

# Mixing Modality and Probability

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## My Main Question

What *are* individuals and structures in  
Modal Logic?

## A Possible Answer

*Lattice-valued models* give a wealth of examples of naturally defined types with well structured individuals and operations and relations on them. Experience with lattices (and sheaves) can then suggest further generalizations. In fact, we can extend the modeling to a *modal ZF* where every formula has a probability.

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## What is a Lattice?

$$0 \leq x \leq 1$$

$$x \leq x$$

$$x \leq y \ \& \ y \leq z \Rightarrow x \leq z$$

$$x \leq y \ \& \ y \leq x \Rightarrow x = y$$

$$x \vee y \leq z \Leftrightarrow x \leq z \ \& \ y \leq z$$

$$z \leq x \wedge y \Leftrightarrow z \leq x \ \& \ z \leq y$$

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## What is a Complete Lattice?

$$\bigvee_{i \in I} x_i \leq y \Leftrightarrow (\forall i \in I) x_i \leq y$$

$$y \leq \bigwedge_{i \in I} x_i \Leftrightarrow (\forall i \in I) y \leq x_i$$

**Note:**

$$\bigwedge_{i \in I} x_i = \bigvee \{y \mid (\forall i \in I) y \leq x_i\}$$

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## What is a Heyting Algebra?

$$x \leq y \rightarrow z \Leftrightarrow x \wedge y \leq z$$

## What is a Boolean Algebra?

$$x \leq (y \rightarrow z) \vee w \Leftrightarrow x \wedge y \leq z \vee w$$

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**Theorem:** Every Heyting algebra is *distributive*:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

**Theorem:** Every complete Heyting algebra is *completely distributive*:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

**Note:** The dual law does not follow for complete Heyting algebras.

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## What is a Lewis (S4) Algebra?

A Boolean algebra plus

$$\Box 1 = 1$$

$$\Box \Box x = \Box x \leq x$$

$$\Box (x \wedge y) = \Box x \wedge \Box y$$

The second two laws can be combined:

$$\Box x = \bigvee \{y \mid y = \Box y \leq x\}$$

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## Some Abbreviations

Ha = Heyting Algebra

cHa = Complete Heyting Algebra

Ba = Boolean Algebra

cBa = Complete Boolean Algebra

La = Lewis Algebra

cLa = Complete Lewis Algebra

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## What is a Frame?

**Definition.** A frame is complete lattice which is  $(\wedge \vee)$ -*distributive*.

**Theorem.** In a cLa the  $\Box$ -stable elements form a *subframe*.

**Theorem.** In a cBa *any* subframe creates a cLa.

**We can define:**  $\Box x = \bigvee \{y \in H \mid y \leq x\}$ ,  
where **H** is the subframe.

## An Important Theorem

**Theorem.** *Every* frame can be made into a cHa.

**Hint:**  $y \rightarrow z = \bigvee \{x \mid x \wedge y \leq z\}$

**Corollary.** In a cHa every subframe can be regarded as a cHa (but *not* with the same  $\rightarrow$ ).

Whence comes the **topological interpretation** of intuitionistic logic (Tarski/Stone).

## Topology vs. Probability

**Proposition.** For every topological space **X**, the *powerset*  $P(\mathbf{X})$  is a cBa, and the *lattice of open subsets*  $Op(\mathbf{X})$  is a cHa and a *subframe*.

**Note:** These examples include the **Kripke models**.

**Theorem.** For the standard *probability space*  $(\text{Borel}([0, 1]), \mu)$  with *Lebesgue measure*  $\mu$ , the *measure algebra*  $\text{Borel}([0, 1])/\text{Null}$  is a cBa, and the quotient  $Op([0, 1])/\text{Null}$  is a cHa and is a *proper subframe*.

**Note:** Call this cLa **M**. Think of it as a **pointless space**.

## First-Order Algebraic Semantics

$\langle\langle aRb \rangle\rangle = \text{given}$

$\langle\langle \Phi \wedge \Psi \rangle\rangle = \langle\langle \Phi \rangle\rangle \wedge \langle\langle \Psi \rangle\rangle$

$\langle\langle \Phi \vee \Psi \rangle\rangle = \langle\langle \Phi \rangle\rangle \vee \langle\langle \Psi \rangle\rangle$

$\langle\langle \Phi \rightarrow \Psi \rangle\rangle = \langle\langle \Phi \rangle\rangle \rightarrow \langle\langle \Psi \rangle\rangle$

$\langle\langle \Box \Phi \rangle\rangle = \Box \langle\langle \Phi \rangle\rangle$

$\langle\langle \exists x. \Phi(x) \rangle\rangle = \bigvee_{a \in A} \langle\langle \Phi(a) \rangle\rangle$

$\langle\langle \forall x. \Phi(x) \rangle\rangle = \bigwedge_{a \in A} \langle\langle \Phi(a) \rangle\rangle$

## Extension vs. Intension

$$\exists y [x = y \wedge \Phi(y)] \text{ vs. } \Box \Phi(x)$$

Different principles hold in different contexts:

$$\sigma = \tau \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

vs.

$$\Box[\sigma = \tau] \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

The prime example of an intensional mapping:

$$\Box \langle \Phi \leftrightarrow \Psi \rangle \leq \langle \Box \Phi \leftrightarrow \Box \Psi \rangle$$

## What is an L-Set?

**Definition.** An L-set is a set A equipped with an L-valued equality  $\langle x = y \rangle$ , where for all  $x, y, z \in A$

$$(i) \langle x = x \rangle = 1 ;$$

$$(ii) \langle x = y \rangle = \langle y = x \rangle ; \text{ and}$$

$$(iii) \langle x = y \rangle \wedge \langle y = z \rangle \leq \langle x = z \rangle .$$

**Note:** There is a useful notion of **complete L-set** and a process of **completion**.

**Note:** Mappings of L-sets can be either **extensional** or **intensional**.

## Boolean-valued Reals

**Theorem.** The set  $\mathbb{R}_{\mathbf{L}} = \{\alpha \mid \alpha: \text{Op}(\mathbb{R}) \rightarrow_{\text{frm}} \mathbf{L}\}$  can be made into a complete L-set by defining;

$$\langle \alpha = \beta \rangle = \bigwedge_{U \in \text{Op}(\mathbb{R})} (\alpha(U) \leftrightarrow \beta(U)).$$

**Theorem.** The frame  $\text{Op}(\mathbb{R} \times \mathbb{R})$  is the **frame-coproduct** of  $\text{Op}(\mathbb{R})$  with itself.

**Theorem.** Using  $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $(+) : \text{Op}(\mathbb{R}) \rightarrow_{\text{frm}} \text{Op}(\mathbb{R} \times \mathbb{R})$ , then for  $\alpha, \beta : \text{Op}(\mathbb{R}) \rightarrow_{\text{frm}} \mathbf{L}$  we have  $(\alpha, \beta) : \text{Op}(\mathbb{R} \times \mathbb{R}) \rightarrow_{\text{frm}} \mathbf{L}$ , and so we can define  $(\alpha + \beta) = (\alpha, \beta) \circ (+) : \text{Op}(\mathbb{R}) \rightarrow_{\text{frm}} \mathbf{L}$ .

**Note:** Other continuous functions can be handled in the same way. Many laws of algebra then follow automatically.

## Random Variables as Reals

**Theorem.** For the cLa  $\mathbf{M}$  we can identify

$$\mathbb{R}_{\mathbf{M}} = \{(f)/\text{Null} \mid (f): \text{Op}(\mathbb{R}) \rightarrow_{\sigma\text{-frm}} \text{Borel}([0, 1])\}$$

where the  $f: [0, 1] \rightarrow \mathbb{R}$  are measurable functions and  $(f)$  means inverse image; moreover, we can set:

$$\langle (f)/\text{Null} = (g)/\text{Null} \rangle = \{t \in \mathbb{R} \mid f(t) = g(t)\}/\text{Null}.$$

In this representation we find:

$$(f)/\text{Null} + (g)/\text{Null} = (f + g)/\text{Null}.$$

**Note:** We can similarly treat other measurable operations on the  $\mathbf{M}$ -valued reals.

## Intensional Powersets

**Definition:** Given a complete  $\mathbf{L}$ -set  $A$  the *intensional powerset* of  $A$  is the collection of  $P: A \rightarrow \mathbf{L}$  where, for all  $x, y \in A$ , we have  $P(x) \wedge \Box \langle x = y \rangle \leq P(y)$ .

And we use the definition

$$\langle P = Q \rangle = \bigwedge_{x \in A} (P(x) \leftrightarrow Q(x))$$

**Theorem:** The intensional powerset of  $A$  is a complete  $\mathbf{L}$ -set.

**Note:** A Principle of Comprehension follows.

**Question:** Should we be able to iterate this notion of powerset?

## A Modal Boolean-valued Universe

$$V^{(\mathbf{L})} = \{v: \text{dom } v \rightarrow \mathbf{L} \mid \text{dom } v \subseteq V^{(\mathbf{L})} \ \& \ \forall x, y \in \text{dom } v. v(x) \wedge \Box \langle x = y \rangle \leq v(y)\}$$

$$\langle u \in v \rangle = \bigvee \{v(y) \wedge \Box \langle u = y \rangle \mid y \in \text{dom } v\}$$

$$\langle u = v \rangle = \bigwedge \{u(x) \rightarrow \langle x \in v \rangle \mid x \in \text{dom } u\} \wedge \bigwedge \{v(y) \rightarrow \langle y \in u \rangle \mid y \in \text{dom } v\}$$

**The new insight:**

intensional  
 $\swarrow$   
 $u \in v$   
 $\nwarrow$   
extensional

## What is MZF?

### Substitution

$$\Box [u = v] \wedge \Phi(u) \rightarrow \Phi(v)$$

### Extensionality & Comprehension

$$\forall u, v [u = v \leftrightarrow \forall x [x \in u \leftrightarrow x \in v]]$$

$$\forall u \exists v \forall x [x \in v \leftrightarrow x \in u \wedge \Phi(x)]$$

### Singleton

$$\forall u \exists v \forall x [x \in v \leftrightarrow \Box [x = u]]$$

### Leibnitz' Law

$$\forall x, y [[\Box [x = y] \leftrightarrow \forall u [x \in u \rightarrow y \in u]]]$$

### Definable Modality

$$\{\emptyset\} = \{\emptyset \mid \Phi\} \leftrightarrow \Phi$$

$$\Box \Phi \leftrightarrow \forall u [\{\emptyset\} \in u \rightarrow \{\emptyset \mid \Phi\} \in u]$$

## Two Membership Relations?

### Extensional Membership

$$u \in v \leftrightarrow \exists y [y \in v \wedge u = y]$$

### Extensional Comprehension

$$\forall u \exists v \forall x [x \in v \leftrightarrow x \in u \wedge \exists y [\Phi(y) \wedge x = y]]$$

### Extensional Singleton

$$\forall u \exists v \forall x [x \in v \leftrightarrow x = u]$$

### Extensional Leibnitz' Law

$$\forall x, y [x = y \leftrightarrow \forall u [x \in u \rightarrow y \in u]]$$

### Intensional Powerset

$$\forall v \exists w \forall u [u \in w \leftrightarrow \Box [u \subseteq v]]$$

### Extensional Powerset

$$\forall v \exists w \forall u [u \in w \leftrightarrow u \subseteq v]$$

## Some Problems

**Note:** In general these are non-reversible implications:

$$\Box[x = y] \rightarrow [x = y] \rightarrow \Diamond[x = y]$$

So, are these statements sometimes **refutable**?

$$\forall u \exists v \forall x [x \in v \leftrightarrow x = u]$$

$$\forall v \exists w \forall u [u \in w \leftrightarrow u \subseteq v]$$

$$\forall u \exists v \forall x [x \in v \leftrightarrow \Diamond[x = u]]$$

**Note:** In the measure-algebra model using  $M$ , every formula of set theory has a **probability**. Owing to the automorphisms of  $M$ , every **pure statement** without free variables thus has truth value either 0 or 1.

## A Refutation

**Theorem.** In  $V^{(M)}$  the following has truth value 0:

$$\forall u, v [u = v \leftrightarrow \forall x [x \in u \leftrightarrow x \in v]].$$

**Proof:** Find  $p \in M$  with  $0 < p < 1$  and  $\Box p = 0$ . (**How?**)

Let  $a = \{\emptyset\}$  and  $b = \{\emptyset \mid p\}$ , and  $u = \{a \mid p\}$  and  $v = \{b \mid p\}$ .

We have  $\langle a = b \rangle = p$ , and  $\langle a \in u \rangle = p$  and  $\langle a \in v \rangle = 0$ .

It follows that  $\langle u = v \rangle = \neg p$ . We also calculate that

$$\langle x \in u \rangle = \langle x = a \rangle \wedge p \text{ and } \langle x \in v \rangle = \langle x = b \rangle \wedge p.$$

But then  $\langle x \in v \rangle = \langle x = a \rangle \wedge p$  as well. From this we get:

$$\langle u = v \leftrightarrow \forall x [x \in u \leftrightarrow x \in v] \rangle = \langle u = v \rangle = \neg p.$$

The conclusion of the theorem follows by the 0-1 Law.

## Random Numbers?

Alex Simpson (Edinburgh) has argued that the quotient frame  $Op([0,1]) \rightarrow_{\text{frm}} Op([0,1])/\text{Null}$  (N.B. another **pointless space**) can be considered as modeling **random reals**.

**Note:** This space has many  $M$ -valued points, and it can be taken as a subset of  $\mathbb{R}_M$ . In fact, we can identify

$$\text{Rand}_M = \{ (f)/\text{Null} \mid f: [0, 1] \rightarrow_{\text{meas}} [0, 1] \text{ \& } \{ t \in \mathbb{R} \mid f(t) \in N \} \in \text{Null for all } N \in \text{Null} \}.$$

**Question:** Do the random reals have interesting modal properties?

## Applying Ergodic Theory?

**Note:** In the measure-algebra model of MZF, every automorphism of  $L$  preserving the modal operator induces an automorphism of the whole universe.

**Furstenberg's Multiple Recurrence Theorem:**

Let  $\tau$  be a measure-preserving modal automorphism of  $L$ , and let  $\langle \Phi(a) \rangle \neq 0$ , then for all  $k$  there exists an  $n$  such that  $\langle \Phi(a) \wedge \Phi(\tau^n(a)) \wedge \Phi(\tau^{2n}(a)) \wedge \Phi(\tau^{3n}(a)) \wedge \dots \wedge \Phi(\tau^{kn}(a)) \rangle \neq 0$ .