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Lecture 3: Orlov's theorem and reconstruction results

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Notation. Throughout we work with varieties over a field k, and fibre products are always taken over k. By $D(X)$ we mean the bounded derived category of coherent sheaves on X. If X is a smooth k-variety then we write $\omega_X := \Omega^n_{X/k}$ for the canonical bundle, where $n = \dim(X)$. We shall look at $D(X)$ with its structure of a k-linear triangulated category; so if we say that $D(X)$ and $D(Y)$ are equivalent then we mean that there is an equivalence of categories between them that respects these structures. Such an equivalence need not, a priori, respect the finer structures that we have, such as the tensor structure on $D(X)$.

§ 1. Serre functors and Fourier-Mukai transforms

Definition. Let $\mathscr{D}$ be a k-linear category. Then a Serre functor is a k-linear functor $S\colon \mathscr{D} \to \mathscr{D}$ such that for any two objects A, B in $\mathscr{D}$ we have an isomorphism

$$\eta_{A,B}\colon \mathrm{Hom}(A,B) \xrightarrow{\sim} \mathrm{Hom}\bigl(B, S(A)\bigr)^\vee ,$$

functorially in A and B. (In other words, we should have an isomorphism of bifunctors $\mathrm{Hom}(-,*) \xrightarrow{\sim} \mathrm{Hom}\bigl(*, S(-)\bigr)$.)

This definition is modelled on classical Serre duality: If X is a smooth projective k-variety then the functor S_X given by

$$S_X := \; - \overset{\mathrm{L}}{\otimes} \omega_X \bigl[\dim(X)\bigr]$$

is a Serre functor on $D(X)$. For instance, if E and F are coherent sheaves on X then we have isomorphisms

$$\begin{aligned} \mathrm{Ext}^i(E,F) = \mathrm{Hom}_{D(X)}\bigl(E, F[i]\bigr) &\xrightarrow{\sim} \mathrm{Hom}_{D(X)}\bigl(F[i], E\otimes\omega_X[\dim(X)]\bigr)^\vee \\ &= \mathrm{Hom}_{D(X)}\bigl(F, E\otimes\omega_X[\dim(X)-i]\bigr)^\vee \\ &= \mathrm{Ext}^{\dim(X)-i}(F, E\otimes\omega_X)^\vee , \end{aligned}$$

which is the version of Serre duality you'll find in Hartshorne.

Lemma. *Suppose $\mathscr{D}_1$ and $\mathscr{D}_2$ are k-linear categories with finite dimensional Hom's. Suppose S_i is a Serre functor on $\mathscr{D}_i$. Then any k-linear equivalence $F\colon \mathscr{D}_1 \to \mathscr{D}_2$ "commutes with the Serre functors", that is, we have $F\circ S_1 \cong S_2 \cong F$.*

In particular, up to isomorphism a category with finite dimensional Hom's can have at most one Serre functor. In the lemma the condition that F is an equivalence is necessary.

Fourier-Mukai functors. Given an object $P \in D(X \times Y)$, we define $\Phi_P\colon D(X) \to D(Y)$ to be the functor given by

$$\Phi_P(E) := R\mathrm{pr}_{Y,*}\big(\mathrm{pr}_X^*(E) \overset{\mathrm{L}}{\otimes} P\big)\,,$$

and we call this the Fourier-Mukai (FM) transform with kernel P. If we need to indicate in which direction the functor goes (as we can also view P as an object in $D(Y \times X)$), we write $\Phi_{P,X\to Y}$.

Examples.

(i) If $f\colon X \to Y$ is a morphism, take $P = O_\Gamma$, where $\Gamma = \Gamma_f$ is the graph of f. Then $\Phi_{P,X\to Y} = Rf_*$ and $\Phi_{P,Y\to X} = Lf^*$.

(ii) If L is a line bundle on X then the functor $E \mapsto E \otimes L$ is an auto-equivalence of $D(X)$. It is the FM-transform with kernel $\Delta_* L$, where $\Delta\colon X \to X \times X$.

(iii) The shift functor T that sends E to $E[1]$ is is an auto-equivalence of $D(X)$. It is the FM-transform with kernel $\Delta_* O_X[1]$.

Exact functors. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are triangulated categories then a functor $\Phi\colon \mathscr{D}_1 \to \mathscr{D}_2$ is called *exact* if it satisfies the following two conditions:

(a) Φ commutes with shift functors; more formally: there is an isomorphism of functors $\Phi \circ T_1 \xrightarrow{\sim} T_2 \circ \Phi$.

(b) Φ maps distinguished triangles to distinguished triangles; more formally: if $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is distinguished in $\mathscr{D}_1$ then $\Phi(A) \longrightarrow \Phi(B) \longrightarrow \Phi(C) \longrightarrow \Phi(A)[1]$ should be distinguished in $\mathscr{D}_2$. Here we use the isomorphism in (a) to identify $\Phi\big(A[1]\big)$ and $\Phi(A)\big[1\big]$.

Examples. Any Serre functor is exact. A composition of exact functors is exact. If $F\colon \mathscr{A}_1 \to \mathscr{A}_2$ is a left exact functor between abelian categories that has a right derived functor $RF\colon D(\mathscr{A}_1) \to D(\mathscr{A}_2)$ then RF is exact; similarly for left derived functors. Hence any FM-transform $\Phi_P\colon D(X) \to D(Y)$ is exact.

Adjoints. Any FM-functor $\Phi = \Phi_P$ admits a left adjoint Φ^* and a right adjoint $\Phi^!$. Explicitely:

$$\Phi^* = \Phi_{P^\vee} \circ S_Y \qquad \text{and} \qquad \Phi^! = S_X \circ \Phi_{P^\vee}\,,$$

where $P^\vee := R\mathcal{H}om(P, \mathscr{O}_{X\times Y})$.

§ 2. Orlov's theorem

Theorem (Orlov, 1997). *Let X and Y be smooth projective varieties over a field k. Let $F\colon D(X) \to D(Y)$ be an exact additive fully faithful functor. Then there exists an object $P \in D(X \times Y)$, unique up to isomorphism, such that F is isomorphic to Φ_P.*

Brief sketch of the proof. Start by choosing a projective embedding $i\colon X \hookrightarrow \mathbb{P}^m$ such that the algebra $A := \oplus_{n\geqslant 0} H^0\big(X, \mathscr{O}(n)\big)$ has nice properties (to be used in step 3). Write $\mathscr{C}$ for the full subcategory of $D(X)$ whose objects are the sheaves $\mathscr{O}_X(n)$ for $n \in \mathbb{Z}$. Let $\iota\colon \mathscr{C} \hookrightarrow D(X)$ be the inclusion functor. By a theorem of Bondal and van den Bergh, the functor F has left and right adjoints F^* and $F^!$.

Step 1, lemma: Suppose $G\colon D(X) \to D(X)$ is an equivalence such that $G_{|\mathscr{C}}$ is isomorphic to ι. Then $G \cong \mathrm{id}_{D(X)}$.

Step 2. Consider the functor $F'\colon D(\mathbb{P}^m) \to D(Y)$ given by $F' := F \circ Li^*$. Orlov constructs an object P' in $D(\mathbb{P}^m \times Y)$ for which there are functorial isomorphisms $F'\big(\mathscr{O}_{\mathbb{P}^m}(n)\big) \cong \Phi_{P'}\big(\mathscr{O}_{\mathbb{P}^m}(n)\big)$ for $n \in \mathbb{Z}$. The construction of P' is rather explicit. It starts with Beilinson's resolution of the diagonal in $\mathbb{P}^m \times \mathbb{P}^m$; this is a complex of sheaves of the following form:

$$\mathscr{O}(-m) \boxtimes \Omega^m(m) \xrightarrow{d_{-m}} \mathscr{O}(m-1) \boxtimes \Omega^{m-1}(m-1) \xrightarrow{d_{-m+1}} \cdots \longrightarrow \mathscr{O}(-1) \boxtimes \Omega^1(1) \xrightarrow{d_{-1}} \mathscr{O} \boxtimes \mathscr{O}$$

with differentials d_i that we shall not make explicit. Let $F'\colon D(\mathbb{P}^m) \to D(Y)$ be the functor given by $F' = F \circ Lj^*$. Now one constructs from the previous complex a complex of objects (!) in $D(\mathbb{P}^m \times Y)$ of the form

$$\mathscr{O}(-m)\boxtimes F'\big(\Omega^m(m)\big) \longrightarrow \mathscr{O}(m-1)\boxtimes F'\big(\Omega^{m-1}(m-1)\big) \longrightarrow \cdots \longrightarrow \mathscr{O}(-1)\boxtimes F'\big(\Omega^1(1)\big) \longrightarrow \mathscr{O}\boxtimes F'\big(\mathscr{O}\big)\,.$$

Call this complex C. Intuitively, we could think of C as a double complex of sheaves on $\mathbb{P}^m \times Y$, and then we should like to pass to the associated total complex, which should be an object in $D(\mathbb{P}^m \times Y)$. In general, such a construction makes no sense. However, Orlov shows that under suitable technical assumptions, which are satisfied in this case, there is a well-defined object $P' \in D(\mathbb{P}^m \times Y)$ that plays the role of a total complex associated to C. This is the object we take.

Step 3. Orlov proves that there exists an object $P \in D(X \times Y)$ such that $R(i \times \mathrm{id})_*\big(P\big) \cong P'$.

Step 4. We get an isomorphism of functors $f\colon F_{|\mathscr{C}} \to (\Phi_P)_{|\mathscr{C}}$, noting that $F\big(\mathscr{O}_X(n)\big) = F'\big(\mathscr{O}_{\mathbb{P}^m}(n)\big)$ and $\Phi_P\big(\mathscr{O}_X(n)\big) = \Phi_{P'}\big(\mathscr{O}_{\mathbb{P}^m}(n)\big)$. Further, using the ampleness of $\mathscr{O}_X(1)$ Orlov shows that $F^!F_{|\mathscr{C}} \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$ and that the functor $G := F^! \circ \Phi_P\colon D(X) \to D(X)$ is an equivalence. Hence we can apply the lemma from step 1; this gives that $F^! \circ \Phi_P \xrightarrow{\sim} \mathrm{id}_{D(X)}$. From this Orlov proves that $F \cong \Phi_P$, again using ampleness of $\mathscr{O}_X(1)$.

Step 5. Finally on proves the uniqueness of P. After all the preceding steps this is not so hard.

Remark. One of the reasons this is such an important result, is that it allows us to associate to an equivalence $\Phi\colon D(X) \to D(Y)$ maps on K-theory, Chow groups, cohomology, etc. For instance, if "H" is a Weil cohomology then we get an induced map

$$\Phi^H\colon H(X) \to H(Y)\,;$$

just write $\Phi = \Phi_P$ and define Φ^H to be the map given by

$$\xi \mapsto \mathrm{pr}_{Y,*}\big(\mathrm{pr}_X^*(\xi) \cup \mathrm{ch}(P)\big)\,,$$

where $\mathrm{ch}(P) \in H(X \times Y)$ is the Chern character of the object P. These maps are typically not compatible with the product structures and typically do not respect the gradings! (Think of the classical case of an abelian variety.)

Further remarks. (1) It is not known if the assumption that F is fully faithful can be dropped. There are, however, positive results on this (B. Toën, Inventiones Math., 2007) in the context of dg-categories.

(2) There is a generalisation of the theorem to orbifolds, due to Kawamata.

(3) There is no known analogue of this theorem for affine varieties or derived categories of modules over an algebra, except for some very special cases.

§ 3. Recovering information from $D(X)$

3.1. *Recovering the dimension.* If X and Y are smooth projective k-varieties such that $D(X)$ and $D(Y)$ are equivalent then $\dim(X) = \dim(Y)$.

Proof. By Orlov's theorem there exists an object $P \in D(X \times Y)$ such that $\Phi_P\colon D(X) \to D(Y)$ is an equivalence. We know that its left and right adjoints are isomorphic; this gives $S_Y \circ \Phi_P \cong \Phi_P \circ S_X$. But

$$S_Y \circ \Phi_P = \Phi_{P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_Y^* \omega_Y[\dim(Y)]} \qquad \text{and} \qquad \Phi_P \circ S_X = \Phi_{P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_X^* \omega_X[\dim(X)]}\,.$$

Uniqueness of the Fourier-Mukai kernel (for equivalences) implies that

$$P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_Y^* \omega_Y[\dim(Y)] \cong P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_X^* \omega_X[\dim(X)]\,.$$

But since we are in the *bounded* derived category, this readily implies that $\dim(X) = \dim(Y)$. (Look at the degrees in which an object has nonzero homology.) □

Remark. The argument shows that if $\Phi_P\colon D(X) \to D(Y)$ is an equivalence, then

$$P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_X^* \omega_X \cong P \overset{\mathrm{L}}{\otimes} \mathrm{pr}_Y^* \omega_Y\,.$$

In the context of birational geometry we shall frequently encounter correspondences $X \xleftarrow{p} Z \xrightarrow{q} Y$ such that $p^*\omega_X \cong q^*\omega_Y$. Note the similarity! (Take $P = i_*\mathscr{O}_Z$, with $i\colon Z \hookrightarrow X \times Y$.)

3.2. *Recovering the canonical ring, and the Hochschild (co)homology.* Given a variety X we define a bigraded k-algebra $\mathrm{HH}(X) = \oplus_{i,j} \mathrm{HH}_{i,j}(X)$ by

$$\mathrm{HH}_{i,j}(X) := \mathrm{Ext}^i_{\mathscr{O}_{X\times X}}\bigl(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j\bigr) = \mathrm{Hom}_{D(X\times X)}\bigl(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j[i]\bigr)\,.$$

Here $\Delta\colon X \to X \times X$ is the diagonal morphism, and the ring structure is given by the maps

$$\mathrm{Hom}\bigl(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j[i]\bigr) \times \mathrm{Hom}\bigl(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^{j'}[i']\bigr) \longrightarrow \mathrm{Hom}\bigl(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j[i] \overset{\mathrm{L}}{\otimes} \Delta_*\omega_X^{j'}[i']\bigr)\,,$$

noting that

$$\Delta_*\omega_X^j[i] \overset{\mathrm{L}}{\otimes} \Delta_*\omega_X^{j'}[i'] \cong \bigl(\Delta_*\omega_X^j \overset{\mathrm{L}}{\otimes} \Delta_*\omega_X^{j'}\bigr)[i+i'] \cong \Delta_*\bigl(\omega_X^{j+j'}\bigr)[i+i']\,.$$

The algebra $\mathrm{HH}(X)$ contains some substructures that are of interest:

(1) the canonical ring: $R(X) := \oplus_{j\geqslant 0} H^0(X,\omega^j) = \oplus_{j\geqslant 0} \mathrm{HH}_{0,j}(X)$;
(2) the Hochschild cohomology: $\mathrm{HH}^*(X) := \oplus_{i\geqslant 0} \mathrm{Ext}^i_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X) = \oplus_{i\geqslant 0} \mathrm{HH}_{i,0}(X)$;
(3) the Hochschild homology: $\mathrm{HH}_*(X) := \oplus_{i\geqslant 0} \mathrm{Ext}^i_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\omega_X) = \oplus_{i\geqslant 0} \mathrm{HH}_{i,1}(X)$, which has the structure of a graded module over $\mathrm{HH}^*(X)$.

Theorem (Orlov). *Let X and Y be smooth projective k-varieties. If $\Phi\colon D(X) \to D(Y)$ is an equivalence then there is an induced isomorphism of bigraded k-algebras*

$$\Phi^{\mathrm{HH}}\colon \mathrm{HH}(X) \to \mathrm{HH}(Y)\,.$$

Corollary. *If X and Y are smooth projective with equivalent derived categories then we have:*

(i) $R(X) \cong R(Y)$*; in particular X and Y have the same Kodaira dimension, and if they both have ample canonical bundle then $X \cong Y$;*

(ii) $\mathrm{HH}^*(X) \cong \mathrm{HH}^*(Y)$*; in particular, if* $\mathrm{char}(k) = 0$ *then this gives that*

$$\sum_{p-q=n} h^{p,q}(X) = \sum_{p-q=n} h^{p,q}(Y)\,,$$

where $h^{p,q} = \dim H^q(X, \Omega^p)$*. (See below for explanation.)*

Sketch of the proof of the theorem. Let $d := \dim(X) = \dim(Y)$.

Step 1: The assumptions imply, by Orlov's theorem in § 2, that there exists an object $P \in D(X \times Y)$ such that $\Phi_{P,X\to Y}\colon D(X) \to D(Y)$ is an equivalence. The left (resp. right) adjoint of Φ_P is the FM-transform associated to $P^\vee \otimes^{\mathrm{L}} \mathrm{pr}_Y^*\omega_Y[d]$ (resp. $P^\vee \otimes^{\mathrm{L}} \mathrm{pr}_X^*\omega_X[d]$), and because these adjoints are isomorphic the corresponding objects are isomorphic. So we can define $Q \in D(X \times Y)$ by

$$Q := P^\vee \overset{\mathrm{L}}{\otimes} \mathrm{pr}_X^*\omega_X[d] \cong P^\vee \overset{\mathrm{L}}{\otimes} \mathrm{pr}_Y^*\omega_Y[d]\,.$$

We are going to look at $\Phi_{Q,X\to Y}$. (Note the direction! We already know that $\Phi_{Q,Y\to X}$ is a quasi-inverse of Φ_P; now we look at the FM-transform in the other direction.)

Step 2: One proves that $\Phi_{Q,X\to Y}\colon D(X) \to D(Y)$ is again an equivalence. If we define an object $Q \boxtimes P$ in $D(X \times X \times Y \times Y)$ by

$$Q \boxtimes P := \mathrm{pr}_{1,3}^*Q \overset{\mathrm{L}}{\otimes} \mathrm{pr}_{2,4}^*P$$

then this gives that $\Phi_{Q\boxtimes P}\colon D(X \times X) \to D(Y \times Y)$ is an equivalence.

Step 3: Choose an integer k and define $R := \Phi_{Q\boxtimes P}\big(\Delta_{X,*}\omega_X^k\big)$. Then one can show that the associated FM-transform $\Phi_R\colon D(Y) \to D(Y)$ is isomorphic to the composition

$$D(Y) \xrightarrow{\Phi_Q} D(X) \xrightarrow{\Phi_{\Delta_{X,*}\omega_X^k}} D(X) \xrightarrow{\Phi_P} D(Y)\,.$$

Here the map in the middle is given by $E \mapsto E \otimes^{\mathrm{L}} \omega_X^k$; so this is just the functor $E \mapsto S_X^k(E)\big[-kd\big]$. But we know that Φ_P commutes with Serre functors and that $\Phi_P \circ \Phi_Q \cong \mathrm{id}_{D(Y)}$. So Φ_R is isomorphic to

$$\Phi_{\Delta_{Y,*}\omega_Y^k}\colon D(Y) \to D(Y)\,,$$

which is given by $E \mapsto E \otimes^{\mathrm{L}} \omega_Y^k$. By uniqueness of the FM-kernel it follows that

$$\Phi_{Q\boxtimes P}\big(\Delta_{X,*}\omega_X^k\big) \cong \Delta_{Y,*}\omega_Y^k$$

for all $k \in \mathbb{Z}$. This gives the desired bijection $\Phi^{\mathrm{HH}}\colon \mathrm{HH}(X) \to \mathrm{HH}(Y)$, compatible with bigradings.

Step 4: It remains to be checked that this map Φ^{HH} respects the ring structures. For this the idea is that the ring structure is in fact given by compositions in the derived category (using Serre functors). Explicitely, if

$$s \in \mathrm{Ext}^i_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j) = \mathrm{Hom}_{D(X\times X)}\big(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^j[i]\big)$$

and

$$t \in \mathrm{Ext}^k_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^l) = \mathrm{Hom}_{D(X\times X)}\big(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^l[k]\big)$$

then $S^j_{X\times X}[i-dj](t)$ is a morphism from

$$S^j_{X\times X}[i-dj]\big(\Delta_*\mathscr{O}_X\big) = \Delta_* S^j_X[i-dj]\big(\mathscr{O}_X\big) = \Delta_*\omega^j_X[i]$$

to

$$S^j_{X\times X}[i-dj]\big(\Delta_*\omega^l_X[k]\big) = \Delta_* S^j_X[i-dj]\big(\omega^l_X[k]\big) = \Delta_*\omega^{j+l}_X[i+k]$$

so we can compose it with s. The relation with the ring multiplication in $\mathrm{HH}(X)$ is that

$$s\cdot t = S^j_{X\times X}[i-dj]\big(t\big)\circ s\colon \Delta_*\mathscr{O}_X \longrightarrow \Delta_*\omega^{j+l}_X[i+k]\,.$$

3.3 *More on Hochschild cohomology.* First note that Hochschild (co)homology is not functorial in an obvious way. One can get further insight in $\mathrm{HH}^*(X)$ and $\mathrm{HH}_*(X)$ by using that the spectral sequences

$$E_2^{p,q} = H^p\big(X\times X, \mathit{Ext}^q_{X\times X}(\Delta_*\mathscr{O}_X,\Delta_*\mathscr{O}_X)\big) \quad\Rightarrow\quad \mathrm{Ext}^{p+q}_{X\times X}(\Delta_*\mathscr{O}_X,\Delta_*\mathscr{O}_X) = \mathrm{HH}^{p+q}(X)$$

and

$$E_2^{p,q} = H^p\big(X\times X, \mathit{Ext}^q_{X\times X}(\Delta_*\mathscr{O}_X,\Delta_*\omega_X)\big) \quad\Rightarrow\quad \mathrm{Ext}^{p+q}_{X\times X}(\Delta_*\mathscr{O}_X,\Delta_*\omega_X) = \mathrm{HH}_{p+q}(X)$$

both degenerate at E_2. Further it can be shown that

$$\mathit{Ext}^q(\Delta_*\mathscr{O}_X,\Delta_*\mathscr{O}_X) \cong \Delta_*\big(\wedge^q\mathscr{N}\big) = \Delta_*\big(\wedge^q\mathscr{T}_X\big)$$

with $\mathscr{N}$ the normal bundle of the diagonal, and

$$\mathit{Ext}^q(\Delta_*\mathscr{O}_X,\Delta_*\omega_X) \cong \Delta_*\big(\wedge^q\mathscr{T}_X\otimes\omega_X\big)\,.$$

So we get

$$\mathrm{HH}^n(X) \cong \oplus_{p+q=n} H^p\big(X,\wedge^q\mathscr{T}_X\big)$$

and

$$\mathrm{HH}_n(X) \cong \oplus_{p+q=n} H^p\big(X,\wedge^q\mathscr{T}_X\otimes\omega_X\big) \cong \oplus_{p+q=n} H^{d-p}\big(X,\Omega^q_{X/k}\big)^\vee \cong \oplus_{p+q=n} H^p\big(X,\Omega^{d-q}_{X/k}\big)\,.$$

Hence the dimension of HH_n is the sum of the Hodge numbers $h^{p,q}$ over the nth *column* of the Hodge diamond. (The Betti numbers are the sums over the rows.)

§ 4. Varieties with ample or anti-ample canonical bundle.

Theorem (Bondal-Orlov). *Let X and Y be smooth projective varieties over k, and assume that either ω_X or ω_X^{-1} is ample. If $D(X)$ and $D(Y)$ are equivalent then $X\cong Y$.*

Remarks. (i) By the preceding results we are done is we can show that ω_Y (resp. ω_Y^{-1}) is ample. But how to prove this...?

(ii) The condition on ω_X cannot be omitted. For example, in general an abelian variety X is not isomorphic to its dual X^t, but we do have an equivalence between $D(X)$ and $D(X^t)$.

Sketch of the proof. For simplicity we assume that $k = \overline{k}$, and we only do the case that ω_X is ample. By Orlov's theorem we know there exists an object $P \in D(Y \times X)$ such that $\Phi = \Phi_{P,Y\to X}\colon D(Y) \to D(X)$ is an equivalence.

Notation: if $x \in X(k)$ then we simply write $k(x)$ for the skyscraper sheaf $x_*(k)$, where $x\colon \mathrm{Spec}(k) \to X$ is the point x.

Step 1: The idea is that we try to characterize the objects of the form $k(x)\big[m\big]$ purely in terms of the derived category. Namely: an object $E \in D(X)$ is said to be *point-like* if it satisfies the following conditions:

(i) $E \cong S_X(E)\big[i\big]$ for some $i \in \mathbb{Z}$;
(ii) $\mathrm{Ext}^j(E, E) = \mathrm{Hom}_{D(X)}\big(E, E[j]\big) = 0$ for all $j < 0$;
(iii) $\mathrm{Hom}(E, E) = k$.

The following facts are then easy to prove:

— all objects $k(x)\big[m\big]$ are point-like;
— if $F\colon D(Y) \to D(X)$ is an equivalence then it sends point-like objects to point-like objects;
— if ω_X or ω_X^{-1} is ample then the $k(x)\big[m\big]$ are the only point-like objects.

(But note that there are in general many more point-likes if ω_X is trivial.)

The conclusion is that our equivalence $\Phi\colon D(Y) \to D(X)$ sends any object $k(y)$ to an object of the form $k(x)\big[m\big]$ for some $x = x(y)$. On the other hand, by definition we have

$$\Phi\big(k(y)\big) = R\mathrm{pr}_{X,*}\big(\mathrm{pr}_Y^*(k(y)) \overset{\mathrm{L}}{\otimes} P\big)\,,$$

and because $\mathrm{pr}_Y^*(k(y)) \otimes^{\mathrm{L}} P$ is an object with support in $\{y\} \times X$, it follows that for any $y \in Y(k)$ we have:

$$\mathrm{pr}_Y^*(k(y)) \overset{\mathrm{L}}{\otimes} P \cong k(z)\big[m\big] \qquad \text{for some } z = (y, x) \in Y \times X \text{ and some } m \in \mathbb{Z}.$$

Step 2, Lemma: Let $p\colon Z \to Y$ be a flat morphism of k-varieties. Let $P \in D(Z)$, and suppose that for every $y \in Y(k)$ there exists a point $z = z(y) \in Z(k)$ and an integer $m = m(y)$ such that $p^*\big(k(y)\big) \otimes^{\mathrm{L}} P \cong k(z)\big[m\big]$. Then there is a section $\sigma\colon Y \to Z$ of p, a line bundle $\mathscr{L}$ on Y and an integer m such that $P \cong R\sigma_*\mathscr{L}[m]$.

Step 3: Applying this to the projection $\mathrm{pr}_Y\colon Y \times X \to Y$ we find that there is a morphism $\gamma\colon Y \to X$, a line bundle $\mathscr{L}$ on Y, and an integer m such that $P \cong R(\mathrm{id}, \gamma)_*\mathscr{L}[m]$. But then $\Phi = \Phi_{P,Y\to X}$ is the functor given by $E \mapsto R\gamma_*(E \otimes \mathscr{L})\big[m\big]$. This is an equivalence if and only if $R\gamma_*$ is an equivalence, and this is the case if and only if $\gamma\colon Y \to X$ is an isomorphism. □

The argument sketched here (which borrows from Hille-van den Bergh) also proves another result. Before we state it, let us remark that for any variety X there are a couple of "obvious" auto-equivalences of $D(X)$, namely:

— the shifts $E \mapsto E[m]$;
— taking tensor product with a line bundle;
— automorphisms of X itself.

If $\omega_X^{\pm}$ is ample, these are all:

Theorem (Bondal-Orlov). *Let X be a smooth projective k-variety such that ω_X or ω_X^{-1} is ample. Then*

$$\mathrm{Aut}\big(D(X)\big) \cong \mathbb{Z} \times \big(\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)\big)\,.$$