TOPOLOGICAL DUALITY AND LATTICE EXPANSIONS PART I: A TOPOLOGICAL CONSTRUCTION OF CANONICAL EXTENSIONS

M. ANDREW MOSHIER AND PETER JIPSEN

1. Introduction

The two main objectives of this paper are (a) to prove topological duality theorems for semilattices and bounded lattices, and (b) to show that the topological duality from (a) provides a construction of canonical extensions of bounded lattices. The paper is first of two parts. The main objective of the sequel is to establish a characterization of lattice expansions, i.e., lattices with additional operations, in the topological setting built in this paper.

Regarding objective (a), consider the following simple question:

Is there a subcategory of **Top** that is dually equivalent to **Lat**?

where **Top** is the category of topological spaces and continuous maps and **Lat** is the category of bounded lattices and bounded lattice homomorphisms.

To date, the question has been answered positively either by specializing **Lat** or by generalizing **Top**. The earliest examples are of the former sort.

Tarski [Tar29] (treated in English, e.g., in [BD74]) showed that every complete atomic Boolean lattice is represented by a powerset. Birkhoff [Bir37] showed that every finite distributive lattice is represented by the lower sets of a finite partial order. With some historical licence, we can now say that these show that the categories of complete atomic Boolean lattices and of finite distributive lattices are dually equivalent to the categories discrete spaces and of finite T_0 spaces, respectively. In the seminal papers, [Sto36, Sto37], Stone generalized Tarski and then Birkhoff, showing that (a) the category of Boolean lattices and lattice homomorphisms is dually equivalent to the category of zero-dimensional, regular spaces and continuous maps and then (b) the category of distributive lattices and lattice homomorphisms is dually equivalent to the category of *spectral spaces* and *spectral maps*. All of these results can be viewed as specializing **Lat** and obtaining a subcategory of **Top**. In the case of distributive lattices, the topological category is not full because spectral maps are special continuous maps.

As a conceptual bridge, Priestley [Pri70] showed that distributive lattices can also be dually represented in a category of certain topological spaces augmented with a partial order. This is an example of the latter sort of result, namely, a duality between a category of lattices and a subcategory of a generalization of **Top**.

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Urquhart [Urq78], Hartung [Har92] and Hartonas [HD97] developed similar dualities for arbitrary bounded lattices. It is fair to say that they follow in the spirit of Priestley duality for distributive lattices in that their dual objects are certain topological spaces equipped with additional (partial order) structure. The dual morphisms are continuous maps that suitably preserve the additional structure. This is in contrast to the spirit of Stone duality, in which the dual category is simply a subcategory of **Top**.

Another approach to dualities for arbitrary lattices is given an exposition in Chapters 1 and 4 of Gierz *et al*, [GHK$^+$80]. There, the duality between inf complete semilattices and sup complete semilattices arising from adjoint pairs of maps is specialized to various categories of algebraic and arithmetic lattices (reviewed in the full paper).

We take a different path via purely topological considerations. At the end of this path, we find algebraic and arithmetic lattices characterized as topological spaces. This establishes an affirmative answer to our original question with no riders: the dual category to **Lat** is a subcategory of **Top** *simpliciter*. This leads directly to a very natural topological characterization of canonical extensions for arbitrary bounded lattices.

Because the sequel paper applies topological duality to lattices with additional "quasioperators" such as modal operators, residuals, etc., the sense in which a map between lattices is "structure preserving" must be considered carefully. We characterize the topological duals of meet semilattice homomorphisms and lattice homomorphisms. In the sequel, we use these to represent quasioperators.

2. Background and Definitions

In this paper, lattices are always bounded; semilattices always have a unit. Also, we designate semilattices as meet or join semilattices according to which order we intend. Lattice and semilattice homomorphisms preserve bounds.

In a T_0 space, order-theoretic notions apply to the *specialization order*: $x \sqsubseteq y$ if every neighborhood of x is a neighborhood of y. In particular, a *saturated set* is an upper set with respect to specialization; a *filter* is a saturated, downward directed set, i.e., a $\sqsubseteq$-filter in the usual order-theoretic sense.

We will be interested in special sorts of subsets of a space X: compact saturated sets, open sets and filters. In that light, we define

- $\mathsf{K}(X)$: the collection of compact saturated subsets of X;
- $\mathsf{O}(X)$: the collection of open subsets of X; and
- $\mathsf{F}(X)$: the collection of filters of X.

Intersections of these are denoted by concatenation, e.g., $\mathsf{OF}(X) = \mathsf{O}(X) \cap \mathsf{F}(X)$. In particular, OF, KO and KOF will be important.

Recall that a *canonical extension* of a lattice L is a complete lattice C satisfying (i) L is a sublattice of C, (ii) every element of C is a meet of joins of elements of L and a join of meet of elements of L, and (iii) for any $U, V \subseteq L$, if $\bigwedge U \leq \bigvee V$, then $\bigwedge U_0 \leq \bigvee V_0$ for some finite $U_0 \subseteq U$ and $V_0 \subseteq V$.

In the full paper we prove, via topological duality, the following theorem, originally due to Gehrke and Harding [GH01].

Theorem 2.1 ([GH01]). *Every lattice L has a canonical extension, denoted by L^σ, unique up to isomorphism, i.e. if C is also a canonical extension of L, then there is a lattice isomorphism between L^σ and C that keeps L fixed.*

3. Outline of Results

The first main result of the paper captures a useful class of topological spaces.

Theorem 3.1. *For a topological space X, the following are equivalent:*

(1) *X is spectral and $\mathsf{OF}(X)$ forms a basis that is closed under finite intersection;*
(2) *X is spectral, $\mathsf{OF}(X)$ forms a basis, X is a meet semilattice with respect to specialization and X has a least element;*
(3) *X is sober and $\mathsf{KOF}(X)$ forms a basis that is closed under finite intersection.*

We refer to a topological space satisfying these conditions as a *semilattice (SL) space*.

Define
$$\mathsf{fsat}(A) := \bigcap \{F \in \mathsf{OF}(X) \mid A \subseteq F\}.$$
and let $\mathsf{FSat}(X)$ denote the collection of sets of the form $\mathsf{fsat}(A)$. Since fsat clearly is a *closure operator*, $\mathsf{Fsat}(X)$ is a complete lattice. Rereading Theorem 3.1, we now see that *SL* spaces are those sober spaces for which $\mathsf{KOF}(X)$ forms a sub-meet semilattice of $\mathsf{FSat}(X)$. We are interested in characterizing those *SL* spaces for which it actually forms a sub-lattice.

Theorem 3.2. *For an SL space, the following are equivalent.*

(1) *$\mathsf{OF}(X)$ forms a sublattice of $\mathsf{FSat}(X)$;*
(2) *$\mathsf{KOF}(X)$ forms a sublattice of $\mathsf{FSat}(X)$;*
(3) *if $U \subseteq X$ is open, then $\mathsf{fsat}(U)$ is also open.*

We refer to the spaces satisfying the conditions of the theorem as *bounded lattice (BL) spaces*.

For a meet semilattice M, let $\mathsf{Filt}(M)$ be the space of filters in M with basic opens of the form
$$\varphi_a := \{F \in \mathsf{Filt}(M) \mid a \in F\}.$$
for $a \in M$. Since filters of M correspond to ideals of M^∂, every algebraic lattice occurs as $\mathsf{Filt}(M)$.

Theorem 3.3. *Any meet semilattice M is isomorphic to $\mathsf{KOF}(\mathsf{Filt}(M))$. Any SL space X is homeomorphic to $\mathsf{Filt}(\mathsf{KOF}(X))$. These constructions restrict to lattices and BL spaces.*

Notice that these results also tell us that the *SL* spaces are exactly the algebraic lattices and the *BL* spaces are exactly the arithmetic lattices, both with their topologies given by the usual Stone construction.

Now in the full paper, we are prepared to prove the existence and uniqueness of canonical extensions.

Theorem 3.4. *For a BL space X, $\mathsf{FSat}(X)$ is a canonical extension of $\mathsf{KOF}(X)$.*

Corollary 3.5. *Every lattice has a canonical extension, unique up to isomorphism.*

Clearly, the next thing to do is to characterize those continuous functions between *SL* (*BL*) spaces that correspond to meet semilattice (resp., lattice) homomorphisms

Lemma 3.6. *For a function $f\colon X \to Y$ between SL spaces, the following are equivalent.*

1. *f^{-1} restricted to $\mathsf{KOF}(Y)$ co-restricts to $\mathsf{KOF}(X)$.*
2. *f is spectral (f^{-1} preserves compact opens) and f^{-1} restricted to $\mathsf{OF}(Y)$ co-restricts to $\mathsf{OF}(X)$.*
3. *f is spectral and $\mathsf{fsat}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{fsat}(B))$ for all $B \subseteq Y$.*
4. *f is spectral and $\mathsf{fsat}(f^{-1}(U)) \subseteq f^{-1}(\mathsf{fsat}(U))$ for all opens $U \subseteq Y$.*

A function f is called *F-continuous* if it satisfies the equivalent conditions of the lemma.

Lemma 3.7. *For an F-continuous map $f\colon X \to Y$ between BL spaces, the following are equivalent.*

1. *f^{-1} preserves finite joins of compact open filters.*
2. *f^{-1} preserves finite joins of open filters.*
3. *f^{-1} preserves all joins of open filters.*
4. *$f^{-1}(\mathsf{fsat}(U)) \subseteq \mathsf{fsat}(f^{-1}(U))$ for any open $U \subseteq Y$.*

Theorem 3.8. *The category* **SLat** *of semilattices and semilattice homomorphisms is dually equivalent to the category* **SL** *of SL spaces and F-continuous maps. Also the category* **Lat** *of lattices and lattice homomorphisms is dually equivalent to the category* **BL** *of BL spaces and F-stable functions.*

In the full paper, we illustrate these dualities with non-trivial, non-distributive lattices.

4. CONCLUSION

We have established a dual equivalence between **Lat** and **BL**, an easily described subcategory of **Top**, and have shown that the very natural construction of the complete lattice of F-saturated subsets produces the canonical extension of $\mathsf{KOF}(X)$ for a *BL* space.

In the sequel paper, we use the other topological duality for semilattice reducts of lattices to handle n-ary operations that are join reversing or meet preserving in each argument, or dually that are meet reversing or join preserving in each argument. Such operations are called quasioperators, and we consider several examples to illustrate the general case. Similar extensions have been discussed by Hartonas, but our topological duality for the underlying lattices simplifies the description of morphisms in the dual category and permits us to extend the representation to quasioperator preserving lattice homomorphisms.

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