Amalgamation and Interpolation in Ordered Algebras

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Notation

$\mathcal{L}$: a language of algebras (All symbols in $\mathcal{L}$ have finite arity, at least one of them has arity zero, and $|\mathcal{L}| \leq \aleph_0$.)

$\mathcal{U}_{\mathcal{L}}$: the variety of all algebras of signature $\mathcal{L}$

$\mathbb{X}$: an infinite countable set

$\mathbf{Fm}(\mathbb{X})$: $\mathcal{U}_{\mathcal{L}}$-free algebra over $\mathbb{X}$ (We refer to this algebra as the formula algebra or the term algebra of signature $\mathcal{L}$ over $\mathbb{X}$.)

$\mathcal{U}$: an arbitrary subvariety of $\mathcal{U}_{\mathcal{L}}$

$\mathbf{F}_{\mathcal{U}}(\mathbb{X})$ or $\mathbf{F}(\mathbb{X})$: the $\mathcal{U}$-free algebra over $\mathbb{X}$

$Eq(\mathbb{X})$: $Fm(\mathbb{X}) \times Fm(\mathbb{X})$

The Relation $\models_{\mathcal{U}}$

The Consequence Relation $\models_{\mathcal{U}}$

Let $\mathcal{K} \subseteq \mathcal{U}_{\mathcal{L}}$. For $\Sigma \cup \{\varepsilon\} \subseteq Eq(\mathbb{X})$, we say that ε is a $\mathcal{K}$-consequence of Σ – $\Sigma \models_{\mathcal{K}} \varepsilon$ – if for every $\mathbf{A} \in \mathcal{K}$ and every homomorphism $\varphi\colon \mathbf{Fm}(\mathbb{X}) \to \mathbf{A}$, if $\Sigma \subseteq \mathrm{Ker}(\varphi)$, then $\varepsilon \in \mathrm{Ker}(\varphi)$.

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If $\mathcal{K}$ is a (quasi)variety, then $\models_{\mathcal{K}}$ is finitary.

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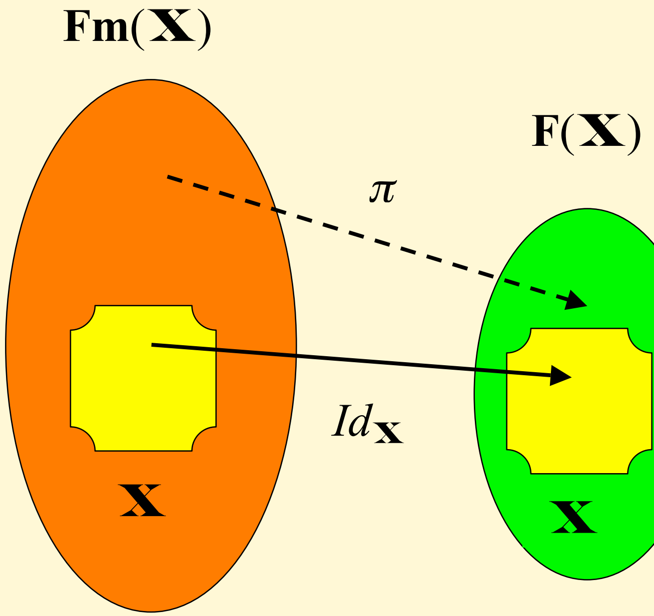

$$t \mapsto \pi(t) = \bar{t}$$

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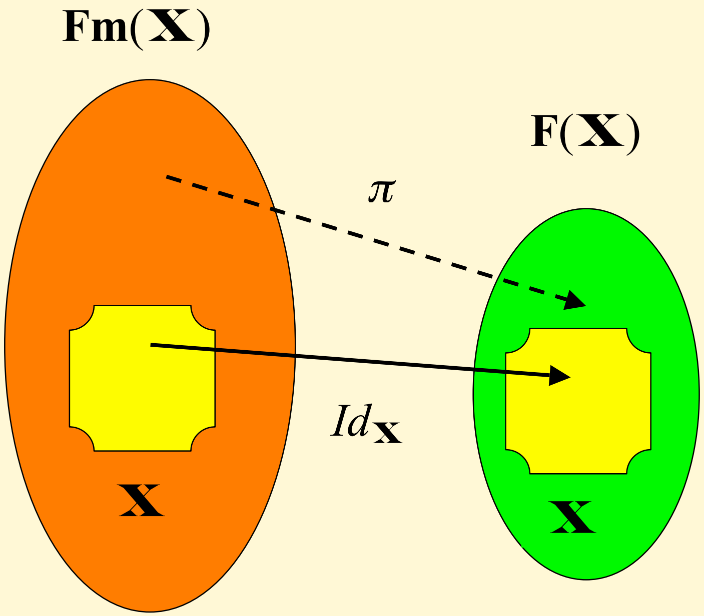

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$$\begin{aligned} \mathrm{Ker}(\pi) &= \Theta_{\mathcal{U}} \\ &= \bigcap\{\Theta \in Con(\mathbf{Fm}(\mathbb{X})) : \mathbf{Fm}(\mathbb{X})/\Theta \in \mathcal{U}\} \end{aligned}$$

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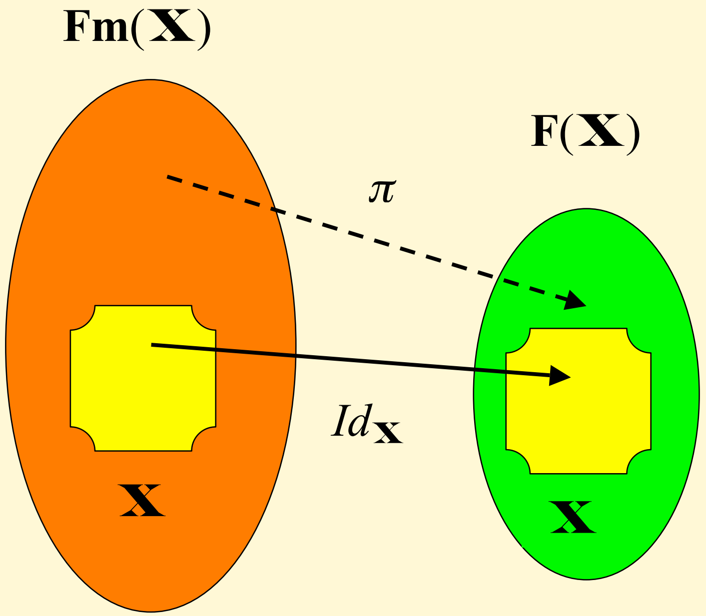

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For $\Sigma \cup \{\varepsilon\} \subseteq Eq(\mathbb{X})$, TFAE:

(1) $\Sigma \models_{\mathcal{U}} \varepsilon$;

(2) $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(\mathbb{X})}(\bar{\Sigma})$ in $\mathrm{Con}(\mathbf{F}(\mathbb{X}))$; and

(3) $\varepsilon \in \mathrm{Cg}_{\mathbf{Fm}(\mathbb{X})}(\Sigma) \vee \Theta_{\mathcal{U}}$ in $\mathrm{Con}(\mathbf{Fm}(\mathbb{X}))$.

The Amalgamation Property

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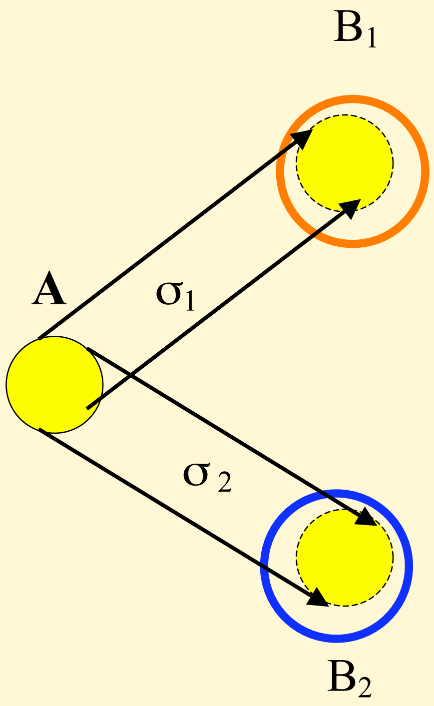

$\mathcal{U}$ is said to satisfy the amalgamation property (AP) if for any algebras $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{U}$, and embeddings $\sigma_1 : \mathbf{A} \to \mathbf{B}_1$, $\sigma_2 : \mathbf{A} \to \mathbf{B}_2$,

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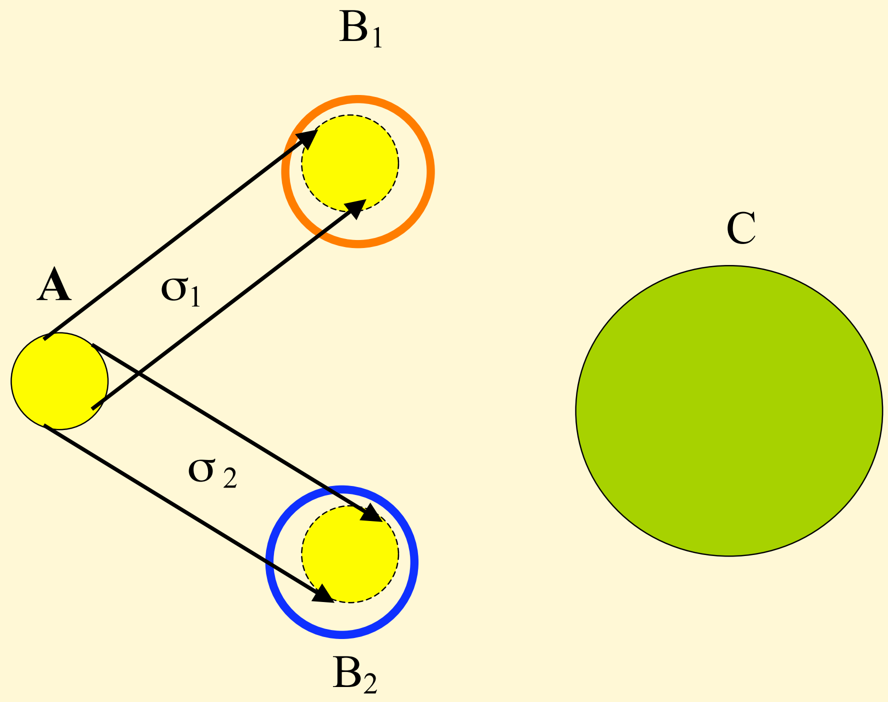

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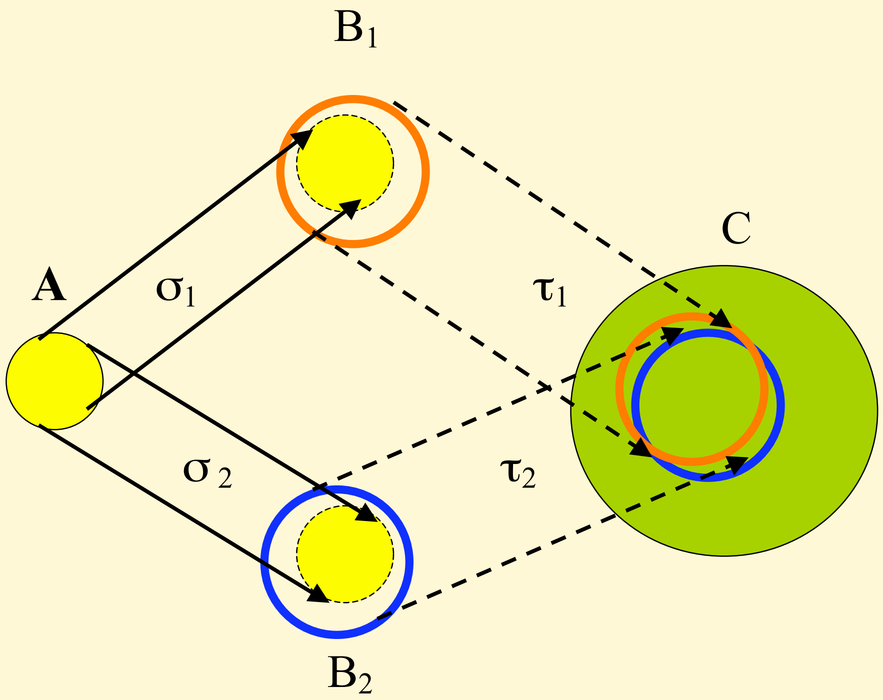

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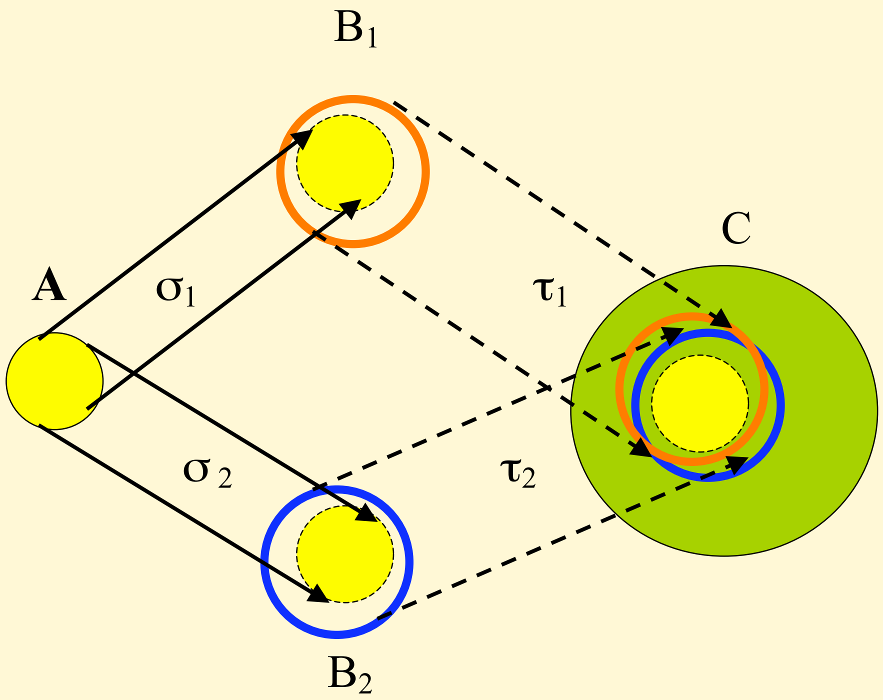

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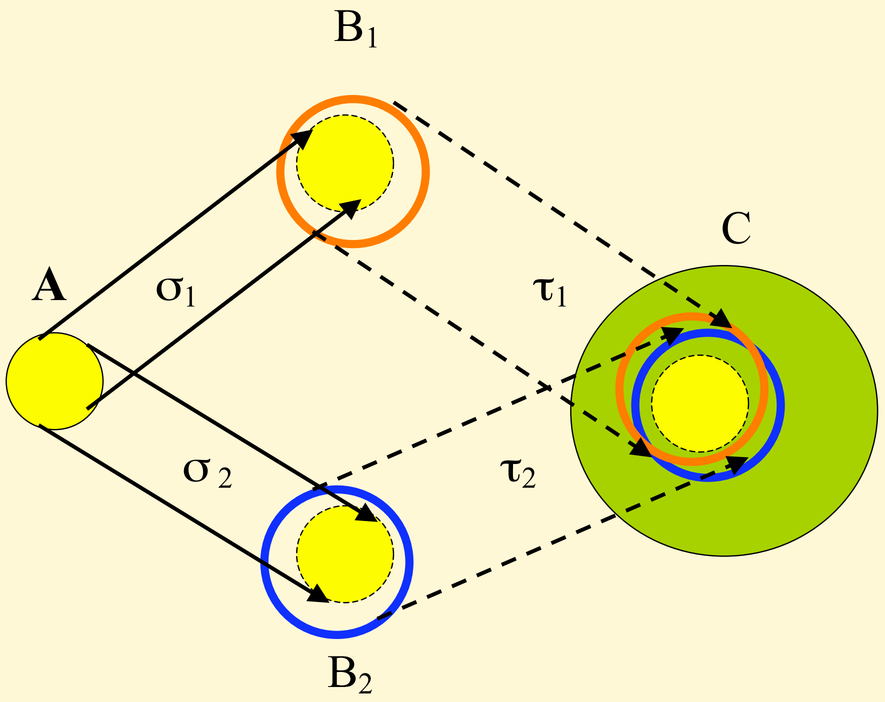

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The triple $(\tau_1, \tau_2, \mathbf{C})$ is called an amalgam in $\mathcal{U}$ for the V-formation $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \sigma_1, \sigma_2)$.

Notation
The Relation $\models_{\mathcal{U}}$
The Amalgamation Property
AP (1)
AP (2)
AP (3)
AP (4)
Interpolation Properties
Residuated Lattices
DIP for $\mathcal{CRL}$
DIP for $\mathcal{A}$
Failure of the AP
EXTRAS
GMV-algebras
The Calculus RL

The Amalgamation Property

Historical Remarks

The AP was first considered by O. Schreier (1927) – and later by H. Neumann (1949) – in the context of amalgamated free products of groups. The general form of the AP was first formulated by M. R. Fraïssè (1954), and its significance to the study of algebraic systems was further demonstrated by B. Jónsson's pioneering work Jónsson [1956, 1960, 1961, 1962].

Significant contributions in the general study of the AP include D. Pigozzi [1972], P. D. Bacsich [1975], L. L. Maksimova [1977, 1979, 1999], H. Ono [1986], J. Madarász [1998, 1998], and J. Czelakowski-D. Pigozzi [1999].

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$$\begin{tikzcd}
 & \mathbf{F}(\mathbf{X}) & \\
\mathbf{F}(\mathbf{Y}) \arrow[ur, hook] & \mathbf{Y} \cup \mathbf{Z} = \mathbf{X} & \mathbf{F}(\mathbf{Z}) \arrow[ul, hook'] \\
 & \mathbf{F}(\mathbf{Y} \cap \mathbf{Z}) \arrow[ul, hook] \arrow[ur, hook'] &
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Let $\mathbb{X} = Y \cup Z, \Theta_Y \in \mathrm{Con}(\mathbf{F}(Y)), \Theta_Z \in \mathrm{Con}(\mathbf{F}(Z))$ s.t. $\Theta_Y \cap F(Y \cap Z)^2 = \Theta_Z \cap F(Y \cap Z)^2$

Then there exists $\Theta \in \mathrm{Con}(\mathbf{F}(\mathbb{X}))$ s.t. $\Theta \cap F(Y)^2 = \Theta_Y$ and $\Theta \cap F(Z)^2 = \Theta_Z$.

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Question:Why does $|\mathbb{X}| = \aleph_0$ suffice?

The Amalgamation Property

The PP – and hence the AP – is trivially equivalent to the next syntactic condition, referred to as the Robinson Property (Pigozzi [1972]; Ono [1986]).

The Robinson Property (RP) for $\models_{\mathcal{U}}$

Let $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(\mathbb{X})$. Suppose that

(i) $\Sigma, \Pi \models_{\mathcal{U}} \varepsilon$;

(ii) $\Sigma \models_{\mathcal{U}} \delta$ iff $\Pi \models_{\mathcal{U}} \delta$, for all $\delta \in Eq(\mathbb{X})$ with $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$; and

(iii) $Var(\varepsilon) \cap (Var(\Pi) - Var(\Sigma)) = \emptyset$.

Then,

$$\Sigma \models_{\mathcal{U}} \varepsilon$$

Interpolation Properties

The Deductive Interpolation Property

The relation $\models_{\mathcal{U}}$ is said to have the **deductive interpolation property** (**DIP**) if whenever $\Sigma \models_{\mathcal{U}} \varepsilon$, there exists Π such that $\Sigma \models_{\mathcal{U}} \Pi$, $\Pi \models_{\mathcal{U}} \varepsilon$ and $Var(\Pi) \subseteq Var(\Sigma) \cap Var(\varepsilon)$.

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Assume that $\mathcal{U}$ admits free-products. We say that $\mathcal{U}$ has the Flat Amalgamation Property (FAP) if for every embedding $\mathbf{A} \to \mathbf{B}$ in $\mathcal{U}$ and every set Y, the induced homomorphism $\mathbf{A} * \mathbf{F}(Y) \to \mathbf{B} * \mathbf{F}(Y)$ from the free product $\mathbf{A} * \mathbf{F}(Y)$ to the free product $\mathbf{B} * \mathbf{F}(Y)$ is an embedding.

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$\models_{\mathcal{U}}$ has the DIP iff $\mathcal{U}$ has the FAP.

(Jónsson [1965]; Bacsich [1975])

It is again sufficient to establish the satisfaction of the FAP for finitely generated algebras in $\mathcal{U}$.

RP $\Longrightarrow$ DIP

RP implies DIP

(Jónsson [1965]; attributed to H. J. Keisler)

Proof Sketch

Suppose that $\mathcal{U}$ has the RP, and let $\Sigma \models_{\mathcal{U}} \varepsilon$, for $\Sigma \cup \{\epsilon\} \subseteq Eq(\mathbb{X})$. A straightforward application of the RP shows that

$$\Pi = \{\delta \in Eq(\mathbb{X}) : \Sigma \models_{\mathcal{U}} \delta, Var(\delta) \subseteq Var(\Sigma) \cap Var(\varepsilon)\}$$

is the required interpolant.

The Congruence Extension Property

$\mathcal{U}$ has the congruence extension property (CEP) if for every $\mathbf{B} \in \mathcal{U}$, every subalgebra $\mathbf{A}$ of $\mathbf{B}$, and every congruence Θ on $\mathbf{A}$, there exists a congruence $\tilde{\Theta}$ on $\mathbf{B}$ such that $\tilde{\Theta} \cap A^2 = \Theta$.

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The following statements are equivalent:

- $\mathcal{U}$ satisfies the CEP.
- Whenever $Y \subseteq \mathbb{X}, \Theta, \Phi \in \mathrm{Con}(\mathbf{F}(\mathbb{X}))$, then $(\Theta \vee \Phi) \cap F(Y)^2 = (\Theta \cap F(Y)^2) \vee' (\Phi \cap F(Y)^2)$. Here $\vee$ denotes the join in $\mathrm{Con}(\mathbf{F}(\mathbb{X}))$, and $\vee'$ denotes the join in $\mathrm{Con}(\mathbf{F}(Y))$.

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DIP
RP $\Longrightarrow$ DIP
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DIP + ExtIP
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The relation $\models_{\mathcal{U}}$ is said to have the Extension Interpolation Property (ExtIP) if whenever $\Sigma, \Pi \models_{\mathcal{U}} \varepsilon$, there exists Δ such that $\Sigma, \Delta \models_{\mathcal{U}} \varepsilon, \Pi \models_{\mathcal{U}} \Delta$, and $Var(\Delta) \subseteq Var(\Sigma \cup \{\varepsilon\})$.

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$\mathcal{U}$ has the CEP iff $\models_{\mathcal{U}}$ has the ExtIP.

(Ono, 1986)

DIP $+$ ExtIP $\implies$ RP

If $\models_{\mathcal{U}}$ has the DIP and ExtIP, then it has the RP.

Proof:

Suppose that $\Sigma \cup \Pi \cup \{\epsilon\} \subseteq Eq(\mathbb{X})$, $\Sigma, \Pi \models_{\mathcal{U}} \varepsilon$, $Var(\varepsilon) \cap (Var(\Pi) - Var(\Sigma)) = \emptyset$, and $\Sigma \models_{\mathcal{U}} \delta$ iff $\Pi \models_{\mathcal{U}} \delta$ whenever $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$.

By the ExtIP, there exists $\Delta_1 \subseteq Eq(\mathbb{X})$ such that: $\Sigma, \Delta_1 \models_{\mathcal{U}} \varepsilon; \Pi \models_{\mathcal{U}} \Delta_1; Var(\Delta_1) \subseteq Var(\Sigma \cup \{\varepsilon\})$.

Applying the DIP to $\Pi \models_{\mathcal{U}} \Delta_1$, we obtain $\Delta_2 \subseteq Eq(\mathbb{X})$ such that $\Pi \models_{\mathcal{U}} \Delta_2 \models_{\mathcal{U}} \Delta_1$, and $Var(\Delta_2) \subseteq Var(\Pi) \cap [Var(\Sigma) \cup Var(\varepsilon)] \subseteq Var(\Pi) \cap Var(\Sigma)$, since $Var(\Pi) \cap Var(\varepsilon) \subseteq Var(\Pi) \cap Var(\Sigma)$.

Hence $\Sigma, \Delta_2 \models_{\mathcal{U}} \varepsilon$. Also, since $\Pi \models_{\mathcal{U}} \Delta_2$, $\Sigma \models_{\mathcal{U}} \Delta_2$, by RP. So $\Sigma \models_{\mathcal{U}} \varepsilon$, as required.

Residuated Lattices

Residuated Lattices

A **residuated lattice** (or a **residuated lattice-ordered monoid**) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that:

(i) $\langle A, \wedge, \vee\rangle$ is a lattice;

(ii) $\langle A, \cdot, 1\rangle$ is a monoid; and

(iii) the operation $\cdot$ is residuated with residuals $\backslash$ and $/$. This means that that, for all $x, y, z \in A$,

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An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is said to be an FL algebra provided: (i) $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is an RL; and (ii) 0 is a distinguished element of A.

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The classes $\mathcal{RL}$ and $\mathcal{FL}$ of residuated lattices and FL-algebras, respectively, are finitely based varieties.

Historical Remarks

Residuated lattices appeared explicitly in the work of Krull, Ward and Dilworth as abstractions of ideal lattices of rings in the early 1930's. Their study, however, goes back to Hilbert's foundational studies of geometry, Riesz's development of the theory of operators and their spaces, and Dedekind's work in algebra and number theory.

Heyting Algebras and Boolean Algebras

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The variety of MV-algebras is term-equivalent to the subvariety, $\mathcal{MV}$, of $\mathcal{FL}$ satisfying the (additional) identities $x \vee y \approx (x \to y) \to y$, $xy \approx yx$ and $x \wedge 0 \approx 0$.

Lattice-Ordered Groups

A lattice-ordered group (ℓ-group) is (in additive notation) an algebra $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0\rangle$ such that (i) $\langle G, \wedge, \vee\rangle$ is a lattice; (ii) $\langle G, +, -, 0\rangle$ is a group; and (iii) addition is order-preserving.

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The variety of ℓ-groups is term-equivalent to the subvariety, $\mathcal{LG}$, of $\mathcal{RL}$ defined by the equations $0/x + x \approx 0 \approx x + x\backslash 0$; the term equivalence is given by $x/y = x - y, y\backslash x = -y + x$, and $-x = 0/x$.

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The variety of Abelian ℓ-groups: $\mathcal{A}b\mathcal{LG} = \mathcal{V}(\mathbb{Z})$

Lattice-Ordered Groups

A lattice-ordered group (ℓ-group) is (in additive notation) an algebra $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0\rangle$ such that (i) $\langle G, \wedge, \vee\rangle$ is a lattice; (ii) $\langle G, +, -, 0\rangle$ is a group; and (iii) addition is order-preserving.

The variety of ℓ-groups is term-equivalent to the subvariety, $\mathcal{LG}$, of $\mathcal{RL}$ defined by the equations $0/x + x \approx 0 \approx x + x\backslash 0$; the term equivalence is given by $x/y = x - y, y\backslash x = -y + x$, and $-x = 0/x$.

The variety of Abelian ℓ-groups: $\mathcal{AbLG} = \mathcal{V}(\mathbb{Z})$

[W. C. Holland] Every ℓ-group can be embedded into the ℓ-group of order-automorphisms of a chain.

DIP for $\mathcal{CRL}$

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The Gentzen Calculus RL_e

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RL_e satisfies Craig's interpolation property.

S. Maehara's Lemma: Suppose that $\Gamma, \Pi \Rightarrow \alpha$ is RL_e-derivable. Then there exists a formula β with $\mathrm{Var}(\beta) \subseteq \mathrm{Var}(\Gamma) \cap \mathrm{Var}(\Pi, \alpha)$ such that $\Gamma \Rightarrow \beta$ and $\Pi, \beta \Rightarrow \alpha$ are RL_e-derivable.

The proof is by induction on the height $h(d)$ of a cut-free derivation d of $\Gamma, \Pi \Rightarrow \alpha$. ([S. Maehara, 1960]; [H. Takamura, 2004]).

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$\mathcal{CRL}$ satisfies the AP ([H. Takamura, 2004]; [K. Terui, 200?]).

DIP for $\mathcal{A}$

The Basic Idea

Given Σ and ε, pick $x \in \mathrm{Var}(\varepsilon) - \mathrm{Var}(\Sigma)$.
Construct δ_1 with $\mathrm{Var}(\delta_1) \subseteq \mathrm{Var}(\varepsilon) - \{x\}$ such that:

$$\delta_1 \models_{\mathcal{A}} \varepsilon \quad \text{and} \quad \Sigma \models_{\mathbb{Z}} \varepsilon \;\Rightarrow\; \Sigma \models_{\mathbb{Z}} \delta_1$$

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Repeating, we get $\delta_2, \ldots, \delta_n$ where:

(1) $\Sigma \models_{\mathbb{Z}} \varepsilon \;\Rightarrow\; \Sigma \models_{\mathbb{Z}} \delta_1 \;\Rightarrow\; \ldots \;\Rightarrow\; \Sigma \models_{\mathbb{Z}} \delta_n$;

(2) $\delta_n \;\models_{\mathcal{A}}\; \delta_{n-1} \;\models_{\mathcal{A}}\; \ldots \;\models_{\mathcal{A}}\; \delta_1 \;\models_{\mathcal{A}}\; \varepsilon$; and

(3) $\mathrm{Var}(\delta_n) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$.

$\mathbb{Z}$ Generation

Weinberg's Theorem

$\mathcal{A}$ is generated by $\mathbb{Z}$.

Proof Sketch

Suppose $\models_{\mathbb{Z}} \varepsilon$. In view of the preceding analysis, there exists a zero variable equation δ such that:

$$\delta \models_{\mathcal{A}} \varepsilon \quad \text{and} \quad \models_{\mathbb{Z}} \delta$$

But then:

$$\models_{\mathbb{Z}} \varepsilon \;\Rightarrow\; \models_{\mathbb{Z}} \delta \;\Rightarrow\; \models_{\mathcal{A}} \delta \Rightarrow \models_{\mathcal{A}} \varepsilon$$

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$\mathcal{A}$ has a decidable equational theory.

Theorem

$\mathcal{A}$ is generated by $\mathbb{Z}$ as a quasi-variety.

DIP for $\mathcal{A}$

Theorem

$\mathcal{A}$ has the DIP.

Proof Sketch

Suppose that $\Sigma \models_{\mathcal{A}} \varepsilon$. (We may assume that Σ is finite.) Then eliminating variables one by one, we get δ such that:

(1) $\Sigma \models_{\mathbb{Z}} \delta$ (and hence, $\Sigma \models_{\mathcal{A}} \delta$, since $\mathbb{Z}$ generates $\mathcal{A}$ as a quasi-variety);

(2) $\delta \models_{\mathcal{A}} \varepsilon$; and

(3) $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$.

Corollary

$\mathcal{A}$ has the AP.

Justification

Notice that:

- $\Sigma \models_{\mathcal{A}} s \approx t$ iff $\Sigma \models_{\mathcal{A}} 0 \leq (s - t) \wedge (t - s)$

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For any term t and variable x, we can find

$$s = \bigwedge_{i \in I} [(p_i + \lambda x) \vee (q_i - \lambda x) \vee u_i]$$

(one or more of the disjuncts may not be present), such that $x \notin \mathrm{Var}(p_i, q_i, u_i)$ for each $i \in I$, and $0 \leq s \models_{\mathcal{A}} 0 \leq t$ and $\Sigma \models_{\mathbb{Z}} 0 \leq t \;\Rightarrow\; \Sigma \models_{\mathbb{Z}} 0 \leq s$.

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Treating each conjunct separately, we can consider

$$\varepsilon : 0 \leq (p + \lambda x) \vee (q - \lambda x) \vee u,$$

where $x \notin \mathrm{Var}(\Sigma) \cup \mathrm{Var}(\{p, q, u\})$, and define

$$\delta : 0 \leq (p + q) \vee u.$$

Then $\delta \models_{\mathcal{A}} \varepsilon \quad \text{and} \quad \Sigma \models_{\mathbb{Z}} \varepsilon \;\Rightarrow\; \Sigma \models_{\mathbb{Z}} \delta$

Failure of the AP

Failure of the AP for $\mathcal{R}ep\mathcal{RL}$

We use algebraic means to show that the variety $\mathcal{R}ep\mathcal{RL}$ fails the AP. In fact, we show that there exists an uncountable interval $[\mathcal{W}, \mathcal{R}ep\mathcal{RL}]$ of subvarieties of $\mathcal{R}ep\mathcal{RL}$ that fail AP.

The proof involves the following steps:

- We show that the variety $\mathcal{R}ep\mathcal{LG}$ of representable ℓ-groups fails AP.
- The ℓ-groups appearing in the V-formation can be chosen to belong to a variety $\mathcal{W}$ for which the interval $[\mathcal{W}, \mathcal{R}ep\mathcal{RL}]$ is uncountable.
- We show that the V-formation above cannot be amalgamated in $\mathcal{R}ep\mathcal{RL}$. (Requires a Holand-type representation theorem.)

Proof Sketch of Step 1

Note that the algebras in $\mathcal{R}ep\mathcal{LG}$ have unique extraction of roots: $x^n = y^n \Rightarrow x = y, \forall n \in \mathbb{Z}_{>0}$.

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We will establish the failure of the AP for $\mathcal{R}ep\mathcal{LG}$ by constructing a V-formation

$$\begin{tikzcd} \mathrm{B}_1 & & \mathrm{B}_2 \\ & \mathrm{A} \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] & \end{tikzcd}$$

of totally ordered groups $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$, with σ_1, σ_2 inclusions, that satisfy the following property:

> there exist elements $a \in A$, $b_1 \in B_1$ and $b_2 \in B_2$ such that $b_1^2 = b_2^2 \in A$, but $a^{b_1} = {b_1}^{-1} a b_1$ and $a^{b_2} = {b_2}^{-1} a b_2$ are distinct elements of A.

Thus, any amalgam of $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \sigma_1, \sigma_2)$ will not be in $\mathcal{R}ep\mathcal{LG}$ since it will satisfy $b_1^2 = b_2^2$ for $b_1 \neq b_2$.

Proof Sketch of Step 1

Let $\mathbf{G}$ be a totally ordered group and let φ be an automorphism of $\mathbf{G}$. Let $\mathbf{G}(\varphi)$ be the totally ordered group based on $G \times \{\varphi^n : n \in \mathbb{Z}\}$, defined as follows:

$$(x, \varphi^n)(y, \varphi^m) = (x\varphi^n(y), \varphi^{n+m})$$

$(x, \varphi^n) \leq (y, \varphi^m) \iff n < m$, or $n = m$ and $x \leq y$

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Suppose that φ, ψ, χ are automorphisms of $\mathbf{G}$ such that $\varphi = \psi^2 = \chi^2$ and $\psi \neq \chi$. Then any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{G}(\psi)$ and $\mathbf{G}(\chi)$ fails the AP.

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Suppose that φ, ψ, χ are automorphisms of $\mathbf{G}$ such that $\varphi = \psi^2 = \chi^2$ and $\psi \neq \chi$. Then any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{G}(\psi)$ and $\mathbf{G}(\chi)$ fails the AP.

Indeed, $\mathbf{G}(\varphi)$ is a subalgebra of $\mathbf{G}(\psi)$ and $\mathbf{G}(\chi)$. Further, for all $g \in G$, $g^\psi = \psi(g)$ in $\mathbf{G}(\psi)$ and $g^\chi = \chi(g)$ in $\mathbf{G}(\chi)$. Hence, if $\psi \neq \chi$, then $g^\psi \neq g^\chi$ in $\mathbf{G}(\varphi)$, for some $g \in G$.

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We complete the proof by exhibiting a totally ordered group G and automorphisms ψ, χ of G such that $\psi^2 = \chi^2$, but $\psi \neq \chi$.

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We complete the proof by exhibiting a totally ordered group $\mathbf{G}$ and automorphisms ψ, χ of $\mathbf{G}$ such that $\psi^2 = \chi^2$, but $\psi \neq \chi$.

We first construct distinct ordered automorphisms of a totally set $\mathbf{I}$ whose squares are equal.

Let $\mathbf{I} = \mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ (ordered by the anti-lexicographic order) and let $\bar{\psi}, \bar{\chi}$ be defined on I as follows:

$$\bar{\psi}(x,i) = \begin{cases} (x+1, i+1) & \text{if } i \in 2\mathbb{Z} \\ (x, i+1) & \text{if } i \notin 2\mathbb{Z} \end{cases}$$

$$\bar{\chi}(x,i) = \begin{cases} (x, i+1) & \text{if } i \in 2\mathbb{Z} \\ (x+1, i+1) & \text{if } i \notin 2\mathbb{Z} \end{cases}$$

It is clear that both $\bar{\psi}$ and $\bar{\chi}$ are distinct order automorphisms of $\mathbf{I}$ such that $\bar{\psi}^2 = \bar{\chi}^2$.

Proof Sketch of Step 1

Finally, let $\mathbf{G} = \overset{\leftarrow}{\bigoplus_{i \in I}} \mathbb{Z}_i$, $\mathbb{Z}_i \cong \mathbb{Z}$. This totally ordered group has the required properties.

Note that each order-automorphism $\bar{f}$ of $\mathbf{I}$ induces an automorphism f of $\mathbf{G}$, defined by $f((x_i)_{i\in I}) = (x_{\bar{f}(i)})_{i\in I}$. Thus, $\bar{\psi}$ and $\bar{\chi}$ above induce distinct automorphisms ψ and χ of $\mathbf{G}$ such that $\psi^2 = \chi^2$. Thus, $\mathcal{Rep}\mathcal{LG}$ fails the AP.

Thank you for your attention.

EXTRAS

Open Problems

- Provide a purely algebraic proof of the AP for the variety $\mathcal{CRL}$ of commutative residuated lattices.
- Determine which of the varieties below satisfy the DIP (and hence the AP): $\mathcal{CRepRL}$ (commutative representable RLs), $\mathcal{CGBL}$ (commutative GBL-algebras), $\mathcal{CIGBL}$ (commutative integral GBL-algebras), $\mathcal{CCanRL}$ (commutative cancellative RLs), etc.
- Prove or disprove that the variety $\mathcal{RL}$ of all RLs satisfies the AP.
- Are there any non-commutative subvarieties of $\mathcal{RL}$ that satisfy the AP?
- Give an example of a subvariety of $\mathcal{RL}$ that satisfies the DIP (FAP), but fails the RP (AP).
- Investigate universal-homogeneous algebras in selected subvarieties of $\mathcal{RL}$ satisfying the AP.

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The Amalgamation Property

We assume that σ_1 and σ_2 are inclusions.

$$\begin{tikzcd}
B_1 & & B_2 \\
& A \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] &
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$$\begin{tikzcd}
 & & F(B_1 \cup B_2) & & \\
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 & & C & & \\
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GMV-algebras

Generalized MV-algebras

The variety $\mathcal{IGMV}$ of integral GMV-algebras is axiomatized, relative to $\mathcal{RL}$, by the equations

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The variety $\mathcal{IGMV}$ of integral GMV-algebras is axiomatized, relative to $\mathcal{RL}$, by the equations

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The variety $\mathcal{GMV}$ of generalized MV-algebras (GMV-algebras) is defined as the subvariety of $\mathcal{RL}$ satisfying the identities

$$x/(y\backslash x \wedge 1) \approx x \vee y \approx (x/y \wedge 1)\backslash x.$$

Equivalently, it is the subclass of $\mathcal{RL}$ whose members satisfy the quasi-identity

$$x \leq y \Rightarrow x/(y\backslash x) = y = (x/y)\backslash x.$$

The AP for Varieties of GMV-algebras

Theorem

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- Let $\mathcal{U}$ be a subvariety of $\mathcal{CGMV}$ that does not contain $\mathcal{CIGMV}$. Then $\mathcal{U}$ has the AP iff it is generated by a single chain; two chains, one of which is either $\mathbb{Z}^-$ or $\mathbb{Z}$; or by three chains, two of which are $\mathbb{Z}^-$ and $\mathbb{Z}$.

The Calculus RL

The Calculus RL

- Signature: $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1\}$
- The succedent of each sequent in $\mathbf{RL}$ is a single formula.

Initial Sequents

$$\alpha \Rightarrow \alpha \text{ and } \quad \Rightarrow 1$$

Structural Rules

1-Weakening Rule

$$\frac{\Gamma, \Delta \Rightarrow \beta}{\Gamma, 1, \Delta \Rightarrow \beta}\ (1w \Rightarrow)$$

Cut Rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Delta \Rightarrow \beta}{\Sigma, \Gamma, \Delta \Rightarrow \beta}$$

The Calculus RL

Rules for Logical Connectives

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma \qquad \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \gamma}\ (\Rightarrow\ \vee 1) \qquad \frac{\Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \alpha \vee \gamma}\ (\Rightarrow\ \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

Notation
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The Amalgamation Property
Interpolation Properties
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DIP for $\mathcal{CRL}$
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Failure of the AP
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The Calculus RL
RL Calculus (1)
RL Calculus (2)
RL Calculus (3)
FL Calculus

The Calculus RL

Rules for Logical Connectives (continued)

Monoid Operation and Implications (Residuals)

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}\ (\cdot \Rightarrow\ ) \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\ \Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \Rightarrow \gamma}\ (\backslash \Rightarrow\ ) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\ \Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \Rightarrow \gamma}\ (/ \Rightarrow\ ) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\ \Rightarrow /)$$

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Rules for Logical Connectives (continued)

Monoid Operation and Implications (Residuals)

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \Rightarrow \gamma}\ (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \Rightarrow \gamma}\ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

The concepts of a provable sequent and of a provable formula (theorem) are defined as before. The set of theorems of $\mathbf{RL}$ will be denoted by $Thm_{\mathbf{RL}}$.

The Calculus FL

The succedent of each sequent in $\mathbf{FL}$ is a single formula or empty.

One way to add negation(s) to $\mathbf{RL}$ is to introduce a constant 0 to the signature and define

$$\sim\alpha = \alpha\backslash 0 \;\text{ and }\; -\alpha = 0/\alpha.$$

The calculus obtained from $\mathbf{RL}$ by adding the axiom and the rule below is called FL (Full Lambek Calculus).

$$0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}(0 \text{ weakening})$$