be able to maintain the maximum speed reached after 20 m for another 80 m. In other words, the predictive power of the Keller model is apparently not strong for this running event. The Tibshirani model predicts a seemingly

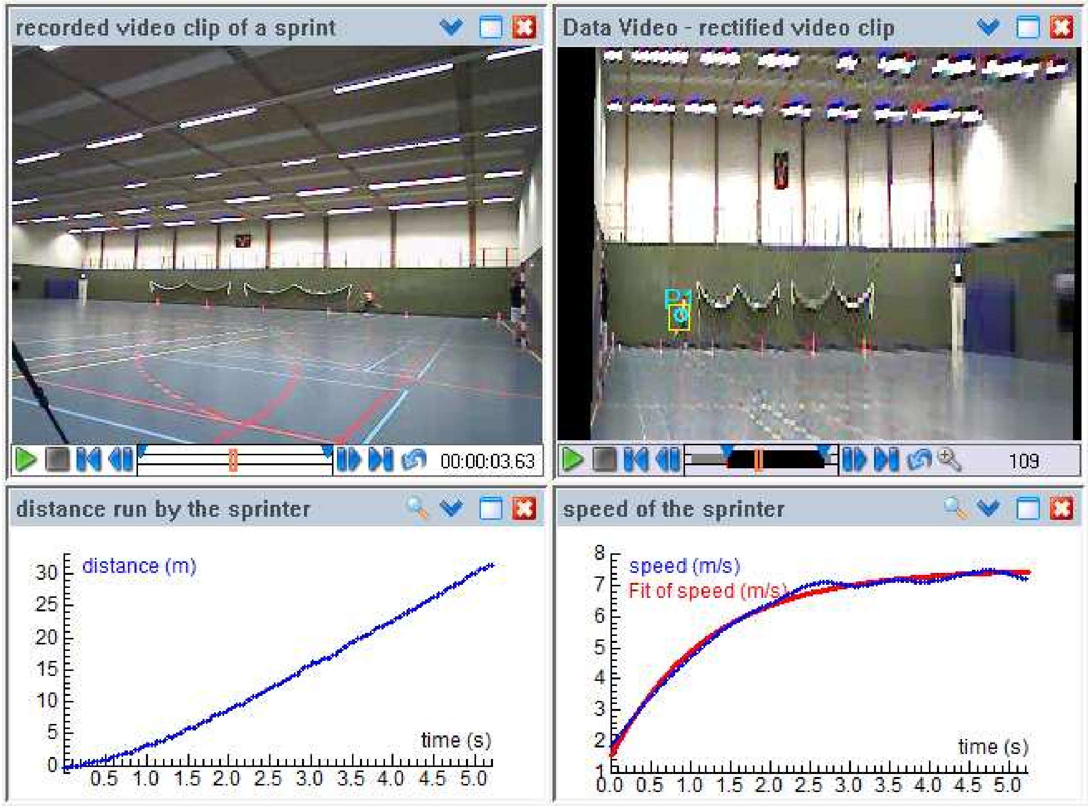


FIG. 8: A screen shot of a COACH activity showing the video analysis of a 25-meter sprint of a student. The upper left window shows the movie recorded with a webcam. Correction of the perspective distortion of the plane of motion and additional image processing has been applied to get the movie in the upper right window. Distance data (plotted in the lower left window) have been collected through automated tracking of the sprinter. Because the position data are noisy, penalized spline smoothing has been used to compute numerical derivatives. The speed-time data plot is shown in the lower right window. It also contains the best curve fit according to the Keller model of sprinting.

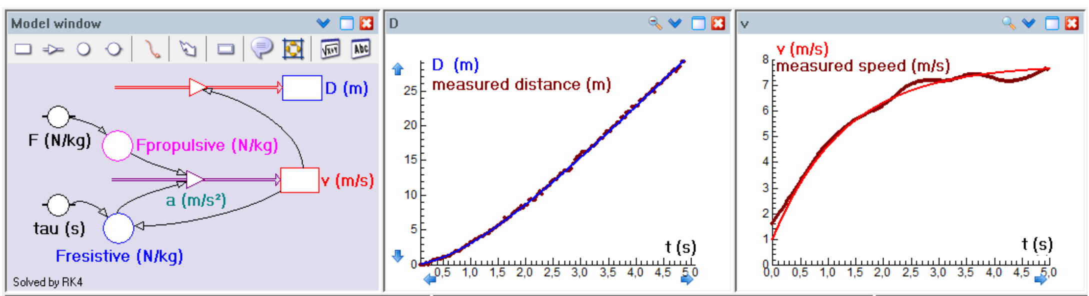


FIG. 9: A screen shot of a simulation of the Keller model for the 25-meter sprint of a student, comparing model results with experimental results.

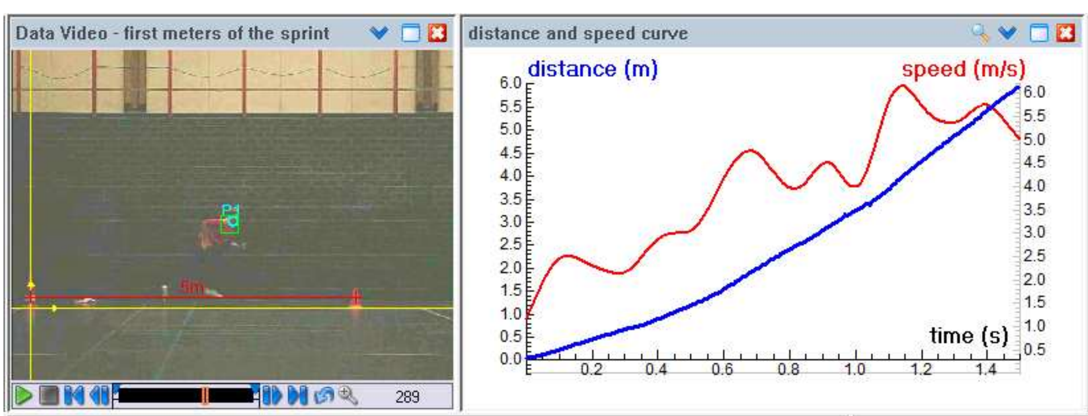


FIG. 10: A screen shot of a video analysis of the first 6 meters of the sprint of a student, showing the video clip recorded with a high-speed camera at a frame rate of 300 fps, and the distance and speed curves.

more realistic race time of 15.4 s. In practical work by students, it would be interesting to compare this theoretical result with the result of a real 100-meter sprint of the same student. We are of opinion that it is important that students get acquainted with this way of looking at the quality of their (computer) model and going back and forth between theory (model results) and experiment (measurement results).

More details of the motion of a sprinting student during take-off can be found in the video recording of the first 6 meters using a high-speed camera with a frame rate of 300 fps. In this case, point tracking becomes indispensable for collecting kinematical data. Figure 10 shows the distance and speed-time graphs obtained. The speed curve, which was determined by penalized spline smoothing, is the most interesting curve: increasing and decreasing parts of the graph correspond with propulsion during foot contact and with the flight phase, respectively. Differences in consecutive increases in the speed-time graph indicate that the propulsive force of the right leg is stronger than that of the left leg. This may be connected with left- or right-handedness of the student, but it may also have to do with the runner's attempts to balance during take-off. In other words, the speed curve is more than just a bumpy graph: its shape is closely related to the real motion. The goal of this part of the investigative work is to let students experience make clear to the students that in mathematics and science a graph is a means, rather than an end in itself.

V. CONCLUSION

The practical, investigative work discussed in this article is an illustrative example of how students can

- collect high-quality, real-time data;
- construct and use computer models of dynamic systems in much the same way as professionals do;
- carry out authentic work by comparing results from experiments, models, and theory.

Authenticity is interpreted in two ways:

1. students are engaged in data collection, data analysis, and modeling in a challenging, cross-disciplinary research setting, which is personalized by doing experiments with their own bodies;
2. students use modern technology and an integrated computer environment to work with collected data.

The broad set of rather sophisticated tools present in a single computer environment enables them to look at a realistic application of mathematics and science and to do challenging investigative work that resembles professional practice. Students can develop and practice many research skills through the activities. The fact that they must meaningfully apply their knowledge of biology, mathematics, and physics in a concrete sports context leads at the same time to deepening and consolidation of this knowledge. In other words, the activities contribute to the science literacy of the students. The research methods learned and practiced in the context of modeling of running are prototypical and will be useful when students do practical investigations in other areas.

The diversity of the models of running offers students the opportunity to practice evaluation and revision of their models. In the process of evaluating a model on the basis of experimental data, parameter estimation also plays an important role, and the complexity of finding suitable parameter values must not be underestimated. Initial guessing on mathematical and other grounds can be practiced in the student activities. Once the descriptive quality of a model is considered satisfactory, it can be used to make predictions. Comparing predictions with experimental results makes it explicit to students that both the descriptive and predictive power of a model determine the quality of the model.

We are of opinion that by looking at various models of the same phenomenon, a critical attitude of students is promoted and the importance of theoretical underpinning of a mathematical model comes to the fore. Furthermore, by linking modeling directly with experimental work, students get the important message that ideally there exists a synergy between theoretical and empirical scientific work. Just as empirical science cannot do without theory, theoretical sciences benefit from empirical work.

Acknowledgments

The authors thank Maurice Maas and Erik Landman of the University Sports Center and their colleague Peter Uylings at the AMSTEL Institute for the collaboration in the organization of the students' activities. They are also grateful to the anonymous reviewers for their helpful comments.

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