

THE GROWTH OF A C_0 -SEMGROUP CHARACTERISED BY ITS COGENERATOR

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ABSTRACT. We characterise contractivity, boundedness and polynomial growth for a C_0 -semigroup in terms of its cogenerator V (or the Cayley transform of the generator) or its resolvent. In particular, we extend results of Gomilko and Brenner, Thomée and show that polynomial growth of a semigroup implies polynomial growth of its cogenerator. As is shown by an example, the result is optimal. For analytic semigroups we show that the converse holds, i.e., polynomial growth of the cogenerators implies polynomial growth of the semigroup. In addition, we show by simple examples in $(\mathbb{C}^2, \|\cdot\|_p)$, $p \neq 2$, that our results on the characterization of contractivity are sharp. These examples also show that the famous Foiaş-Sz.-Nagy theorem on cogenerators of contractive C_0 -semigroups on Hilbert spaces fails in $(\mathbb{C}^2, \|\cdot\|_p)$ for $p \neq 2$.

1. INTRODUCTION

In the theory of C_0 -semigroups, it is a major task to characterise the asymptotics of the semigroup $(T(t))_{t \geq 0}$ in terms of its generator A . Since A usually is an unbounded operator, one uses its resolvent $R(\lambda, A)$ as a family of bounded operators. On the other hand, the (negative) Cayley transform of A defined as

$$V := (A + I)(A - I)^{-1},$$

whenever $1 \in \rho(A)$, is a bounded operator and determines A and hence the semigroup uniquely. This operator is called the *cogenerator* of the semigroup $T(\cdot)$.

In 1960, Sz.-Nagy and Foiaş characterised cogenerators of contractive C_0 -semigroups on Hilbert spaces in the following way. Here and later we denote by $\mathcal{L}(H)$ the space of linear bounded operators on H .

Theorem 1.1. (Foiaş, Sz.-Nagy [18] or [19, Theorem III.8.1]) *Let H be a Hilbert space and $V \in \mathcal{L}(H)$. Then V is the cogenerator of a contractive C_0 -semigroup if and only if V is contractive and $1 \notin P_\sigma(V)$.*

In Hilbert spaces, not only contractivity is preserved by the cogenerator. Sz.-Nagy and Foiaş showed also that a C_0 -semigroup is normal, unitary, self-adjoint, isometric, completely non-unitary and strongly stable if and only if its cogenerator is normal, unitary, self-adjoint, isometric, completely non-unitary and strongly stable, respectively (see Sz.-Nagy, Foiaş [18, Sections III.8-9]). Note that all these results strongly depend on Hilbert space techniques.

The question whether on Hilbert spaces boundedness of a C_0 -semigroup implies power boundedness of its cogenerator is still open. Gomilko [8] and Guo, Zwart [11] showed that this holds for analytic semigroups and for semigroups such that the semigroup generated by the inverse of

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the generator is bounded as well. Gomilko [8] proved that boundedness of a semigroup implies that the powers V^n of its cogenerator grow at most like $\ln(1+n)$.

For Hilbert spaces Guo and Zwart [11] proved that boundedness of the semigroup implies uniform boundedness of $V^n(V-I)$. On Banach spaces one can only show that $\|V^n\| \leq c(1+\sqrt{n})$ for some c , see Brenner, Thomée [1]. Recently, Piskarev, Zwart [16] proved that this result (and even the estimate $\|V^n(V-I)\| \leq c(1+\sqrt{n})$) is optimal. On the other side, Gomilko, Zwart, Tomilov [9] showed that on every l^p -space, $1 < p < \infty$, $p \neq 2$ there exists a contraction V with $1 \notin P_\sigma(V)$ which is not the cogenerator of a C_0 -semigroup. An analogous example on the space c_0 follows from Komatsu [13, pp. 343–344], see Section 4.

In this paper we study the connection between contractivity, boundedness and polynomial growth of a C_0 -semigroup on a Banach space and analogous properties of its cogenerator.

We first characterise cogenerators of contractive and bounded semigroups in terms of the behaviour of the resolvent of the cogenerator near the point 1 (Section 2). This is an analogue to the Hille–Yosida theorem for generators. Then, in Section 3, we generalise the above Foias–Sz.-Nagy theorem to Banach spaces using the cogenerator itself and some naturally related operators. Note that although the proofs in this section are easy, the presented method seems to be promising. We also discuss the connection with the inverse of a generator and growth of the corresponding semigroup (see Zwart [21, 22] Gomilko, Zwart [10], Gomilko, Zwart, Tomilov [9], de Laubenfels [14] for this aspect).

In Section 4 we present elementary examples of non-contractive semigroups with contractive cogenerators and conversely contractive semigroups with non-contractive cogenerators. In particular, we show that the Foias–Sz.-Nagy theorem fails for semigroups on $(\mathbb{C}^2, \|\cdot\|_p)$ with $p \neq 2$.

Finally, in Section 5 we show that polynomial growth of a C_0 -semigroup on a Banach space implies polynomial growth of (the powers of) its cogenerator. We also show that the provided growth is the best possible. This generalises the result of Brenner, Thomée [1] and extends the result of Gomilko [8] mentioned above. Conversely, we prove that for analytic semigroups polynomial growth of the cogenerator is also sufficient for polynomial growth of the semigroup.

2. CHARACTERISATIONS VIA THE RESOLVENT

In this section we give a resolvent characterisation of cogenerators of bounded and contractive C_0 -semigroups on Banach spaces. This can be viewed as an analogue of the Hille–Yosida theorem for generators.

We first need the following easy lemma.

Lemma 2.1. *Let X be a Banach space and $V \in \mathcal{L}(X)$ such that $1 \notin P_\sigma(V)$. Then the operator $A : \text{rg}(V-I) \rightarrow X$ defined by $A := (V+I)(V-I)^{-1}$ is closed and satisfies the following:*

- 1) $\rho(A) \setminus \{1\} = \left\{ \frac{\mu+1}{\mu-1}, 1 \neq \mu \in \rho(V) \right\}$;
- 2) For $\lambda \in \rho(A) \setminus \{1\}$ one has

$$(1) \quad R(\lambda, A) = \frac{1}{\lambda-1}(I-V)R\left(\frac{\lambda+1}{\lambda-1}, V\right).$$

Proof. By $A = I + 2(V-I)^{-1}$, A is closed and $1 \in \rho(A)$. Assertion 1) follows from the spectral mapping theorem for $(V-I)^{-1}$ (see e.g. Engel, Nagel [6, Theorem IV.1.13]), while assertion 2)

follows from

$$\begin{aligned}\lambda I - A &= (\lambda V - \lambda I - V - I)(V - I)^{-1} \\ &= (V(\lambda - 1) - (\lambda + 1))(V - I)^{-1} = (\lambda - 1) \left(\frac{\lambda + 1}{\lambda - 1} - V \right) (V - I)^{-1}\end{aligned}$$

for every $\lambda \neq 1$. \square

We now characterise cogenerators of contractive C_0 -semigroups on Banach spaces. The necessary and sufficient condition uses the behaviour of the resolvent of V near the point 1.

Theorem 2.2. *Let X be a Banach space and $V \in \mathcal{L}(X)$. Then the following conditions are equivalent.*

- (i) V is the cogenerator of a contraction C_0 -semigroup on X .
- (ii) $V - I$ is injective and has dense range; $(1, \infty) \in \rho(V)$ and

$$(2) \quad \|(I - V)R(\mu, V)\| \leq \frac{2}{\mu + 1} \quad \text{for all } \mu > 1.$$

- (iii) $V - I$ is injective and has dense range; there exists $\mu_0 > 1$ such that $(1, \mu_0) \in \rho(V)$ and

$$(3) \quad \|(I - V)R(\mu, V)\| \leq \frac{2}{\mu + 1} \quad \text{for all } \mu \in (1, \mu_0).$$

Proof. We first note that injectivity and dense range of the operator $V - I$ is necessary for every cogenerator V . Assume now $V - I$ to be injective and to have dense range.

Define $A := (V + I)(V - I)^{-1}$ which is densely defined. By the Hille–Yosida theorem A generates a contraction semigroup if and only if some $(\lambda_0, \infty) \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for all $\lambda > \lambda_0 \geq 0$. Note that for $\mu := \frac{\lambda+1}{\lambda-1}$, $\lambda > \lambda_0 > 1$ holds if and only if $1 < \mu < \mu_0$ for $\mu_0 := \frac{\lambda_0+1}{\lambda_0-1}$. Moreover, by Lemma 2.1 we have for $1 < \mu \in \rho(V)$ that $1 < \lambda := \frac{\mu+1}{\mu-1} \in \rho(A)$ and

$$(4) \quad \lambda R(\lambda, A) = \frac{\lambda}{\lambda - 1} (I - V)R(\mu, V) = \frac{\mu + 1}{2} (I - V)R(\mu, V).$$

This proves the equivalence of (i) and (iii). Using the same arguments one shows (i) \Leftrightarrow (ii). \square

Analogously, one proves the following resolvent characterisation of cogenerators of bounded C_0 -semigroups on Banach spaces.

Theorem 2.3. *Let X be a Banach space, $V \in \mathcal{L}(X)$ and $M \geq 1$. Then the following conditions are equivalent.*

- (i) V is the cogenerator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on X satisfying $\|T(t)\| \leq M$ for all $t \geq 0$.
- (ii) $V - I$ is injective and has dense range; $(1, \infty) \in \rho(V)$ and

$$(5) \quad \|[(I - V)R(\mu, V)]^n\| \leq \frac{2^n M}{(\mu + 1)^n} \quad \text{for all } \mu > 1, n \in \mathbb{N}.$$

- (iii) $V - I$ is injective and has dense range; there exists $\mu_0 > 1$ such that $(1, \mu_0) \in \rho(V)$ and

$$(6) \quad \|[(I - V)R(\mu, V)]^n\| \leq \frac{2^n M}{(\mu + 1)^n} \quad \text{for all } \mu \in (1, \mu_0), n \in \mathbb{N}.$$

Remarks 2.4.

- (1) One can replace the intervals $(1, \infty)$ and $(1, \mu_0)$ in Theorems 2.2 and 2.3 by a sequence of numbers $\mu_n > 1$ converging to 1. Indeed, from the proof of the Hille–Yosida theorem follows that it suffices to check the condition on the resolvent only for a sequence $\{\lambda_n\}_{n=0}^{\infty} \subset (0, \infty)$ converging to infinity.
- (2) Note that $(I - V)R(\mu, V) = I - (\mu - 1)R(\mu, V)$. Therefore one can replace the estimates in (ii) and (iii) in Theorems 2.2 and 2.3 by

$$\|I - (\mu - 1)R(\mu, V)\| \leq \frac{2}{\mu + 1}$$

or

$$\|[I - (\mu - 1)R(\mu, V)]^n\| \leq \frac{2^n M}{(\mu + 1)^n},$$

respectively.

- (3) Conditions (ii) and (iii) in Theorem 2.3 involve all powers of the resolvent of V and are therefore difficult to check. Therefore it is desirable to find a simpler (sufficient) condition on a bounded operator V to be the cogenerator of a bounded C_0 -semigroup.

3. CHARACTERISATION VIA COGENERATORS OF THE RESCALED SEMIGROUPS

In this section we study the direct connection between the semigroup and its cogenerator without using the resolvent. The simplest example of such a connection is the Foias–Sz.-Nagy theorem. However, the analogous assertion does not hold on Banach spaces (see Section 4 for elementary examples). In our approach we consider the cogenerators of the rescaled semigroups.

We begin with the following observation. If A generates a contractive or bounded C_0 -semigroup, then also all operators τA do for $\tau > 0$. However, as we will see in Section 4, it is not always true that the operators

$$(7) \quad V_\tau := (\tau A + I)(\tau A - I)^{-1}, \quad \tau > 0,$$

remain contractive when V is.

The following proposition characterises generators of contraction semigroups in terms of V_τ .

Proposition 3.1. *Let A be a densely defined operator on a Banach space X . Then the following assertions are equivalent.*

- (i) A generates a contraction C_0 -semigroup on X .
(ii) $(0, \infty) \subset \rho(A)$ and the operators V_τ satisfy

$$\|V_\tau - I\| \leq 2 \quad \text{for all } \tau > 0.$$

- (iii) There exists $\tau_0 > 0$ such that $(\frac{1}{\tau_0}, \infty) \subset \rho(A)$ and the operators V_τ satisfy

$$\|V_\tau - I\| \leq 2 \quad \text{for all } 0 < \tau < \tau_0.$$

Proof. By the formula

$$V_\tau = (A + tI)(A - tI)^{-1} = I - 2tR(t, A)$$

for $t := \frac{1}{\tau}$ we immediately obtain that

$$(8) \quad tR(t, A) = \frac{I - V_\tau}{2}.$$

Then the proposition follows from the Hille–Yosida theorem. \square

We now obtain the Foias–Sz.-Nagy theorem as a corollary of the above proposition.

Corollary 3.2. *Let $V \in \mathcal{L}(H)$ for a Hilbert space H . Then V is the cogenerator of a contractive C_0 -semigroup if and only if V is contractive and $1 \notin P_\sigma(V)$.*

Proof. Assume that V is contractive and $1 \notin P_\sigma(V)$. By Lemma 2.1, the operator $A := (V + I)(V - I)^{-1}$ satisfies $(0, \infty) \subset \rho(A)$. Moreover, it is densely defined by the mean ergodic theorem, see Yosida [20, Theorem VIII.3.2].

By Proposition 3.1 it is enough to show that contractivity of $V = V_1$ implies that all operators V_τ , $\tau > 0$, are contractive. Take $\tau > 0$ and $x \in X$. For $t := \frac{1}{\tau}$ and $y := -R(t, A)x = (A - tI)^{-1}x$ we have

$$\begin{aligned} (9) \quad \|V_\tau x\|^2 - \|x\|^2 &= \|(A + tI)(A - tI)^{-1}x\|^2 - \|x\|^2 \\ &= \langle (A + tI)y, (A + tI)y \rangle - \langle (A - tI)y, (A - tI)y \rangle = 4t \operatorname{Re} \langle Ay, y \rangle, \end{aligned}$$

so contractivity of V_τ is independent of τ .

Conversely, if V is the cogenerator of a contractive C_0 -semigroups with generator A , then contractivity of V follows from (9) for $\tau = 1$ and the Lumer–Phillips theorem. Moreover, by $I - V = 2R(1, A)$ we have $1 \notin P_\sigma(V)$. \square

Remark 3.3. As we see from the above proof, the following nice property holds for cogenerators of C_0 -semigroups on Hilbert spaces: Contractivity of V automatically implies contractivity of every V_τ , $\tau > 0$. As we will see in the next sections, on Banach spaces this property fails in general.

A result analogous to Proposition 3.1 holds for generators of bounded C_0 -semigroups as well.

Proposition 3.4. *Let A be a densely defined operator on a Banach space X . Then the following assertions are equivalent.*

- (i) A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ satisfying $\|T(t)\| \leq M$ for every $t \geq 0$.
- (ii) $(0, \infty) \subset \rho(A)$ and the operators V_τ satisfy

$$\left\| \left[\frac{V_\tau - I}{2} \right]^n \right\| \leq M \quad \text{for all } \tau > 0 \text{ and } n \in \mathbb{N}.$$

- (iii) There exists τ_0 such that $(\frac{1}{\tau_0}, \infty) \subset \rho(A)$ and the operators V_τ satisfy

$$\left\| \left[\frac{V_\tau - I}{2} \right]^n \right\| \leq M \quad \text{for all } 0 < \tau < \tau_0 \text{ and } n \in \mathbb{N}.$$

The proof follows from formula (8) and the Hille–Yosida theorem for bounded semigroups.

Propositions 3.1 and 3.4 imply the following sufficient condition on Banach spaces being analogous to the one of Foias and Sz.-Nagy.

Theorem 3.5. *Let A be densely defined on a Banach space X . Then the following assertions hold.*

- (a) *If there exists $\tau_0 > 0$ such that $(\frac{1}{\tau_0}, \infty) \subset \rho(A)$ and the operators V_τ are contractive for every $\tau \in (0, \tau_0)$, then A generates a contractive C_0 -semigroup.*
- (b) *If there exists $\tau_0 > 0$ such that $(\frac{1}{\tau_0}, \infty) \subset \rho(A)$ and the operators V_τ satisfy $\|V_\tau^n\| \leq M$ for all $\tau \in (0, \tau_0)$ and $n \in \mathbb{N}$, then A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ with $\|T(t)\| \leq M$ for all $t \geq 0$.*

Proof. Assertion (a) follows immediately from Proposition 3.1. To prove (b) assume that $\|V_\tau^n\| \leq M$. Then we have

$$\left\| \left[\frac{V_\tau - I}{2} \right]^n \right\| \leq \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} \|V_\tau^j\| \leq \frac{M \cdot 2^n}{2^n} = M$$

and (b) follows from Proposition 3.4. \square

Remark 3.6. In Proposition 3.1, Proposition 3.4 and Theorem 3.5 it suffices to consider $\{V_{\tau_n}\}_{n=1}^\infty$ for a sequence $\tau_n > 0$ converging to zero. This again follows from the fact that in the Hille–Yosida theorem it suffices to check the resolvent condition only for a sequence $\{\lambda_n\}_{n=1}^\infty \subset (0, \infty)$ converging to infinity, which follows directly from its proof.

We finish this section by the following observation. If V is contractive or power bounded, then so is the operator $-V$. Note that $-V$ is the cogenerator of the semigroup generated by A^{-1} if A^{-1} generates a C_0 -semigroup. However, contractivity or boundedness of $(e^{tA})_{t \geq 0}$ does not imply the same property of $(e^{tA^{-1}})_{t \geq 0}$ (see Zwart [21] and also Section 4 for elementary examples). We refer to Zwart [21, 22], Gomilko, Zwart [10], Gomilko, Zwart, Tomilov [9], de Laubenfels [14] for further information on this aspect.

Moreover, we have the following relation.

Remark 3.7. Assume that $(0, \infty) \subset \rho(A)$ and A^{-1} exists as a densely defined operator. Then we have

$$(10) \quad V_{\tau, A^{-1}} = (\tau A^{-1} + I)(\tau A^{-1} - I)^{-1} = (\tau I + A)(\tau I - A)^{-1} = -V_{\frac{1}{\tau}}.$$

So we see that contractivity (uniform power boundedness) of V_τ for *all* $\tau > 0$ or even for some sequences $\tau_{n,1} \rightarrow 0$ and $\tau_{n,2} \rightarrow \infty$ implies that A and A^{-1} both generate a contractive (bounded) C_0 -semigroup.

Conversely, Gomilko [8] and Guo, Zwart [11] showed for Hilbert spaces that if A and A^{-1} both generate bounded semigroups, then the cogenerator V is power bounded (and hence so are all operators V_τ by the rescaling argument).

It is an interesting and open question whether contractivity (boundedness) of the semigroups generated by A and A^{-1} implies contractivity (power boundedness) of the cogenerator V on Banach spaces.

4. EXAMPLES

In [9] Gomilko, Zwart and Tomilov show that for every $p \in [1, \infty)$, $p \neq 2$, there exists a contractive operator V on l^p such that $(V - I)^{-1}$ exists as a densely defined operator, but V is not a cogenerator of a C_0 -semigroup.

The idea of their construction is the following. One considers the generator $A := S_l - I$ for the left shift S_l given by $S_l(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$. The corresponding cogenerator $V = S_l R(2, S_l)$ is a contraction by contractivity of S_l and the Neumann series for the resolvent. Further, one shows that A^{-1} does not generate a C_0 -semigroup which is the hard part. As a consequence one obtains that the contraction $-V$ is not a cogenerator of a C_0 -semigroup.

Komatsu [13, pp. 343–344] showed that the operator $A := S_r - I$ for the right shift S_r given by $S_r(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$ on c_0 satisfies the same properties, i.e., A^{-1} does not generate a C_0 -semigroup. Since the cogenerator V corresponding to A is contractive as well, we have a contraction on c_0 which is not a cogenerator of a C_0 -semigroup.

The following example shows that even for $X = \mathbb{C}^2$ the semigroup cogenerated by a contraction need not to be contractive. Note that the cogeneration property is no problem here.

In particular, this example and Example 4.3 show that none of the implications in the Foias–Sz.-Nagy theorem holds even on two-dimensional Banach spaces.

Example 4.1. Take $X = \mathbb{C}^2$ considered with $\|\cdot\|_p$, $p \neq 2$, and $A := \begin{pmatrix} -1 & \beta \\ 0 & -2 \end{pmatrix}$ for $\beta > 0$. The semigroup generated by A is

$$T(t) = \begin{pmatrix} e^{-t} & \beta(e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{pmatrix}, \quad t \geq 0.$$

We first show that $(T(t))_{t \geq 0}$ is not contractive for appropriate β . Consider first $p = \infty$ and $\beta > 1$. We have $\|T(t)\| = (1 + \beta)e^{-t} - \beta e^{-2t} =: f(t)$. Since $f(0) = 1$ and $f'(0) = \beta - 1 > 0$, the semigroup is not contractive.

Let now $2 < p < \infty$ and define $\beta := (3^p - 1)^{\frac{1}{p}}$. Then

$$\left\| T(t) \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|_p^p = (e^{-t}x + \beta(e^{-t} - e^{-2t}))^p + e^{-2pt} =: f_x(t) \quad \text{for } x > 0.$$

We have $f_x(0) = x^p + 1 = \|(x, 1)\|_p^p$. Further, $f'_x(0) = px^{p-1}(\beta - x) - 2p$, so the semigroup is not contractive if $x^{p-1}(\beta - x) > 2$ for some $x > 0$. This is the case for $x := \frac{\beta}{2}$. Indeed, $x^{p-1}(\beta - x) = \left(\frac{\beta}{2}\right)^{p-1}(\beta - \frac{\beta}{2}) = \frac{3^p - 1}{2^p} > 2$ for $p > 2$.

We now show that the cogenerator V is contractive for $\beta \leq 3$ if $p = \infty$ and $\beta := (3^p - 1)^{\frac{1}{p}}$ if $p \in (2, \infty)$. The cogenerator is given by

$$V = (I + A)(A - I)^{-1} = \begin{pmatrix} 0 & \beta \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\beta}{6} \\ 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\beta}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

So for $p = \infty$ we have $\|V\| = \max\{\frac{1}{3}, \frac{\beta}{3}\} \leq 1$ for $\beta \leq 3$. For $p \in (2, \infty)$ we have $\|V\|^p = \left\| \begin{pmatrix} -\frac{\beta}{3} & \frac{1}{3} \end{pmatrix} \right\|_p^p = \frac{(\beta^p + 1)}{3^p} \leq 1$ if and only if $\beta \leq (3^p - 1)^{\frac{1}{p}}$.

We see that for $p \in (2, \infty]$ there exists a contraction such that the cogenerated semigroup is not contractive. The analogous assertion for $p \in [1, 2)$ follows by duality.

Remark 4.2. From Theorem 3.5, Remark 3.6 and the above example we see that there exist contractions V (even on \mathbb{C}^2 with l^p -norm, $p \neq 2$) such that V_τ are not contractive for every τ in a small interval $(0, \tau_0)$.

The following example gives a class of contractive semigroups with non-contractive cogenerators and shows that such semigroups exist even on $(\mathbb{C}^2, \|\cdot\|_\infty)$. In particular, this provides a two-dimensional counterexample to the converse implication in the Foias–Sz.-Nagy theorem.

Example 4.3. Every operator A generating a contractive C_0 -semigroup such that A^{-1} generates a C_0 -semigroup which is not contractive leads to an example of a contractive semigroup with non-contractive cogenerator. Indeed, by the previous remark, there exists $\tau > 0$ such that V_τ is not contractive. Therefore, the operator τA generates a contractive semigroup with non-contractive cogenerator.

For a concrete example consider $X := \mathbb{C}^2$ endowed with $\|\cdot\|_\infty$ and A as in Example 4.1. Then $(e^{tA})_{t \geq 0}$ is not contractive for $\beta > 1$. We show that the semigroup generated by A^{-1} is contractive if and only if $\beta \leq 2$.

Indeed, we have $A^{-1} = \begin{pmatrix} -1 & -\frac{\beta}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$ and

$$e^{tA^{-1}} = \begin{pmatrix} e^{-t} & \beta(e^{-t} - e^{-\frac{t}{2}}) \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \quad t \geq 0.$$

Therefore $\|e^{tA^{-1}}\|_\infty = \sup\{e^{-t} + \beta(e^{-\frac{t}{2}} - e^{-t}), e^{-\frac{t}{2}}\}$. Hence $e^{tA^{-1}}$ is contractive if and only if $g(t) := e^{-t} + \beta(e^{-\frac{t}{2}} - e^{-t}) \leq 1$ for every $t > 0$. We have $g(0) = 1$ and $g'(t) = -e^{-t} + \beta(e^{-t} - \frac{1}{2}e^{-\frac{t}{2}}) = e^{-t}[\beta(1 - \frac{1}{2}e^{\frac{t}{2}}) - 1]$. Since the function $t \rightarrow 1 - \frac{1}{2}e^{\frac{t}{2}}$ is monotonically decreasing, we obtain that $g'(t) \leq 0$ for every $t \geq 0$ is equivalent to $g'(0) \leq 0$, i.e., $\beta \leq 2$.

So we see that for $1 < \beta \leq 2$ the semigroup generated by A^{-1} is contractive while the semigroup generated by A is not contractive. The rescaling procedure described above leads to a contractive semigroup (generated by τA for some τ) with non-contractive cogenerator.

Zwart [21] gives another example of an operator A generating a contractive C_0 -semigroup such that the semigroup generated by A^{-1} is not contractive and even not bounded. He takes a nilpotent semigroup on $X = C_0[0, 1]$ such that the semigroup generated by A^{-1} grows like $t^{1/4}$. By the rescaling procedure we again obtain a contractive semigroup with non-contractive cogenerator.

Remark 4.4. The above example for $2 < \beta \leq 3$ yields a contractive cogenerator V such that the semigroups generated by both operators A and A^{-1} are not contractive. This gives an example of a contraction V on $(\mathbb{C}^2, \|\cdot\|_\infty)$ such that operators V_τ are not contractive for every $\tau \in (0, \tau_1) \cup (\tau_2, \infty)$, $0 < \tau_1 < 1 < \tau_2$, by Remark 3.6.

5. POLYNOMIAL GROWTH

In this section we investigate the connection between polynomial growth of a C_0 -semigroup and of its cogenerator.

We first recall that a C_0 -semigroup $T(\cdot)$ (a bounded operator V) is said to be of *polynomial growth* if $\|T(t)\| \leq p(t)$ ($\|V^n\| \leq p(n)$) holds for some polynomial p and every $t \geq 0$ ($n \in \mathbb{N}$).

This property has been characterised via the resolvent of the generator in Malejki [15], Eisner [4], Eisner, Zwart [5]. See also Gomilko [7], Shi and Feng [17] for the boundedness case.

The following result shows that polynomial growth of a C_0 -semigroup on a Banach space implies polynomial growth of its cogenerator. This generalises a result of Hersch and Kato [12] and Brenner and Thomée [1] on bounded semigroups. For the proof of this result, we need the following estimate.

Lemma 5.1. *Let $L_n^1(t)$ denote the first generalised Laguerre polynomial, i.e.,*

$$(11) \quad L_n^1(t) = \sum_{m=0}^n \frac{(-1)^m}{m!} \binom{n+1}{n-m} t^m.$$

Then we have for fixed $k \in \mathbb{N}$ that

$$(12) \quad C_1 n^k \sqrt{n} \leq \int_0^\infty |L_n^1(2t)| e^{-t} t^k dt \leq C_2 n^k \sqrt{n}$$

for some constants C_1, C_2 and all $n \in \mathbb{N}$.

Proof. We begin by showing that the integral can be bounded from below by a constant times $n^k \sqrt{n}$. For this we use the idea in [2].

The Laplace transform of $g(t) := -2L_{n-1}^1(2t)e^{-t}$ equals

$$G(s) = \left(\frac{s-1}{s+1} \right)^n - 1.$$

Thus the Laplace transform of $f(t) := g(t)t^k$, $k \geq 1$, is given by

$$(13) \quad F(s) = (-1)^k \frac{d^k}{ds^k} G(s) = \sum_{m=1}^k \left(\frac{s-1}{s+1} \right)^{n-m} P_{n,m} \left(\frac{1}{s+1} \right)$$

where $P_{n,m}$ are polynomials with order $\leq 2^k$.

Since $\frac{s-1}{s+1}$ has norm one on the imaginary axis, we can write $F_m(s) := \left(\frac{s-1}{s+1} \right)^{n-m} \frac{1}{(s+1)^{\ell_m}}$ on the imaginary axis as $\frac{1}{(i\omega+1)^{\ell_m}} e^{i(n-m)\Phi(\omega)}$ for some real-valued function Φ . Furthermore, $\Phi''(\omega)$ is non-zero for almost all ω . Note that $\Phi(\omega)$ equals $-2 \arctan(\omega) + \pi$. From Corollary 1.5.1 of [2], we know that the induced multiplier norm¹ of $\frac{1}{(s+1)^{\ell_m}} \left(\frac{s-1}{s+1} \right)^{n-m}$ on L^∞ is larger or equal to $c_m \sqrt{n}$ for some constant c_m . This induced norm equals the $L^1(0, \infty)$ -norm of the function $f_m(t)$, i.e., the inverse Laplace transform of $F_m(s)$. The constant $C_{n,k} = \frac{n!}{(n-k)!} \alpha_k$ appearing in $P_{n,k}$ is the one with the highest power of n . Thus for n large, the $L^1(0, \infty)$ -norm of f behaves like the $L^1(0, \infty)$ -norm of $C_{n,k} f_k(t)$, which is larger or equal to $C_1(n^k \sqrt{n})$.

This proves that there exists a lower bound for the $L^1(0, \infty)$ -norm of f which is of the order $n^k \sqrt{n}$. To prove the upper bound, we use the Carlson estimate, see [1]:

$$(14) \quad \|f\|_1 \leq 2\sqrt{\|f\|_2 \|tf\|_2},$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ denote the $L^1(0, \infty)$ -norm and $L^2(0, \infty)$ -norm, respectively. For completeness we include the proof of this estimate. Take $c > 0$ and observe

$$\begin{aligned} \int_0^\infty |f(t)| dt &= \int_0^c |f(t)| dt + \int_c^\infty |f(t)| dt \\ &\leq \sqrt{c} \|f\|_2 + \sqrt{\int_c^\infty \frac{1}{t^2} dt \int_c^\infty |tf(t)|^2 dt} \leq \sqrt{c} \|f\|_2 + \sqrt{\frac{1}{c} \|tf(t)\|_2}, \end{aligned}$$

where we have used the Cauchy-Schwarz estimate twice. Choosing $c = \frac{\|tf\|_2}{\|f\|_2}$, we find (14).

So to obtain a $L^1(0, \infty)$ -norm estimate of $f(t) := L_n^1(2t)e^{-t}t^k$ we must estimate the $L^2(0, \infty)$ -norm of this function and of t times it.

In order to estimate the $L^2(0, \infty)$ -norms, we use Parseval identity for the Fourier transform, i.e., $\|f\|_2^2 = \frac{1}{2\pi} \|\hat{f}\|_2^2$. Furthermore, the Fourier transform of f equals the Laplace transform of f restricted to the imaginary axis. Finally, we have that the Fourier transform of tf equals $i(\hat{f})'(\omega)$.

¹In the formulation of this corollary it is assumed that the function in front of $e^{i(n-m)\Phi(\omega)}$ has compact support. However, this is not used in the proof of this corollary. In the proof it is assumed that there exists a C^∞ function, χ , with compact support such that χ divided by the function (in front of the exponential) has compact support, lies in C^∞ , and Φ'' has no zeros in the support of χ

Using (13), we see that

$$(15) \quad \|\hat{f}\|_2 \leq \sum_{m=1}^k \left\| \left(\frac{i\omega - 1}{i\omega + 1} \right)^{n-m} P_{n,m} \left(\frac{1}{i\omega + 1} \right) \right\|_2 = \sum_{m=1}^k \left\| P_{n,m} \left(\frac{1}{i\omega + 1} \right) \right\|_2,$$

where we have used that $\frac{i\omega-1}{i\omega+1}$ has absolute value one. So we must estimate the $L^2(-\infty, \infty)$ -norm of $P_{n,m} \left(\frac{1}{i\omega+1} \right)$. Since we are interested in the behaviour with respect to n , and since the order of this polynomial is independent of n , we may look at the coefficients. Again we have that $P_{n,k}$ has the coefficient $C_{n,k} = \frac{n!}{(n-k)!} \alpha_k$ with of the highest power of n . So combining this with (15) we see that

$$(16) \quad \|f\|_2 = \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_2 \leq \gamma_1 n^k$$

for some constant γ_1 .

We further have that $F'(s)$ equals

$$F'(s) = \sum_{m=1}^k \left(\frac{s-1}{s+1} \right)^{n-m-1} \frac{1}{(s+1)^2} \left[2(n-m)P_{n,m} \left(\frac{1}{s+1} \right) - \left(\frac{s-1}{s+1} \right) P'_{n,m} \left(\frac{1}{s+1} \right) \right].$$

By a similar argument as above, we have that the L^2 -norm on the imaginary axis of this function is bounded by a constant times n^{k+1} . Hence

$$(17) \quad \|tf\|_2 = \frac{1}{\sqrt{2\pi}} \|i(\hat{f})'\|_2 \leq \gamma_2 n^{(k+1)}.$$

Combining (14), (16) and (17) shows that $\|f\|_1 \leq C_2 n^k \sqrt{n}$. \square

Theorem 5.2. *Let $T(\cdot)$ be a C_0 -semigroup on a Banach space with cogenerator V . If $\|T(t)\| \leq M(1+t^k)$ for some M and $k \in \mathbb{N} \cup \{0\}$ and every $t \geq 0$, then $\|V^n\| \leq C_1 n^{k+\frac{1}{2}}$ some C_1 and every $n \in \mathbb{N}$.*

Furthermore, this estimate cannot be improved, i.e., for every $k \in \mathbb{N} \cup \{0\}$ there exists a Banach space and a C_0 -semigroup satisfying $\|T(t)\| = O(t^k)$, t large, such that $\|V^n\| \geq C_2 n^{k+\frac{1}{2}}$ for some $C_2 > 0$ and every $n \in \mathbb{N}$.

Proof. The proof is based on the following relation between the semigroup and the cogenerator

$$(18) \quad V^n h = h - 2 \int_0^\infty L_{n-1}^1(2t) e^{-t} T(t) h dt,$$

where $L_n^1(t)$ is again the first generalised Laguerre polynomial, see e.g. Gomiiko [8] or Butzer and Westpal [3].

Using the fact that the semigroup is of polynomial growth we find that

$$(19) \quad \|V^n\| \leq 1 + 2M \int_0^\infty |L_{n-1}^1(2t)| e^{-t} (1+t^k) dt.$$

Now by Lemma 5.1 we conclude that $\|V^n\| \leq C_1 n^{k+\frac{1}{2}}$ some C_1 and every $n \in \mathbb{N}$.

It remains to show that this estimate is sharp. Let $X_0 := C_0([0, \infty))$ be the Banach space of continuous functions on $[0, \infty)$ vanishing at infinity, considered with the maximum-norm, $\|\cdot\|_\infty$. Let $(T_0(t))_{t \geq 0}$ be the left-shift semigroup on X_0 , i.e., $(T_0(t)f)(\eta) = f(t+\eta)$. This is a strongly continuous, contractive semigroup on X_0 .

As Banach space X we take now $(k+1)$ copies of X_0 , again with the maximum norm $\|x\|_X = \max_{m=1,\dots,k+1} \|x_m\|_\infty$. The infinitesimal generator A is given by

$$A = \begin{pmatrix} A_0 & I & 0 & \cdots & 0 \\ 0 & A_0 & I & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & A_0 & I \\ 0 & \cdots & 0 & 0 & A_0 \end{pmatrix},$$

where A_0 is the infinitesimal generator of $T_0(t)$. The semigroup generated by A is given by

$$(20) \quad T(t) = \begin{pmatrix} T_0(t) & tT_0(t) & \frac{t^2}{2!}T_0(t) & \cdots & \frac{t^k}{k!}T_0(t) \\ 0 & T_0(t) & tT_0(t) & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & T_0(t) & tT_0(t) \\ 0 & \cdots & 0 & 0 & T_0(t) \end{pmatrix}.$$

Since $T_0(\cdot)$ is a contraction semigroup, it is easy to see that $\|T(t)\| \approx t^k$ for t large.

In order to give the idea of the further construction, we first choose as h in (18) the following function $h(\eta) = (0, \dots, 0, \text{sign}(L_{n-1}^1(2\eta)))^T$. Using the definition of $T(\cdot)$, $T_0(\cdot)$ and (18), we have

$$(V^n h)(\eta) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \text{sign}(L_{n-1}^1(2\eta)) \end{pmatrix} - 2 \int_0^\infty L_{n-1}^1(2t) e^{-t} \begin{pmatrix} \frac{t^k}{k!} \cdot \text{sign}(L_{n-1}^1(2(t+\eta))) \\ \frac{t^{k-1}}{(k-1)!} \cdot \text{sign}(L_{n-1}^1(2(t+\eta))) \\ \vdots \\ t \cdot \text{sign}(L_{n-1}^1(2(t+\eta))) \\ \text{sign}(L_{n-1}^1(2(t+\eta))) \end{pmatrix} dt.$$

So

$$(21) \quad \begin{aligned} \|V^n h\|_X &\geq \|(V^n h)_1\|_\infty \geq |(V^n h)_1(0)| = 2 \int_0^\infty L_{n-1}^1(2t) e^{-t} \frac{t^k}{k!} \cdot \text{sign}(L_{n-1}^1(2t)) dt \\ &= \frac{2}{k!} \int_0^\infty |L_{n-1}^1(2t)| e^{-t} t^k dt. \end{aligned}$$

Since by Lemma 5.1 this last integral behaves like $n^k \sqrt{n}$, we see that the estimate is sharp. Since h is not continuous and is not vanishing at infinity, we see that the above construction is not finished. However, for every $\varepsilon > 0$ one can find a $h_\varepsilon \in X$ such that the equality in (21) holds within an error margin of ε , see Example 3.5 of [21]. \square

Remark 5.3. For Hilbert spaces one can obtain a sharper result. For $k = 0$, i.e., for bounded semigroups, Gomilko [8] proved that on Hilbert spaces $\|V^n\|$ grows at most like $\ln(n+1)$. It is unknown whether this result is optimal.

Our next result shows that for analytic semigroups the converse implication in Theorem 5.2 holds, i.e., polynomial growth of the cogenerator implies polynomial growth of the semigroup.

Theorem 5.4. *Let $T(\cdot)$ be an analytic C_0 -semigroup on a Banach space with cogenerator V . If $\|V^n\| \leq Cn^k$ for some $C, k \geq 0$ and every $n \in \mathbb{N}$, then $\|T(t)\| \leq M(1 + t^{2k+1})$ for some M and every $t \geq 0$.*

Proof. Assume $\|V^n\| \leq Cn^k$ for some $C, k \geq 0$ and every $n \in \mathbb{N}$. Then $r(V) \leq 1$ and therefore $\lambda \in \rho(A)$ for $\operatorname{Re} \lambda > 0$ by Lemma 2.1. Our aim is to show that there exist $a_0, M > 0$ such that

$$(22) \quad \|R(\lambda, A)\| \leq \frac{M}{(\operatorname{Re} \lambda)^{k+1}} \text{ for all } \lambda \text{ with } 0 < \operatorname{Re} \lambda < a_0,$$

$$(23) \quad \|R(\lambda, A)\| \leq M \text{ for all } \lambda \text{ with } \operatorname{Re} \lambda \geq a_0.$$

By Eisner, Zwart [5, Thm. 2.1] this implies growth at most like t^{2k+1} for analytic semigroups.

Take some $a_0 > \max\{0, \omega_0(T)\}$. Then (23) automatically holds and we only have to show (22).

Since $T(\cdot)$ is analytic, $R(\lambda, A)$ is uniformly bounded on $\{\lambda : |\operatorname{Im} \lambda| > b_0, 0 < \operatorname{Re} \lambda < a_0\}$ for some $b_0 \geq 0$. Moreover, $R(\lambda, A)$ is also uniformly bounded on $\{\lambda : |\operatorname{Im} \lambda| \leq b_0, \frac{1}{3} \leq \operatorname{Re} \lambda < a_0\}$ as well. Take now λ with $0 < \operatorname{Re} \lambda < \frac{1}{3}$ and $-b_0 < \operatorname{Im} \lambda < b_0$.

By Lemma 2.1 we have

$$(24) \quad \|R(\lambda, A)\| \leq \frac{1 + \|V\|}{|\lambda - 1|} \left\| R\left(\frac{\lambda + 1}{\lambda - 1}, V\right) \right\|.$$

By Eisner, Zwart [5, Thm. 2.4], growth of $\|V^n\|$ like n^k implies

$$(25) \quad \|R(\mu, V)\| \leq \frac{\tilde{C}}{(|\mu| - 1)^{k+1}} \text{ for all } \mu \text{ with } 1 < |\mu| \leq 2$$

for some constant \tilde{C} . For $\mu := \frac{\lambda+1}{\lambda-1}$ and $0 < \operatorname{Re} \lambda < \frac{1}{3}$ we have

$$|\mu| - 1 = \frac{|\lambda + 1| - |\lambda - 1|}{|\lambda - 1|} = \frac{4\operatorname{Re} \lambda}{|\lambda - 1|(|\lambda + 1| + |\lambda - 1|)} < 1.$$

Then we use (25) to obtain

$$\left\| R\left(\frac{\lambda + 1}{\lambda - 1}, V\right) \right\| \leq \frac{\tilde{C}|\lambda - 1|^{k+1}(|\lambda + 1| + |\lambda - 1|)^{k+1}}{4^{k+1}(\operatorname{Re} \lambda)^{k+1}} \leq \frac{C_1}{(\operatorname{Re} \lambda)^{k+1}}$$

for $C_1 := \tilde{C}(b_0^2 + \frac{5}{4})^{k+1}$. So, by (24),

$$\|R(\lambda, A)\| \leq \frac{C_1(1 + \|V\|)}{|\lambda - 1|(\operatorname{Re} \lambda)^{k+1}} \leq \frac{C_1(1 + \|V\|)}{2(\operatorname{Re} \lambda)^{k+1}}$$

which proves (22). □

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