SEISMIC INVERSE SCATTERING IN THE DOWNWARD CONTINUATION APPROACH

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ABSTRACT. Seismic data are commonly modeled by a linearization around a smooth background medium in combination with a high frequency approximation. The perturbation of the medium coefficient is assumed to contain the discontinuities. This leads to two inverse problems, first the linearized inverse problem for the perturbation, and second the estimation of the background, which is a priori unknown (velocity estimation). Here we give a reconstruction formula for the linearized problem using the downward continuation approach. The reconstruction is done microlocally, up to an explicitly given pseudodifferential factor that depends on the aperture. Our main result is a characterization of the wave-equation angle transform that generates the common image point gathers as an invertible Fourier integral operator, microlocally. We show that the common image point gathers are free of so called kinematic artifacts, even in the presence of caustics. The assumption is that the rays in the background that are associated with the reflections due to the medium perturbation are nowhere horizontal. Finally, pseudodifferential annihilators of the data are constructed. These annihilators detect whether the data are contained in the range of the modeling operator, which is the precise criterion in migration velocity analysis to determine whether a background medium is acceptable.

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1. INTRODUCTION

In reflection seismology one places point sources and point receivers on the earth's surface. Each source generates acoustic waves in the subsurface, that are reflected where the medium properties vary discontinuously. The recorded reflections that can be observed in the data are used to reconstruct these discontinuities. This paper is the second of a pair of papers on the modeling and inversion of seismic data in the downward continuation approach. The notion of downward continuation refers to the continuation of surface seismic data into the subsurface. The first paper of the pair concerned the modeling, and will be referred to as Paper I [24].

We study the inversion of seismic data with an acoustic model. Let $z \in \mathbb{R}$ denote the vertical (depth) coordinate, $x \in \mathbb{R}^{n-1}$ the horizontal coordinate and t the time, and let $c = c(z, x)$ be the wave speed, then the scalar wave equation for acoustics is given by

$$Pu = f \ , \quad P = c(z,x)^{-2}\partial_t^2 + D_z^2 + \sum_{j=1}^{n-1} D_{x_j}^2 \ , \tag{1.1}$$

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where $D_z = -\mathrm{i}\partial_z, D_x = -\mathrm{i}\partial_x$. We denote the Green's function of (1.1) by G. When the wavespeed c is smooth, the singularities of the solution propagate along curved trajectories, while discontinuities in the wavespeed lead to reflections.

In practice, (1.1) is not used directly, but a linearization is invoked, together with a separation of 'scales'. The linearization is in the coefficient c around a smooth background c_0, $c = c_0 + \delta c$. The perturbation δc is assumed to contain the short wavelengths variations of c, and has no long wavelengths components. Note that this is only a partial linearization: The dependence of the wavefield on c_0 is still nonlinear. The high-frequency part of the perturbation in the Green's function, which we denote by δG, models the once reflected waves, whence this approximation is called the high-frequency single scattering approximation. The data are assumed to be given by $\delta G(z_r, r, t, z_s, s)$, for $z_r = z_s = 0$. Here $s \in \mathbb{R}^{n-1}$, $r \in \mathbb{R}^{n-1}$ denote the source and receiver position, respectively. We assume that the data are available for (s, r, t) in a bounded open subset Y, called the acquisition manifold, of $\mathbb{R}^{2n-2} \times \mathbb{R}_+$. Except for Remark 2.5 we assume that no further restrictions apply to Y; the acquisition geometry is maximal. The modeling map F is defined as the map from δc to δG restricted to Y. Since Y is bounded and the waves propagate with finite speed we may assume that δc is supported in a bounded open subset X of $\mathbb{R}^{n-1} \times \mathbb{R}_+$. We furthermore assume that $\overline{X} \cap \{z = 0\} = \emptyset$.

In the seismic inverse problem, both the smooth background c_0 and the perturbation δc are unknown and have to be reconstructed. This leads to two related inverse problems. The first one is the linear inverse problem of reconstructing δc given c_0. The solution of this problem is used in the estimation of c_0, the second problem, which is highly nonlinear. The purpose of this paper is to develop and analyze methods based on downward continuation for these two inverse problems.

We discuss some of the known results on the reconstruction of δc given c_0. The bicharacteristics that describe the propagation of singularities by the wave equation are the solutions to a Hamilton system and will be parameterized by initial position (z_0, x_0), takeoff direction $\alpha \in S^{n-1}$, frequency τ, which together define the initial cotangent vector $(\zeta_0, \xi_0) = -\tau c(z_0, x_0)^{-1}\alpha$, and time t,

$$\eta(t, z_0, x_0, \alpha, \tau) = (\eta_z(t, z_0, x_0, \alpha, \tau), \eta_x(t, z_0, x_0, \alpha, \tau), t, \eta_\zeta(t, z_0, x_0, \alpha, \tau), \eta_\xi(t, z_0, x_0, \alpha, \tau), \tau). \tag{1.2}$$

Here the evolution parameter is the time t. Note that τ is invariant along the Hamilton flow. Under certain conditions, the modeling operator F is a Fourier integral operator [19] with canonical relation

(1.3)
$$\big\{(\eta_x(t_\mathrm{s}, z, x, \beta, \tau), \eta_x(t_\mathrm{r}, z, x, \alpha, \tau), t_\mathrm{s}+t_\mathrm{r}, \eta_\xi(t_\mathrm{s}, z, x, \beta, \tau), \eta_\xi(t_\mathrm{r}, z, x, \alpha, \tau), \tau; z, x, \zeta, \xi) \mid t_\mathrm{s}, t_\mathrm{r} > 0,\ \eta_z(t_\mathrm{s}, z, x, \beta, \tau) = \eta_z(t_\mathrm{r}, z, x, \alpha, \tau) = 0,\ (\zeta, \xi) = -\tau c_0(z, x)^{-1}(\alpha + \beta), (z, x, \alpha, \beta, \tau) \in \text{ subset of } X \times (S^{n-1})^2 \times \mathbb{R}\backslash 0\big\} \subset T^*\mathbb{R}^{2n-1}_{(s,r,t)} \times T^*\mathbb{R}^n_{(z,x)},$$

see also the discussion in the introduction of Paper I; for a general reference of Fourier integral operators, see [8].

To image the singularities of δc from the data we consider the adjoint $F^*\psi_Y$, where $\psi_Y \in C_0^\infty(Y)$ is a function that goes smoothly to zero near the boundary of Y. The operator $F^*\psi_Y$ is a Fourier integral operator also. Let $N := F^*\psi_Y F$. Applying the adjoint $F^*\psi_Y$ to the data, d say, yields the normal equation,

$$N(z, x, D_z, D_x)\delta c = F^*\psi_Y d. \tag{1.4}$$

The following condition is known as the Bolker condition, see the work by Guillemin [9]

Assumption 1. *The projection of the canonical relation (1.3) on* $T^*Y\backslash 0$ *is an embedding.*

If N is pseudodifferential, the operator $F^*\psi_Y$ images singularities of δc. This is guaranteed by the Bolker condition [10, 15]

Theorem 1.1. *With Assumption 1 the operator* N *is pseudodifferential of order* $n - 1$.

Since (1.3) is a canonical relation that projects submersively on the subsurface variables (z, x, ζ, ξ), the projection of (1.3) on $T^*Y\backslash 0$ is immersive [13, 25.3.6]. Therefore only the injectivity in the assumption needs to be verified [15].

In general, by (1.3) not all singularities in δc are mapped to singularities in the data (the projection of (1.3) on the subsurface variables is not surjective), so that only part of the singularities in δc will be reconstructed. Whether a singularity is reconstructed is determined by the principal symbol of N: It is nonzero at (z, x, ζ, ξ) whenever there is a point $(s, r, t, \sigma, \rho, \tau; z, x, \zeta, \xi)$ in the canonical relation (1.3) with $(s, r, t, \sigma, \rho, \tau)$ in the support of ψ_Y (i.e. whenever there is illumination). In the reconstruction of δc, seismologists distinguish between imaging, which produces a function that has singularities at the same position as δc, and methods that also correctly compute the size of the singularities or the discontinuities. In the latter case they speak of inversion, or of true-amplitude imaging. Thus $F^*\psi_Y$ is an imaging operator, while $\langle N\rangle^{-1}F^*\psi_Y$ is an inversion operator, where $\langle N\rangle^{-1}$ is the regularized parametrix of N, that is an inverse microlocally for a subset of $T^*\mathbb{R}^n_{(z,x)}$ where the symbol of N is nonzero. The inversion operator reconstructs microlocally δc.

Other results concerning the normal operator have been obtained under various assumptions on the acquisition manifold, and the background medium, concerning the presence of caustics and the geometry of the rays [1, 10, 16, 20].

The main topic of this paper is the so-called wave-equation or downward continuation approach [4, 3, 17] to seismic inverse scattering. In this approach, the data are downward continued, leading to data from fictitious experiments below the surface at varying depths. We summarize the results concerning the forward map from Paper I before outlining the results of this paper on inverse scattering.

The singular part of the data can be described by the solution to an inhomogeneous pseudodifferential evolution equation in depth, provided that the rays in the background that are associated with the reflections are nowhere horizontal. To determine whether the velocity vector at some point of the ray is close to horizontal we use the angle, θ, with the vertical, given by $\tan(\theta) = \frac{\|\xi\|}{|\zeta|}$. We use some angle less than, but close to $\pi/2$ in the following

Assumption 2. *(DSR assumption) If* $(z,x) \in X$ *and* $\alpha, \beta \in S^{n-1}$*,* $t_{\rm s}, t_{\rm r} > 0$ *depending on* (z,x,α,β) *are such that* $\eta_z(t_{\rm s}, z, x, \beta, \tau) = \eta_z(t_{\rm r}, z, x, \alpha, \tau) = 0$ *and* $(\eta_x(t_{\rm s}, z, x, \beta, \tau), \eta_x(t_{\rm r}, z, x, \alpha, \tau), t_{\rm s} + t_{\rm r}) \in Y$ *(cf. (1.3)), then*

$$c(z,x)^{-1} \frac{\partial \eta_z}{\partial t}(t, z, x, \beta, \tau) < -\cos(\theta),\ t \in [0, t_{\rm s}], \tag{1.5}$$

$$c(z,x)^{-1} \frac{\partial \eta_z}{\partial t}(t, z, x, \alpha, \tau) < -\cos(\theta),\ t \in [0, t_{\rm r}]. \tag{1.6}$$

Solutions to the wave equation are approximated by solutions to a one-way wave equation, i.e. an equation of the form

$$(\partial_z - \mathrm{i}B_- - C)u_- = 0, \quad z < z_0, \quad u_-(z_0, \cdot, \cdot) = v_-, \tag{1.7}$$

where the principal symbol of B_- is given by $-b(z,x,\xi,\tau) = \tau\sqrt{c_0(z,x)^{-2} - \tau^{-2}\|\xi\|^2}$, and C is a dissipation operator as in [22]. The solution operator that maps initial values at z_0 to values at z is denoted by $G_-(z, z_0)$. The approximation to the Green's function, G, of the original wave equation (1.1) is by $-\frac{1}{2}\mathrm{i}Q_-^*(z)G_-(z,z_0)Q_-(z_0)$, where the operator Q_- is defined in Paper I and has principal symbol $\tau^{-1/2}(c_0^{-2} - \tau^{-2}\|\xi\|^2)^{-1/4}$. We use the notation $\gamma(z, z_0, x_0, \xi_0, \tau)$ for the bicharacteristics of $\partial_z - \mathrm{i}B_-$, parameterized by z, initiated at (z_0, x_0) and ξ_0. In components, we write them as (note that they are time translation invariant)

$$\begin{aligned}\gamma(z, z_0, x_0, t_0, \xi_0, \tau) = (z, \gamma_x(z, z_0, x_0, \xi_0, \tau), \gamma_t(z, z_0, x_0, \xi_0, \tau) + t_0,\\ - b(z, \gamma_x, \gamma_\xi, \tau), \gamma_\xi(z, z_0, x_0, \xi_0, \tau), \tau).\end{aligned} \tag{1.8}$$

The upward continuation operator $H(z, z_0)$, acting on distributions of (s, r, t), was defined in Paper I by

$$H(z, z_0) = (\mathrm{Id}_s \otimes G_{-,r}(z, z_0)) \circ (G_{-,s}(z, z_0) \otimes \mathrm{Id}_r). \tag{1.9}$$

Here, $G_{-,s}$ acts in the s, t variables, and Id_s is the identity operator on functions of s, and a similar notation is assumed for $G_{-,r}$ and Id_r. We recall that $H(z, z_0)$ is the solution operator of the initial value problem for the so-called double-square-root equation

$$\begin{aligned}(\tfrac{\partial}{\partial z} - \mathrm{i}B_-(z, s, D_s, D_t) - \mathrm{i}B_-(z, r, D_r, D_t) - C(z, s, D_s, D_t) - C(z, r, D_r, D_t))u = 0,\\ 0 \le z < z_0,\end{aligned} \tag{1.10}$$

i.e. the solution to (1.10) with initial value $u(z_0, s_0, r_0, t_0) = v(s_0, r_0, t_0)$ is given by $H(z, z_0)v$. This equation is used to compute the downward continued data (for example, by method of generalized screen expansion [7]). The bicharacteristics associated with (1.10) are given by

$$\begin{aligned}\Gamma(z, z_0; s_0, r_0, t_0, \sigma_0, \rho_0, \tau) = (\gamma_x(z, z_0, s_0, \sigma_0, \tau), \gamma_x(z, z_0, r_0, \rho_0, \tau), t_0\\ + \gamma_t(z, z_0, s_0, \sigma_0, \tau) + \gamma_t(z, z_0, r_0, \rho_0, \tau), \gamma_\xi(z, z_0, s_0, \sigma_0, \tau), \gamma_\xi(z, z_0, r_0, \rho_0, \tau), \tau).\end{aligned} \tag{1.11}$$

We define two mappings,

$$E_1 : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{2n-1}) : c(z,x) \mapsto \delta(x-\bar{x})c(z, \tfrac{\bar{x}+x}{2}), \tag{1.12}$$

$$E_2 : \mathcal{D}'(\mathbb{R}^{2n-1}) \to \mathcal{D}'(\mathbb{R}^{2n}) : h(z,\bar{x},x) \mapsto \delta(t)h(z,\bar{x},x), \tag{1.13}$$

and mappings $L : \mathcal{E}'(\mathbb{R}^{2n}_{(z,s,r,t)}) \to \mathcal{D}'(\mathbb{R}^{2n-1}_{(s,r,t)})$ and $K : \mathcal{E}'(\mathbb{R}^{2n-1}_{(z,s,r)}) \to \mathcal{D}'(\mathbb{R}^{2n-1}_{(s,r,t)})$ by

$$Lg = Q^*_{-,s}(0)Q^*_{-,r}(0) \int_0^Z (H(0,z)Q_{-,s}(z)Q_{-,r}(z)g(z,\cdot,\cdot,\cdot))(s,r,t)\,\mathrm{d}z, \tag{1.14}$$

and

$$K = LE_2. \tag{1.15}$$

The operators L and K are Fourier integral operators, see Paper I, Lemma 3.1 and Theorem 4.2. We recall that K is microlocally invertible.

The main result of Paper I concerns what we call the downward continuation or double-square-root modeling map

$$F_{\mathrm{D}} \ : \ \tfrac{1}{2}c_0^{-3}\delta c \to D_t^2 K E_1(\tfrac{1}{2}c_0^{-3}\delta c). \tag{1.16}$$

(Note that we have absorbed the multiplication by $\frac{1}{2}c_0^3$ in F_{D} unlike the definition in Paper I.) We found (Theorem 5.1) that

$$F_{\mathrm{D}}(\tfrac{1}{2}c_0^{-3}\delta c) \equiv \psi_{\mathrm{D}} F \delta c \quad \text{or} \quad F_{\mathrm{D}}\, \tfrac{1}{2}c_0^{-3} \equiv \psi_{\mathrm{D}} F, \tag{1.17}$$

where ψ_{D} is a cutoff that is 1 on the set Ω_{θ_1}, and in $S^{-\infty}$ outside Ω_{θ_2}, with $0 < \theta_1 < \theta_2 < \pi/2$ and where Ω_θ is given by

$$\Omega_\theta = \{(s,r,t,\sigma,\rho,\tau) \,|\, t < T_{\max}(0,s,r,\sigma,\rho,\tau,\theta)\}. \tag{1.18}$$

Here the maximal time $T_{\max}$ is determined by the canonical relation of K from the maximal depth, $Z_{\max}(0,s,r,\sigma,\rho,\tau,\theta)$, that is the upper boundary of the maximal interval containing $z=0$ for which the source and receiver rays satisfy the DSR assumption.

Substituting K back into (1.16) we identify the composition E_2E_1; we observe that

$$E_2E_1(\tfrac{1}{2}c_0^{-3}\delta c) = (2\pi)^{-n} \int (\tfrac{1}{2}c_0^{-3}\delta c)\left(z, \frac{\bar{x}+x}{2}\right) \\ \times \exp[\mathrm{i}\langle(\bar{x}-x),p\rangle\,\tau] \exp(\mathrm{i}t\tau)\,|\tau|^{n-1}\mathrm{d}p\mathrm{d}\tau =: \Lambda E_3 E_4(\tfrac{1}{2}c_0^{-3}\delta c), \tag{1.19}$$

where Λ is a pseudodifferential operator with symbol $|\tau|^{n-1}$, and $E_3 : \mathcal{D}'(\mathbb{R}^{2n-1}) \to \mathcal{D}'(\mathbb{R}^{2n})$, $E_4 : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{2n-1})$ are given by

$$E_3 f(z,\bar{x},x,t) = (2\pi)^{-(n-1)} \int \delta(t - \langle(x-\bar{x}),p\rangle)\, f(z, \frac{\bar{x}+x}{2}, p)\,\mathrm{d}p, \tag{1.20}$$

$$(E_4(\tfrac{1}{2}c_0^{-3}\delta c))(z,x,p) = (\tfrac{1}{2}c_0^{-3}\delta c)(z,x), \tag{1.21}$$

respectively, so that

$$F_{\mathrm{D}} = D_t^2 L \Lambda E_3 E_4.$$

By the map E_4 the perturbation is viewed as being p dependent as if it were a reflection coefficient. In fact, p is, given the singular direction from the wavefront set of δc, through trigonometric formulae related to scattering angles [6].

The first result of this paper concerns the reconstruction of δc. In section 2 we derive the normal equation in the downward continuation approach from the Born DSR modeling operator, F_{D}, and derive a reconstruction equation in Theorem 2.2. This theorem is the downward continuation counterpart of Theorem 1.1. The adjoint F_{D}^* is factorized to emphasize the different steps that compose an algorithm for reconstruction.

Because of the cutoff ψ_{D} in (1.17), fewer singularities in δc will be reconstructed, than if the full operator F^* is used. For example, vertical reflectors that could be illuminated by turning rays are not reconstructed. In many practical cases this disadvantage is however not so important.

The results of section 3 concern the determination of c_0 through migration velocity analysis. Here it is exploited that the reconstruction of δc is an overdetermined problem. Indeed δc is a function of n variables, and d of $2n-1$ variables. Beylkin [1] gives conditions so that δc can be reconstructed by an operator A_{ss} from a subset of dimension n of the data, given c_0. In conventional migration velocity analysis the data are viewed as an $(n-1)$-parameter family of such subsets, which results in an $(n-1)$-dimensional family of reconstructions. If we denote the $(n-1)$ parameters by p, and the family of reconstructions of δc by $A_{\mathrm{ss}}[c_0]d(z,x,p)$, then this gives the following criterion for the determination of c_0:

$$(A_{\mathrm{ss}}[c_0]d)(z,x,p) \text{ is independent of } p, \tag{1.22}$$

at least microlocally. In migration velocity analysis some c_0 is constructed based on this criterion. Note that c_0 need not be uniquely determined by the singular part of the data.

In the presence of caustics the conditions of [1] are generally violated, and the application of A_{ss} (which can still be defined in this case) results into images with so called kinematic artifacts, that correspond to *nonlocal singular contributions to* $A_{\mathrm{ss}}F$ [16]. As a result, (1.22) is no longer valid. To remedy this, the parameter p can be chosen to no longer parameterize a subset of data, but instead to parameterize the angle between in- and outgoing rays at the image point. In the Kirchhoff approach to seismic inverse scattering it has been proposed to use a generalized Radon transform to generate a set of images parameterized by angle (such an operator will be referred to as angle transform) [2, 29]. However, in the presence of caustics, artifacts were observed in numerical examples [2, 25], also in this case. Such artifacts are attributed to contributions from nonlocal operators to the composition of the generalized Radon transform with the modeling operator F. By microlocal analysis of the Kirchhoff approach the presence of artifacts was shown in [21].

The main result of this paper is to show that there is a downward continuation based angle transform A_{WE} that is artifact free under much weaker conditions than those of Beylkin [1] (Theorem 3.1). In particular the presence of caustics is allowed. The map A_{WE} is defined in section 3, similarly as an operator introduced in [5] to study angle dependent reflection coefficients. We show that $A_{\mathrm{WE}}F$ is a p-family of pseudodifferential, which implies that nonlocal singular contributions are absent and data is mapped to a set of images (Theorem 3.1). In addition we show that A_{WE} is an invertible Fourier integral operator. We introduce an appropriate modification, denoted by $\widetilde{A}_{\mathrm{WE}} = \widetilde{A}_{\mathrm{WE}}[c_0]$ such that $\widetilde{A}_{\mathrm{WE}}[c_0]F[c_0]$ is a p-family of pseudodifferential operators with symbol 1, microlocally, hence mapping data to a set of reconstructions (Proposition 3.2). In [6] it was confirmed numerically that this angle transform does not generate artifacts in the presence of caustics.

The background model c_0 is acceptable if the data are in the range of the modeling operator $F_{\mathrm{D}}[c_0]$. The criterion that the data $d \in \mathrm{range}(F_{\mathrm{D}}[c_0])$ becomes equivalent to

$$(\widetilde{A}_{\mathrm{WE}}[c_0]d)(z,x,p) \text{ is independent of } p, \tag{1.23}$$

microlocally, where p can be identified with the integration variables in E_3 (cf. (1.20)). Thus, the criterion for migration velocity analysis (1.22) is extended to allow for background media with caustics.

Equation (1.23) reveals the redundancy in the data. The redundancy also leads to the existence of pseudodifferential operators that annihilate the singular part of the data [23]. These are related to the differential semblance measure in the framework of Beylkin's conditions, for estimating the background medium [26]. Annihilators of the data in the downward continuation approach can be derived from $\widetilde{A}_{\mathrm{WE}}$. In addition, we construct in section 4 a different annihilator, W, such that $\|Wd\|_{L^2}$ measures the *focusing* at $r = s$ of the downward continued data restricted to $t = 0$.

2. Imaging and reconstruction in the downward continuation approach

Conceptually, the first step in the reconstruction of the perturbation δc given the background, c_0, is applying the adjoint of the linear modeling map to the data. We present a normal equation similar to (1.4) based on the double-square-root (DSR) modeling operator (1.16). The operator F^* in the right hand side of (1.4) is replaced by F_{D}^*, so that the right hand side can be computed using the downward continuation approach. This leads, again, to reconstruction modulo a pseudodifferential operator for which an explicit expression is given.

The operator F_{D} is a Fourier integral operator whose canonical relation is a subset of that of F containing the elements with (z, x, α, β) such that Assumption 2 applies, with angle given by θ_2 below (1.17). It follows that in this case Assumption 1 can be replaced by the following weaker assumption

Assumption 3. *If several elements of (1.3) project to the same point in T^*Y, then none of the (z, x, α, β) that describe these elements are such that Assumption 2 is satisfied.*

In effect this assumption implies that the cutoff ψ_{D} has symbol in $S^{-\infty}$ for points in T^*Y where the projection from (1.3) is not injective. A cartoon of two ray pairs such that Assumption 3 is violated is given in Figure 1. As discussed in Paper I, the DSR assumption 2 is stronger than Assumption 1, so is also sufficient for Theorem 1.1 to hold.

We now discuss the different adjoints of operators that make up F_{D}^*. The adjoint operator $H(0,z)^*$ propagates the data downward and backward in time. The adjoint of operator L is given by

$$(L^*d)(z,s,r,t) = Q_{-,s}^*(z) Q_{-,r}^*(z) H(0,z)^* Q_{-,s}(0) Q_{-,r}(0) d. \tag{2.1}$$

The adjoint of extension operator E_2 is given by the restriction R_2 defined by

$$g(z,s,r,t) \mapsto (R_2 g)(z,s,r) = g(z,s,r,0), \tag{2.2}$$

while the adjoint of extension operator E_1 is given by the restriction R_1 defined by

$$h(z,s,r) \mapsto (R_1 h)(z,x) = h(z,x,x). \tag{2.3}$$

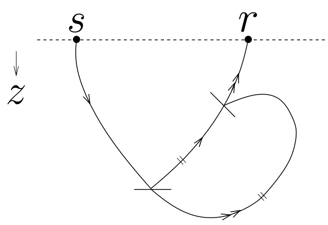

FIGURE 1. Cartoon of two ray pairs such that Assumption 3 is violated. The two trajectories between the reflection points are assumed to have equal traveltime.

The adjoint of K hence equals

$$K^* = R_2 L^*. \tag{2.4}$$

The canonical relation of K maps points $(z, s, r, \zeta, \sigma, \rho)$ in a subset of $T^*\mathbb{R}^{2n-1}\backslash 0$ diffeomorphically to points $(s_0, r_0, t_0, \sigma_0, \rho_0, \tau)$ in a subset of the cotangent acquisition space $T^*\mathbb{R}^{2n-1}\backslash 0$ (Theorem 4.2 of Paper I). We denote this map by Σ. The adjoint K^* maps singularities according to the inverse Σ^{-1}. Furthermore, for given (z, s, r, σ, ρ), we have a mapping $\tau \mapsto \zeta = \Theta(z, s, r, \sigma, \rho, \tau)$ with inverse $\tau = \Theta^{-1}(z, s, r, \zeta, \sigma, \rho)$ (Lemma 4.1 of Paper I).

In the downward continuation version of (1.4), the operator on the left hand side is given by $F_{\mathrm{D}}^* \psi_Y F$. Here $\psi_Y = \psi_Y(s, r, t)$ is a smooth cutoff function on Y that is zero near the boundary of Y, that we introduced to account for the limited acquisition aperture (cf. (1.4)). Suppose now that K' is defined as K, but with angles for the cutoff given by θ_1', θ_2' instead of θ_1, θ_2, such that $0 < \theta_1 < \theta_2 < \theta_1' < \theta_2' < \pi/2$. Then K is supported in the region where the cutoff that forms part of K' is equal to 1. Define an operator F_{D}' by $F_{\mathrm{D}}' = D_t^2 K' E_1$ (cf. (1.16)-1.17)). Then F_{D} can be written as $F_{\mathrm{D}} = \psi_{\mathrm{D}} F_{\mathrm{D}}'$, modulo a regularizing operator. Hence

$$F_{\mathrm{D}}^* \psi_Y F = F_{\mathrm{D}}'^* \psi_{\mathrm{D}}^* \psi_Y F_{\mathrm{D}}' (\tfrac{1}{2} c_0^{-3} \;\;), \tag{2.5}$$

modulo a regularizing term. Thus can we compute the symbol of $F_{\mathrm{D}}^* \psi_Y F$ using the downward continuation expression for F. We have that

$$F_{\mathrm{D}}^* \psi_Y F_{\mathrm{D}} = R_1\, K^* D_t^2 \psi_Y D_t^2 K\, E_1. \tag{2.6}$$

Our first step in the computation of the symbol of this operator will be to evaluate $K^* D_t^2 \psi_Y D_t^2 K$. To simplify the evaluation, we introduce operators $\bar{L}$ and $\bar{K}$ that correspond with operators L and K, respectively, with the Q_- operators removed,

$$\bar{K} = \bar{L} E_2 \tag{2.7}$$

with

$$\bar{L} g = \int_0^Z H(0, z) g(z, \cdot, \cdot, \cdot)\, \mathrm{d}z. \tag{2.8}$$

Lemma 2.1. *The composition* $K^* D_t^2 \psi_Y D_t^2 K$ *is a pseudodifferential operator of order* 0 *with principal symbol* $\Sigma^*(\psi_Y |\psi_{\mathrm{D}}|^2)(z,s,r,\zeta,\sigma,\rho)\Xi(z,s,r,\zeta,\sigma,\rho)$*, where* Ξ *is given by*

$$\Xi(z,s,r,\zeta,\sigma,\rho) = (\Sigma^* a_0)(z,s,r,\zeta,\sigma,\rho) \\ a_z(s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))\bar{\Xi}(z,s,r,\zeta,\sigma,\rho) \tag{2.9}$$

where

$$a_z(s,r,\sigma,\rho,\tau) = \tau^2 b(z,s,\sigma,\tau)^{-1} b(z,r,\rho,\tau)^{-1} \tag{2.10}$$

and

$$\begin{aligned}\bar{\Xi}(z,s,r,\zeta,\sigma,\rho)^{-1} &= \left|\frac{\partial\Theta}{\partial\tau}(z,s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))\right| \\ &= \big[c_0(z,s)^{-2}(c_0(z,s)^{-2} - \tau^{-2}\|\sigma\|^2)^{-1/2} \\ &\quad + c_0(z,r)^{-2}(c_0(z,r)^{-2} - \tau^{-2}\|\rho\|^2)^{-1/2}\big]_{\tau=\Theta^{-1}(z,s,r,\zeta,\sigma,\rho)}.\end{aligned} \tag{2.11}$$

Proof. From the fact that the canonical relation of K is invertible (the graph of a diffeomorphism) between subsets of $T^*\mathbb{R}^{2n-1}\backslash 0$, it follows that $K^*\psi_Y K$ is a pseudodifferential operator. We calculate microlocally the principal symbol of $K^* D_t^2 \psi_Y D_t^2 K$.

First, we evaluate the symbol of $\bar{K}^*\bar{K}$ microlocally, ignoring the cutoffs; its principal part we denote by $\bar{\Xi}(z,s,r,\zeta,\sigma,\rho)$. We recall that the kernel of the operator $H(0,z)$ has microlocally an oscillatory integral representation with amplitude $A = A(z,y_0,\eta_{0J},s,r,t)$ and a phase function associated with generating function, $S = S(z,y_{0I},\eta_{0J},s,r,t)$, where $y_0 = (s_0,r_0,t_0)$ and η_0 is the corresponding cotangent vector, and $\{I,J\}$ is a partition of $\{1,\ldots,2n-1\}$ (Lemma 3.1 of Paper I). The kernel of operator $\bar{K}$ has an oscillatory integral representation similar to the one of $H(0,z)$,

$$\bar{K}(y_0,z,s,r) = (2\pi)^{-(2n-1+|I|)/2} \int A(z,y_0,\eta_{0J},s,r,0)\mathrm{e}^{\mathrm{i}(S(z,y_{0I},\eta_{0J},s,r,0)+\langle\eta_{0J},y_{0J}\rangle)}\,\mathrm{d}\eta_{0J} \tag{2.12}$$

which follows upon considering the action of $\bar{L}$ on $\delta(t)\cdot\text{distribution}(z,s,r)$ and carrying out the t integration.

We evaluate, microlocally, $\bar{K}^*\bar{K}$. To this end, we identify the gradient

$$-\frac{\partial S}{\partial(z,s,r)}(z,y_{0I},\eta_{0J},s,r,0) = \\ (\zeta(z,y_{0I},\eta_{0J},s,r,0),\sigma(z,y_{0I},\eta_{0J},s,r,0),\rho(z,y_{0I},\eta_{0J},s,r,0)). \tag{2.13}$$

Applying a change of variables of integration, $(y_{0I},\eta_{0J}) \mapsto (\zeta,\sigma,\rho)$, the phase in the oscillatory integral representation of the kernel of $\bar{K}^*\bar{K}$ takes the form

$$\langle(\zeta,\sigma,\rho),(z'-z,s'-s,r'-r)\rangle; \tag{2.14}$$

a Jacobian, $\left|\frac{\partial(\zeta,\sigma,\rho)}{\partial(y_{0I},\eta_{0J})}\right|^{-1}$, appears in the amplitude, so that the amplitude of $\bar{K}^*\bar{K}$ has principal part (using the expression for A from Lemma 3.1 of Paper I)

$$\left|\frac{\partial(\zeta,\sigma,\rho)}{\partial(y_{0I},\eta_{0J})}\right|^{-1}\left|\frac{\partial(\sigma,\rho,\tau)}{\partial(y_{0I},\eta_{0J})}\right|_{t=0}. \tag{2.15}$$

For fixed (z,s,r,σ,ρ) the map $\tau \mapsto \zeta = \Theta(z,s,r,\sigma,\rho,\tau)$ is invertible on a set given by $|\tau|$ sufficiently large: $\tau = \Theta^{-1}(z,s,r,\zeta,\sigma,\rho)$ (Lemma 4.1 of Paper I). Carrying out the multiplication of determinants, it follows that the principal part of the symbol of $\bar{K}^*\bar{K}$ is microlocally given by

$$\bar{\Xi}(z,s,r,\zeta,\sigma,\rho) = \left| \frac{\partial\Theta}{\partial\tau}(z,s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho)) \right|^{-1}. \tag{2.16}$$

To obtain $K^* D_t^2 \psi_Y D_t^2 K$ involves the composition of $\bar{K}^*\bar{K}$ with pseudodifferential operators. First, the composition

$$\bar{K}^* Q_{-,s}(0) Q_{-,r}(0)\, D_t^2 \psi_Y D_t^2\, Q_{-,s}^*(0) Q_{-,r}^*(0) \bar{K} \tag{2.17}$$

is carried out with the aid of Egorov's theorem. This leads to a factor $\Sigma^*(\tau^2\psi_Y a_0)$ in the principal symbol. Secondly, having obtained a pseudodifferential operator, the inclusion of the factors $Q_{-,s}^*(z), Q_{-,r}^*(z)$ and $Q_{-,s}(z), Q_{-,r}(z)$ is carried out with the standard calculus of pseudodifferential operators. This leads to a factor $(\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))^{-2} a_z(s,r,\sigma,\rho,\Theta^{-1}(z,s,r,\zeta,\sigma,\rho))$ in the principal symbol. It follows that the principal part of the symbol of $K^* D_t^2 \psi_Y D_t^2 K$ is, microlocally, given by $\Xi\, \Sigma^* \psi_Y$. With the pseudodifferential cutoff ψ_{D} taken into account we obtain the result of the lemma. □

We define an operator Ψ using the operator K' introduced above (2.5), by $\Psi = K'^* D_t^2 \psi_{\mathrm{D}}^* \psi_Y D_t^2 K'$. It follows from the lemma that Ψ is pseudodifferential in $S_\rho^0(\mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1})$ (for some ρ, $\frac{1}{2} < \rho < 1$, see section 2 of paper I) with principal symbol, that we denote by Ψ_0, given by

$$\Psi_0(z,s,r,\zeta,\sigma,\rho) = \Xi(z,s,r,\zeta,\sigma,\rho)(\Sigma^*(\overline{\psi_{\mathrm{D}}}\psi_Y))(z,s,r,\zeta,\sigma,\rho). \tag{2.18}$$

Starting from the DSR Born modeling (1.16), we obtain the following reconstruction result

Theorem 2.2. *Suppose Assumption 3 is satisfied. Then there is a pseudodifferential operator* $\Phi = \Phi(z,x,D_z,D_x)$ *of order* $n-1$, $\Phi \in S_\rho^{n-1}$ ($\frac{1}{2} < \rho < 1$)*, with principal symbol*

$$\Phi_{n-1}(z,x,\zeta,\xi) = \int_{\mathbb{R}^{n-1}} \Psi_0(z,x,x,\zeta,\tfrac{1}{2}\xi - \theta, \tfrac{1}{2}\xi + \theta)\, \mathrm{d}\theta, \tag{2.19}$$

such that

$$\Phi(z,x,D_z,D_x)(\tfrac{1}{2}c_0^{-3}\delta c) = R_1 K^* D_t^2 \psi_Y d, \tag{2.20}$$

where $d = F\delta c$ *is the Born modeled data (1.16).*

Proof. Because of Theorem 1.1 and the modeling formula (1.16), the right-hand side of (2.20) is equal to a pseudodifferential operator acting on $\frac{1}{2}c_0^{-3}\delta c$. It remains therefore to compute its symbol. By (2.5) we compute the symbol of the composition of operators $R_1 \Psi E_1$. The kernel of this composition has an oscillatory integral representation,

$$\begin{aligned} &(2\pi)^{-(2n-1)} \int_{\mathbb{R}^{2n-1}} \Psi(z,x,x,\zeta,\sigma,\rho)\, \mathrm{e}^{\mathrm{i}\langle (z,x,x)-(z',x',x'),(\zeta,\sigma,\rho)\rangle}\, \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}\zeta \\ &= (2\pi)^{-(2n-1)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \Psi(z,x,x,\zeta,\tfrac{1}{2}\xi - \theta, \tfrac{1}{2}\xi + \theta)\, \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}((z-z')\zeta + \langle (x-x'),\xi\rangle)}\, \mathrm{d}\xi \mathrm{d}\zeta, \end{aligned} \tag{2.21}$$

upon changing variables of integration, $\sigma = \frac{1}{2}\xi - \theta$, $\rho = \frac{1}{2}\xi + \theta$. The domain of the θ integral is bounded depending on (ζ, ξ), since $\psi_{\rm D}$ is a cutoff in $(s, r, t, \sigma, \rho, \tau)$. Because of the integration over θ, the inner integral in the second line is homogeneous of order $n-1$ in (ζ, ξ) and $R_1 \Psi E_1$ is a pseudodifferential operator of order $n-1$ with principal symbol (2.19).

Applying the above results to the expression (1.16) for the Born modeled data leads to the statement of the theorem. □

The symbol of $\frac{1}{2}c_0^{-3} F_{\rm D}^* \psi_Y F = F^* \psi_{\rm D}^* \psi_Y F$ also follows from the computation of F^*F of Ten Kroode et al. [15, theorem 4.1], upon inserting $\overline{\psi_{\rm D}}(s, r, t, \sigma, \rho, \tau)\psi_Y(s, r, t)$ in their formula (64). Our purpose is to show that this computation is independent of the results of [15]. Also, our method is easily modified for the result (2.27) below.

Remark 2.3. Note that in (2.20) the operator $\Phi(z, x, D_z, D_x)$ on the left-hand side depends on the Hamiltonian flow associated with the background medium in the depth interval $[0, z]$. To account for the operator $\Phi(z, x, D_z, D_x)$ on the left-hand side one thus requires a ray computation in addition to the downward continuation with K^*.

Remark 2.4. Theorem 2.2 follows a least-squares data fitting approach to linearized inverse scattering. However, departing from this approach, we can derive a reconstruction equation the pseudodifferential operator in which only accounts for illumination effects. To this end, we replace the normal equation in the downward continuation approach by the following equation,

$$\bar{K}^* \bar{K} Q_{-,s}(z) Q_{-,r}(z)\, E_1(\tfrac{1}{2}c_0^{-3}\delta c) = \bar{K}^* Q_{-,s}^*(0)^{-1} Q_{-,r}^*(0)^{-1} D_t^{-2} d. \tag{2.22}$$

Using Lemma 2.1, we obtain the equation

$$\begin{aligned} Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{\Xi}^{-1} \bar{K}^* \bar{K} Q_{-,s}(z) Q_{-,r}(z)\, E_1(\tfrac{1}{2}c_0^{-3}\delta c) \\ = Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{\Xi}^{-1} \bar{K}^* Q_{-,s}^*(0)^{-1} Q_{-,r}^*(0)^{-1} D_t^{-2} d, \end{aligned} \tag{2.23}$$

microlocally, which reduces to

$$\bar{\Psi}\, E_1(\tfrac{1}{2}c_0^{-3}\delta c) = Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{\Xi}^{-1} \bar{K}^* Q_{-,s}^*(0)^{-1} Q_{-,r}^*(0)^{-1} D_t^{-2} \psi_Y d \tag{2.24}$$

in which $\bar{\Psi}$ is a pseudodifferential operator with principal symbol $\bar{\Psi}_0 = \Sigma^*(\overline{\psi_{\rm D}}\psi_Y)$ (compare (2.18)), so that

$$R_1\, \bar{\Psi}\, E_1(\tfrac{1}{2}c_0^{-3}\delta c) = R_1 Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{\Xi}^{-1} \bar{K}^* Q_{-,s}^*(0)^{-1} Q_{-,r}^*(0)^{-1} D_t^{-2} \psi_Y d. \tag{2.25}$$

Substituting $\bar{\Psi}$ for Ψ in (2.19) then leads to the reconstruction equation

$$\begin{aligned} \bar{\Phi}(z, x, D_z, D_x)(\tfrac{1}{2}c_0^{-3}\delta c) \\ = R_1 Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{\Xi}^{-1} \bar{K}^* Q_{-,s}^*(0)^{-1} Q_{-,r}^*(0)^{-1} D_t^{-2} \psi_Y d \end{aligned} \tag{2.26}$$

with

$$\bar{\Phi}(z, x, \zeta, \xi) = \int_{\mathbb{R}^{n-1}} \bar{\Psi}(z, x, x, \zeta, \tfrac{1}{2}\xi - \theta, \tfrac{1}{2}\xi + \theta)\, \mathrm{d}\theta. \tag{2.27}$$

This expression is simpler than the expression for Φ, and appears less sensitive to the precise ray geometry near the surface. Therefore we prefer this equation for the development of a practical algorithm.

Remark 2.5. Depending on the background medium, the reconstruction can also be done using data on a submanifold Y' of Y. Let R' be the restriction of a function on Y to Y', so that the forward map for this case is given by $R'F$. In suitable local coordinates (y', y'') on Y such that $y'' = 0$ defines Y', the adjoint E' of R' is given by the map $(E'f)(y', y'') = f(y')\delta(y'')$. Conditions such that $F^* E' \psi'_Y R' F$ is pseudodifferential are given in [16]. Reconstruction modulo a pseudodifferential operator is done in this case by first applying the map E' to the data, and then applying the previous procedure. Applying E' to the data simply means adding zeroes where there is no data in Y.

3. The wave-equation angle transform, common-image-point gathers

With the aid of (1.19)-(1.20), the modeling operator takes the form

$$F_{\mathrm{D}} = D_t^2 L \Lambda E_3 E_4. \tag{3.1}$$

We define the wave-equation angle transform first as an adjoint, denoted by A_{WE}, with an additional cutoff function χ

$$A_{\mathrm{WE}} = R_3 \chi L^* D_t^2, \tag{3.2}$$

where the composition of χ and the adjoint, R_3 of E_3 is

$$R_3\chi \;:\; g(z,s,r,t) \mapsto (R_3\chi g)(z,x,p) = \int_{\mathbb{R}^{n-1}} g(z, x - \tfrac{h}{2}, x + \tfrac{h}{2}, ph) \chi(z,x,h) \,\mathrm{d}h. \tag{3.3}$$

Here, $\chi(z,x,h)$ is a compactly supported cutoff function the support of which contains $h = 0$. First we analyze the properties of A_{WE} mostly from a geometrical perspective. Then we derive a p-dependent reconstruction equation that replaces (2.20) or (2.26). A map similar to A_{WE} was introduced in [5] for the purpose of imaging angle dependent reflection coefficients, see also [18]. For each x, the function $(z,p) \mapsto (A_{\mathrm{WE}} \psi_Y d)(z,x,p)$ is a so-called common-image-point gather.

Theorem 3.1. *Suppose Assumption 2 holds. Let C_0 be an upper bound for c_0. Assume that*

$$\|p\| < p_{\max} < \tfrac{1}{2} C_0^{-1}. \tag{3.4}$$

Then A_{WE} is a Fourier integral operator such that $A_{\mathrm{WE}} F$ is a smooth p-family of pseudo-differential operators in (z,x). Let C_1 be an upper bound for $\frac{\partial c_0^{-2}}{\partial x}$, C_2 an upper bound for c_0^{-1}. If in addition the function $h \mapsto \chi(z,x,h)$, contained in A_{WE}, is supported in $B(0,R)$, where R depends on θ_2, C_0, C_1, C_2, then the canonical relation of A_{WE} corresponds to an invertible map from a subset of $T^*\mathbb{R}^{2n-1}_{(s,r,t)}$ to a subset of $T^*\mathbb{R}^{2n-1}_{(z,x,p)}$ that has nonempty intersection with the set $\vartheta = 0$ (where ϑ denotes the p-covector).

Proof. By Assumption 2 we have that $F = F_{\mathrm{D}}(\frac{1}{2} c_0^{-3}\ \ )$, modulo a regularizing term. So for the first statement it is sufficient to show that $A_{\mathrm{WE}} F_{\mathrm{D}}$ is a p-family of pseudodifferential operators.

The Schwartz kernel of the map $R_3\chi$ equals

$$\begin{aligned} &\delta(x - \tfrac{s+r}{2})\, \delta(p(r-s) - t)\, \delta(z - z')\, \chi(z,x,r-s) \\ &\qquad = (2\pi)^{-n-1} \int \chi(z,x,r-s)\, \mathrm{e}^{\mathrm{i}(\langle \xi, x - \frac{s+r}{2}\rangle + \tau(p(r-s)-t) + \zeta(z-z'))} \,\mathrm{d}\xi \,\mathrm{d}\tau \,\mathrm{d}\zeta. \end{aligned} \tag{3.5}$$

It defines a Fourier integral operator with canonical relation

$$\{(z, \tfrac{s+r}{2}, p, \zeta, \xi, (r-s)\tau; z, s, r, p(r-s), \zeta, \tfrac{\xi}{2} + p\tau, \tfrac{\xi}{2} - p\tau, \tau) \mid (z, s, r, p, \zeta, \xi, \tau) \in \text{subset of } \mathbb{R}^{4n-1}\} \subset T^*\mathbb{R}^{2n-1}_{(z,x,p)}\backslash 0 \times T^*\mathbb{R}^{2n}_{(z,s,r,t)}\backslash 0. \tag{3.6}$$

Using the change of variables, $\xi = \sigma + \rho$, $p = \frac{\sigma-\rho}{2\tau}$, this canonical relation can be parameterized by the coordinates of $T^*\mathbb{R}^{2n}_{(z,s,r,t)}\backslash 0$ except t, that is $(z, s, r, \zeta, \sigma, \rho, \tau)$. The projection of canonical relation (3.6) on $T^*\mathbb{R}^{2n}_{(z,s,r,t)}\backslash 0$ is a hypersurface defined by

$$t = \left\langle \frac{\sigma - \rho}{2\tau}, (r-s) \right\rangle. \tag{3.7}$$

The map $d \mapsto L^*\psi_Y d$ is a Fourier integral operator with canonical relation

$$\big\{(z, \gamma_x(z,0,s_0,\sigma_0,\tau), \gamma_x(z,0,r_0,\rho_0,\tau), t_0 + \gamma_t(z,0,s_0,\sigma_0,\tau) + \gamma_t(z,0,r_0,\rho_0,\tau), -b(z,s_0,\sigma_0,\tau) - b(z,r_0,\rho_0,\tau), \gamma_\xi(z,0,s_0,\sigma_0,\tau), \gamma_\xi(z,0,r_0,\rho_0,\tau), \tau; s_0, r_0, t_0, \sigma_0, \rho_0, \tau) \mid (s_0, r_0, t_0, \sigma_0, \rho_0, \tau) \in T^*\mathbb{R}^{2n-1}\backslash 0, z \in \mathbb{R}_+\big\} \subset T^*\mathbb{R}^{2n}_{(z,s,r,t)} \times T^*\mathbb{R}^{2n-1}_{(s_0,r_0,t_0)} \tag{3.8}$$

The canonical relation is parameterized by $(z, s_0, r_0, t_0, \sigma_0, \rho_0, \tau)$, and is time translation invariant (in t_0). The line in the projection of this canonical relation on $T^*\mathbb{R}^{2n}_{(z,s,r,t)}\backslash 0$ parameterized by t_0 for fixed $(s_0, r_0, \sigma_0, \rho_0, \tau)$ intersects the hypersurface (3.7) transversally. It follows that the composition of the canonical relations (3.8) and (3.6) is transversal. This composition is parameterized by $(z, s_0, r_0, \sigma_0, \rho_0, \tau)$. It follows that $R_3\chi L^*$, and hence $A_{\text{WE}} = R_3\chi L^* D_t^2$, is a Fourier integral operator.

To analyze the composition $A_{\text{WE}}F_{\text{D}}$, we first analyze the composition $L^* D_t^4 \psi_Y L E_2 E_1$, that maps the perturbation $\frac{1}{2}c_0^{-3}\delta c = (\frac{1}{2}c_0^{-3}\delta c)(z', x)$ to the downward continued data as a function of (z, s, r, t). This composition is a Fourier integral operator with canonical relation

$$\{(z, \gamma_x(z,z',x,\sigma',\tau), \gamma_x(z,z',x,\rho',\tau), \gamma_t(z,z',x,\sigma',\tau) + \gamma_t(z,z',x,\rho',\tau), \zeta, \gamma_\xi(z,z',x,\sigma',\tau), \gamma_\xi(z,z',x,\rho',\tau), \tau; z', x, \zeta', \sigma' + \rho') \mid (z, z', x, \zeta', \sigma', \rho') \in \text{ a subset of } \mathbb{R}^{3n}, \tau \text{ such that } \zeta' = \Theta(z', x, x, \sigma', \rho', \tau), \zeta = \Theta(z, \gamma_x(z,z',x,\sigma',\tau), \gamma_x(z,z',x,\rho',\tau), \gamma_\xi(z,z',x,\sigma',\tau), \gamma_\xi(z,z',x,\rho',\tau), \tau)\}. \tag{3.9}$$

The propagation of singularities upward by LE_2E_1 and downward by L^* is along the same DSR bicharacteristics.

We now show that the composition $A_{\text{WE}}D_t^2 L E_2 E_1$ is a Fourier integral with canonical relation contained in

$$\{(z, x, p, \zeta, \xi, 0; z, x, \zeta, \xi) \mid (z, x, \zeta, \xi) \in T^*\mathbb{R}^n_{(z,x)}\backslash 0, \|p\| < p_{\max}\}. \tag{3.10}$$

From that it follows that $A_{\text{WE}}F_{\text{D}}$ is a p-family of pseudodifferential operators. Indeed, the projection of (3.9) on $T^*\mathbb{R}^{2n}_{(z,s,r,t)}$ intersects the hypersurface (3.7) at $z' = z$, since only then $r - s = 0$ and $t = 0$, leading to elements in (3.10). Since singularities propagate with speed less than C_0, if $(1, v_{\text{s}}, v_{\text{r}}, v_{\text{t}}, v_\zeta, v_\sigma, v_\rho, 0)$ derived from $\frac{\mathrm{d}\Gamma}{\mathrm{d}z}$ is a tangent vector to the DSR bicharacteristic Γ (the first component $v_{\text{z}} = 1$), then $\frac{\|v_{\text{s}} - v_{\text{r}}\|}{v_{\text{t}}} \leq 2C_0$. By integrating this inequality and using (3.4) it follows that the composition of (3.9) with (3.6) is transversal and contains no elements outside (3.10).

Finally, we show that A_{WE} is invertible. The projection of the canonical relation of A_{WE} on the second component $T^*\mathbb{R}^{2n-1}_{(s,r,t)}\backslash 0$ is invertible if each DSR bicharacteristic with initial values $(s_0, r_0, t_0, \sigma_0, \rho_0, \tau)$, parameterized by z, intersects the hypersurface (3.7) at most once and transversally. Let $\boldsymbol{p}(z)$ denote $\frac{\sigma-\rho}{2\tau}$ along a certain DSR bicharacteristic and let $\boldsymbol{t}(z)$ denote the time and $\boldsymbol{h}(z)$ denote the value of $r - s$. The elements of the canonical relation of A_{WE} correspond to solutions $(z, s_0, r_0, \sigma_0, \rho_0, \tau)$ of $\boldsymbol{t}(z) - \langle \boldsymbol{p}(z), \boldsymbol{h}(z)\rangle = 0$. To estimate the derivative of the left-hand side, we will argue that

$$\frac{\partial \boldsymbol{t}}{\partial z} - \langle \boldsymbol{p}(z), \frac{\partial \boldsymbol{h}}{\partial z}(z)\rangle < -\epsilon_0 \tag{3.11}$$

for some $\epsilon_0 > 0$ depending on $p_{\max}$ and R as in the theorem. Indeed, note that the factor

$$1 - \langle \boldsymbol{p}(z), \frac{\partial \boldsymbol{h}}{\partial z}(z) / \frac{\partial \boldsymbol{t}}{\partial z}\rangle$$

is bounded away from zero. Also, note that $\frac{\partial \boldsymbol{t}}{\partial z}$ is strictly negative while $\left|\frac{\partial \boldsymbol{t}}{\partial z}\right| \geq \frac{1}{2C_0}$. Since according to the Hamilton equations,

$$\frac{\mathrm{d}\gamma_\xi}{\mathrm{d}z} = \frac{\partial b}{\partial x} = -\frac{\tau}{2\sqrt{c_0^{-2} - \tau^{-2}\|\xi\|^2}} \frac{\partial c_0^{-2}}{\partial x}, \tag{3.12}$$

it follows that $\left\|\frac{\partial \boldsymbol{p}}{\partial z}(z)\right\| \leq \frac{C_2}{2}\frac{C_1}{\cos(\theta_2)}$. So, if $\|h\| < C_3 \frac{\cos(\theta_2)}{C_0 C_1 C_2}$, it follows that

$$\left|\langle \frac{\partial \boldsymbol{p}}{\partial z}(z), \boldsymbol{h}(z)\rangle\right| < C_3 \frac{1}{2C_0} < \epsilon_0 \tag{3.13}$$

by an appropriate choice of C_3. This implies that the function $z \mapsto \boldsymbol{t}(z) - \langle \boldsymbol{p}(z), \boldsymbol{h}(z)\rangle$ is monotone. Hence the projection of the canonical relation of A_{WE} on $T^*\mathbb{R}^{2n-1}_{(s,r,t)}\backslash 0$ is invertible.

From canonical relation (3.6) we now establish that, given $z, \frac{s+r}{2} = x, \sigma, \rho$, the map $(h = s-r, \tau) \mapsto (\vartheta = (r-s)\tau, \zeta = \Theta(z, s, r, \sigma, \rho, \tau))$ is invertible for h sufficiently small. Assume that (h_1, τ_1) maps to (ϑ_1, ζ_1) and that (h_2, τ_2) maps to (ϑ_2, ζ_2) with $\vartheta_2 = \vartheta_1$. We show that $\zeta_1 > \zeta_2$ if $\tau_1 > \tau_2$. Estimating the difference

$$\Theta(z, x + \tfrac{1}{2}h_1, x - \tfrac{1}{2}h_1, \sigma, \rho, \tau_1) - \Theta(z, x + \tfrac{1}{2}h_2, x - \tfrac{1}{2}h_2, \sigma, \rho, \tau_2), \tag{3.14}$$

using the bounds

$$\left|\frac{\partial b}{\partial \tau}\right| = \left|\frac{c_0^{-2}}{\sqrt{c_0^{-2} - \tau^{-2}\|\xi\|^2}}\right| \geq \frac{C_5}{C_4}, \tag{3.15}$$

in which C_4 depends on the lower and upper bounds of c_0, and $C_5 = \frac{1}{C_2^2}$ is a lower bound for c_0^{-2}, and (cf. (3.12))

$$\left\|\frac{\partial b}{\partial x}\right\| = \left\|\frac{\tau}{2\sqrt{c_0^{-2} - \tau^{-2}\|\xi\|^2}} \frac{\partial c_0^{-2}}{\partial x}\right\| \leq \frac{\tau \|\frac{\partial c_0^{-2}}{\partial x}\|_{L^\infty}}{C_6}, \tag{3.16}$$

where C_6 depends on θ_2 and the upper and lower bounds of c_0, yields the estimate

$$\zeta_1 - \zeta_2 \geq \frac{C_5(\tau_1 - \tau_2)}{C_4} - \frac{\|\frac{\partial c_0^{-2}}{\partial x}\|_{L^\infty} \tau_1 \|h_1 - h_2\|}{C_6}. \tag{3.17}$$

Since $\vartheta_2 = \vartheta_1$, $(h_1 - h_2)\tau_1 = h_2(\tau_2 - \tau_1)$, so that

$$\frac{C_5(\tau_1 - \tau_2)}{C_4} - \frac{\|\frac{\partial c_0^{-2}}{\partial x}\|_{L^\infty} \tau_1 \|h_1 - h_2\|}{C_6} = (\tau_1 - \tau_2)\left[\frac{C_5}{C_4} - \frac{\|\frac{\partial c_0^{-2}}{\partial x}\|_{L^\infty}}{C_6}\|h_2\|\right]. \tag{3.18}$$

The right-hand side of this equality is strictly greater than zero for $\|h_2\|$ sufficiently small, since $\tau_1 - \tau_2 > 0$. It follows from this that $\zeta_1 > \zeta_2$. We conclude that the projection of the canonical relation of A_{WE} on $T^*\mathbb{R}^{2n-1}_{(z,x,p)} \backslash 0$ is invertible as well. It follows also that the linearization of this projection is invertible. This establishes the last statement of the theorem. □

To conclude this section we determine, at the principal symbol level, the modification of (3.2) that leads to reconstruction of singularities of δc microlocally. Like the reconstruction in (2.20) of Theorem 2.2, the reconstruction is microlocal. The three cutoffs, χ and $\psi_Y, \psi_{\mathrm{D}}$, must be taken into account. The canonical relation of A_{WE} defines a map $(s, r, t, \sigma, \rho, \tau) \rightarrow (z, x, p, \zeta, \xi, \vartheta)$ (where ϑ is the p-covector); there is also an associated value of $h = r - s$ through (3.6). By pull back with the inverse of the mentioned map, one can map the symbols $\psi_Y, \psi_{\mathrm{D}}$ to symbols in the variables $(z, x, p, \zeta, \xi, \vartheta)$. By the mentioned evaluation of h one obtains by pull back the cutoff χ in these variables also. We define $\Psi_{\mathrm{WE}} = \Psi_{\mathrm{WE}}(z, x, p, \zeta, \xi, \vartheta)$ as the product of these symbols. With this definition we have

Proposition 3.2. *Define $\widetilde{A}_{\mathrm{WE}}$ by*

$$(\widetilde{A}_{\mathrm{WE}} d)(z, x, p) = \jmath^{-1} R_3 \bar{\Xi}^{-1} Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} \bar{L}^* Q^*_{-,s}(0)^{-1} Q^*_{-,r}(0)^{-1} D_t^{-2} \psi_Y d, \tag{3.19}$$

in which $\bar{\Xi} = \bar{\Xi}(z, s, r, \Theta(z, s, r, \sigma, \rho, \tau), \sigma, \rho)$ is independent of t (the corresponding operator is convolutional in time), and where the symbol of $\jmath$ is defined in (3.32) below. Suppose Assumption 2 holds. Suppose that χ is 1 on a neighborhood of $h = 0$ and $h \mapsto \chi(z, x, h)$ is supported in $B(0, R)$ (cf. Theorem 3.1), then $\widetilde{A}_{\mathrm{WE}}$ is an invertible Fourier integral operator. Let the symbol Ψ_{WE} be as defined above. The composition $\widetilde{A}_{\mathrm{WE}} F$ is a p-family of pseudodifferential operators with principal symbol $\Psi_{\mathrm{WE}}(z, x, p, \zeta, \xi, 0)$.

The microlocal reconstruction hence follows from

$$(\Psi_{\mathrm{WE}}(z, x, p, D_z, D_x, 0) + \mathrm{order}(-1))(\tfrac{1}{2} c_0^{-3} \delta c) = \widetilde{A}_{\mathrm{WE}} d. \tag{3.20}$$

Proof. We consider the operator $R_3 \bar{\Xi}^{-1} \bar{L}^* \bar{L} E_2 E_1$ or the map

$$(\tfrac{1}{2} c_0^{-3} \delta c) \mapsto \int \left(\bar{\Xi}(z)^{-1} H(0, z)^* \int H(0, z') E_2 E_1 (\tfrac{1}{2} c_0^{-3} \delta c) \, \mathrm{d}z' \right) (x - \tfrac{h}{2}, x + \tfrac{h}{2}, ph) \, \mathrm{d}h, \tag{3.21}$$

and evaluate, microlocally, its principal symbol. This principal symbol will be the principal symbol of operator $\jmath$ in the theorem. In this proof, we will omit the cutoff functions that are part of the symbols; the calculations will be valid microlocally on the support of a cutoff.

Using an oscillatory integral representation of H similar to the one in the proof of Theorem 2.2, we find that the principal contribution to the kernel of this map, as a function of $(z, x, p; z', x')$, can be written as

$$(3.22)\quad (2\pi)^{-(2n-1)} \int \bar{\Xi}\left(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \Theta\left(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, -\frac{\partial S}{\partial s}, -\frac{\partial S}{\partial r}, -\frac{\partial S}{\partial t}\right), -\frac{\partial S}{\partial s}, -\frac{\partial S}{\partial r}\right)^{-1} \times \overline{A(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, ph, y_0, \eta_{0J})} A(z', x', x', 0, y_0, \eta_{0J}) \times \mathrm{e}^{\mathrm{i}[\Phi_R(z,x,p,z',x',y_{0I},\eta_{0J})]}\, \mathrm{d}y_{0I} \mathrm{d}\eta_{0J}\, \mathrm{d}h,$$

in which $\frac{\partial S}{\partial s}, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial t}$ are evaluated at $(z, x - \frac{1}{2}h, x + \frac{1}{2}h, ph, y_{0I}, \eta_{0J})$, and where

$$(3.23)\quad \Phi_R(z, x, p, z', x', y_{0I}, \eta_{0J}) = -S(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, ph, y_{0I}, \eta_{0J}) + S(z', x', x', 0, y_{0I}, \eta_{0J}).$$

We expand this phase in a Taylor series about $(z', x', h) = (z, x, 0)$ and identify the gradient at $(z, x, 0)$,

$$(3.24)\quad -\frac{\partial \Phi_R}{\partial x}(z, x, x, 0, y_{0I}, \eta_{0J}) = \sigma(z, x, x, 0, y_{0I}, \eta_{0J}) + \rho(z, x, x, 0, y_{0I}, \eta_{0J}),$$

$$(3.25)\quad -\frac{\partial \Phi_R}{\partial z}(z, x, x, 0, y_{0I}, \eta_{0J}) = \zeta(z, x, x, 0, y_{0I}, \eta_{0J}),$$

$$(3.26)\quad -\frac{\partial \Phi_R}{\partial h}(z, x, x, 0, y_{0I}, \eta_{0J}) = -\tfrac{1}{2}\sigma(z, x, x, 0, y_{0I}, \eta_{0J}) + \tfrac{1}{2}\rho(z, x, x, 0, y_{0I}, \eta_{0J}) + p\tau(z, x, x, 0, y_{0I}, \eta_{0J}),$$

where

$$(3.27)\quad -\frac{\partial S}{\partial(s, r, t, z)}(z, x, x, 0, y_{0I}, \eta_{0J}) = (\sigma(z, x, x, 0, y_{0I}, \eta_{0J}), \rho(z, x, x, 0, y_{0I}, \eta_{0J}), \tau(z, x, x, 0, y_{0I}, \eta_{0J}), \zeta(z, x, x, 0, y_{0I}, \eta_{0J})).$$

Applying a change of variables, $(y_{0I}, \eta_{0J}) \mapsto (\zeta, \sigma, \rho)$, the phase takes the form

$$(3.28)\quad \zeta(z - z') + \langle \sigma + \rho, x - x' \rangle + \langle \tfrac{1}{2}(\rho - \sigma) + p\tau, h \rangle,\ \tau = \Theta^{-1}(z, x, x, \zeta, \sigma, \rho).$$

The amplitude factor $\bar{\Xi}^{-1}\overline{A}A$ (at $h = 0$) becomes equal to one by the calculations in the proof of Theorem 2.2. Upon changing integration variables, $\sigma = \frac{1}{2}\xi - \vartheta, \rho = \frac{1}{2}\xi + \vartheta$, the oscillatory integral (3.22) takes the leading-order form

$$(3.29)\quad (2\pi)^{-(2n-1)} \int \mathrm{e}^{\mathrm{i}(\zeta(z-z') + \langle \xi, x - x' \rangle + \langle \vartheta + \tau p, h \rangle)}\, \mathrm{d}\zeta \mathrm{d}\xi \mathrm{d}\vartheta \mathrm{d}h, \quad \tau = \Theta^{-1}(z, x, x, \zeta, \tfrac{1}{2}\xi - \vartheta, \tfrac{1}{2}\xi + \vartheta).$$

By the method of stationary phase (see e.g. [8, section 1.2]) the following integral is a symbol

$$(2\pi)^{-(n-1)} \int \mathrm{e}^{\mathrm{i}\langle \vartheta + \Theta^{-1}(z,x,x,\zeta,\frac{1}{2}\xi-\vartheta,\frac{1}{2}\xi+\vartheta)p,h\rangle} \, \mathrm{d}\vartheta \, \mathrm{d}h \, ; \tag{3.30}$$

it follows from (3.29) that this equals to highest order the symbol of $\widetilde{A}_{\mathrm{WE}}F$. By the method of stationary phase the principal part of this symbol is given by

$$\jmath(z,x,p,\zeta,\xi) := \left| \frac{\partial(\langle \vartheta + \Theta^{-1}p, h\rangle)}{\partial(h,\vartheta)} \right|^{-\frac{1}{2}} , \tag{3.31}$$

evaluated where $h = 0$ and ϑ is such that $\vartheta + \Theta^{-1}p = 0$. By evaluating this, we find that

$$\begin{aligned} \left| \frac{\partial(\langle \vartheta + \Theta^{-1}p, h\rangle)}{\partial(h,\vartheta)} \right|^{\frac{1}{2}} = \left| \frac{\partial(\vartheta + \Theta^{-1}p)}{\partial\vartheta} \right| &= \left| \det\left(I + p \otimes \frac{\partial\Theta^{-1}}{\partial\vartheta} \right) \right| \\ &= \left| \det\left(I + p \otimes \left(\frac{\partial\Theta^{-1}}{\partial\rho} - \frac{\partial\Theta^{-1}}{\partial\sigma} \right) \right) \right| . \end{aligned} \tag{3.32}$$

This completes the proof. □

Remark 3.3. Since $\widetilde{A}_{\mathrm{WE}}$ is invertible, microlocally, we have obtained the following diagram as suggested by Symes [27]

$$\begin{array}{ccc} \mathcal{E}'(X \times E) & \overset{\widetilde{A}_{\mathrm{WE}}^{-1}}{\longrightarrow} & \mathcal{D}'(Y) \\ \Psi_{\mathrm{WE}} \uparrow & & \uparrow \mathrm{Id} \\ \mathcal{E}'(X) & \overset{F_{\mathrm{D}}}{\longrightarrow} & \mathcal{D}'(Y) \end{array} \tag{3.33}$$

where $E = \{p \in \mathbb{R}^{n-1} \mid \|p\| < p_{\max}\}$ as in Theorem 3.1 (cf. (3.4)).

4. ANNIHILATORS

As discussed in the introduction, the inverse problem of determining the background medium c_0 can be addressed by making use of the redundancy in the data. The background medium must be such that the data is in the range of the operator F. The data are in the range of F_{D}, if the angle transform generates a collection of identical images of $\frac{1}{2}c_0^{-3}\delta c$, parameterized by $p = (p_1, \ldots, p_{n-1})$, microlocally. It follows that we have the following criterion

$$(\widetilde{A}_{\mathrm{WE}}[c_0]d)(z,x,p) \text{ is independent of } p, \tag{4.1}$$

microlocally, or

$$\frac{\partial}{\partial p_i} (\widetilde{A}_{\mathrm{WE}}[c_0]d)(z,x,p) = 0, \tag{4.2}$$

microlocally. Seismologists recognize this as 'alignment' of the singularities in common-image-point gathers. Of course it must be taken into acount that $\widetilde{A}_{\mathrm{WE}}$ is only a microlocal inverse, so (4.2) is not valid globally.

The criterion (4.2) can be restated as that certain pseudodifferential operators annihilate the singular part of the data, see [23]. With the approach based on inversion for δc using subsets of data (discussed in the introduction) this is closely related to differential semblance [26]. A construction of such operators, annihilators, follows straightforwardly from the angle transform introduced in the previous section. On transformed data $\widetilde{A}_{\mathrm{WE}}d$, i.e. on *common-image-point gathers*, annihilators are given by $\frac{\partial}{\partial p_i}$, $i = 1, \ldots, n-1$. Using Proposition 3.2, it follows that annihilators of the *data* are given by

$$\left(\frac{\partial}{\partial t}\right)^{-1} \langle \widetilde{A}_{\mathrm{WE}} \rangle^{-1} \frac{\partial}{\partial p_i} \widetilde{A}_{\mathrm{WE}} \tag{4.3}$$

as pseudodifferential operators of order zero, where $\langle \widetilde{A}_{\mathrm{WE}} \rangle^{-1}$ is a regularized inverse for $\widetilde{A}_{\mathrm{WE}}$, that is supported microlocally on a subset of $T^*\mathbb{R}^{2n-1}_{(z,x,p)}$ where $\widetilde{A}_{\mathrm{WE}}$ is invertible. Such annihilators are not uniquely defined. Indeed (4.3) can be composed on the left by any invertible order 0 pseudodifferential operator and we still have an annihilator.

We now derive an alternative definition of an annihilator. This definition is motivated by the observation that

$$\frac{\partial}{\partial p_i} R_3 \left(\frac{\partial}{\partial t}\right)^{-1} g \bigg|_{p=0} = \int_{\mathbb{R}^{n-1}} M_j (R_2 g)(z, x - \tfrac{h}{2}, x + \tfrac{h}{2}) \chi(z, x, h) \, \mathrm{d}h,$$

where M_j denotes the multiplication by $(r_j - s_j)$. We have $M_j E_1 = 0$, since $(r_j - s_j)\delta(r - s) = 0$. Observe first that with Assumption 2

$$M_j K^* D_t^2 d = K^* D_t^4 K M_j E_1(\tfrac{1}{2} c_0^{-3} \delta c) + [M_j, K^* D_t^4 K] E_1(\tfrac{1}{2} c_0^{-3} \delta c). \tag{4.4}$$

The first term vanishes, and the second term is of lower order, so M_j is to highest order an annihilator for $K^* D_t^2 d$.

To derive from (4.4) a pseudodifferential annihilator of the data, $M_j K^* D_t^2$ is multiplied on the left by $D_t^2 K$. We must also have the support of the operator on the left smaller than the support of the operator on the right of M_j. So suppose ψ_Y' is in $C_0^\infty(Y)$ and is 1 on $\operatorname{supp}(\psi_Y)$, and K' is defined as K but with pseudodifferential cutoff at larger angles of propagation θ_1', θ_2' as introduced above (2.5). Define

$$W_j = \psi_Y D_t^2 K M_j \langle K'^* D_t^2 \psi_Y' D_t^2 K' \rangle^{-1} K'^* D_t^2 \psi_Y', \tag{4.5}$$

where we recognize $N_{K'} := K'^* D_t^2 \psi_Y' D_t^2 K'$ from Lemma 2.1. Then we have

Theorem 4.1. *With Assumption 2, the W_j and the operators (4.3) are annihilators of the data.*

We could define $\widetilde{K} = K N_K^{-1/2}$ as a modification of K so that $D_t^2 \widetilde{K}$ is unitary – at least where ψ_{D} is 1. Then $\frac{1}{2}\sum_j \|W_j[c_0]d\|^2$ simplifies approximately to

$$\tfrac{1}{2} \sum_j \|W_j[c_0]d\|^2 \approx \tfrac{1}{2} \int \left(\|r - s\|^2 |\widetilde{K}^* D_t^2 d|^2 + \text{lower order terms} \right) \mathrm{d}z \, \mathrm{d}s \, \mathrm{d}r, \tag{4.6}$$

This expression is small when the time-to-depth converted data $\widetilde{K}^* D_t^2 d$ are 'focused' at $r = s$.

Migration velocity analysis is the estimation of c_0 (the background or 'velocity model') based upon the alignment expressed by (4.2). Traditionally, the updating of the velocity model is carried out interactively. The problem of updating c_0 so that the data will be contained in the range of $F_{\rm D}$ has been cast in an optimization problem. Liu and Bleistein [14] developed an automated method for updating c_0 on the basis of the curvature of misalignment at $p = 0$ using ray perturbation theory. It was also done using the data subset based annihilators (differential semblance [26]). Annihilators, $W_j = W_j[c_0]$, replace the necessity to estimate this curvature from the p-family of images. They depend on the background medium. The semi-norm $\left(\sum_j \|W_j[c_0]d\|^2\right)^{\frac{1}{2}}$ detects whether c_0 was an acceptable choice or not. The functional $\frac{1}{2}\sum_j \|W_j[c_0]d\|^2$ can be viewed as the downward continuation analog of the differential semblance functional of Symes [26] with the advantage that our annihilator admits the formation of caustics. Of course, the same is true with the W_j replaced by the operators in (4.3).

REFERENCES

[1] G. Beylkin, Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, J. Math. Phys., 26 (1985), pp. 99-108.

[2] S. Brandsberg-Dahl, M. V. de Hoop, and B. Ursin, Focusing in dip and AVA compensation on scattering-angle/azimuth common image gathers, Geophysics, 68 (2003), pp. 232-254.

[3] J. F. Claerbout, Imaging the Earth's Interior. Blackwell Scientific Publications, Oxford, 1985.

[4] R. W. Clayton, Common midpoint migration, Technical Report SEP-14, Stanford University, 1978.

[5] C. G. M. de Bruin, C. P. A. Wapenaar, and A. J. Berkhout, Angle-dependent reflectivity by means of prestack migration, Geophysics, 55 (1990), pp. 1223-1234.

[6] M. V. de Hoop, J. H. Le Rousseau, and B. Biondi, Symplectic structure of wave-equation imaging: A path-integral approach based on the double-square-root equation, Geoph. J. Int., 153 (2003), pp. 52-74.

[7] M. V. de Hoop, J. H. Le Rousseau, and R.-S. Wu, Generalization of the phase-screen approximation for the scattering of acoustic waves, Wave Motion, 31 (2000), pp. 43-70.

[8] J. J. Duistermaat, Fourier Integral Operators, Birkhäuser, Boston, MA, 1996.

[9] V. Guillemin, On some results of Gel′fand in integral geometry, in Pseudodifferential operators and applications (Notre Dame, Ind., 1984), Amer. Math. Soc., Providence, RI, 1985, pp. 149-155.

[10] S. Hansen, Solution of a hyperbolic inverse problem by linearization, Comm. Partial Differential Equations, 16 (1991), pp. 291-309.

[11] L. Hörmander, The Analysis of Linear Partial Differential Operators, Volume 1. Springer-Verlag, Berlin, 1983.

[12] L. Hörmander, The Analysis of Linear Partial Differential Operators, Volume 3. Springer-Verlag, Berlin, 1985.

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators, Volume 4, Springer-Verlag, Berlin, 1985.

[14] Z. Liu, and N. Bleistein, Velocity analysis by inversion, Geophysics, 60 (1995), pp. 142-153.

[15] A. P. E. ten Kroode, D. J. Smit, and A. R. Verdel, A microlocal analysis of migration, Wave Motion, 28 (1998), pp. 149-172.

[16] C. J. Nolan and W. W. Symes, Global solution of a linearized inverse problem for the wave equation, Comm. Partial Differential Equations, 22 (1997), pp. 919-952.

[17] A. M. Popovici, Prestack migration by split-step DSR, Geophysics, 61 (1996), pp. 1412-1416.

[18] M. Prucha, B. Biondi, and W. W. Symes, Angle-domain common image gathers by wave-equation migration, in 69th Ann. Internat. Mtg., Soc. Of Expl. Geophys., 1999, pp. 824-827.

[19] Rakesh, A linearized inverse problem for the wave equation, Comm. Partial Differential Equations, 13 (1988), pp. 573-601.

[20] C. C. Stolk, Microlocal analysis of a seismic linearized inverse problem, Wave Motion, 32 (2000), pp. 267-290.

[21] C. C. Stolk, Microlocal analysis of the scattering angle transform, Comm. Partial Differential Equations. 27 (2002), pp. 1879-1900.

[22] C. C. Stolk, A pseudodifferential equation with damping for one-way wave propagation in inhomogeneous acoustic media, to appear in Wave Motion. http://www.math.polytechnique.fr/˜stolk.
[23] C. C. Stolk and M. V. de Hoop, Microlocal analysis of seismic inverse scattering in anisotropic elastic media, Comm. Pure Appl. Math., 55 (2002), pp. 261-301.
[24] C. C. Stolk and M. V. de Hoop, Modeling of seismic data in the downward continuation approach, SIAM J. Appl. Math., submitted (2003).
[25] C. C. Stolk and W. W. Symes, Kinematic artifacts in prestack depth migration, Geophysics, 2004, In Press.
[26] W. W. Symes, A differential semblance algorithm for the inverse problem of reflection seismology, Comput. Math. Appl., 22 (1991), pp. 147-178.
[27] W. W. Symes, personal communication.
[28] F. Treves, Introduction to Pseudodifferential and Fourier Integral Operators, Volume 2. Plenum Press, New York, NY, 1980.
[29] S. Xu, H. Chauris, G. Lambaré, and M. Noble, Common angle migration: A strategy for imaging complex media, Geophysics, 66 (2001), pp. 1877-1894.

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