

Static Analysis of a Sleeper with Mathematica

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Abstract

Showing the power of Mathematica for the static analysis of a sleeper is the main goal of this paper. It is part of a case study for the Engineering Office of the Dutch Railways to investigate the applicability of computer algebra in their engineering practice.

1 Introduction

A classical problem in structural analysis is the study of beams in static equilibrium. The beam analysis is regarded as completed when the internal stresses and the displacements have been determined, i.e., once the shear force, bending moment, slope, and deflection functions have been computed. In this paper we show that Mathematica ([1]) is powerful and flexible enough to solve the symbolic, numerical, and graphical problems concerning beam deflections. The BeamStatics package of Levent Kitiş ([2]), already shows some of the possibilities. However, it does not allow studies of beams on foundations. In this paper we will study such beams. In particular, we will analyze a sleeper, which is modeled as a symmetrically loaded beam with partial elastic support.

2 Acknowledgements

Ing. T. Miedema and drs. G. Kouwijzer from the Engineering Office of the Dutch Railways raised the question of how good Mathematica is as a working environment for studies in beam analysis. The analysis of a sleeper is one of the case studies carried out to arrive at a positive answer to this question.

3 The Problem

We consider a sleeper in static equilibrium under the following conditions (see Figure 1 on the next page):

- the sleeper is a symmetrical system of loads and supports;
- there are only two concentrated loads;
- the support is elastic, but the sleeper is in the middle unsupported.

The task is to describe the bending of the sleeper under these conditions.

4 The Mathematical Model

The mathematical model for the sleeper comes from classical beam analysis. In this section we collect main results of the theory, which can be found in almost any book on beam analysis.

The beam coordinate system is shown in Figure 1, i.e., downward beam deflections are positive. The deflection of the beam is denoted by w . The distance x is measured along the axis of the beam from the origin at the left end of the beam to the right end of the beam at distance L . Figure 2 shows the sign convention for the bending moment M and the shear force Q .

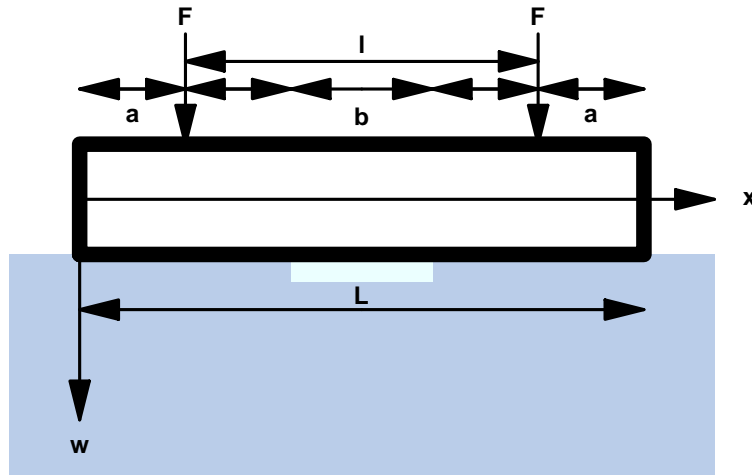


Figure 1: Partially Supported Sleeper

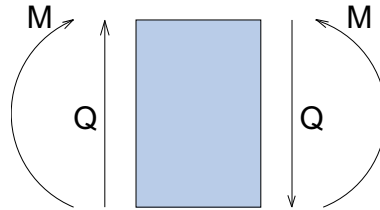


Figure 2: Sign Convention — Positive Shear and Bending Moment

Let H denote the slope of the beam and EI be the flexural rigidity of the beam, which is supposed to be constant along the beam axis.

The basis of the analysis are the following equations:

$$-\frac{\partial Q}{\partial x} = \text{load} - \text{reaction force} \quad (1)$$

$$\frac{\partial M}{\partial x} = Q(x) \quad (2)$$

$$\frac{\partial^2 w}{\partial x^2} = -\frac{M(x)}{EI} \quad (3)$$

From these equations follows

$$EI \frac{\partial^4 w}{\partial x^4} = \text{load} - \text{reaction force} \quad (4)$$

The applied load is conveniently expressed in terms of

- the unit step function

$$\text{UnitStep}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

- the Dirac delta function denoted by `DiracDelta` in Mathematica

The contribution of a concentrated load of magnitude F acting at $x = a$ to the load is equal to the product $F \cdot \text{DiracDelta}(x - a)$.

The action of an elastic foundation may be taken into account by introducing a restoring force $k \cdot w(x)$ acting in the upward direction. The constant k is called the modulus of the foundation. We can then rewrite Equation 4 as

$$EI \frac{\partial^4 w}{\partial x^4} + k w(x) = load \quad (5)$$

If we now introduce β via

$$4\beta^4 = \frac{k}{EI}$$

the equation 5 is transformed to

$$\frac{\partial^4 w}{\partial x^4} + 4\beta^4 w(x) = \frac{load}{EI} \quad (6)$$

As we will consider a beam with free ends, the boundary conditions are that the moment and shear force the ends of the beam vanish, i.e.,

$$M(0) = 0, \quad Q(0) = 0, \quad M(L) = 0, \quad \text{and} \quad Q(L) = 0 \quad (7)$$

5 The Computer Algebra

We consider two cases:

1. the elastically supported beam with two concentrated loads F symmetrically placed at distance l of each other (see Figure 3);
2. the partially supported beam with concentrated loads as shown in Figure 1.

5.1 Beam on Elastic Foundation

We can use Mathematica to sketch the situation:

```
<<Graphics`Arrow`
beam = { Thickness[0.02],
         Line[{ -4,-1}, {4,-1}, {4,1}, {-4,1}, {-4,-1} ] ];
foundation = { GrayLevel[0.8], Rectangle[ {-5,-5}, {5,-1} ] };
xArrow = Arrow[{-4,0}, {5, 0}];
wArrow = Arrow[{-4,0}, {-4,-4}];
leftPointLoad = Arrow[{-2.5,2.5}, {-2.5,1}];
rightPointLoad = Arrow[{ 2.5,2.5}, { 2.5,1}];
LArrow = { Arrow[{0,-2}, {4, -2}], Arrow[{0,-2}, {-4,-2} ] };
lArrow = { Arrow[{0, 2}, {2.5,2}], Arrow[{0, 2},{-2.5,2} ] };
aArrows = {
  Arrow[{ 2.5,2}, { 4,2}], Arrow[{ 4,2}, { 2.5,2}],
  Arrow[{-2.5,2}, {-4,2}], Arrow[{-4,2}, {-2.5,2}
];
text = {
  Text[ FontForm["x"],{"Helvetica-Bold",14}], {5.5,0} ],
  Text[ FontForm["w"],{"Helvetica-Bold",14}], {-4,-4.5} ],
  Text[ FontForm["L"],{"Helvetica-Bold",14}], {0,-1.75} ],
  Text[ FontForm["l"],{"Helvetica-Bold",14}], {0,1.7} ],
```

```

Text[ FontForm["a",{"Helvetica-Bold",14}], {3.25,1.7} ],
Text[ FontForm["a",{"Helvetica-Bold",14}], {-3.25,1.7} ],
Text[ FontForm["F",{"Helvetica-Bold",14}], {-2.5,2.8} ],
Text[ FontForm["F",{"Helvetica-Bold",14}], { 2.5,2.8} ]
];
Show[Graphics[ { foundation, beam, xArrow, wArrow, leftPointLoad,
rightPointLoad, LArrow, lArrow, aArrows, text },
PlotRange -> All ]];

```

It may look overwhelming, but it is merely a specification of the graphics objects and it will produce Figure 3.

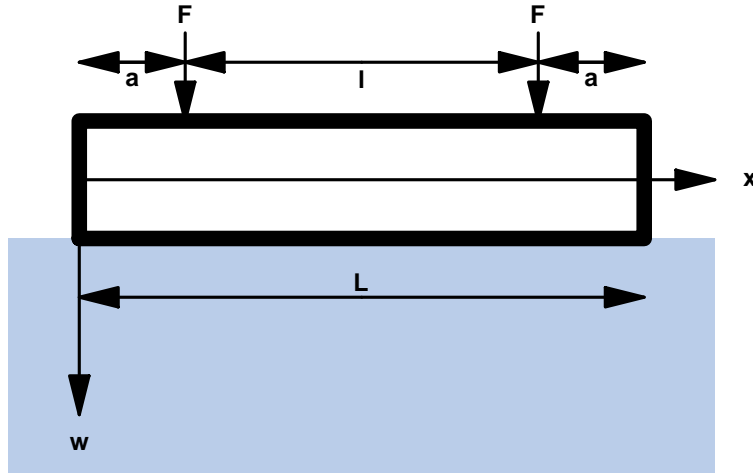


Figure 3: Beam on Elastic Foundation

By the way, this is also how Figure 1 was created.

The differential equation 6 is entered in Mathematica as

```
4 beta^4 w[x] + D[w[x],{x,4}] == load / EI
```

and displays as

$$4 \beta^4 w[x] + w^{(4)}[x] == \frac{\text{load}}{EI}$$

The load is in this case

```
<<Calculus'DiracDelta'
load = F DiracDelta[x-a] + F DiracDelta[x-a-l];
```

The Mathematica command DSolve is powerful enough to solve the differential equation. the result is after some simplification

$$\frac{d1 \cos[\beta x]}{\beta x E} + d3 E \cos[\beta x] + \frac{d2 \sin[\beta x]}{\beta x E} + d4 E \sin[\beta x] +$$

$$\begin{aligned}
& \frac{E \beta (a-x) F (\cos[\beta(-a+x)] - \frac{2\beta(-a+x)}{E} \cos[\beta(-a+x)] + \sin[\beta(-a+x)] + \frac{2\beta(-a+x)}{E} \sin[\beta(-a+x)]) \text{UnitStep}[-a+x]}{3} \\
& + \frac{F (-\cos[\beta(a+1-x)] + \frac{2\beta(a+1-x)}{E} \cos[\beta(a+1-x)] - \sin[\beta(a+1-x)] - \frac{2\beta(a+1-x)}{E} \sin[\beta(a+1-x)]) \text{UnitStep}[-a-1+x]}{3 E \beta (a+1-x) EI}
\end{aligned}$$

We can then define the deflection, slope, bending moment, and shear force functions.

```

W[x_] = %;
H[x_] = D[ W[x], x ];
M[x_] = -EI D[ H[x], x ];
Q[x_] = D[ M[x], x ];

```

The ends of the beam are free so that the bending moment and shear force are at these points equal to zero. This gives us 4 equations in the four unknowns d1, d2, d3, and d4, which Mathematica can solve. Even after simplification the results are large: typically formulae that fill one A4 page.

Instead of a we may also use in the formulae the indeterminates l and L via the relation $a = \frac{(L-l)}{2}$. Furthermore, if we take concrete values for the unknowns, says $L = 2600$ mm, $l = 1500$ mm, $EI = 6.33 \cdot 10^{11}$ N mm², $F = 56.25 \cdot 10^3$ N, and $k = 25$ N/mm, then the general solution specializes to

```

{d1 -> 0.537024, d2 -> 0.216867, d3 -> 0.970758,
 d4 -> 0.216867}

```

We define a new plot routine to plot the graphs of the beam functions in our downward oriented coordinate system. This shows a bit of Mathematica's programming facilities.

```

myPlot[ function_, domain_, options___ ] :=
Module[ {picture, range, markers},
picture = Plot[ function, domain, options,
DisplayFunction -> Identity ];
range = FullOptions[ picture, PlotRange ];

```