On domain algebras

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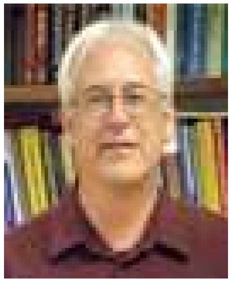
Collaborators



M. Andrew Moshier
Chapman University

Steven Vickers
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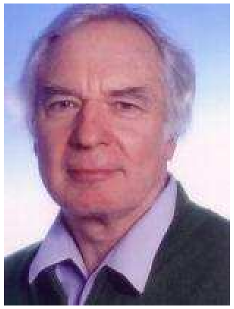

Klaus Keimel
Darmstadt University

Jimmie Lawson
Louisiana State University

I. Motivation and Examples

II. The Main Results

III. Applications

IV. Technical Details

V. The Topological View and Generalisation

VI. Open Problems

Presentations

- Groups: $G = \{a, b, c\}$, $a^2 = b^2 = e$, $ab = ba$

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Question: Can the same be done for dcpos?

Two problems: "Directed supremum" has unbounded arity and is partial.

Free complete lattices

Theorem. [Hales, 1964]
The free complete lattice on three generators does not exist.

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Theorem. [Bénabou, 1958]
The free frame on any set of generators exists.

Theorem. [Johnstone, 1972]
Any presentation by generators and relations (a site) defines a frame.

Free complete lattices — cont'd

Note that

$$\bigvee \; = \; \bigvee^{\uparrow} \vee$$

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and we will argue that free frames exist because they can be constructed as dcpo completions of free lattices.

The bitopological view of Stone Duality

A careful analysis of the various classical Stone dualities shows that the logical operations

$$\vee \ , \ \wedge \ , \ tt \ , \ f\!f$$

are orthogonal to the approximation of predicates by partial predicates:

$$\bot \ , \ \bigsqcup^{\uparrow}$$

(Jung and Moshier, 2006)

Denotational Semantics

Since Dana Scott's seminal 1969 paper, A type-theoretical alternative to CUCH, ISWIM, OWHY, denotational semantics of programming languages has employed directed-complete partial orders (dcpo s) to construct meanings for languages with iteration and recursion.

In order to model other constructions of programming languages, one has to augment dcpos with algebraic operations.

Example (Hennessy and Plotkin, 1979): Powerdomains are free dcpo algebras with respect to an operation $+$ subject to equations

$$\begin{aligned} x + x &= x & \qquad [[x + y \sqsubseteq x]] \\ x + (y + z) &= (x + y) + z & \qquad [[x + y \sqsupseteq x]] \\ x + y &= y + x \end{aligned}$$

Domain and dcpo algebras

Definition. *A domain is a continuous dcpo, i.e., a dcpo in which for all x,*

$$x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x$$

Theorem. [Jung and Abramsky, 1994]
Free domain algebras over domains exist for all finitary signatures and systems of inequalities.

Proof. A nice concrete (but non-trivial) construction via abstract bases.

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Theorem. *Free dcpo algebras over domains exist for all finitary signatures and systems of inequalities.*

Proof. An application of Freyd's Adjoint Functor Theorem.

Order vs. topology

Every T_0-topological space (X, τ) carries an order, called specialization order, defined by

$$x \leq_\tau y \quad :\Leftrightarrow \quad x \in \mathrm{cl}\{y\}$$

On any ordered set $(P, \leq)$ one can define the Scott topology $\sigma_\leq$: $A \subseteq P$ is closed if $A = {\downarrow}A$ and A is closed under existing directed suprema.

Proposition.

- *For any sober topological space* (X, τ), $\tau \subseteq \sigma_{\leq_\tau}$
- *For any ordered set* $(P, \leq)$, $\quad \leq \, = \, \leq_{\sigma_\leq}$
- *For any dcpo,* $\quad \sigma = \sigma_{\leq_\sigma}$

Dcpo cones

(Lawson, Saheb-Djahromi, Jones, Plotkin, Kirch, Tix, Keimel, Heckmann, and many others)

Borel measures on dcpos can be replaced (usefully and meaningfully) by valuations, and in many cases a one-to-one correspondence can be shown.

Valuations can be presented as algebras with respect to addition and scalar multiplication with non-negative scalars, subject to a suitable adaptation of vector space axioms.

In addition, the set of valuations carries a dcpo order. Together, this is an example of a dcpo-ordered cone.

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DCPO presentations

Definition. *A* dcpo presentation *consists of*

- *a set* P *of* generators
- *a* preorder $\precsim$ *on* P
- *a subset* C *of* $P \times \mathcal{P}P$*, whose elements are called* covers *and written* $[a \lhd U]$*, subject to the requirement that* U *is directed with respect to* $\precsim$

Write this as a triple $(P; \precsim, C)$.

The category DCPO-pres of DCPO presentations

objects are dcpo presentations

morphisms $f\colon (P; \precsim, C) \to (P'; \precsim', C')$ are

monotone maps from $(P; \precsim)$ to $(P'; \precsim')$ such that

$$[a \lhd U] \ \in C \qquad \Rightarrow \qquad [fa \lhd fU] \ \in C'$$

From DCPO to DCPO presentations

For every dcpo D we can define a canonical presentation by setting

- generators: D
- preorder: $\sqsubseteq$
- covers: all $[a \lhd U]$ where $a \sqsubseteq \bigsqcup^{\uparrow} U$

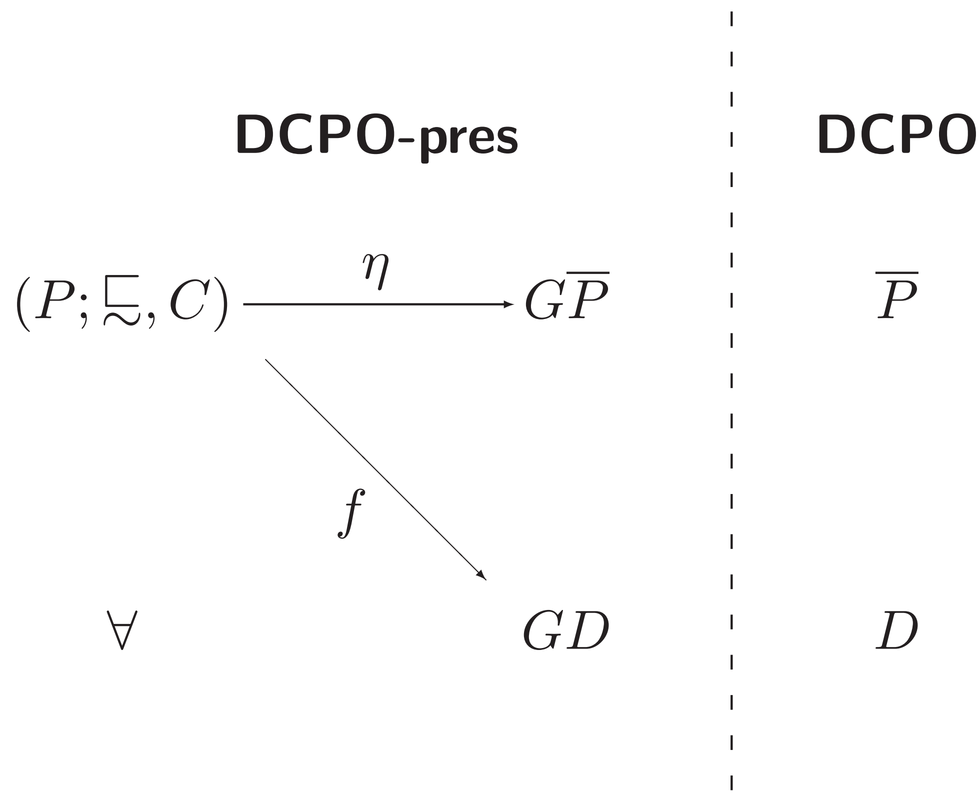
This defines a functor G from **DCPO** to **DCPO-pres**.

The universal property

DCPO-pres | **DCPO**

$$(P; \precsim, C) \xrightarrow{\eta} G\overline{P} \qquad \overline{P}$$

The universal property



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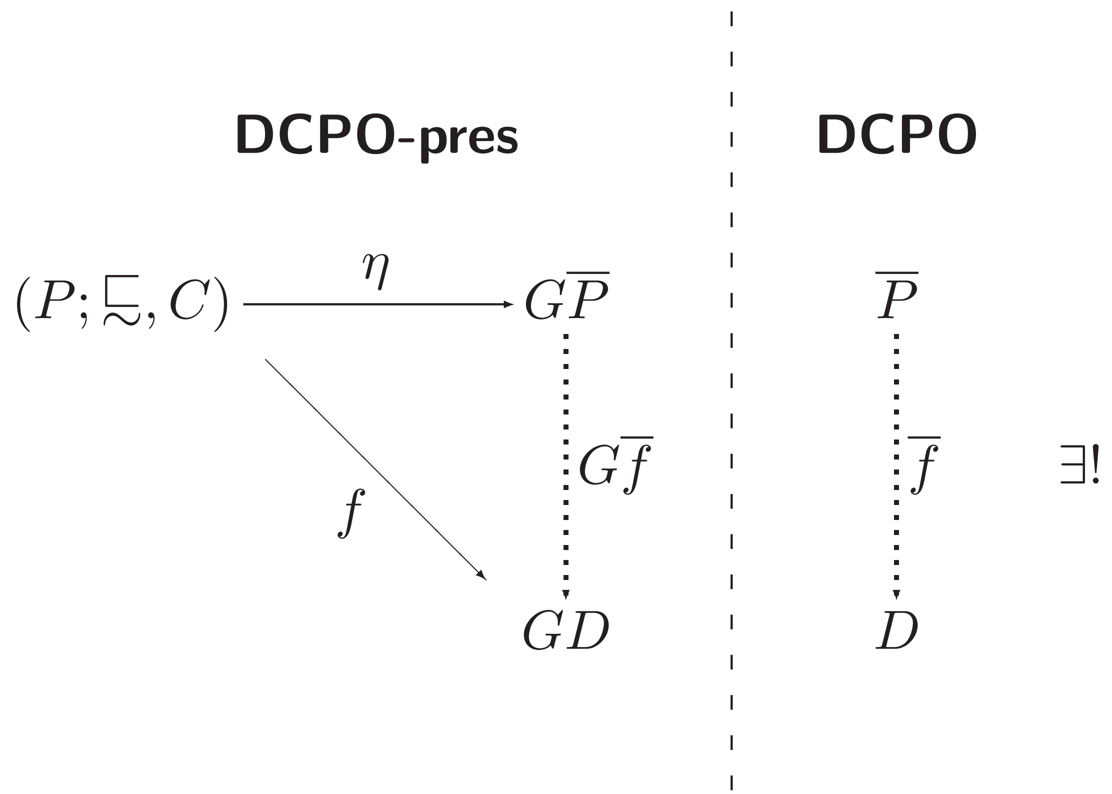
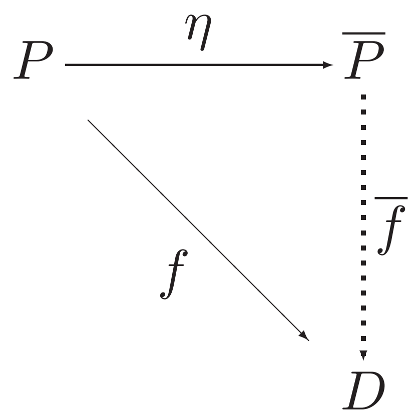


The universal property — simplified



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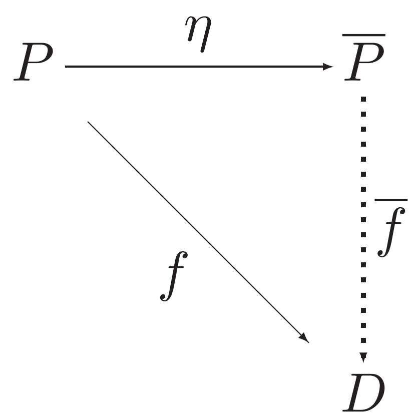

The maps η and f convert covers to joins in the sense that for every $[a \lhd U]$ in C

$$\eta a \sqsubseteq \bigsqcup^{\uparrow} \eta U \quad \text{and} \quad f a \sqsubseteq \bigsqcup^{\uparrow} f U$$

Theorem 1

DCPO presentations present.

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or

The functor G has a left adjoint.

Algebras

- Ω: a set of operation symbols
- $\alpha\colon \Omega \to \mathbb{N}$: an arity function (i.e., each operation is of finite arity)
- A preordered algebra w.r.t. Ω is given by
  - a preorder P as the carrier, and
  - order-preserving operations $\omega_P\colon P^{\alpha(\omega)} \to P$

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- A dcpo algebra w.r.t. Ω is given by
  - a dcpo D as the carrier, and
  - Scott-continuous operations $\omega_D\colon D^{\alpha(\omega)} \to D$

Completing preordered algebras

Definition. *A dcpo algebra presentation consists of:*

- *a set P of generators*
- *a preorder $\sqsubseteq$ on P*
- *an order-preserving operation ω_P for each $\omega \in \Omega$*
- *a set C of covers, subject to the stability condition*

$$[a \lhd U] \in C \qquad \Rightarrow \qquad [\omega_P(-, a, -) \lhd \omega_P(-, U, -)] \in C$$

Theorem 2

Given a dcpo algebra presentation $(P; \Omega_P, \sqsubseteq\!\!\!\sim, C)$ let $\overline{P}$ be the dcpo presented by the reduct $(P; \sqsubseteq\!\!\!\sim, C)$ (according to Theorem 1).

Then $\overline{P}$ carries Scott-continuous operations $\overline{\omega}$ for each $\omega \in \Omega$, i.e., it is an dcpo Ω algebra. Furthermore, $P \to \overline{P}$ is universal.

Algebras: inequations

- An inequation is given by a pair of Ω terms over a set X of variables, written as $t_1 \sqsubseteq t_2$.

- An inequation $t_1 \sqsubseteq t_2$ is valid in a preordered algebra D if the associated term functions satisfy $f_{t_1} \precsim f_{t_2}$.

Theorem 3

Given a dcpo algebra presentation $(P; \Omega_P, \precsim, C)$, the free algebra $(\overline{P}; \Omega_{\overline{P}}, \sqsubseteq)$ constructed in Theorem 2 satisfies all the inequations satisfied by P.

Summary

Theorem 1.
Every dcpo presentation can be "completed" to a dcpo in a universal way.

Theorem 2.
Algebraic operations "lift" to the completion.

Theorem 3.
The lifting of operations preserves all (in)equations.

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but

$$E/\approx \text{ is not a dcpo, in fact, not even preordered}$$

Colimits in DCPO — cont'd

The problem is resolved by using a suitable dcpo presentation:

generators: elements of E

covers: all pairs $(fx, \{gx\})$ and $(gx, \{fx\})$

all pairs (a, U) where $a \sqsubseteq \bigsqcup^{\uparrow} U$ in E.

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Our results give an alternative (and more concrete) construction:

- Construct the free *preordered algebra* over the given dcpo considered as a preordered set. (This is easy: Take the term algebra and construct the smallest congruence-preorder that contains the order on the generators and all instances of inequations.)

- Then complete the preorder using the technique from Theorem 1, encoding the directed suprema of the given dcpo in the covers.

Free frames

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Stage 2: Normalize the given frame inequalities into the form

$$s \leq \bigvee^{\uparrow} t_i$$

where s, t_i are elements of $D(G)$.
This uses frame distributivity and the split of $\bigvee$ into $\bigvee^{\uparrow}$ and $\vee$.

Free frames — cont'd

Stage 3: Set up the dcpo algebra presentation

$$(D(G); \leq, \{\vee, \wedge, 0, 1\}, C)$$

where C contains the cover $[s \lhd \{t_i \mid i \in I\}]$ for each inequation $s \leq \bigvee^{\uparrow} t_i$.

The resulting dcpo algebra is the desired frame, because...

Free frames — cont'd

The extended lattice operations are still distributive, by Theorem 3. They also distribute over $\bigvee^{\uparrow}$ because they are Scott-continuous by Theorem 2. Together this amounts to frame distributivity.

The dcpo order coincides with the order derived from the extended lattice operations; show that $\wedge$ computes the largest lower bound:

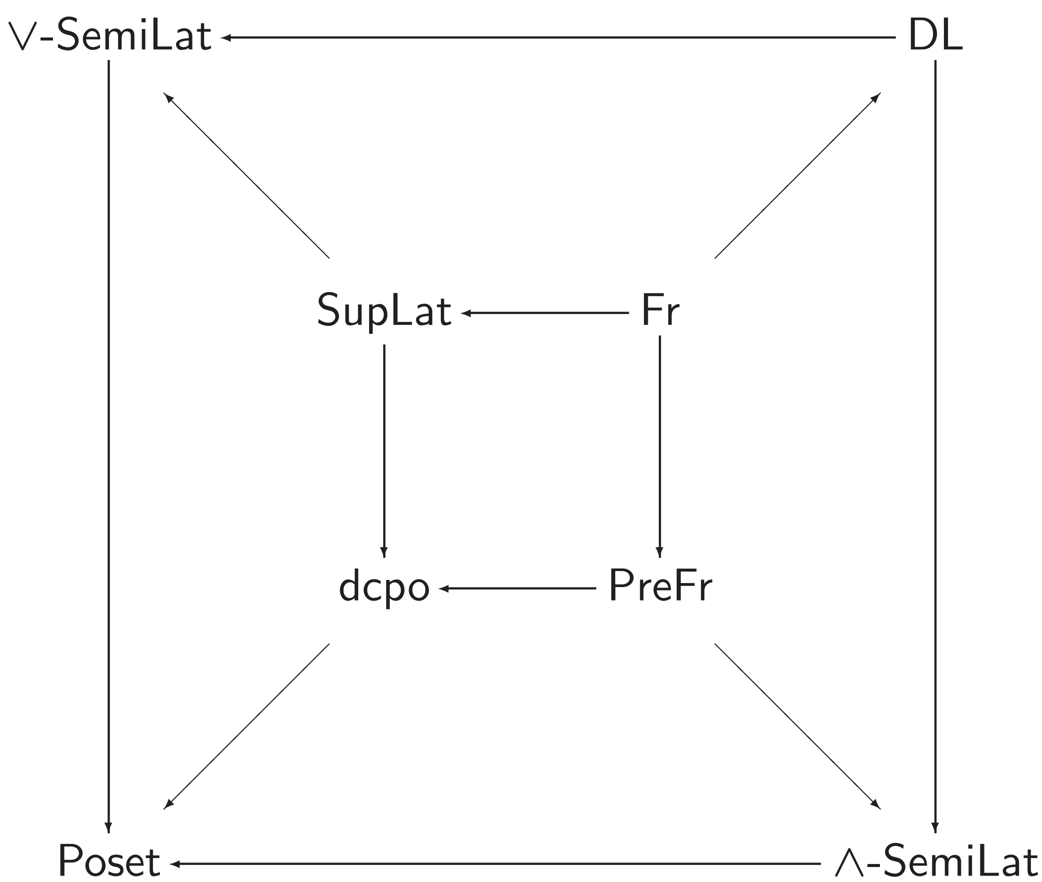
$$x \wedge y \sqsubseteq x$$

holds because inequations remain true. If z is a lower bound for x, y, then

$$z = z \wedge z \sqsubseteq x \wedge y$$

because inequations remain true and the extended operations are order-preserving.

Steve's "coverage theorems"



Every arrow denotes a forgetful functor.

Each of these has a left adjoint.

The left adjoints to the four "diagonal" functors are all constructed as dcpo completions of ordered algebras.

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Given a DCPO presentation $(P; \sqsubseteq\!\!\!\sim, C)$, a C-ideal is a lower set I of P such that

$$a \in I \quad \text{whenever} \quad U \subseteq I$$

for all covers $[a \lhd U]$ in C.

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for all covers $[a \lhd U]$ in C.

This defines a closure system on $\mathcal{P}P$ and so the set of C-ideals forms a complete lattice $C\text{-Idl}(P)$. The supremum in $C\text{-Idl}(P)$ is union followed by C-ideal closure.

Proof of Theorem 1 — cont'd

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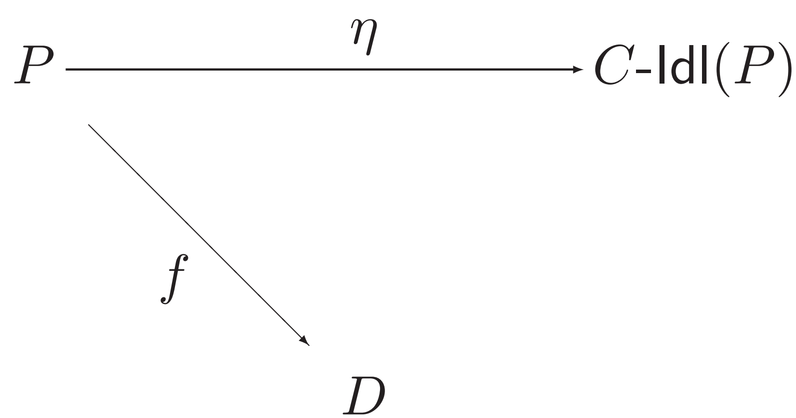
$$P \xrightarrow{\quad\eta\quad} C\text{-}\mathsf{Idl}(P)$$

$\eta(x) := \langle x \rangle$ the C-ideal closure of $\{x\}$

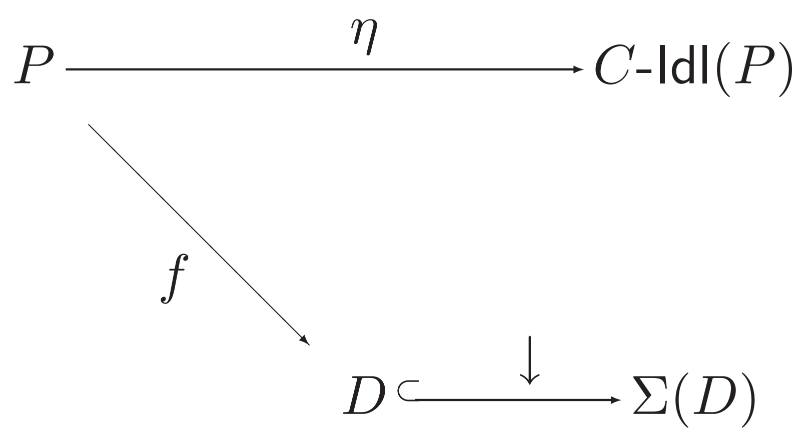
Note that η converts covers to joins in C-$\mathsf{Idl}(P)$.

Proof of Theorem 1 — cont'd



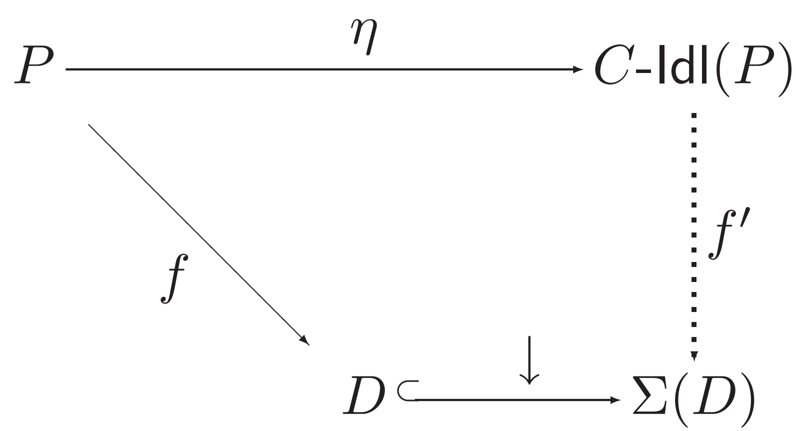
Unfortunately, D is only a dcpo.

Proof of Theorem 1 — cont'd



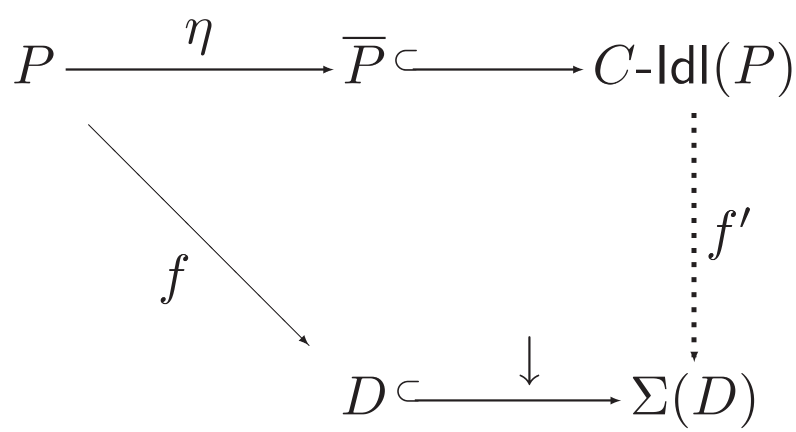
$\Sigma(D)$ is the complete lattice of Scott-closed subsets of D, and $\downarrow$ maps elements to their principal ideal $\downarrow x$.

Proof of Theorem 1 — cont'd



Lemma. C-Idl(P) is the free sup-lattice generated by $(P; \sqsubseteq, C)$.

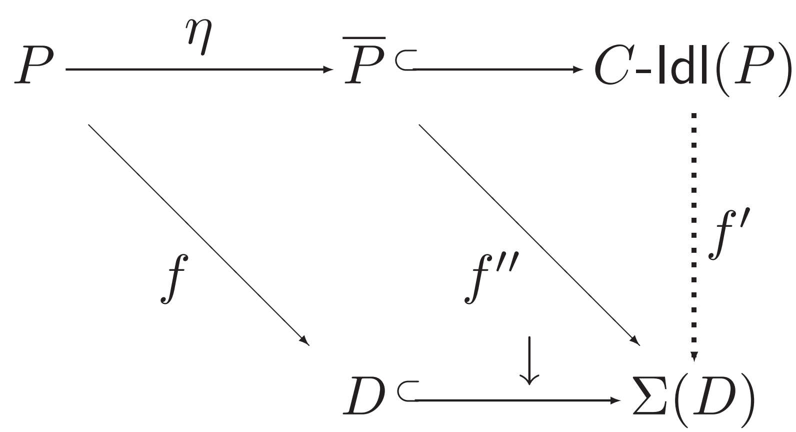
Proof of Theorem 1 — cont'd



$\overline{P}$ is the smallest sub-dcpo of C-Idl(P) containing ηP.

The directed sups in $\overline{P}$ are the sups of C-Idl(P).

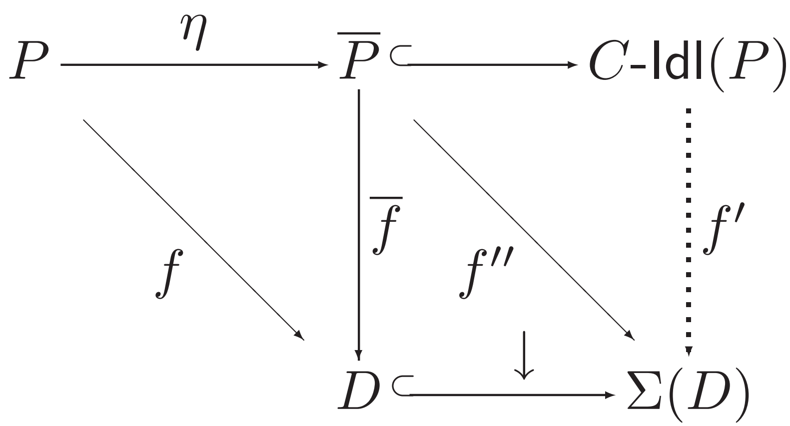
Proof of Theorem 1 — cont'd



f'' is just the restriction of f' to $\overline{P}$.

It is Scott-continuous.

Proof of Theorem 1 — cont’d



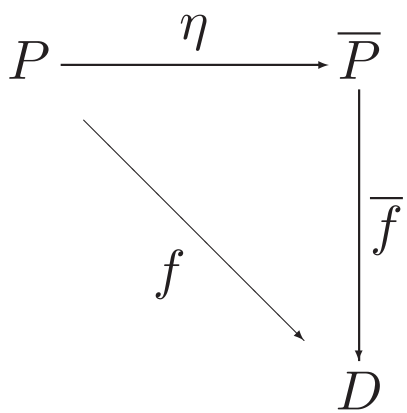
Using

— im$\downarrow$ is a sub-dcpo of $\Sigma(D)$

— $f'(\text{im}\,\eta) \subseteq$ im$\downarrow$, hence im$\,f'' \subseteq$ im$\downarrow$

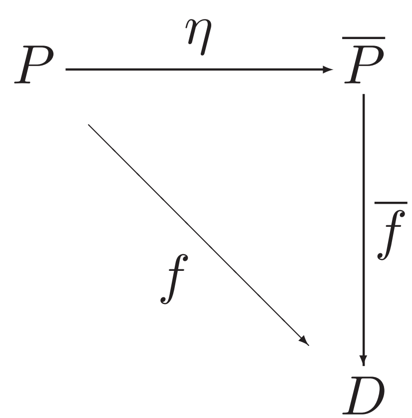
— $\downarrow$ is a Scott-continuous injection

Proof of Theorem 1 — cont'd



...which is the required universal property.

Proof of Theorem 1 — cont'd



NB: This is essentially the same construction as in

Johnstone, Vickers: *Preframe presentations present*, 1991.

(But their proof wasn't animated...)

An analysis of the presented dcpo

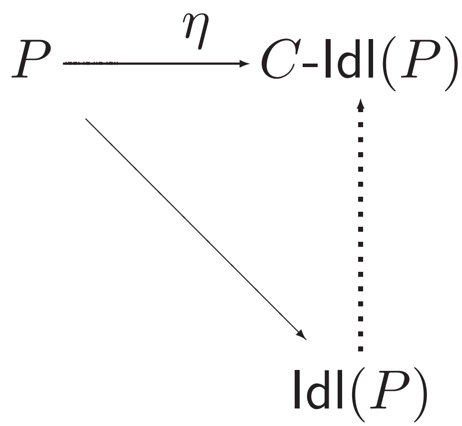
$$\overline{P} \quad \subseteq \quad C\text{-Idl}(P) \quad = \quad \text{the set of } C\text{-ideals}$$

Question. Which C-ideals belong to $\overline{P}$?

(Transfinitely) iterated ideal completion

$$P \xrightarrow{\eta} C\text{-}\mathsf{Idl}(P)$$

(Transfinitely) iterated ideal completion



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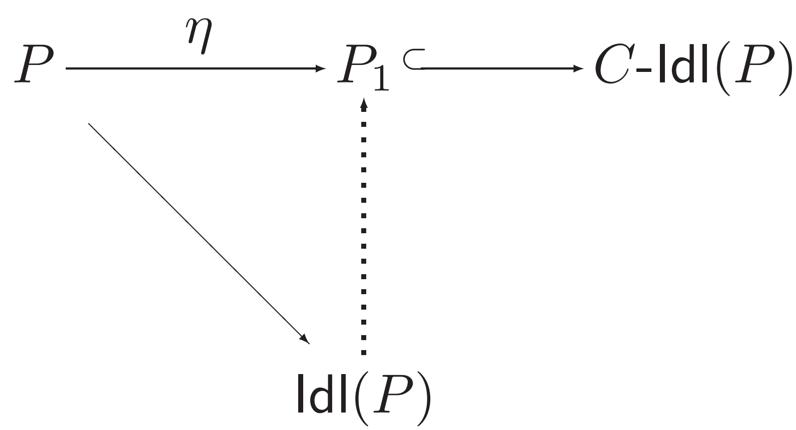

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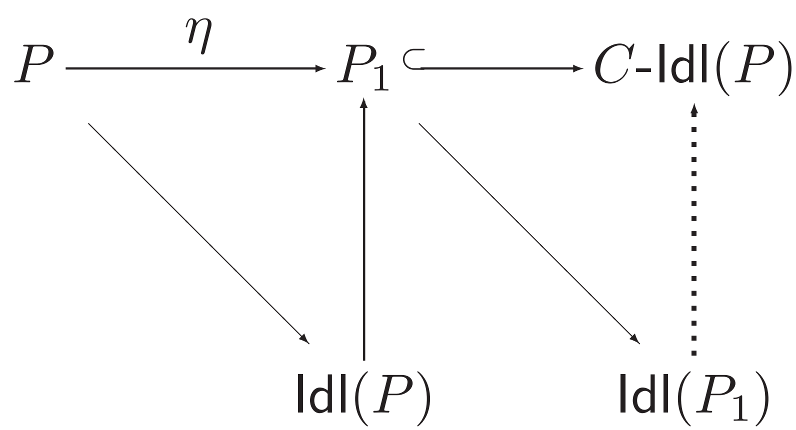

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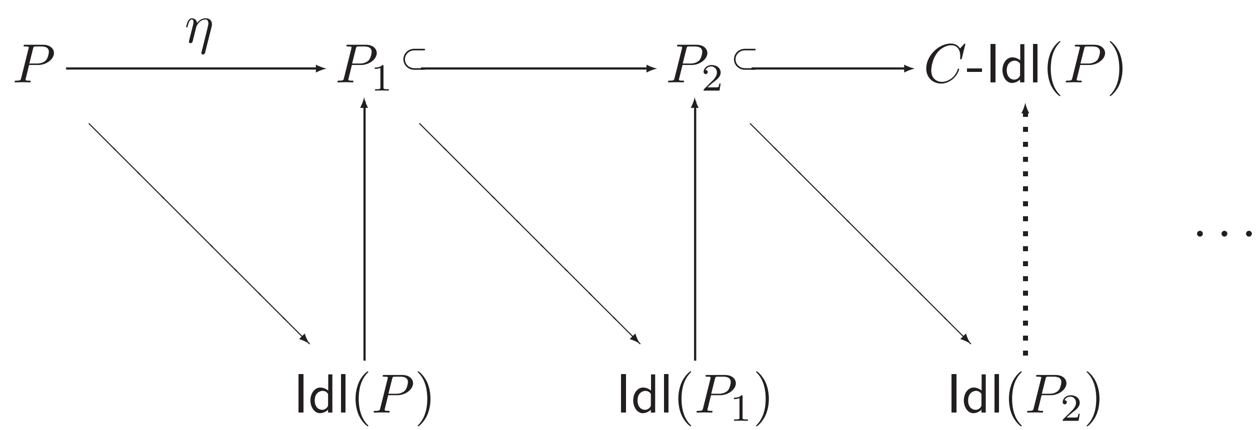

The algebra part

The essential core of Theorems 2 and 3 is the following:

Proposition. *The left adjoint functor from* **DCPO-pres** *to* **DCPO** *preserves finite products.*

Proof. Induction over the number of factors.

Note. For this to be meaningful, we need to extend the notion of morphism in **DCPO-pres**. Never mind.

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Nota bene

All results in this part are due to Klaus Keimel and Jimmie Lawson:

- K. Keimel and J. D. Lawson. D-completions and the d-topology. *Annals of Pure and Applied Logic*, 159:292–306, 2008
- K. Keimel and J. D. Lawson. Extending algebraic operations to D-completions. In S. Abramsky, M. Mislove, and C. Palamidessi, editors, *Proceedings of the 25th Annual Conference on Mathematical Foundations of Programming Semantics*, volume ?? of *Electronic Notes in Theoretical Computer Science*. Elsevier Science Publishers B.V., 2009. To appear

From dcpo presentations to topological spaces

Every dcpo presentation $(P; \precsim, C)$ gives rise to a (non-T_0) topological space $(P; \tau)$ whose closed sets are exactly the C-ideals.

This uses the fact that finite unions of C-ideals are again C-ideals, which follows from the restriction to directed sets U in our definition of covers $[a \lhd U]$.

From dcpo algebra presentations to semitopological algebras

Every dcpo algebra presentation $(P; \Omega_P, \sqsubseteq\!\!\!\sim, C)$ gives rise to a semitopological algebra, meaning that the operations in Ω_P are continuous in each argument separately with respect to the topology of C-ideals.

Completing semitopological algebras — first attempt

Every topological space $(P;\tau)$ is topologically embedded in the lattice Γ of its closed subsets:

$$\eta\colon x \quad\mapsto\quad \mathsf{cl}(x) = {\downarrow}x$$

where Γ is equipped with the topology $\mathcal{T}$ of opens

$$\Diamond O \quad := \quad \{A \in \Gamma \mid O \cap A \neq \emptyset\}, \quad O \in \tau$$

Completing semitopological algebras — first attempt

Every topological space $(P;\tau)$ is topologically embedded in the lattice Γ of its closed subsets:

$$\eta\colon x \;\;\mapsto\;\; \mathsf{cl}(x) = {\downarrow}x$$

where Γ is equipped with the topology $\mathcal{T}$ of opens

$$\Diamond O \;\;:=\;\; \{A \in \Gamma \mid O \cap A \neq \emptyset\}, \quad O \in \tau$$

In our setting,

$$\begin{aligned} \Gamma &= C\text{-}\mathsf{Idl}(P) \\ \eta &= \eta \end{aligned}$$

Completing semitopological algebras — first attempt

Theorem. *The operations of a semitopological algebra P can be extended uniquely to its lattice Γ of closed subsets such that they preserve suprema in each argument:*

$$\omega(A_1, \ldots, A_n) := \mathsf{cl}\{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\}$$

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However:

Theorem. *Γ satisfies only the linear inequations valid in P.*

$$x + (y + z) = (x + y) + z \text{ is linear}$$
$$x + x = x \text{ is not linear}$$

Completing semitopological algebras — second attempt

The sobrification of a topological space is known to produce a dcpo in the specialization order. There is a standard way to construct the sobrification as a subspace of Γ:

Definition. [Grothendieck and Dieudonné, 1971]
The strong topology on a topological space (X, τ) is given as the join of τ and the lower Alexandroff topology wrt the order of specialization.
(generic open: $O \cap {\downarrow}x$)

If $(X; \tau)$ is T_0 then the strong topology is Hausdorff and indeed totally disconnected because ${\downarrow}x = \mathrm{cl}\{x\}$ is closed and open for all $x \in X$.

Completing semitopological algebras — second attempt

Theorem. [Grothendieck and Dieudonné, 1971]

The closure $\overline{\overline{P}}$ *of* $\eta(P)$ *in* Γ *wrt the strong topology is the sobrification of* $(P;\tau)$.

Completing semitopological algebras — second attempt

Theorem. [Grothendieck and Dieudonné, 1971]

The closure $\overline{\overline{P}}$ *of* $\eta(P)$ *in* Γ *wrt the strong topology is the sobrification of* $(P;\tau)$.

Proposition. *The (extended) algebraic operations on* Γ *can be restricted to* $\overline{\overline{P}}$.

Completing semitopological algebras — second attempt

Theorem. [Grothendieck and Dieudonné, 1971]

The closure $\overline{\overline{P}}$ *of* $\eta(P)$ *in* Γ *wrt the strong topology is the sobrification of* $(P;\tau)$*.*

Proposition. *The (extended) algebraic operations on* Γ *can be restricted to* $\overline{\overline{P}}$*.*

However:

Theorem.

$\overline{\overline{P}}$ *still only satisfies the linear inequations valid in* P*.*

Completing semitopological algebras — third attempt

Definition. *A topological space* $(P; \leq, \tau)$ *is called a monotone convergence space if the specialization order is directed-complete and directed sets converge to their suprema wrt* τ*.*
(Alternatively, if τ *is contained in the Scott topology* $\sigma_{\leq}$*.)*

Proposition. *The forgetful functor from* **MCS** *to* **Top** *has a left adjoint* F*.*

We call the image $F(X)$ of a topological space X its D-completion.

Completing semitopological algebras — third attempt

Again, there is a standard way to construct this left adjoint as a subspace of Γ:

Definition. [Wyler, 1981]
The closed sets of the d-topology on a dcpo are the sub-dcpos. (Alternatively, the "admissible predicates".)

The d-topology is always Hausdorff and indeed totally disconnected because $\downarrow x$ is always clopen.

Completing semitopological algebras — third attempt

Theorem. [Wyler, 1981, Keimel & Lawson, 2008]
The closure $\overline{P}$ *of* $\eta(P)$ *in* Γ *wrt the* d*-topology is the* D*-completion of* $(P;\tau)$.

Completing semitopological algebras — third attempt

Theorem. [Wyler, 1981, Keimel & Lawson, 2008]
The closure $\overline{P}$ of $\eta(P)$ in Γ wrt the d-topology is the D-completion of $(P;\tau)$.

Proposition. *The (extended) algebraic operations on Γ can be restricted to $\overline{P}$.*

Completing semitopological algebras — third attempt

Theorem. [Wyler, 1981, Keimel & Lawson, 2008]

The closure $\overline{P}$ *of* $\eta(P)$ *in* Γ *wrt the* d*-topology is the* D*-completion of* $(P;\tau)$*.*

Proposition. *The (extended) algebraic operations on* Γ *can be restricted to* $\overline{P}$*.*

Finally:

Theorem.

$\overline{P}$ *satisfies all inequations valid in* P*.*

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Finally:

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$\overline{P}$ *satisfies all inequations valid in* P*.*

Aside: The proof uses Scott continuity wrt the specialization order.

Comparison

C-Idl(P)	$(\Gamma, \mathcal{T})$	linear inequations
$\cup\!\mid$	$\cup\!\mid$	
$\overline{\overline{P}}$	sobrification	linear inequations
$\cup\!\mid$	$\cup\!\mid$	
$\overline{P}$	D-completion	all inequations
$\cup\!\mid$	$\cup\!\mid$	
$(P; \underset{\sim}{\sqsubset}, C)$	$(P; \tau)$	

I. Motivation and Examples

II. The Main Results

III. Applications

IV. Technical Details

V. The Topological View and Generalisation

VI. Open Problems

Beyond inequations

Are there other logical formulae that are preserved in the completion process?

Beyond Universal Algebra

Under which conditions can these techniques be extended to more general notions of "algebra"?

C-spaces and domain algebras

Observation. [Ershov, Erné]
Abstract bases can be equivalently described as C-spaces: every point has a neighbourhood basis of sets of the form $\uparrow x$.

Proposition. *The D-completion, the sobrification, and the round ideal completion of a C-space all coincide.*

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Abstract bases can be equivalently described as C-spaces: every point has a neighbourhood basis of sets of the form $\uparrow x$.

Proposition. *The D-completion, the sobrification, and the round ideal completion of a C-space all coincide.*

Is there a simple argument that shows that free algebras exist for C-spaces?
(Replacing the construction of Jung & Abramsky 1994.)

Bicompletions

Can we give a similar analysis for canonical extensions?

Papers

- A. Jung, M. A. Moshier, and S. J. Vickers. Presenting dcpos and dcpo algebras. In A. Bauer and M. Mislove, editors, *Proceedings of the 24th Conference on the Mathematical Foundations of Programming Semantics*, volume 218 of *Electronic Notes in Theoretical Computer Science*, pages 209–229. Elsevier Science Publishers B.V., 2008

- K. Keimel and J. D. Lawson. D-completions and the d-topology. *Annals of Pure and Applied Logic*, 159:292–306, 2008

- K. Keimel and J. D. Lawson. Extending algebraic operations to D-completions. In S. Abramsky, M. Mislove, and C. Palamidessi, editors, *Proceedings of the 25th Annual Conference on Mathematical Foundations of Programming Semantics*, volume ?? of *Electronic Notes in Theoretical Computer Science*. Elsevier Science Publishers B.V., 2009. To appear