Topological duality and canonical extensions for lattices Part I: A topological construction of canonical extensions

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Main objectives

- prove topological dualities for semilattices and bounded lattices
- give a new construction of canonical extensions of lattices
- use the duality for lattice expansions (Part II)

Consider the following simple question:

Is there a subcategory of **Top** *that is dually equivalent to* **Lat**?

Top is the category of topological spaces and continuous maps

Lat is the category of *bounded* lattices and lattice homomorphisms that preserve the bounds

So far the question has been answered positively either by specializing **Lat** or by generalizing **Top**

Tarski 1929: every complete atomic Boolean lattice is represented by a powerset

i.e. the category of complete atomic Boolean lattices with complete lattice homomorphisms is dually equivalent to the category of discrete topological spaces

Birkhoff 1937: every finite distributive lattice is represented by the lower sets of a finite partial order

i.e. the category of finite distributive lattices is dually equivalent to the category of finite T_0 spaces and continuous maps

Stone 1936: the category of Boolean lattices and lattice homomorphisms is dually equivalent to the category of compact Hausdorff zero-dimensional spaces and continuous maps

Stone 1937: the category of distributive lattices and lattice homomorphisms is dually equivalent to the category of *spectral spaces* and *spectral maps*

These results are specializing **Lat** and obtaining a subcategory of **Top**

In the last case the topological category is not full because spectral maps are special continuous maps

Priestley 1970: distributive lattices can also be dually represented in a category of certain topological spaces augmented with a partial order

A duality between lattices and a subcategory of a generalization of **Top**

Urquhart 1978, Hartung 1992, Hartonas 1997: developed similar dualities for arbitrary bounded lattices

They generalize Priestley duality so the dual objects are certain topological spaces equipped with additional (partial order) structure

Morphisms are continuous maps that suitably preserve the additional structure

Want a purely topological duality eliminating any auxiliary relation

Will get algebraic and arithmetic lattices characterized as topological spaces

The dual category to **Lat** that we obtain is a subcategory of **Top**

Bonus: get a simple construction of the canonical extension for lattices

All lattices are assumed to be bounded

All semilattices are assumed to have a unit

Lattice and semilattice homomorphisms preserve these bounds

Let X be a topological space, with $\mathrm{O}(X)$ the set of opens

$\mathrm{N}^\circ(x)$ is the *filter of open neighborhoods* of x

Define the *specialization (pre)order* $x \sqsubseteq y \quad \text{iff} \quad \mathrm{N}^\circ(x) \subseteq \mathrm{N}^\circ(y)$

X is $T_0 \quad$ iff $\quad \sqsubseteq$ is a partial order $\quad$ (i.e. $\mathrm{N}^\circ(x)$ determines x)

${\downarrow}x$ and ${\uparrow}x$ are the principal ideal and filter in the specialization order

${\downarrow}y$ is the closure of $\{y\}$: $\qquad x \in {\downarrow}y \iff (x \in U \Rightarrow y \in U)$

$$\iff (x \notin C \Rightarrow y \notin C) \iff (y \in C \Rightarrow x \in C) \iff x \in \overline{\{y\}}$$

A set D is *(up)-directed* if for all $x, y \in D$ there exist $z \in D$ with $x, y \sqsubseteq z$

Use “square” symbols for topological situations

E.g. $x \sqcap y$ is the meet with respect to specialization (if it exists)

The *Scott topology* is defined on a poset by:

Open sets are upper sets U that are inaccessible by directed joins, i.e., if D is directed, $\bigsqcup^{\uparrow} D$ exists and $\bigsqcup^{\uparrow} D \in U$, then $D \cap U \neq \emptyset$

X is *sober* if the map $x \mapsto \mathrm{N}^{\circ}(x)$ is a bijection between X and the collection of completely prime filters of the lattice $\mathrm{O}(X)$ [$\Rightarrow T_0$]

A *spectral space* is a sober space X in which the compact open sets form a basis that is closed under finite intersections (in particular, X is itself compact)

A *spectral function* is a continuous function f for which f^{-1} also preserves compact opens

Stone's representation theorem for bounded distributive lattices:

Theorem

The category **DL** *of distributive lattices and lattice homomorphisms is dually equivalent to the category* **Spec** *of spectral spaces and spectral functions.*

A *filter of X* is a nonempty down-directed upper set

A subset of X is *saturated* if it is an intersection of open sets

In a T_0 space, saturated sets are precisely the upper sets

- $\mathrm{K}(X)$: the collection of compact saturated subsets of X
- $\mathrm{O}(X)$: the collection of open subsets of X
- $\mathrm{F}(X)$: the collection of filters of X

Intersections of these are denoted by concatenation, e.g.,
$\mathrm{OF}(X) = \mathrm{O}(X) \cap \mathrm{F}(X)$

In particular, OF, KO and KOF will be important

Compact filters are principal (in any topological space) so $\mathrm{KF}(X) \cong X^\partial$

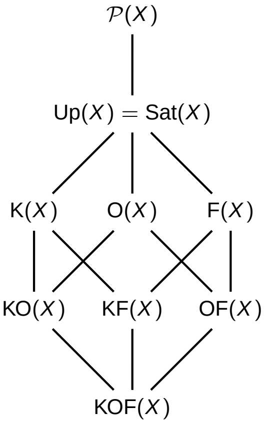
Here P^∂ denotes the order dual of poset P

An element $a \in X$ is called *finite (or compact)* if $\uparrow a$ is open

$\mathrm{Fin}(X)$ is the collection of finite points of X ordered by specialization

Hence $\mathrm{KOF}(X) \cong \mathrm{Fin}(X)^\partial$

Inclusion relations among collections of subsets



X:	T_0 space ordered by specialization
$\mathcal{P}(X)$:	all subsets of X
$\mathrm{Up}(X)$:	up-closed subsets of X
$\mathrm{Sat}(X)$:	saturated subsets of X
$\mathrm{K}(X)$:	compact saturated subsets of X
$\mathrm{O}(X)$:	open subsets of X
$\mathrm{F}(X)$:	filters of X

Theorem

For a topological space X, the following are equivalent:

1. *X is spectral and* $\mathrm{OF}(X)$ *forms a basis that is closed under finite intersection*
2. *X is spectral,* $\mathrm{OF}(X)$ *forms a basis, X is a meet semilattice with respect to specialization and X has a least element*
3. *X is sober and* $\mathrm{KOF}(X)$ *forms a basis that is closed under finite intersection*

An *SL space* (for "semilattice") is a space X satisfying these conditions

Note: $\mathrm{KOF}(X)$ is a semilattice with intersection as meet

Definition

A set is *F-saturated* if it is an intersection of open filters

$\mathrm{FSat}(X) =$ the complete lattice of F-saturated subsets of X ordered by inclusion

$$\mathrm{fsat}(A) := \bigcap\{F \in \mathrm{OF}(X) \mid A \subseteq F\}$$

Arbitrary meets in $\mathrm{FSat}(X)$ are intersections

Joins are defined by $\bigvee \mathcal{A} := \mathrm{fsat}(\bigcup \mathcal{A})$

fsat is a *closure operator*

In $\mathrm{FSat}(X)$ a directed join of open filters is simply a union

When will $\mathrm{KOF}(X)$ form a lattice, not just a semilattice?

Theorem

For an SL space, the following are equivalent.

1. $\mathrm{OF}(X)$ *forms a sublattice of* $\mathrm{FSat}(X)$
2. $\mathrm{KOF}(X)$ *forms a sublattice of* $\mathrm{FSat}(X)$
3. $\mathrm{fsat}(U)$ *is again open for any open* U

A *BL space* (for "bounded lattice") is a space satisfying these conditions

Want to show that every semilattice and every lattice occurs as $\mathrm{KOF}(X)$ for some *SL* space and some *BL* space, respectively

A complete lattice C is *algebraic* if it is isomorphic to $\mathrm{Idl}(J)$ for some join semilattice J

$\mathrm{Idl}(J)$ is the lattice of ideals of J

A complete lattice C is *arithmetic* if it is isomorphic to $\mathrm{Idl}(L)$ for some lattice L

For a meet semilattice M, $\mathrm{Filt}(M)$ is the space of filters of M with the Scott topology

Since filters of M correspond to ideals of M^∂, every algebraic lattice occurs as $\mathrm{Filt}(M)$

The Scott topology has a natural basis:

For $a \in M$, basic opens are $\phi_a := \{F \in \mathrm{Filt}(M) \mid a \in F\}$

Lemma

For a meet semilattice M, $\mathrm{Filt}(M)$ is an SL space

For a lattice L, $\mathrm{Filt}(L)$ is a BL space

Theorem

Any meet semilattice M is isomorphic to $\mathrm{KOF}(\mathrm{Filt}(M))$

Any SL space X is homeomorphic to $\mathrm{Filt}(\mathrm{KOF}(X))$

These constructions restrict to lattices and BL spaces

Thus the *SL* spaces are exactly the algebraic lattices and the *BL* spaces are exactly the arithmetic lattices, both with their Scott topologies

Will show KOF and Filt are functors for a duality **SLat** $\cong^{\partial}$ **SL**

This is a topological version of a duality **SLat** $\cong^{\partial}$ **AlgLat** given in "Continuous Lattices and domains" by G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott

But first an application to canonical extensions of lattices

A *completion* of a lattice L is a complete lattice C with L as a sublattice (more generally, with L embedded in C)

L is *lattice dense* in C if

$$\mathrm{Meets}_C(\mathrm{Joins}_C(L)) = C = \mathrm{Joins}_C(\mathrm{Meets}_C(L)),$$

where

$$\begin{aligned} \mathrm{Meets}_C(A) &= \{\bigwedge B \mid B \subseteq A\} \\ \mathrm{Joins}_C(A) &= \{\bigvee B \mid B \subseteq A\} \end{aligned}$$

L is *lattice compact* in C if for all $U, V \subseteq L$, if $\bigwedge_C U \leq \bigvee_C V$ then there exist finite $U_0 \subseteq U$, $V_0 \subseteq V$ for which $\bigwedge U_0 \leq \bigvee V_0$

A *canonical extension of L* is a completion C such that L is lattice dense and lattice compact in C

Theorem (Gehrke and Harding 2001)

Every lattice L has a canonical extension, denoted by L^σ, unique up to isomorphism, i.e.,

if C is also a canonical extension of L, then there is a lattice isomorphism between L^σ and C that keeps L fixed.

Theorem

For a BL space X, $\mathrm{FSat}(X)$ is a canonical extension of $\mathrm{KOF}(X)$

Proof

One half of lattice density is almost trivial. Consider an open filter $F = \bigcup\{{\uparrow}a \mid a \in F \cap \mathrm{Fin}(X)\}$. This union is directed, so it is the join in $\mathrm{FSat}(X)$. Hence any $S \in \mathrm{FSat}(X)$ takes the form

$$\begin{aligned} S &= \bigcap\{F \in \mathrm{OF}(X) \mid S \subseteq F\} \\ &= \bigcap\{\bigcup\{{\uparrow}a \mid a \in F \cap \mathrm{Fin}(X)\} \mid F \in \mathrm{OF}(X) \textit{ and } S \subseteq F\} \\ &= \bigcap\{\bigvee\{{\uparrow}a \mid a \in F \cap \mathrm{Fin}(X)\} \mid F \in \mathrm{OF}(X) \textit{ and } S \subseteq F\}. \end{aligned}$$

For the other half of density, consider $S \in \mathrm{FSat}(X)$. Then $S = \bigcap_{i \in I} F_i$ for some family of open filters $\{F_i\}$. Each F_i is a directed join, hence union, of compact open filters $\{{\uparrow}a_{ij}\}_{j \in J_i}$.

Proof (continued)

So

$$\begin{aligned} S &= \bigcup_{\gamma \in \prod_{i \in I} J_i} \bigcap_{i \in I} {\uparrow} a_{i,\gamma(i)} \\ &\subseteq \mathrm{fsat}\Big(\bigcup_{\gamma \in \prod_{i \in I} J_i} \bigcap_{i \in I} {\uparrow} a_{i,\gamma(i)} \Big) \\ &= \bigsqcup_{\gamma \in \prod_{i \in I} J_i} \bigcap_{i \in I} {\uparrow} a_{i,\gamma(i)} \\ &\subseteq S. \end{aligned}$$

For lattice compactness, it suffices to show that when $\{F_i\}_i$ is a downward directed family of compact open filters and $\{G_j\}_j$ is an upward directed family of compact open filters,

if $\bigcap_i F_i \subseteq \bigcup_j G_j$, then for some i and j, $F_i \subseteq G_j$.

Proof continued.

Each F_i is a principal filter, so let $a_i = \min F_i$.

Because the family $\{F_i\}_i$ is downward directed, the set of these generators $\{a_i\}_i$ is directed.

By sobriety of X, this directed set has a least upper bound, say x.

Now $x \in \bigcap_i F_i$, and every open neighborhood of x includes some a_i.

In particular, $\bigcup_j G_j$ is such a neighborhood.

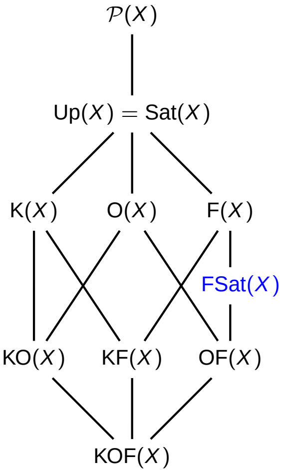
So for some i, $a_i \in \bigcup_j G_j$.

Hence for some j, $a_i \in G_j$. □

Corollary

Every lattice has a canonical extension, unique up to isomorphism

Position of canonical extension



X:	T_0 space ordered by specialization
$\mathcal{P}(X)$:	all subsets of X
Up(X):	up-closed subsets of X
Sat(X):	saturated subsets of X
K(X):	compact saturated subsets of X
O(X):	open subsets of X
F(X):	filters of X
FSat(X):	intersections of open filters of X

Morphisms

Characterize those (continuous) functions between *SL* spaces that correspond to meet semilattice morphisms:

Lemma

For a function $f : X \to Y$ between SL spaces, the following are equivalent

1. f^{-1} *restricted to* $\mathrm{KOF}(Y)$ *co-restricts to* $\mathrm{KOF}(X)$
2. f *is spectral and* f^{-1} *restricted to* $\mathrm{OF}(Y)$ *co-restricts to* $\mathrm{OF}(X)$
3. f *is spectral and* $\mathrm{fsat}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{fsat}(B))$ *for all* $B \subseteq Y$
4. f *is spectral and* $\mathrm{fsat}(f^{-1}(U)) \subseteq f^{-1}(\mathrm{fsat}(U))$ *for all opens* $U \subseteq Y$

f is *F-continuous* if it satisfies the equivalent conditions of the lemma

$\Longrightarrow$ f is continuous

For lattice hom. $h : L \to M$ define Filt $h = h^{-1}$: Filt$(M) \to$ Filt(L)

For F-cont. map $f : X \to Y$ define KOF $f = f^{-1}$: KOF$(Y) \to$ KOF(X)

Theorem

The category of semilattices and meet preserving functions is dually equivalent to the category of SL spaces and F-continuous functions

$$\mathbf{SLat} \quad \overset{\text{Filt}}{\underset{\text{KOF}}{\rightleftarrows}} \equiv^{op} \quad \mathbf{SL\ spaces}$$

This cuts down to the full subcategory of lattices and meet preserving functions and the full subcategory $\mathbf{BL}_c$ *of BL spaces and F-continuous functions*

$$\mathbf{Lat}_\wedge \quad \overset{\text{Filt}}{\underset{\text{KOF}}{\rightleftarrows}} \equiv^{op} \quad \mathbf{BL}_c$$

Next want to restrict this duality to lattice homomorphisms

Lemma

For an F-continuous map $f : X \to Y$ between BL spaces, the following are equivalent.

1. f^{-1} *preserves finite joins of compact open filters*
2. $f^{-1}(\mathrm{fsat}(U)) \subseteq \mathrm{fsat}(f^{-1}(U))$ *for any open* $U \subseteq Y$

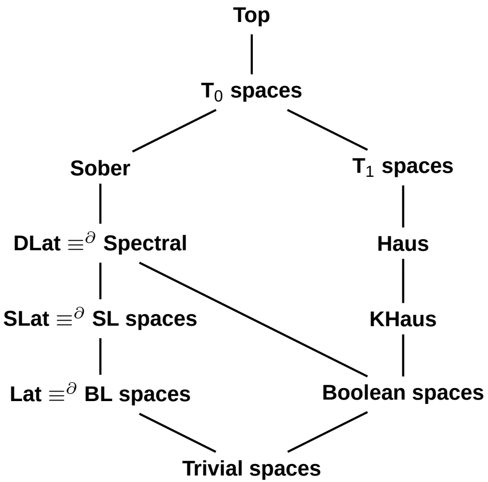
A spectral function is *F-stable* if $f^{-1}(\mathrm{fsat}(U)) = \mathrm{fsat}(f^{-1}(U))$ for any open U

Theorem

The category of lattices and lattice homomorphisms is dually equivalent to the category of BL spaces and F-stable functions

$$\mathbf{Lat} \quad \underset{\overleftarrow{\ \mathrm{KOF}\ }}{\overset{\xrightarrow{\ \mathrm{Filt}\ }}{\equiv^{op}}} \quad \mathbf{BL\ spaces}$$

Relations among categories



The duality given here is “bigger” than Urquhart’s duality

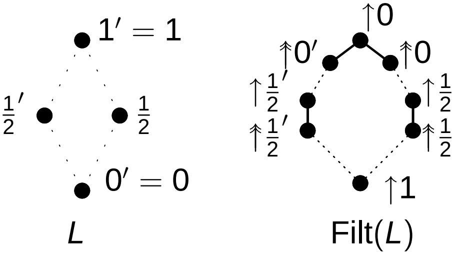
E.g. in a finite lattice, Filt(L) is isomorphic to L^∂

But for any lattice FSat(Filt(L)) $= L^\sigma$ is a perfect lattice, so we can obtain the *reduced* dual space by restricting to completely join irreducibles and completely meet irreducibles with subspace topologies

Then morphisms between dual spaces are given by certain relations

A (nonreduced) example: two copies of $[0, 1]$ with 0’s and 1’s identified

Write x for elements of one copy and x' for the corresponding elements of the other copy



A non-distributive lattice and its filters

There are two types of filter:

1. $\uparrow x$ – the principal filter generated by any $x \in L$
2. $\twoheaduparrow x = \uparrow x \setminus \{x\}$ – the 'round' filter of elements strictly above x by any $x \in L \setminus \{1\}$ In the special case of 0, there is a distinction between $\twoheaduparrow 0$ and $\twoheaduparrow 0'$ according to which copy of $[0, 1]$ is used

In $\mathrm{Filt}(L)$ specialization order is inclusion

There are three types of filters:

$$
\begin{aligned}
\mathcal{H}_x &:= \{F \in \mathrm{Filt}(L) \mid {\uparrow}x \subseteq F\} \qquad [x \neq 1] \\
\mathcal{G}_x &:= \{F \in \mathrm{Filt}(L) \mid {\uparrow}x \subseteq F\} \\
\mathcal{F}_x &:= \bigcup_{y<x} \mathcal{H}_y \qquad [x \neq 0]
\end{aligned}
$$

$\mathcal{F}_x \subseteq \mathcal{G}_x \subseteq \mathcal{H}_x$

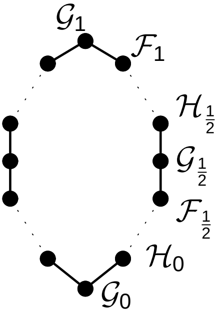
$\mathcal{H}_x$ and $\mathcal{G}_x$ are compact

$\mathcal{G}_x$ and $\mathcal{F}_x$ are open

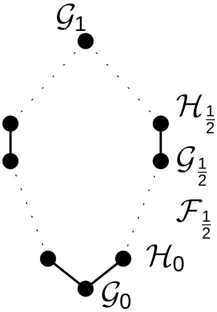
So the filters $\mathcal{G}_x$ constitute $\mathrm{KOF}(X)$

In this example, every member of $\mathrm{F}(\mathrm{Filt}(L))$ is saturated

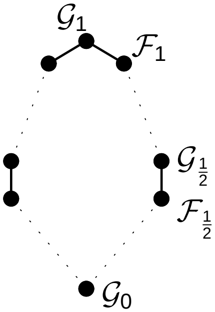
Hence the canonical extension of L is isomorphic to $\mathrm{F}(\mathrm{Filt}(L))$



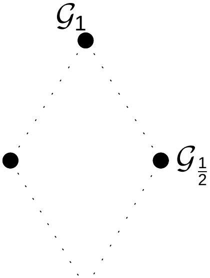
F(Filt(L))



KF(Filt(L))



OF(Filt(L))



Conclusions

SLat and **Lat** are dually equivalent to a subcategory of **Top**

This gives a new construction of canonical extension of lattices

The duality between **Lat** and **BL** can be extended to a duality between lattices with quasioperators and certain expanded *BL* spaces (Part II presented next by Drew Moshier)

The dual objects and morphisms occur in a natural topological framework (morphisms are certain continuous *functions*)

This provides connections with other areas of TACL

E.g. domain theory, positive modal logic, proof theory

Preprints at `www.chapman.edu/~jipsen/preprints.html`