A "TYPICAL" CONTRACTION IS UNITARY

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Abstract. We show that the set of unitary operators on a separable infinite-dimensional Hilbert space is residual (for the weak operator topology) in the set of all contractions. The same holds for unitary operators in the set of all isometries and with respect to the strong operator topology. The continuous versions are discussed as well. These results are applied to the problem of embedding operators into strongly continuous semigroups.

1. Introduction

Unitary operators and unitary groups form an important and natural subclass of contractive operators and operator semigroups on Hilbert spaces. They are quite well-understood and have many nice properties not shared by general contractions and contractive semigroups.

Such nice properties of analytic objects are seldom a rule and often an exception. As an example one may think, e.g., of continuity or differentiability of functions or strong mixing of transformations in ergodic theory. However, there is a series of recent results in operator and ergodic theory on generity of some good properties of operators and operator semigroups or measure preserving transformations, respectively. We refer to Bartoszek, Kuna [2, 3], Lasota, Myjak [14], Rudnicki [16], Eisner [4], Eisner, Serény [5, 6] and Ageev [1], King [12], de la Rue, de Sam Lazaro [17], Stepin, Eremenko [19, 18] for further information.

In this note we show that unitarity is "typical" in the Baire sense for contractions. More precisely, we show that unitary operators form a residual set in the set of all contractions for the weak operator topology on a separable infinite-dimensional Hilbert space (Section 2). Recall that a subset of a Baire space is called residual if its complement is of first category, i.e., can be represented as a countable union of nowhere dense sets. We also show that for the strong operator topology, a "typical" isometry is unitary as well.

To show the power of this result, we conclude in particular that a "typical" contraction T on a separable Hilbert space can be embedded into a C_0-semigroup $(T(t))_{t\geq 0}$, i.e., $T = T(1)$ holds for some strongly continuous representation $(T(t))_{t\geq 0}$ of $\mathbb{R}_+$ (Section 3). As a consequence, a "typical"

2000 *Mathematics Subject Classification.* 47A05, 47D06, 37A05.

Key words and phrases. Hilbert spaces, unitary operators, isometries, contractions, Baire category, residual sets, C_0-semigroups, unitary groups.

contraction on a Hilbert space has roots of all orders. This is an operator theoretic analogue of recent results in ergodic theory for measure preserving transformations by King [12] and de la Rue, de Sam Lazaro [17] mentioned above.

We finally consider the continuous case and show in Section 4 that a "typical" isometric C_0-semigroup on a separable Hilbert space is a unitary group. We also discuss the difficulties of proving the same statement for contractive C_0-semigroups.

2. Main result

Let H be an infinite-dimensional separable Hilbert space. We denote by $\mathcal{C}$ the set of all contractions on H endowed with the weak operator topology, and by $\mathcal{I}$ the set of all isometries on H endowed with the strong operator topology. Both $\mathcal{C}$ and $\mathcal{I}$ are complete metric spaces with respect to the metric

$$d_{\mathcal{C}}(T,S) := \sum_{i,j=1}^{\infty} \frac{|\langle Tx_i, x_j\rangle - \langle Sx_i, x_j\rangle|}{2^{i+j}\|x_i\|\|x_j\|} \quad \text{for } T,S \in \mathcal{C}$$

and

$$d_{\mathcal{I}}(T,S) := \sum_{j=1}^{\infty} \frac{\|Tx_j - Sx_j\|}{2^j\|x_j\|} \quad \text{for } T,S \in \mathcal{I},$$

respectively, where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of $H \setminus \{0\}$. The set of all unitary operators is denoted by $\mathcal{U}$.

We will need the following well-known property of weak convergence, see e.g. Halmos [11]. For the reader's convenience we give its simple proof.

Lemma 2.1. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of linear operators on a Hilbert space H converging weakly to a linear operator S. If $\|T_nx\| \le \|Sx\|$ for every $x \in H$, then $\lim_{n\to\infty} T_n = S$ strongly.*

Proof. For every $x \in H$ we have

$$\begin{aligned}\|T_nx - Sx\|^2 &= \langle T_nx - Sx, T_nx - Sx\rangle = \|Sx\|^2 + \|T_nx\|^2 - 2\mathrm{Re}\,\langle T_nx, Sx\rangle \\ &\le 2\,\langle Sx, Sx\rangle - 2\mathrm{Re}\,\langle T_nx, Sx\rangle = 2\mathrm{Re}\,\langle (S-T_n)x, Sx\rangle \to 0\end{aligned}$$

as $n \to \infty$, and the lemma is proved. □

We now state the main result of this note showing that, in the Baire category sense, a "typical" contraction as well as a "typical" isometry is unitary.

Theorem 2.2. *The set $\mathcal{U}$ of unitary operators is residual in the set $\mathcal{C}$ of all contractions with respect to the weak operator topology. In addition, it is also residual in the set $\mathcal{I}$ of all isometries with respect to the strong operator topology.*

Proof. The proof consists of two parts. We first show that a "typical" isometry is unitary for the strong operator topology and then consider contractions.

Part 1. We prove residuality of $\mathcal{U}$ in the set $\mathcal{I}$ of all isometries on H endowed with the strong operator topology. Let T be a non-invertible isometry. Then $\operatorname{rg} T$ is closed and different from H. Therefore there exists j such that $\operatorname{dist}(x_j, \operatorname{rg} T) > 0$, and hence

$$\mathcal{I} \setminus \mathcal{U} = \bigcup_{k,j=1}^{\infty} M_{j,k} \quad \text{with} \quad M_{j,k} := \left\{ T \text{ isometry} : \operatorname{dist}(x_j, \operatorname{rg} T) > \frac{1}{k} \right\}. \tag{1}$$

We now prove that every set $M_{j,k}$ is nowhere dense in $\mathcal{I}$. Since unitary operators are dense in $\mathcal{I}$, see Eisner, Serény [5, Proposition 3.3], it suffices to show that

$$\mathcal{U} \cap \overline{M_{j,k}} = \emptyset \quad \forall j, k. \tag{2}$$

Assume the contrary, i.e., that there exists a sequence $\{T_n\}_{n=1}^{\infty} \subset M_{j,k}$ for some j, k and a unitary operator U with $\lim_{n\to\infty} T_n = U$ strongly. In particular, $\lim_{n\to\infty} T_n y = Uy = x_j$ for $y := U^{-1}x_j$. This however implies $\lim_{n\to\infty} \operatorname{dist}(x_j, \operatorname{rg} T_n) = 0$, a contradiction. So (2) is proved, every set $M_{j,k}$ is nowhere dense, and $\mathcal{U}$ is residual in $\mathcal{I}$.

Part 2. We first prove that the set $\mathcal{I} \setminus \mathcal{U}$ of non-invertible isometries is of first category in $\mathcal{C}$. We again represent $\mathcal{I} \setminus \mathcal{U}$ as in (1). Since unitary operators are dense in $\mathcal{C}$, see Takesaki [21, p. 99] or Peller [15], it is enough to show that

$$\mathcal{U} \cap \overline{M_{j,k}} = \emptyset \quad \forall j, k.$$

Assume that for some j, k there exists a sequence $\{T_n\}_{n=1}^{\infty} \subset M_{j,k}$ converging weakly to a unitary operator U. Then, by Lemma 2.1, T_n converges to U strongly. As in Part 1, this implies $\lim_{n\to\infty} T_n y = Uy = x_j$ for $y := U^{-1}x_j$. Hence $\lim_{n\to\infty} \operatorname{dist}(x_j, \operatorname{rg} T_n) = 0$ contradicting $\{T_n\}_{n=1}^{\infty} \subset M_{j,k}$, so every $M_{j,k}$ is nowhere dense and $\mathcal{I} \setminus \mathcal{U}$ is of first category.

We now show that the set of non-isometric operators is of first category in $\mathcal{C}$ as well. Let T be a non-isometric contraction. Then there exists x_j such that $\|Tx_j\| < \|x_j\|$, hence

$$\mathcal{C} \setminus \mathcal{I} = \bigcup_{k,j=1}^{\infty} N_{j,k} \quad \text{with} \quad N_{j,k} := \left\{ T : \frac{\|Tx_j\|}{\|x_j\|} < 1 - \frac{1}{k} \right\}.$$

It remains to show that every $N_{j,k}$ is nowhere dense in $\mathcal{C}$. By density of $\mathcal{U}$ in $\mathcal{C}$ it again suffices to show that

$$\mathcal{U} \cap \overline{N_{j,k}} = \emptyset \quad \forall j, k.$$

Assume that for some j, k there exists a sequence $\{T_n\}_{n=1}^{\infty} \subset N_{j,k}$ converging weakly to a unitary operator U. Then T_n converges to U strongly by

Lemma 2.1. This implies that $\lim_{n\to\infty} \|T_n x_j\| = \|Ux_j\| = \|x_j\|$ contradicting $\frac{\|T_n x_j\|}{\|x_j\|} < 1 - \frac{1}{k}$ for every $n \in \mathbb{N}$. □

Remark 2.3. One cannot replace the weak operator topology on $\mathcal{C}$ by the strong one in the above theorem since unitary operators are not dense in $\mathcal{C}$ for the strong operator topology. Moreover, $\mathcal{U}$ and $\mathcal{I}$ become even nowhere dense in $\mathcal{C}$ for this topology. Indeed, any open set contains an operator with norm strictly less than one, leading to a neighbourhood containing no isometric operator.

Remark 2.4. It is an interesting question for which W*-algebras an analogous result holds, i.e., the unitary elements are residual in the unit ball for the weak* topology. Note that l^∞ does not have this property since the unitary elements (i.e., unimodular sequences) are not even dense in the unit ball.

3. Application to the embedding problem

In this section we consider the following problem: Which bounded operators T can be *embedded* into a strongly continuous semigroup, i.e., does there exist a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $T = T(1)$? (For the basic theory of C_0-semigroups we refer to Engel, Nagel [8].) This question is open and difficult and has analogues in ergodic and measure theory. For some classes of operators it has a positive answer, e.g., for operators with spectrum in a certain area using functional calculus, see e.g. Haase [9, Section 3.1], and for isometries on Hilbert spaces with infinite-dimensional kernel, see [4].

We apply the above category results to this problem and show that a "typical" isometry and a "typical" contraction on a separable infinite-dimensional Hilbert space is embeddable.

It is well-known that unitary operators have the embedding property.

Lemma 3.1. *Every unitary operator on a Hilbert space can be embedded into a unitary C_0-group.*

Proof. Let T be a unitary operator on a Hilbert space H. Then by the spectral theorem, see e.g. Halmos [10], T is isomorphic to a direct sum of multiplication operators M given by $Mf(e^{i\varphi}) := e^{i\varphi} f(e^{i\varphi})$ on $L^2(\Gamma, \mu)$ for the unit circle Γ and some measure μ. Each such operator can be embedded into the unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ given by

$$U(t)f(e^{i\varphi}) := e^{it\varphi} f(e^{i\varphi}), \quad \varphi \in [0, 2\pi],\ t \in \mathbb{R}.$$

□

A direct corollary of Theorem 2.2 and the above lemma is the following category result for embeddable operators.

Theorem 3.2. *On a separable infinite-dimensional Hilbert space, the set of all embeddable operators is residual in the set $\mathcal{I}$ of all isometries (for the* *strong operator topology) and in the set $\mathcal{C}$ of all contractions (for the weak operator topology).*

In other words, a "typical" isometry and a "typical" contraction can be embedded into a C_0-semigroup. This is an operator theoretic counterpart to a recent result of de la Rue, de Sam Lazaro [17] in ergodic theory stating that a "typical" measure preserving transformation can be embedded into a continuous measure preserving flow.

Remark 3.3. In particular, a "typical" contraction (on a separable infinite-dimensional Hilbert space) has roots of all orders. This is an operator theoretic analogue of a result of King [12] from ergodic theory. See also Ageev [1] and Stepin, Eremenko [19] for related results.

4. Continuous case

In this section we consider a continuous analogue of Theorem 2.2. More precisely, we show that, in the Baire sense, a "typical" isometric C_0-semigroup on a separable Hilbert space is a unitary group, and discuss the difficulties for contracion semigroups.

Let H be a separable infinite-dimensional Hilbert space. Consider the set $\mathcal{I}_{\mathbb{R}}$ of all isometric C_0-semigroups on H endowed with the metric

$$d(T(\cdot), S(\cdot)) := \sum_{n,j=1}^{\infty} \frac{\sup_{t\in[0,n]} \|T(t)x_j - S(t)x_j\|}{2^j \|x_j\|} \quad \text{for } T(\cdot),\ S(\cdot) \in \mathcal{I},$$

where $\{x_j\}_{j=0}^{\infty}$ is a fixed dense subset of $H \setminus \{0\}$. This metric corresponds to strong convergence of semigroups uniform on compact time intervals. By the first Trotter-Kato theorem, see e.g. Engel, Nagel [8, Theorem III.4.8], this metric space is complete. The set of all unitary C_0-groups on H is denoted by $\mathcal{U}_{\mathbb{R}}$.

The following is a continuous analogue of Theorem 2.2 for isometric C_0-semigroups.

Theorem 4.1. *The set $\mathcal{U}_{\mathbb{R}}$ of all unitary C_0-groups is residual in the set of all isometric C_0-semigroups $\mathcal{I}_{\mathbb{R}}$ for the topology of strong convergence uniform on compact time intervals.*

Proof. Take first a set $\{t_l\}_{l=1}^{\infty}$ dense in $(0,\infty)$, and let $T(\cdot)$ be a non-invertible isometric C_0-semigroup. For every $t > 0$, $\operatorname{rg} T(t)$ is closed and different from H, therefore for every $l \in \mathbb{N}$ there exists $j \in \mathbb{N}$ with $\operatorname{dist}(x_j, \operatorname{rg} T(t_l)) > 0$. In particular this implies

$$\mathcal{I}_{\mathbb{R}} \setminus \mathcal{U}_{\mathbb{R}} = \bigcup_{jkl=1}^{\infty} M_{jkl} \quad \text{with} \quad M_{jkl} := \left\{ T(\cdot) : \operatorname{dist}(x_j, \operatorname{rg} T(t_l)) > \frac{1}{k} \right\}.$$

It remains to show that every set M_{jkl} is nowhere dense in $\mathcal{I}_{\mathbb{R}}$.

As shown in Eisner, Serény [6, Prop. 3.3], one can approximate every isometric C_0-semigroup strongly and uniformly on compact time intervals by periodic unitary C_0-groups. In particular, $\mathcal{U}_\mathbb{R}$ is dense in $\mathcal{I}_\mathbb{R}$, and it suffices to show that

$$\mathcal{U}_\mathbb{R} \cap \overline{M_{jkl}} = \emptyset \quad \text{for all } j,k,l \in \mathbb{N}.$$

Assume the contrary, i.e., there exists a sequence $\{T_n(\cdot)\}_{n=1}^\infty$ of semigroups in some M_{jkl} converging strongly and uniformly on compact time intervals to a unitary C_0-group $U(\cdot)$. Then we have in particular $\lim_{n\to\infty} T_n(t_l)y = U(t_l)y = x_j$ for $y := U(-t_l)x_j$. This however contradicts the condition $\operatorname{dist}(x_j, \operatorname{rg} T(t_j)) > \frac{1}{k}$, and therefore each M_{jkl} is nowhere dense in $\mathcal{I}_\mathbb{R}$. □

Remark 4.2. In contrary to the discrete case, it is not clear whether a "typical" contractive C_0-semigroup is a unitary group. The main difficulty is that the set of all contractive C_0-semigroups fails to be complete (or compact) for the natural metric coming from the weak convergence uniform on compact time intervals, see Eisner, Serény [7], and it is not clear whether this space is at least Čech complete. Note that density of unitary groups in the set of all contractive C_0-semigroups for this topology still holds, see Król [13].

Remark 4.3. We finally mention that by the proof of Theorem 2.2 (first part), the set of all invertible isometries is residual in the set of all isometries for the strong operator topology on every separable Banach space for which it is dense for this topology. The same holds for isometric C_0-semigroups and the topology coming from strong convergence uniform on compact time intervals by the proof of Theorem 4.1. So the question of residuality in this context reduces to the question of density which is formally a much weaker property.

Acknowledgement. The author is grateful to Rainer Nagel for his valuable comments.

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