

Some logical aspects of Priestley duality

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(Outline of the talk presented by A. Pultr)

1 Prime filters in distributive lattices

After learning that

- prime filters in Boolean algebras coincide with the maximal ones,
- and also in a general distributive lattices *regular* prime filters are trivially ordered,

one may get the impression that the inclusion order of prime filters are perhaps not of much interest (certainly I had been so misguided when first encountering the topic). But in fact,

the order structure of prime filters is universally rich.

That is,

each finite poset can be represented as the set of *all* prime filters of a finite distributive lattice,

and general posets can be at least embedded into the lattice of all prime filters of a distributive lattice.

This is where the business of Priestley duality starts.

2 Priestley duality

Priestley ([14] 1970, [15] 1972). This duality is a contravariant equivalence of the category of Priestley spaces (that is, ordered compact Hausdorff spaces,

order-totally-disconnected) and the category of bounded distributive lattices. It can be described by the functors

$$\mathcal{D} : \mathbf{PSp} \rightarrow \mathbf{DLat}^{\text{op}}, \quad \mathcal{P} : \mathbf{DLat} \rightarrow \mathbf{PSp}^{\text{op}}$$

defined by

$$\begin{aligned} \mathcal{D}(X) &= \{U \mid U \subseteq X \text{ clopen decreasing}\}, \\ \mathcal{D}(f)(U) &= f^{-1}[U], \\ \mathcal{P}(L) &= (\{x \mid x \subseteq L \text{ prime ideals}\}, \text{suitable topology}), \\ \mathcal{P}(h)(x) &= h^{-1}[x]. \end{aligned}$$

We will be interested in confronting the “shapes” on the **geometric** (topological) side, and formulas on the **algebraic** one.

The shapes will be finite subposets; if such a finite poset is order connected we speak of a *configuration*.

On the algebraic side we will be interested in the classes of lattices obtained by prohibiting (or requiring) that or other configuration in the associated Priestley space (the connectedness plays just a technical role).

A natural question: Why, in this context, the topology? Do we need it? Surprisingly: YES, very much so, as it will be apparent later.

A few observations. $A \not\hookrightarrow B$ indicates that A cannot be imbedded into B as an induced subposet.

(a) Stone duality is the “flat” part of Priestley duality. This can be expressed by stating that

$$\{0 < 1\} \not\hookrightarrow X \quad \text{iff} \quad \mathcal{D}(X) \text{ is Boolean.}$$

(b) A result by Monteiro ([13] 1974, see also [12]) can be reformulated as

$$\text{For the } V\text{-shape } \{0 < 1, 2\}, \quad V \not\hookrightarrow X \quad \text{iff} \quad \mathcal{D}(X) \text{ is relatively normal,}$$

that is, if it has the property that

$$\forall a_1, a_2 \exists c_1, c_2, \quad a_1 \wedge c_1 = 0, \quad a_2 \wedge c_2 \leq a_1 \vee c_1, \quad a_2 \vee c_2 = 1.$$

(c) (Adams and Beazer, [1] 1991)

$$\{0 < 1 < \dots < n\} \not\hookrightarrow X \quad \text{iff} \quad \forall (a_0, \dots, a_{n-1}) \text{ in } \mathcal{D}(X) \exists (c_0, \dots, c_{n-1}) \text{ s.t.}$$

$$a_0 \wedge c_0 = 0,$$

$$a_k \wedge c_k \leq a_{k-1} \vee c_{k-1} \quad \text{for } 0 < k \leq n, \quad \text{and } c_n = 1$$

The main topic of the talk is the question:

Given a configuration P , how does the class of the distributive lattices corresponding to the spaces X satisfying the condition $P \dashv \vdash X$, look like?

In the mentioned examples, the prohibitions resulted in first order theories (in fact, axiomatizable ones).

Is it a general phenomenon ?

If not, how general it is ?

We will deal, first, with topped configurations.

3 Topped trees

(here: the Computer Science trees: rooted, and upside down.)

Let T be a tree with top t_0 . A T -complement of a mapping (any mapping, not necessarily monotone) $a : T \rightarrow L$ is a $c : T \rightarrow L$ such that

$$\begin{aligned} a(t) \wedge c(t) &\leq a'(t) = \bigvee \{a(\tau) \mid t \not\leq \tau\} \quad \text{for } t \in \min(T), \\ a(t) \wedge c(t) &\leq \bigvee_{\tau \prec t} a(\tau) \vee c(\tau) \quad \text{for } t \in T \setminus \min(T), \\ \text{and } c(t_0) &= 1. \end{aligned}$$

Proposition. ([2] 2004) *For T a finite forest, $T \dashv \vdash \mathcal{P}(L)$ if and only if each $a : T \rightarrow L$ has a T -complement.*

4 Coproducts in the category of Priestley spaces

To get any further we have to turn to another problem (seemingly not quite closely connected).

Observe that

the category **PSp** has coproducts.

This may be not obvious at the first sight, but by the duality it is: the category **DLat** has products. Now how do the coproducts look like? Because of the compactness they cannot be simply the disjoint sums. They have to be a sort of compactification of such sums. Now there are new points somehow intervoven into the order. The possible new configurations will be in the focus of our interest. Do there occur some surprising new ones?

How much the situation differs, if at all, from that of finite coproducts, or general coproducts of posets (or, for that matter, ordered spaces)? Is it true that a configuration P (recall it is connected) occurs in $\coprod X_i$ only if it occurs in some of the individual X_i 's? The P 's for which this is true will be called

coproductive.

Although coproducts of Priestley spaces are still not very well understood there is a description (Koubek and Sichler, [11] 1991) that will be of a help. One has

- $\coprod_{i \in J} X_i = \bigcup_{u \in \beta J} X_u$ disjoint and order independent,
- $X_u = \mathcal{P}(\prod_u \mathcal{D}(X_i))$ where $\prod_u X_i$ is the ultraproduct.

Thus, our question reduces to possible new configurations in the duals of ultraproducts.

5 Main results on the topped case

We have the following ([3] 2006)

Theorem. *Let P be a configuration with top. Then TFAE.*

- (1) P is acyclic.
- (2) P is coproductive.
- (3) $\mathbf{Forb}(P)$ is first order definable.
- (4) $\mathbf{Forb}(P)$ is axiomatizable.

Proof. (1) \Rightarrow (4): the T -complement formulas from 3.

(4) \Rightarrow (3) is trivial.

(3) \Rightarrow (2): Using the fact that X_u are duals of ultraproducts, Łoś theorem.

(2) \Rightarrow (1): A construction creating a new configuration in a coproduct.

For Heyting algebras one has ([2])

Theorem. *Let P be a configuration with top. Then TFAE.*

- (1) P is acyclic.
- (2) P is coproductive.
- (3) $\mathbf{Forb}(P)$ is closed under products.
- (4) $\mathbf{Forb}(P)$ is first order definable.
- (5) $\mathbf{Forb}(P)$ is axiomatizable.
- (6) $\mathbf{Forb}(P)$ is a quasivariety.
- (7) $\mathbf{Forb}(P)$ is a variety.

Remark. It should be noted that the equivalences (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (5) \rightarrow (6) \rightarrow (7) \rightarrow (1) seem to be well known. They can be proved without using the coproducts and Łoś theorem.

One also knows something on prohibiting more than one configuration ([7] 2008).

Theorem. *TFAE for a class \mathbb{P} of configurations.*

- (1) $\mathbf{Forb}(\mathbb{P})$ is a quasivariety.
 - (2) $\mathbf{Forb}(\mathbb{P})$ is a variety.
 - (3) For every $P \in \mathbb{P}$ and for every cover Q of P there is a $P' \in \mathbb{P}$ such that $P' \hookrightarrow Q$.
- (And such \mathbb{P} does not have to consist of trees only.)

6 The not necessarily topped case

6.1. Combinatorial trees, that is, acyclic configurations. In our context one cannot view posets as (ordered) graphs with the relation \leq or $<$ giving the arrows. $\{0 < 1 < 2\}$ is acyclic although it contains a triangle. The trivial cycles following from transitivity have to be disregarded.

In a configuration P write $x \succ y$ if x is the immediate predecessor or immediate successor of y and consider the *Hasse graph* (P, \succ) of P . P is *cyclic* resp. *acyclic* if such is the corresponding Hasse graph.

Again, as in the topped case we have

Proposition. ([4] 2005) *Let T be acyclic. Then there is a first order formula \mathcal{T} in the language of bounded distributive lattices such that $T \dashv\vdash L$ iff $L \models \mathcal{T}$.*

The formulas, however, are by far not so transparent as before:

For n natural, $a : T \rightarrow L$, $t \in T$ and $c \in L$ define, inductively,

$$\begin{aligned} \mathcal{A}(0, a, t, c) &\equiv c \leq \bigvee_{s \not\geq t} a(s), \\ \mathcal{B}(0, a, t, c) &\equiv c \geq a(t), \\ \mathcal{A}(n+1, a, t, c) &\equiv \\ \exists j_s \mathcal{A}(n, a, s, j_s) \exists f_s \mathcal{B}(n, a, s, f_s) \bigwedge_{s \geq t} f_s \wedge c &\leq \bigvee_{s \leq t} j_s \\ \mathcal{B}(n+1, a, t, c) &\equiv \\ \exists j_s \mathcal{A}(n, a, s, j_s) \exists f_s \mathcal{B}(n, a, s, f_s) \bigwedge_{s \geq t} f_s &\leq c \vee \bigvee_{s \leq t} j_s \end{aligned}$$

Then choose $t_0 \in T$ and if d is the largest distance of t from t_0 in (T, \succ) .

$$\mathcal{T} \equiv \forall a : T \rightarrow L, \mathcal{B}(d+1, a, t_0).$$

6.2. The cyclic case. The fact that we have dismissed the cycles created by the transitivity did not play much role in the topped case.

There, P was cyclic iff it contained a diamond. In the general case the situation is more complicated:

P is cyclic iff it contains, as an induced subposet,

- *either a diamond,*
- *or an n -crown with $n \geq 3$*
- *or a proper 2-crown.*

(A crown $\{a_1, a_2 < b_1, b_2\}$ in P is proper if there is no $x \in P$ such that $a_1, a_2 < x < b_1, b_2$.)

We encounter very soon a phenomenon that one *does not* have in graphs:

a subposet of an acyclic poset can be cyclic.

(Take the X -shape $\{a, b < c < e, f\}$; removing c one obtains the 2-crown.) In fact, this particular anomaly did not, after all, create much trouble. But the 2-crown itself did.

It is conjectured that there will be a similar theorem like that on the topped case. So far, however, there is no conclusive proof.

We have that

- prohibition of no P containing diamond makes for a first order theory ([6] 2007),
- neither does a prohibition of a crown ([5] 2005, [6] 2006),
- and there is a very messy more general theorem that does not cover everything, but quite a lot; if you draw a random cyclic poset you can be practically sure it will be in the class ([6] 2007);
- and, of course, we have no counterexample.

6.3. Summary. All that is decided holds in parallel with the topped case, and it is conjectured that the general theorem holds.

Furthermore, there are other phenomena (for which we do not have the time here) of quite analogous proved facts.

BUT: The technical aspects are surprisingly different. The difference in complexity of the parallel proofs is somehow much bigger than what one would expect of a (natural) generalization. For instance, at a moment of a construction where in the topped case practically any sufficiently dense bipartite graph gave satisfactory results, for the 2-crown we needed a very deep result by Noga Alon on projective planes (and it had looked originally even worse: our first proof had been based on the Erdős conjecture on Sidon numbers).

There is one exception in which the necessary and sufficient condition has been equally difficult, or easy,, namely extending surjective monotone maps to retractions ($f : Y \rightarrow P$, where P is a configuration and Y an infinite poset, that is not a retraction can be extended to a retraction $\hat{f} : \hat{Y} \rightarrow P$ where \hat{Y} is the Priestley modification of Y , iff P is cyclic – [10]).

(The conjecture concerns the Diophantine equations $x^k + y^k = u^k + v^k$, and it seems to be in the class of the Fermat theorem.)

7 Problem of prohibiting general configurations. Diamonds

Prohibiting the diamond $D = \{0 < a, b < 1; a \neq b\}$, by one of the Theorems above, does not create a first order class of distributive lattices. Yet one has (BP 2004)

Proposition. $\mathcal{P}(L)$ contains no diamond iff

$$\forall a_1, a_2 \in L \exists k \in \mathbb{N} \exists b_1, \dots, b_k, c_{11}, \dots, c_{1k}, c_{21}, \dots, c_{2k} \text{ such that}$$

$$c_{1j} \vee c_{2j} = 1, c_{1j} \wedge a_1 \leq b_j \vee a_2, c_{2j} \wedge a_2 \leq b_j \vee a_1 \text{ and } b_1 \wedge \dots \wedge b_k = 0.$$

The formulas depart from first order by the quantifying $\exists k \in \mathbb{N}$ (and, of course, by “parallelling” the a ’s, b ’s and c ’s by the floating indices). But in a Priestley duality restricted to spaces of uniformly finitely bounded width (which itself is a first order condition – it will be explained) they become first order. Consequently, for instance, one learns that a diamond cannot come into existence in a coproduct $\coprod X_i$ of diamond-less X_i of bounded width.

It should be noted that the width is a first order property, connected with prohibiting forests, for which we have an extension of the proposition on trees as in 3. Prohibiting of antichain produces the formula “for each a_1, \dots, a_n there is a k such that $\bigwedge_{i \neq k} a_i \leq a_k$ ”. This in combination with formulas like the one about diamond gives a first order theory for the fragment of the Priestley duality restricted to give finite width.

Formulas akin to those in the proposition above are known for some other configurations as well (including the “chinese lanterns”, and many more) and the fact may be fairly general. However, for instance the crowns have so far resisted all efforts.

Extras

(If there is some time left)

E1. A coproduct of acyclic spaces is acyclic. This is another example of the phenomenon mentioned in 6.3: it is very easy in the topped case, and not so easy in the general one. Moreover, in the general case we have a proof for the coproduct of finite spaces only ([8]).

E2. In Koubek and Sichler [11] it was proved that the topology of a coproduct is a Stone-Čech compactification of the topological sum iff there is a natural n such that the height of all but finitely many summands is $\leq n$.

The exam[pl]es of “combinatorially wild” coproducts above are all low in this sense, and hence “topologically tame”.

We have investigated the sum of increasing finite chains. It is acyclic, of course, and all the summands, including the X_u with free ultrafilter u , are linearly ordered. But they are not really very simple. We have

Proposition. ([9]) *Under (CH), all the X_u with free u are isomorphic to the Dedekind-MacNeille completion of*

$$\mathbb{N} \oplus (S \vec{\times} \mathbb{Z}) \oplus \mathbb{N}^{\text{op}}$$

where S is the η_1 -set of cardinality \aleph_1 .

(η_1 -set: a chain C of uncountable cofinality and coinitality s.t. for any two countable subsets $A < B$ there is an intermediate c , $A < c < B$. This concept was introduced by Hausdorff.)

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