Algebraic Proof Theory for Substructural Logics II

Kazushige Terui

RIMS, Kyoto University

(Joint work with Agata Ciabattoni, Nikolaos Galatos and Lutz Straßburger)

Algebraic Proof Theory for Substructural Logics I+α

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宿らぬ水の　いかでなからん

藤原道長 (966 – 1028)

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In the starry night, I wonder why
there exists only one moon in the sky,
while every water retains its reflection.

Fundamental Logical Duality

- Proof Theory: existential $\vdash_{\mathbf{L}} A \iff \exists \pi.(\pi \text{ is a proof of } A)$

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- Algebra: universal $\models_{\mathcal{V}(\mathbf{L})} A \iff \forall \mathbf{P} \in \mathcal{V}(\mathbf{L}).(\mathbf{P} \models A)$
- I am interested in proof theory reflected in algebra:

Proof Theory	Algebra
Argument	tnemugrA
Property	ytreporP

Motivating Facts

- Algebra: The variety of Heyting algebras satisfying prelinearity $(x \to y) \vee (y \to x) = \top$ (or weak excluded middle $\neg\neg x \vee \neg x = \top$) is not closed under Dedekind-MacNeille completions (Bezhanishvili-Harding 04), but is closed under some completions.

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- Algebra: The variety of MV algebras is not closed under any kind of completion (cf. Kowalski-Litak 08)
- Proof Theory: Łukasiewicz logic does not admit any kind of calculus with "strong" cut elimination.

Outline

1. Residuated Lattices and Full Lambek Calculus
2. Substructural Hierarchy
3. Class $\mathcal{N}_2$
4. Class $\mathcal{P}_3$
5. Beyond

Commutative Residuated Lattices ($\mathcal{CRL}$)

- A (bounded pointed) commutative residuated lattice is

$$\mathbf{P} = \langle P, \wedge, \vee, \cdot, \to, \top, \bot, 1, 0 \rangle$$

1. $\langle P, \wedge, \vee, \top, \bot \rangle$ is a lattice with $\top$ greatest and $\bot$ least ($\top$ and $\bot$ can be absent)
2. $\langle P, \cdot, 1 \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \cdot y \leq z \iff y \leq x \to z$
4. $0 \in P$.

- $\neg x \stackrel{def}{=} x \to 0$
- Examples: Heyting algebras (where $x \wedge y = x \cdot y$)
  - Lattice-ordered abelian groups with $x \to y = x^{-1}y$
  - The set of ideals of a commutative ring (Ward-Dilworth 39)

Full Lambek Calculus FLe

- The base system for substructural logics (Ono 90)

$\mathbf{FLe}$ $\approx$ The logic corresponding to CRLs

$\approx$ Intuitionistic logic without weakening and contraction

$\approx$ Intuitionistic linear logic without exponentials

- Formulas: terms over

Substructural Logics	$\wedge$	$\vee$	$\cdot$	$\rightarrow$	$\top$	$\bot$	1	0
Linear Logic	$\&$	$\oplus$	$\otimes$	$\multimap$	$\top$	$\mathbf{0}$	$\mathbf{1}$	$\bot$

- Sequents: $\Gamma \Rightarrow \Pi$

(Γ: multiset of formulas, Π: stoup with at most one formula)

Inference Rules of FLe

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \cdot B, \Gamma \Rightarrow \Pi}\ \cdot l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ \cdot r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi}\ \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \vee r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi}\ \bot l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \wedge A_2, \Gamma \Rightarrow \Pi}\ \wedge l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge r \qquad \frac{}{\Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Gamma \Rightarrow \Pi}{1, \Gamma \Rightarrow \Pi}\ 1l \qquad \frac{}{\Rightarrow 1}\ 1r \qquad \frac{}{0 \Rightarrow}\ 0l \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ 0r$$

Substructural Logics

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- Definition: A (commutative) substructural logic $\mathbf{L}$ is an axiomatic extension of $\mathbf{FLe}$.

$$\mathbf{FLew} = \mathbf{FLe} + \text{(Weakening):} \quad A \to 1, \quad 0 \to A$$

$$\mathbf{Int} = \mathbf{FLew} + \text{(Contraction):} \quad A \to A \cdot A$$

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- Cut Elimination is not preserved when axioms are added. Hence axioms have to be transformed into 'good' structural rules:

$$\frac{\Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\,(w) \qquad \frac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\,(c)$$

Structural Rules

- Definition:

  A structural rule is a rule

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0}$$

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- It corresponds to a quasiequation

$$t_1 \leq u_1 \textbf{ and } \cdots \textbf{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0$$

  where t_i is 1 or a product of variables, and u_i is 0 or a variable.

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  Which axioms can be transformed into ‘good’ structural rules enjoying cut elimination?

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- Fundamental question:
  Which identities are preserved by DM completions?

Dedekind Completion of Rationals

- For any $X \subseteq \mathbb{Q}$,

$$\begin{aligned} X^{\triangleright} &= \{y \in \mathbb{Q} : \forall x \in X. x \leq y\} \\ X^{\triangleleft} &= \{y \in \mathbb{Q} : \forall x \in X. y \leq x\} \end{aligned}$$

- X is Galois-closed if $X = X^{\triangleright\triangleleft}$
- $(\mathbb{Q}, +, \cdot)$ can be embedded into $(\overline{\mathbb{Q}}, +, \cdot)$ with

$$\overline{\mathbb{Q}} = \{X \subseteq \mathbb{Q} : X \text{ is closed}\}$$

- Dedekind completion extends to various ordered algebras (MacNeille 37).

Dedekind-MacNeille Completion

- Theorem (cf. Ono 93): $\mathcal{CRL}$ admits DM-completion.
  Every $\mathbf{A} \in \mathcal{CRL}$ has a completion $\overline{\mathbf{A}}$ in $\mathcal{CRL}$, where
  $\overline{\mathbf{A}} = (\overline{A}, \cap, \cup_\gamma, \cdot_\gamma, \to, \overline{A}, \emptyset^{\rhd\lhd}, \epsilon^{\rhd\lhd}, 0^{\rhd\lhd})$

$$\overline{A} = \{X \subseteq A : X = X^{\rhd\lhd}\} \qquad X \cup_\gamma Y = (X \cup Y)^{\rhd\lhd}$$

$$X \cdot_\gamma Y = \{xy : x \in X, y \in Y\}^{\rhd\lhd} \qquad X \to Y = \{y : \forall x \in X xy \in Y$$

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  - A is $\wedge$-dense in $\overline{A}$.

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Polarity

- Key concept of linear logic
- Positive connectives $1, \bot, \cdot, \vee$ $(1, \mathbf{0}, \otimes, \oplus)$ have invertible left rules:

$$\frac{\alpha, \Gamma \Rightarrow \Pi \quad \beta, \Gamma \Rightarrow \Pi}{\alpha \vee \beta, \Gamma \Rightarrow \Pi}$$

  and right rules admit focalization (Andreoli 90).
- Negative connectives $\top, 0, \wedge, \rightarrow$ $(\top, \bot, \&, \parr)$ have invertible right rules:

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

Polarity

- Connectives of the same polarity associate well.
- Positives:

$$\alpha \cdot (\beta \vee \gamma) = (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$$

$$\alpha \cdot 1 = \alpha \qquad \alpha \cdot \bot = \bot \qquad \alpha \vee \bot = \alpha$$

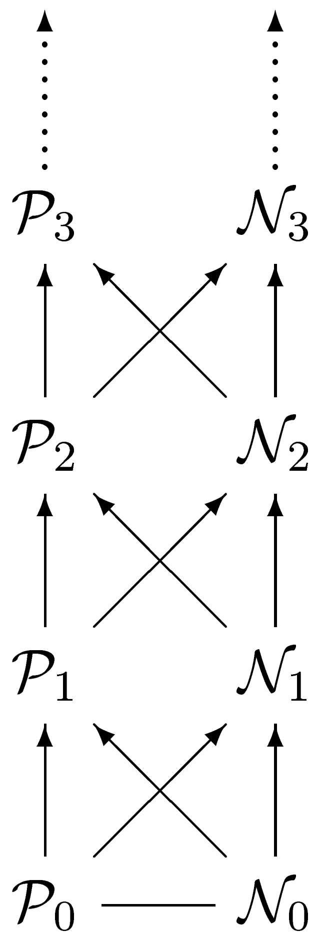
- Negatives:

$$\begin{aligned} \alpha \to (\beta \wedge \gamma) &= (\alpha \to \beta) \wedge (\alpha \to \gamma) \\ (\alpha \vee \beta) \to \gamma &= (\alpha \to \gamma) \wedge (\beta \to \gamma) \end{aligned}$$

$$\alpha \wedge \top = \alpha \qquad \alpha \to \top = \top \qquad 1 \to \alpha = \alpha$$

(polarity reverses on the LHS of an implication)

Substructural Hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

  (0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables

  (P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$

  (P2) $\alpha, \beta \in \mathcal{P}_{n+1} \implies \alpha \vee \beta, \alpha \cdot \beta, 1, \bot \in \mathcal{P}_{n+1}$

  (N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$

  (N2) $\alpha, \beta \in \mathcal{N}_{n+1} \implies \alpha \wedge \beta, 0, \top \in \mathcal{N}_{n+1}$

  (N3) $\alpha \in \mathcal{P}_{n+1}, \beta \in \mathcal{N}_{n+1} \implies \alpha \to \beta \in \mathcal{N}_{n+1}$

  - $\alpha \leq \beta \in \mathcal{N}_n$ if $\alpha \to \beta \in \mathcal{N}_n$

The Zoo of Nonclassical Axioms

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to 1,\ 0 \to \alpha$	weakening
	$\alpha \to \alpha \cdot \alpha$	contraction
	$\alpha \cdot \alpha \to \alpha$	expansion
	$\alpha^n \to \alpha^m$	knotted axioms ($n, m \geq 0$)
	$\neg(\alpha \wedge \neg\alpha)$	no-contradiction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$((\alpha \to \beta) \wedge 1) \vee ((\beta \to \alpha) \wedge 1)$	linearity
	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\bigvee_{i=0}^{k}(p_i \to \bigvee_{j \neq i} p_j)$	Kripke models of width $\leq k$
	$\bigvee_{i=0}^{k}(p_0 \wedge \cdots \wedge p_{i-1} \to p_i)$	Kripke models with k worlds

The Zoo of Nonclassical Axioms

Class	Axiom	Name
$\mathcal{N}_3$	$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
	$(\alpha \to \alpha \cdot \beta) \to \beta$	cancellativity
	$(\alpha \wedge \beta) \to \alpha \cdot (\alpha \to \beta)$	divisibility
		Zakharyaschev's canonical formula

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- Theorem:
  Every $\mathcal{N}_2$-axiom is equivalent to (a set of) structural rules in $\mathbf{FLe}$.

Example

- No contradiction $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

$$\frac{}{\Rightarrow \neg(\alpha \wedge \neg\alpha)}$$

Example

- No contradiction $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

$$\overline{(\alpha \wedge \neg\alpha) \Rightarrow 0}$$

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Example

- No contradiction $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

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Key Lemma

- Lemma: An axiom $A, \Gamma \Rightarrow \Pi$ is equivalent to a rule

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with β a fresh metavariable.

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- Yoneda's Lemma: $Hom(A, B) \cong Nat(Hom(\_, A), Hom(\_, B))$

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- In particular $A \Rightarrow B$ is equivalent to $\dfrac{\beta \Rightarrow A}{\beta \Rightarrow B}$
- Yoneda's Lemma: $Hom(A, B) \cong Nat(Hom(\_, A), Hom(\_, B))$
- Our transformation is sound w.r.t. the categorical interpretation

  formulas = objects

  proofs = morphisms

  for symmetric monoidal closed categories with finite products and coproducts.

Towards Cut Elimination

- Not all structural rules admit cut elimination. They have to be further transformed.
- In absence of (w), cyclic rules are problematic:

$$\frac{\alpha, \gamma \Rightarrow \beta \quad \beta \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \beta}$$

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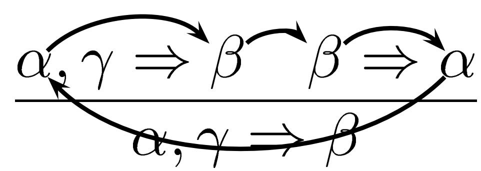

- Theorem (Cut Elimination):
  1. Every acyclic $\mathcal{N}_2$-axiom is equivalent (in $\mathbf{FLe}$) to structural rules enjoying cut elimination.
  2. Every $\mathcal{N}_2$-axiom is equivalent (in $\mathbf{FLew}$) to structural rules enjoying cut elimination.

Example

- (No-contradiction) $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

$$\frac{\beta \Rightarrow \alpha \quad \beta, \alpha \Rightarrow}{\beta \Rightarrow}$$

Example

- (No-contradiction) $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

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- (No-contradiction) $\neg(\alpha \wedge \neg\alpha)$ is equivalent to

$$\frac{\Gamma, \Gamma \Rightarrow}{\Gamma \Rightarrow}$$

- $1 \leq \neg(x \wedge \neg x)$ is equivalent to

$$xx \leq 0 \Longrightarrow x \leq 0.$$

Conservativity and Completion

- Infinitary extension $\mathbf{L}^\infty$ of a substructural logic $\mathbf{L}$: we consider infinitary formulas $\bigwedge_{i \in I} A_i$ and add new rules

$$\frac{\Gamma \Rightarrow A_i \quad \text{for any } i \in I}{\Gamma \Rightarrow \wedge_{i \in I} A_i} \qquad \frac{A_i, \Gamma \Rightarrow \Pi \quad \text{for some } i \in I}{\bigwedge_{i \in I} A_i, \Gamma \Rightarrow \Pi}$$

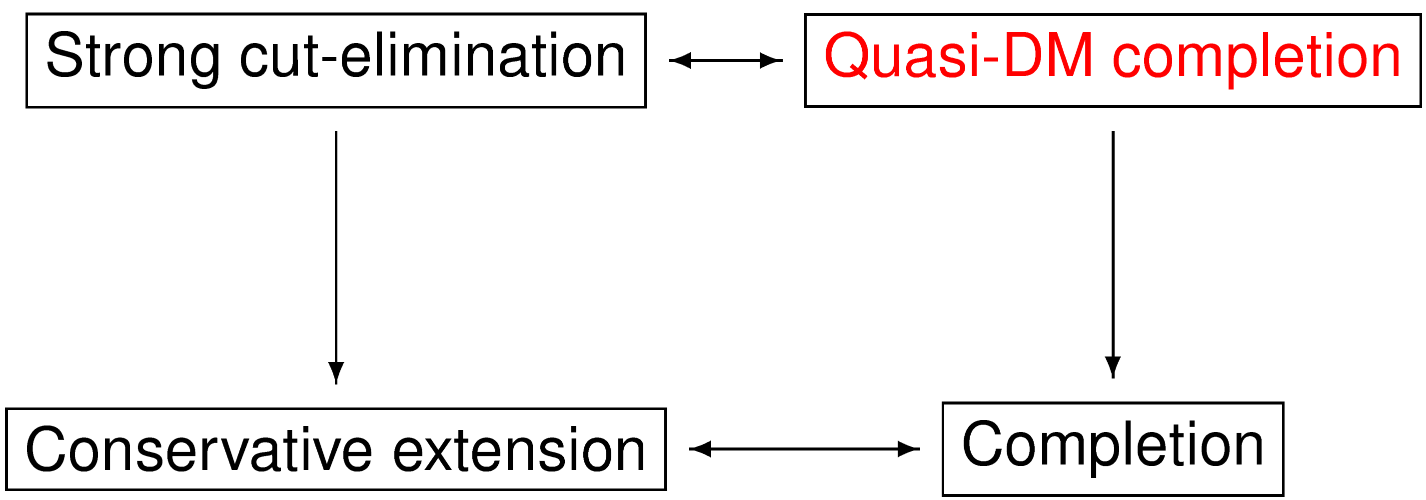
- $\mathbf{L}^\infty$ is a conservative extension of $\mathbf{L}$ if

$$\Phi \vdash_{\mathbf{L}^\infty} A \Longrightarrow \Phi \vdash_{\mathbf{L}} A$$

for any set $\Phi \cup \{A\}$ of finitary formulas.

Conservativity and Completion

- Theorem: For any substructural logic $\mathbf{L}$,
  $\mathbf{L}^\infty$ is a conservative extension of $\mathbf{L}$
  iff $\mathcal{V}(\mathbf{L})$ is closed under completions
  (i.e. any $\mathbf{A} \in \mathcal{V}(\mathbf{L})$ is a subalgebra of a complete $\mathbf{B} \in \mathcal{V}(\mathbf{L})$).
- A strong version of cut-elimination (that works with nonlogical initial sequents) implies conservativity.



Cut Elimination via Quasi-DM Completion

- Syntactic argument:

  Gentzen's procedure

  Cut-ful Proofs $\Longrightarrow$ Cut-free Proofs

- Semantic argument:

  Quasi-DM completion

  CRL $\Longleftarrow$ 'Intransitive' CRL

- Origin: Tait-Girard computability argument.
  Adapted by (Okada 96), (Okada-Terui 99).
  Algebraically reformulated by (Belardinelli-Ono-Jipsen 01), (Galatos-Jipsen).
  Cf. (Titani 65), (Maehara 91)

Cut Elimination via Quasi-DM Completion

- DM completion: Given a CRL $\mathbf{A}$, there is a complete $\overline{\mathbf{A}}$ with an embedding $i : \mathbf{A} \longrightarrow \overline{\mathbf{A}}$ such that
  - $i(A)$ is $\vee$-dense in $\overline{A}$
  - $i(A)$ is $\wedge$-dense in $\overline{A}$.
- Quasi-DM completion (BOJ 01): Given a cut-free Genzen structure $\mathbf{A}$ ('CRL without transitivity'), there is a complete CRL $\overline{\mathbf{A}}$ with two maps $l, u : A \longrightarrow \overline{A}$ such that
  - $l(A)$ is $\vee$-dense in $\overline{A}$
  - $u(A)$ is $\wedge$-dense in $\overline{A}$
  - for every $\star \in \{\wedge, \vee, \cdot, \rightarrow\}$, $a, b \in A$, $x \in [l(a), u(a)]$ and $y \in [l(b), u(b)]$, $x \star y \in [l(a \star b), u(a \star b)]$
  - $l = u$ is an embedding whenever $\mathbf{A}$ is a CRL.

Main Results on $\mathcal{N}_2$

- Theorem: Let $\mathcal{A}$ be a set of $\mathcal{N}_2$-axioms. The following are equivalent.
  1. $\mathcal{A}$ is equivalent to a set $\mathcal{R}$ of acyclic structural rules.
  2. $\mathcal{A}$ is preserved by quasi-DM completions.
  3. $\mathcal{A}$ is equivalent to $\mathcal{R}$ enjoying cut-elimination.
  4. $\mathcal{A}$ is preserved by DM completions.
  5. $\mathcal{CRL}(\mathcal{A})$ is closed under completions.
  6. $(\mathbf{FLe} + \mathcal{A})^\infty$ is conservative over $(\mathbf{FLe} + \mathcal{A})$.
- (Cf. Theunissen-Venema 07)
- Theorem (Gentzen's paradise): In presence of (w), the above holds for any set of $\mathcal{N}_2$-axioms.

Limitation of structural rules

- Theorem: Every structural rule is either derivable in **Int** or inconsistent with **Int**.

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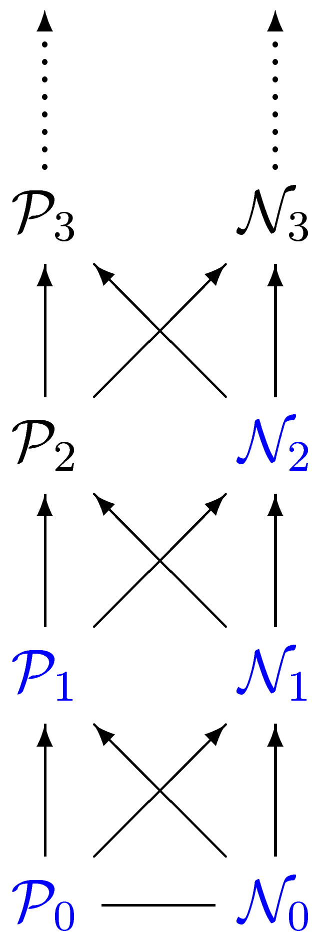
Substructural Hierarchy



- Acyclic $\mathcal{N}_2 \approx$ good structural rules
- Traditional proof theory (sequent calculus) works nicely up to $\mathcal{N}_2$ (Gentzen's paradise).
- Cut-elimination = DM = Completion in $\mathcal{N}_2$
- It does not work for some $\mathcal{P}_2$ and $\mathcal{P}_3$ axioms.

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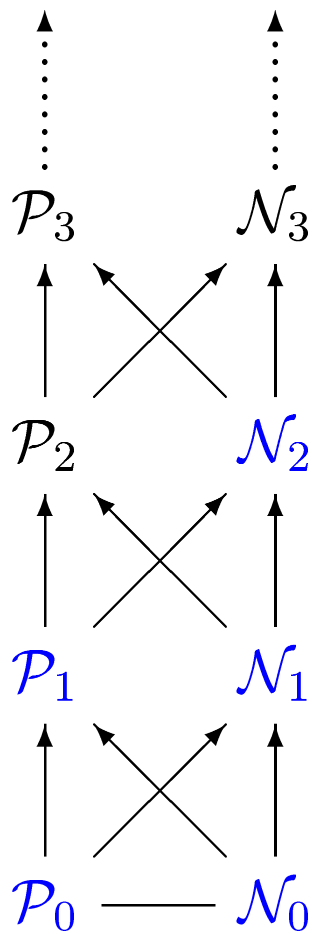

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- Cut-elimination = DM = Completion in $\mathcal{N}_2$
- It does not work for some $\mathcal{P}_2$ and $\mathcal{P}_3$ axioms.
- How to deal with axioms beyond $\mathcal{N}_2$?

Outline

1. Residuated Lattices and Full Lambek Calculus
2. Substructural Hierarchy
3. Class $\mathcal{N}_2$
4. Class $\mathcal{P}_3$
5. Beyond

Hypersequent Calculus (Avron 87)

- Hypersequent:

$$\Gamma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Gamma_n \Rightarrow \Pi_n$$

- Intuition:

$$(\bullet\Gamma_1 \to \Pi_1)_{\wedge 1} \vee \cdots \vee (\bullet\Gamma_n \to \Pi_n)_{\wedge 1}$$

Assuming (w):

$$(\bullet\Gamma_1 \to \Pi_1) \vee \cdots \vee (\bullet\Gamma_n \to \Pi_n)$$

- **HFLe** consists of

Rules of **FLe**	Ext-Weakening	Ext-Contraction
$\dfrac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B}$	$\dfrac{G}{G \mid \Gamma \Rightarrow \Pi}$	$\dfrac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}$

Hypersequent Calculus

- Communication Rule:

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_1 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_2} \ (com)$$

- $\mathbf{HFLe} + (com)$ proves (Prelinearity):

$$\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{\Rightarrow A \to B \mid \ \Rightarrow B \to A}\ (\to r)}{\Rightarrow (A \to B) \vee (B \to A) \mid \ \Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (EC)$$

Hypersequent Calculus

- Theorem (Avron 92):

$$\mathbf{HInt} + (com) = \mathbf{Int} + (Prelinearity)$$

It enjoys cut elimination.

- Which axiom corresponds to a hyperstructural rule?

$$\frac{G \mid \Upsilon_1' \Rightarrow \Psi_1' \quad \cdots \quad G \mid \Upsilon_n' \Rightarrow \Psi_n'}{G \mid \Upsilon_1 \Rightarrow \Psi_1 \mid \cdots \mid \Upsilon_m \Rightarrow \Psi_m}$$

Transformation into Hyperstructural Rule

- A subclass $\mathcal{P}'_n \subseteq \mathcal{P}_n$:

(P1') $\alpha \in \mathcal{N}_n \Longrightarrow \alpha \wedge 1 \in \mathcal{P}'_{n+1}$

(P2') $\alpha, \beta \in \mathcal{P}'_{n+1} \Longrightarrow \alpha \vee \beta, \alpha \cdot \beta, 1, \bot \in \mathcal{P}'_{n+1}$

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- Theorem (Cut Elimination):
  1. Every acyclic $\mathcal{P}'_3$-axiom is equivalent (in $\mathbf{HFLe}$) to hyperstructural rules enjoying cut elimination.
  2. Every $\mathcal{P}_3$-axiom is equivalent (in $\mathbf{HFLew}$) to hyperstructural rules enjoying cut elimination.

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  1. Every acyclic $\mathcal{P}'_3$-axiom is preserved by Hyper-DM completions.
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Example

- Weak nilpotent minimum (Esteva-Godo 01):

$$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$$

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- It is automatically generated.

Other Examples

In $\mathbf{HFLew}$,

$$\alpha \vee \neg\alpha \qquad \neg\alpha \vee \neg\neg\alpha$$

$$\downarrow \qquad \downarrow$$

$$\frac{G|\Gamma, \Delta \Rightarrow \Pi}{G|\Gamma \Rightarrow \ |\Delta \Rightarrow \Pi} \qquad \frac{G|\Gamma, \Delta \Rightarrow}{G|\Gamma \Rightarrow \ |\Delta \Rightarrow}$$

Hyper-DM completion

- Analogy: Canonical extension (Gehrke, Harding, . . .) can be factorized:

$$\mathbf{A} \quad \overset{\text{filters/ideals}}{\longrightarrow} \quad \mathbf{A}' \quad \overset{\text{DM completion}}{\longrightarrow} \quad \overline{\mathbf{A}'}$$

- Given a CRL $\mathbf{A}$, define an intermediate algebra

$$\mathbf{A}_H = (A^2, \sqsubseteq, \wedge, \vee, \cdot, \to, 0, 1, \top, \bot)$$

by:

$$(x, y) \sqsubseteq (z, w) \quad \text{if} \quad 1 \leq_{\mathbf{A}} (x \to z)_{\wedge 1} \vee y_{\wedge 1} \vee w_{\wedge 1}$$

- For $\star \in \{\wedge, \vee, \cdot, \to\}$,

$$(x, y) \star (z, w) = (x \star z, y \vee w)$$

Hyper-DM completion

- $(x, \bot) \sqsubseteq (y, \bot) \iff x \leq_{\mathbf{A}} y$.
- $\sqsubseteq$ is transitive and anti-symmetric only on $A \times \{\bot\}$, but it is residuated:

$$(x_1, x_2) \cdot (y_1, y_2) \sqsubseteq (z_1, z_2) \iff (x_1, x_2) \sqsubseteq (y_1, y_2) \to (z_1, z_2)$$

- Hyper-DM completion

$$\mathbf{A} \longrightarrow \mathbf{A}_H \longrightarrow \overline{\mathbf{A}}_H$$

- Theorem: $\mathcal{CRL}$ is closed under Hyper-DM completions.
- Analogously, one can consider hyper quasi-DM completions.

Main Results on $\mathcal{P}_3$

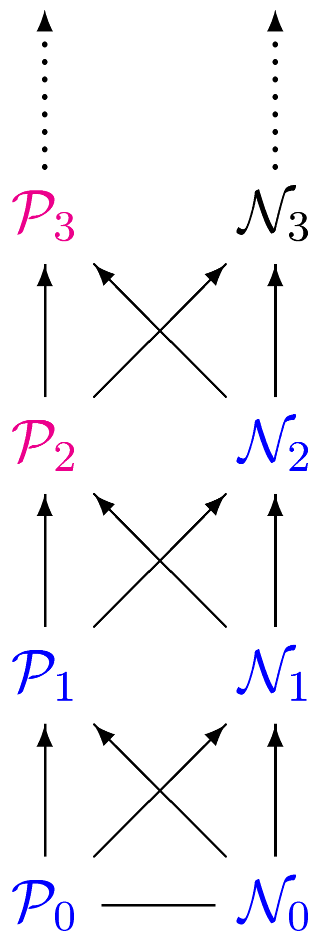
- Theorem:
  1. All acyclic $\mathcal{P}_3'$ axioms are equivalent in $\mathbf{HFLe}$ to hyperstructural rules enjoying cut-elimination.
  2. All $\mathcal{P}_3$ axioms are equivalent in $\mathbf{HFLew}$ to hyperstructural rules enjoying cut-elimination.
- Theorem:
  1. All acyclic $\mathcal{P}_3'$ axioms are preserved by hyper-DM completions.
  2. All $\mathcal{P}_3$ axioms are preserved by hyper-DM completions under the assumption of integrality.
- Examples: (Prelinear), (Weak EM), (Weak Nilpotent Minimum)

Open Problems

- Abstract characterization of Hyper-DM completions
- Is Hyper-DM optimal for $\mathcal{P}_3'$?
- Comparison with DM completions:

  cf. Theorem (Gehrke-Harding-Venema 05): Let $\mathcal{V}$ be a variety of monotone bounded lattice expansions (eg. subvarieties of bounded CRL). If $\mathcal{V}$ is closed under DM completions then is closed under canonical extensions.

$\mathcal{P}_3$ and Hyperstructural Rules



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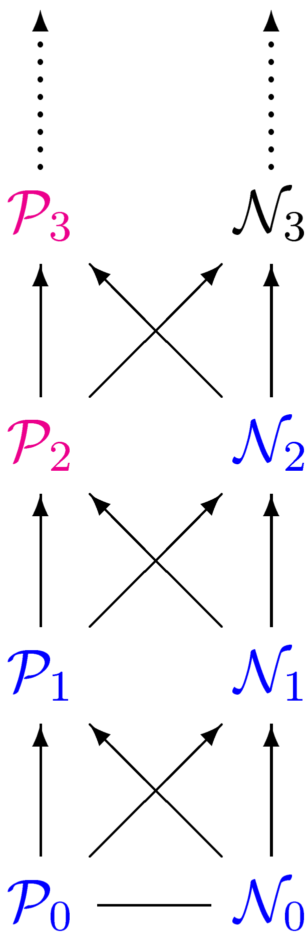

- Acyclic $\mathcal{P}_3'$ $\approx$ good hyperstructural rules
- Extended proof theory (hypersequent calculus) works nicely up to $\mathcal{P}_3'$ (Avron's paradise).
- How to go up higher?

Outline

1. Residuated Lattices and Full Lambek Calculus
2. Substructural Hierarchy
3. Class $\mathcal{N}_2$
4. Class $\mathcal{P}_3$
5. Beyond

Paradise Lost

- Theorem (Funayama 44): Distributivity $\in \mathcal{N}_3$ is not preserved by DM completions.
- However, it can be dealt with by canonical extensions or a variant of DM completions (cf. Kozak; Galatos-Jipsen, Friday).

Paradise Lost

- Theorem (cf. Kowalski-Litak 08): ℓ-groups, BL-algebras and MV-algebras are not closed under completions. Hence

(Inv)	$\alpha \cdot (\alpha \to 1) \leftrightarrow 1$	$\in \mathcal{P}_3$
(Divisibility)	$(\alpha \wedge \beta) \to \alpha \cdot (\alpha \to \beta)$	$\in \mathcal{N}_3$

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  are not conservative with respect to infinitary extensions.

- Absolute limitation: No possibility to obtain strong cut-elimination.

Nevertheless

- Theorem (Metcalfe-Olivetti-Gabbay 05): Abelian Logic (AL) and Łukasiewicz's Logic () can be formalized as hypersequent calculus which enjoys cut-elimination.
- Questions:
  1. Their calculi are discovered by trial and error. Is there a more systematic approach?
  2. Does it correspond to a weaker notion of completion?
- Our Approach:
  1. Shift to Classical Substructural Hierarchy
  2. Counterexample-driven Rule Generation

Abelian Logic

- $\mathsf{AL} = \mathbf{FLe} - \{\top, \bot, 0\} + (Inv)$:

$$\alpha \cdot (\alpha \rightarrow 1) \leftrightarrow 1$$

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Classical Substructural Hierarchy

$\mathcal{P}_0$=$\mathcal{N}_0$ = the set of literals $\alpha, \neg\alpha, \ldots$

$\mathcal{P}_{n+1}$ contains $\mathcal{N}_n$ and closed under $\cdot, \vee, 1, \bot$

$\mathcal{N}_{n+1}$ contains $\mathcal{P}_n$ and closed under $\parr, \wedge, \top, 0$

- Negation defined by: $\neg(A \vee B) = \neg A \wedge \neg B$, $\neg(A \cdot B) = \neg A \parr \neg B$,
- Implication defined by: $A \to B = \neg A \parr B$
- Crucial Fact: $(Inv) \in \mathcal{P}_3$ is equivalent to

$$0 \to 1, \quad A \parr B \to A \cdot B \quad \in \mathcal{N}_2$$

Inference Rules for AL I

$$\frac{\Gamma \Rightarrow \Lambda, A \quad A, \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{A, B, \Gamma \Rightarrow \Sigma}{A \cdot B, \Gamma \Rightarrow \Sigma}\ \cdot l \qquad \frac{\Gamma \Rightarrow \Lambda, A \quad \Delta \Rightarrow \Sigma, B}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma, A \cdot B}\ \cdot r$$

(rules for $\vee, \wedge, 1$)

$$\frac{\Gamma \Rightarrow \Lambda \quad \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Mix \qquad \frac{A, \Gamma \Rightarrow \Lambda, A}{\Gamma \Rightarrow \Lambda}\ Can$$

Not happy with (Can), which is cyclic.

Inference Rules for AL II

$$\frac{\Gamma \Rightarrow \Lambda, A \quad A, \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{A, B, \Gamma \Rightarrow \Sigma}{A \cdot B, \Gamma \Rightarrow \Sigma}\ \cdot l \qquad \frac{\Gamma \Rightarrow \Lambda, A, B}{\Gamma \Rightarrow \Lambda, A \cdot B}\ \cdot r$$

(rules for $\vee, \wedge, 1$)

$$\frac{\Gamma \Rightarrow \Lambda \quad \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Mix$$

- (Cf. Shirahata 1999)
- No cut-elimination. Counterexample: $A \vee \neg A$
- It is in $\mathcal{P}_1$, which can be transformed into a structural rule!

Inference Rules for AL III

$$\frac{\Gamma \Rightarrow \Lambda, A \quad A, \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Cut \qquad \frac{}{A \Rightarrow A}\ Identity$$

$$\frac{A, B, \Gamma \Rightarrow \Sigma}{A \cdot B, \Gamma \Rightarrow \Sigma}\ \cdot l \qquad \frac{\Gamma \Rightarrow \Lambda, A, B}{\Gamma \Rightarrow \Lambda, A \cdot B}\ \cdot r$$

(rules for $\vee, \wedge, 1$)

$$\frac{\Gamma \Rightarrow \Lambda \quad \Delta \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Mix \qquad \frac{\Gamma, \Gamma \Rightarrow \Lambda, \Lambda}{\Gamma \Rightarrow \Lambda}\ WC$$

- No cut-elimination. Counterexample: $(A \wedge 1) \vee (\neg A \wedge 1)$
- It is in $\mathcal{P}_3'$, which can be transformed into a hyperstructural rule!

Inference Rules for AL IV

$$\frac{G \mid \Gamma \Rightarrow \Lambda, A \quad G \mid A, \Delta \Rightarrow \Sigma}{G \mid \Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Cut \qquad \frac{}{G \mid A \Rightarrow A}\ Identity$$

$$\frac{G \mid A, B, \Gamma \Rightarrow \Sigma}{G \mid A \cdot B, \Gamma \Rightarrow \Sigma}\ \cdot l \qquad \frac{G \mid \Gamma \Rightarrow \Lambda, A, B}{G \mid \Gamma \Rightarrow \Lambda, A \cdot B}\ \cdot r$$

(rules for $\vee, \wedge, 1$) $\qquad (EW), (EC)$

$$\frac{G \mid \Gamma \Rightarrow \Lambda \quad G \mid \Delta \Rightarrow \Sigma}{G \mid \Gamma, \Delta \Rightarrow \Lambda, \Sigma}\ Mix \qquad \frac{G \mid \Gamma \Rightarrow \Lambda \quad G \mid \Delta \Rightarrow \Sigma}{G \mid \Gamma \Rightarrow \Lambda \mid \Delta \Rightarrow \Sigma}\ Split$$

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- Theorem (Gabbay-Metcalfe-Olivietti 05): The above system enjoys cut elimination.
- We can embed Ł into AL by $A \Rightarrow B = (A \to B) \wedge 1$. Hence we also obtain a hypersequent calculus for Ł.

Remark

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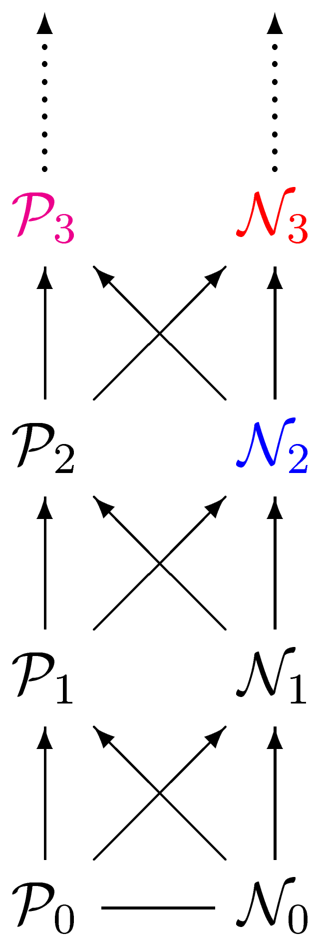
- What does hypersequent mean in terms of categories?
- Does weak cut elimination correspond to a weaker notion of completion?

Conclusion



- Sequent calculus and DM completion work up to $\mathcal{N}_2$.
- Hypersequent calculus and hyper-DM completion work up to $\mathcal{P}_3'$.
- Some absolute limitations are found in $\mathcal{P}_3$ and $\mathcal{N}_3$. But our theory is still useful to obtain partial results for each particular case.
- Is our theory useful for coherence problems in category theory?
- Is there another type of limitation coming from category theory?